Emergent dynamical symmetry at the triple point of nuclear deformations

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Based on the boson realization of the Euclidean algebras, it is shown that the five-dimensional Euclidean dynamical symmetry may emerge at the triple point of the shape phase diagram of the interacting boson model, which thus offers a symmetry-based understanding of this isolated point. It is further shown that the low-lying dynamics in ^{108}Pd, ^{134}Ba, ^{64}Zn, and ^{114}Cd may be dominated by the Euclidean dynamical symmetry.

I. INTRODUCTION

Dynamical symmetries (DSs) provide considerable insight into the nature of quantum many-body dynamical structures. Generally, DS occurs when the Hamiltonian of a system can be written in terms of Casimir operators of a chain of Lie algebras $G \supset G' \supset G'' \cdots$. Typical examples of DS are those associated with the interacting boson model (IBM) [1] for nuclear structure and the vibron model (VM) [2] for molecules and atomic clusters [3].

The IBM possesses an overall U(6) symmetry with three DSs corresponding to three typical collective structures or quadrupole deformations [2]; namely, a spherical vibrator [U(5)], an axially symmetric prolate rotor [SU(3)], and a γ-soft rotor [O(6)]. In addition, an axially symmetric oblate rotor [$\overline{\text{SU(3)}}$] can be involved in the IBM dynamics if adopting an alternative SU(3) quandrupole operator, in contrast to that often used for the prolate rotor [4]. Besides these exact DSs, the partial dynamical symmetries (PDSs) [5–7] and quasidynamical symmetries (QDSs) [8–10] have also been found to occur in the IBM. Indeed, it was found that the SU(3) QDS [11,12] may emerge along the trajectory in the IBM parameter space close to the Alhassid-Whelan arc of regularity [13], which has been empirically confirmed [14]. A link between PDS and QDS has also been established recently via the method of quantum number fluctuation [15]. As there is a link between each DS in the IBM and the quantum (shape) phase or quadrupole deformation [1], the (shape) phase transitions in nuclei may be characterized as the quantum phase transitions (QPTs) in between the different DSs in the IBM [16–18]. Particularly, the QPT from SU(3) to $\overline{\text{SU(3)}}$ may exactly occur at the point of O(6) DS [4]. In addition, an isolate triple point, at which three kinds of quadrupole deformations including the spherical, prolate, and oblate shapes may coexist at the same time [19,20], also emerges in the critical region of the IBM. On the other hand, an algebraic model of the Euclidean dynamical symmetry in five dimensions [Eu(5) DS] [21] has recently been suggested to describe the nuclei in the critical region. Especially, it was shown [21] that the results obtained from the simplest version of the Eu(5) DS (unprojected) in the large-N limit are the same as those from

the E(5) critical point symmetry [E(5) CPS] built from the Bohr Hamiltonian with an infinite well potential [22]. However, the relation between the IBM and the Eu(5) DS still remains to be revealed. In this work, we will present an extensive analysis of the Eu(5) DS and clarify the relation between the Eu(5) DS and the IBM.

II. THE BOSON REALIZATION OF THE EUCLIDEAN ALGEBRA

A Hamiltonian in the IBM framework is constructed from two kinds of boson operators; namely, an s boson with $J^\pi = 0^+$ and a d boson with $J^\pi = 2^+$ [1]. The three DSs in the IBM are characterized by three different chains of the U(6) group [1]:

$$\begin{aligned} &\text{U(6)} \supset \text{U(5)} \supset \text{SO(5)} \supset \text{SO(3)}, \\ &\text{U(6)} \supset \text{O(6)} \supset \text{SO(5)} \supset \text{SO(3)}, \\ &\text{U(6)} \supset \text{SU(3)} \supset \text{SO(3)}. \end{aligned} \quad (1)$$

Then, the Hamiltonian with an explicit DS in the IBM can be written in terms of the Casimir operators of the corresponding group chain. On the other hand, it was shown [21] that one can use the d-boson operator to construct the Casimir operator of the Eu(5) group as

$$\hat{C}_2[\text{Eu(5)}] = \hat{n}_d + \tfrac{5}{2} - \tfrac{1}{2}(\hat{P}_d^\dagger + \hat{P}_d), \quad (2)$$

where $\hat{n}_d = \sum_u d_u^\dagger d_u$, and $\hat{P}_d = \sum_u (-)^u d_u d_{-u}$. Accordingly, the d-boson operator can be also used to construct the fifteen generators of the Eu(5) Lie algebra as

$$\begin{aligned} \hat{Q}_u^{(2)} &= \frac{1}{\sqrt{2}}[\tilde{d}_u - d_u^\dagger], \\ \hat{T}_u^{(\lambda)} &= \sqrt{2}(d^\dagger \tilde{d})_u^{(\lambda)}, \quad \lambda = 1,3, \end{aligned} \quad (3)$$

where $\tilde{d}_u = (-1)^u d_u$. It is evident that the Eu(5) algebra is noncompact as seen from (3). It should be mentioned that a geometric realization of the Eu(5) algebra in the collective model may be constructed by using the quadrupole coordinates q_u and the conjugate momenta $\tilde{p}_u$ [23]. The boson algebraic

and the collective geometric realizations may be linked by implementing the d-boson operator with $\tilde{d}_u = \frac{1}{\sqrt{2}}[q_u + i\,\tilde{p}_u]$ and $d_u^\dagger = \frac{1}{\sqrt{2}}[q_u - i\,\tilde{p}_u]$ [21].

It can be proven that the above Eu(5) generators satisfy the commutation relations

$$\left[\hat{Q}_u^{(2)}, \hat{Q}_v^{(2)}\right] = 0,$$

$$\left[\hat{T}_u^{(\lambda)}, \hat{Q}_v^{(2)}\right] = -\sqrt{\frac{4\lambda+2}{5}}\langle\lambda u 2 v|2u+v\rangle \hat{Q}_{u+v}^{(2)},$$

$$\left[\hat{T}_u^{(\lambda)}, \hat{T}_{u'}^{(\lambda')}\right] = -\sqrt{8(2\lambda+1)(2\lambda'+1)}\sum_{k=\text{odd}}\begin{Bmatrix}\lambda,\lambda',k\\2,2,2\end{Bmatrix}$$
$$\times \langle\lambda u \lambda' u'|ku+u'\rangle \hat{T}_{u+u'}^{(k)}, \quad (4)$$

and

$$\left[\hat{Q}_u^{(2)}, \hat{C}_2[\text{Eu}(5)]\right] = \left[\hat{T}_u^{(\lambda)}, \hat{C}_2[\text{Eu}(5)]\right] = 0. \quad (5)$$

One can further prove that the operators $\{\hat{T}_u^{(\lambda)}\}$ with $\lambda = 1,3$ generate the SO(5) algebra, in which the angular momentum operators defined by $\{\hat{L}_u = \sqrt{5}\hat{T}_u^{(1)}\}$ generate the SO(3) algebra. The Eu(5) algebra may be characterized by the algebraic chain [23]

$$\text{Eu}(5) \supset \text{SO}(5) \supset \text{SO}(3). \quad (6)$$

In addition, $\{\hat{Q}_u^{(2)}\}$ generate the Abelian group T_5 of translations in the five-dimensional space. It is thus realized that the Eu(5) algebra is equivalent to the semidirect sum of T_5 and SO(5), namely, $\text{Eu}(5) = \text{T}_5 \oplus_s \text{SO}(5)$ [24–26]. Accordingly, there is another dynamical symmetry related with the algebraic chain

$$\text{Eu}(5) \supset \text{T}_5 \oplus_s \text{SO}(3) \supset \text{SO}(3), \quad (7)$$

in which the semidirect sum $\text{T}_5 \oplus_s \text{SO}(3)$ is often used to denote the dynamical symmetry of a quadrupole-deformed rigid body [27]. In the following, only the dynamical situation characterized by the algebraic chain (6) will be studied.

III. THE SU(1,1) EXPRESSION OF THE EU(5) DS AND ITS LINK WITH THE E(5) CPS

As analyzed in [21], the spectral structure of (2) in the large-N limit coincides with that generated from the E(5) CPS [22]. Actually, the form of the Eu(5) Casimir operator shown in (2) can be directly translated from the Hamiltonian of the E(5) CPS via the SU(1,1) reformulation. Specifically, the SU(1,1) Lie algebra generated by $\hat{S}_\nu$, $\nu = 0,\pm$, satisfies the commutation relations [28–31]

$$[\hat{S}_-, \hat{S}_+] = 2\hat{S}_0, \quad [\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm. \quad (8)$$

The Casimir operator of SU(1,1) can be written as

$$\hat{C}_2[\text{SU}(1,1)] = \hat{S}_0(\hat{S}_0 - 1) - \hat{S}_+\hat{S}_-. \quad (9)$$

Let $|\lambda,k\rangle$, $k = 0,1,2,\ldots$ be the basis vectors of the irreducible representation λ of SU(1,1), of which the matrix representation

is determined by [28–31]

$$\hat{S}_0|\lambda,k\rangle = \frac{1}{2}(\lambda+2k)|\lambda,k\rangle, \quad (10)$$

$$\hat{S}_+|\lambda,k\rangle = \sqrt{(\lambda+k)(k+1)}|\lambda,k+1\rangle, \quad (11)$$

$$\hat{S}_-|\lambda,k\rangle = \sqrt{(\lambda+k-1)k}|\lambda,k-1\rangle, \quad (12)$$

$$\hat{C}_2[\text{SU}(1,1)]|\lambda,k\rangle = \frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)|\lambda,k\rangle. \quad (13)$$

For the d-boson realization of the SU(1,1) [32],

$$\hat{S}_+^d = \tfrac{1}{2}\hat{P}_d^+, \quad \hat{S}_-^d = \tfrac{1}{2}\hat{P}_d, \quad \hat{S}_0^d = \tfrac{1}{2}(\hat{n}_d + \tfrac{5}{2}), \quad (14)$$

from which the quantum numbers λ and k can be expressed by the seniority quantum number τ of the SO(5) and the number of the d bosons n_d as $\lambda = \tau + \frac{5}{2}$ and $k = \frac{1}{2}(n_d - \tau)$. Then, the Casimir operator of Eu(5) defined in (2) can be expressed as

$$\hat{C}_2[\text{Eu}(5)] = 2\hat{S}_0^d - (\hat{S}_+^d + \hat{S}_-^d). \quad (15)$$

Thus, one may choose to solve the eigenvalue problem of $\hat{C}_2[\text{Eu}(5)]$ within the subspace spanned by the SU(1,1) basis vectors $\{|\lambda,k\rangle\}$. In addition, the Eu(5) with the Casimir operator given in (15) can easily be extended to the n-dimensional case. Specifically, the Casimir operator of the n-dimensional Euclidean group [Eu(n)] with $n = 2l+1$ may be written as

$$\hat{C}_2[\text{Eu}(n)] = 2\hat{S}_0^l - (\hat{S}_+^l + \hat{S}_-^l), \quad (16)$$

of which the SU(1,1) algebra is generated by [33]

$$\hat{S}_+^l = \frac{1}{2}\hat{P}_l^\dagger, \quad \hat{S}_- = \frac{1}{2}\hat{P}_l$$
$$\hat{S}_0^l = \frac{1}{2}\left(\hat{n}_l + \frac{2l+1}{2}\right), \quad (17)$$

where $\hat{n}_l = \sum_m l_m^\dagger l_m$, and $\hat{P}_l = \sum_m (-1)^m l_m l_{-m}$, in which $l_m^\dagger$ (l_m) are the creation (annihilation) operators of the l bosons. It is obvious that the SU(1,1) algebra given in (14) is only a special case of (17) with $l = 2$. Accordingly, one can also construct the Eu(n) algebra with the l-boson operators as

$$\hat{Q}_u^{(l)} = \frac{1}{\sqrt{2}}[\tilde{l}_u - l_u^\dagger],$$
$$\hat{T}_u^{(\lambda)} = \sqrt{2}(l^\dagger\tilde{l})_u^{(\lambda)}, \quad \lambda = 1,3,\ldots,2l-1, \quad (18)$$

of which the $l = 2$ case just corresponds to the Eu(5) algebra given in (3).

On the other hand, the collective Bohr Hamiltonian of the E(5) CPS [22] is written as

$$H_{E(5)} = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\left(\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right.\right.$$
$$\left.\left. -\frac{1}{4}\sum_k \frac{L_k'^2}{\left[\sin\left(\gamma - \frac{2}{3}n\pi\right)\right]^2}\right)\right\} + V(\beta) \quad (19)$$

with

$$V(\beta) = \begin{cases} 0, & \beta \leqslant \beta_W, \\ \infty, & \beta > \beta_W. \end{cases} \quad (20)$$

By writing the eigenfuctions

$$\Phi(\beta,\gamma,\theta) = f(\beta)\Psi(\gamma,\theta), \tag{21}$$

one can get the angular equation

$$\left[-\frac{1}{\sin 3\gamma} \frac{\partial}{\partial\gamma} \sin 3\gamma \frac{\partial}{\partial\gamma} + \frac{1}{4} \sum_k \frac{L_k'^2}{\left[\sin\left(\gamma - \frac{2k\pi}{3}\right)\right]^2} \right] \Psi(\gamma,\theta)$$

$$= \Lambda \Psi(\gamma,\theta), \tag{22}$$

with $\Lambda = \tau(\tau+3)$ and the radial equation

$$\left[-\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Lambda}{\beta^2}\right) + V(\beta) \right] f(\beta) = Ef(\beta). \tag{23}$$

According to the analysis shown in [28–31], the SU(1,1) algebra can be alternatively defined in terms of the differential operators as

$$\hat{S}_\pm^\beta = \frac{1}{4}\left[\frac{\triangle^2}{a^2} - \frac{\Lambda}{(a\beta)^2} + (a\beta)^2 \mp \left(2\beta\frac{\partial}{\partial\beta} + 5\right)\right], \tag{24}$$

$$\hat{S}_0^\beta = \frac{1}{4}\left[-\frac{\triangle^2}{a^2} + \frac{\Lambda}{(a\beta)^2} + (a\beta)^2\right], \tag{25}$$

with $\triangle^2 = \frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}$ and $\frac{1}{a^2} = \frac{\hbar^2}{2B}$. Then, the Hamiltonian associated with (23) in the infinite well can be written as

$$H = 2\hat{S}_0^\beta - (\hat{S}_+^\beta + \hat{S}_-^\beta), \tag{26}$$

which is the same form as that shown in (15). This indicates that the Eu(5) DS can be directly translated from the E(5) CPS at the Hamiltonian level. Thus, solving the differential equation (23) is approximately equivalent to diagonalizing the Hamiltonian (26) within the subspace spanned by the basis vectors of SU(1,1) [28–31]. Though it is not easy to translate the boundary conditions of the infinite well into those in the algebraic description, it can approximately be realized in the diagonalization of (15) with a cutoff in the Hilbert space [21].

IV. THE APPROXIMATE EU(5) DS AT THE TRIPLE POINT IN THE IBM

Because the main composition of the system is the same as that of the IBM with d bosons at the second quantization level, to investigate the relation between the IBM and the Eu(5) DS at the Hamiltonian level, we consider the IBM consistent-Q Hamiltonian [34]

$$\hat{H}(\eta,\chi) = \varepsilon\left[(1-\eta)\hat{n}_d - \frac{\eta}{4N}\hat{Q}^\chi \cdot \hat{Q}^\chi\right], \tag{27}$$

where $\hat{Q}^\chi = (d^\dagger s + s^\dagger \tilde{d})^{(2)} + \chi(d^\dagger\tilde{d})^{(2)}$ is the quadrupole operator, η and χ are the control parameters with $\eta \in [0,1]$ and $\chi \in [-\sqrt{7}/2, \sqrt{7}/2]$, and ε is a scale factor. It can be proven that the Hamiltonian is in the U(5) DS when $\eta = 0$; it is in the O(6) DS when $\eta = 1$ and $\chi = 0$; it is in the SU(3) DS when $\eta = 1$ and $\chi = -\frac{\sqrt{7}}{2}$; and it is in the $\overline{SU(3)}$ DS when $\eta = 1$ and $\chi = \frac{\sqrt{7}}{2}$. The two-dimensional parameter space of (27) can be mapped onto a symmetric triangle (see Fig. 1), called the extended Casten triangle [4]. To identify the QPTs
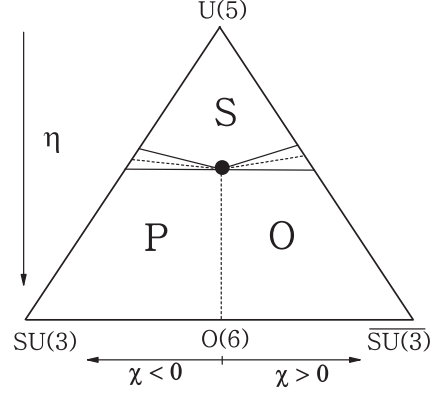

FIG. 1. Shape phase diagram in the IBM parameter space, where S represents the region with $\beta_e = 0$ corresponding to the spherical, P represents the region with $\beta_e > 0$ corresponding to the prolate, and O represents the region with $\beta_e < 0$ corresponding to the oblate. In addition, the dashed lines correspond to the critical points of the first-order QPTs, the two thin regions involving the dashed lines represent the two-phase coexisting regions [36], and the solid dot in the center represents the triple point.

in the IBM, one may use the coherent state defined as [1]

$$|\beta,\gamma,N\rangle = \frac{1}{\sqrt{N!(1+\beta^2)^N}}\left[s^\dagger + \beta\cos\gamma d_0^\dagger \right.$$

$$\left. + \frac{1}{\sqrt{2}}\beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger)\right]^N |0\rangle \tag{28}$$

to obtain the scaled potential surface corresponding to the Hamiltonian (27) in the large-N limit, which is given as

$$V_s(\beta,\gamma) = \frac{1}{\varepsilon N}\langle\beta,\gamma,N|H|\beta,\gamma,N\rangle|_{N\to\infty}$$

$$= (1-\eta)\frac{\beta^2}{1+\beta^2} - \frac{\eta}{4(1+\beta^2)^2}$$

$$\times \left[4\beta^2 - 4\sqrt{\frac{2}{7}}\chi\beta^3\cos 3\gamma + \frac{2}{7}\chi^2\beta^4\right]. \tag{29}$$

To illustrate the type and the order of the QPTs, one should minimize the potential function (29) by varying β and γ for η and χ. The optimal values are denoted as β_e and γ_e, from which one can get the ground state energy per boson defined as $E_g = V_s(\eta,\chi,\beta_e,\gamma_e)$. It can be found that the γ dependence in (29) yields either $\gamma_e = 0°$ or $\gamma_e = 60°$, of which the case with $\gamma_e = 60°$ can be equivalently described by substituting $\gamma_e = 0°$ and $\beta = -\beta_e$. Therefore, β_e may serve as the order parameter [35] to identify the order and the type of the QPTs. For the second-order QPT, the order parameter β_e changes continuously, but with a discontinuous in the second derivative of (29). In contrast, the first-order QPT may involve a discontinuous jump in the order parameter β_e itself [35]. In addition, $\beta_e = 0$, $\beta_e > 0$, and $\beta_e < 0$ represent the spherical, prolate, and oblate deformations, respectively. Based on the criteria mentioned above, one can prove that the system may experience the first-order QPTs in two directions with changing of the control parameters η and χ [19]. Specifically,

the critical points of the first-order QPTs occurring in the η direction are given as $\eta_c = \frac{14}{28+\chi^2}$ and $\chi \in [-\sqrt{7}/2, \sqrt{7}/2]$, and in the χ direction are characterized by $\chi_c = 0$ with $\eta \in (0.5, 1]$. Particularly, the crossing point of the first-order QPTs occurring in the two directions, namely ($\eta = 0.5, \chi = 0$), may be recognized as the single triple point as it is the junction point of the spherical, prolate, and oblate deformations [19]. Meanwhile, this point is also proven to be the critical point of the second-order phase transition in the η direction. The whole shape phase diagram corresponding to (29) is clearly shown in Fig. 1.

As is known, if a system has an underlying symmetry of the group G, the corresponding Hamiltonian should commutes with the generators of the group G. Along this line, it has been proven [12] that there exists a parametrization trajectory preserving the approximate SU(3) symmetry [SU(3) QDS] inside the symmetry triangle of the IBM in the the large-N limit. To identify the underlying Eu(5) DS in the IBM parameter space, we examine the commutation relations between the generators of the Eu(5) defined in (3) and the IBM Hamiltonian $\hat{H}(\eta, \chi)$ given in (27). Firstly, it is easy to know that the IBM Hamiltonian does commute with the angular momentum operators $\hat{L}_u = \sqrt{5}\hat{T}_u^{(1)}$ since the Hamiltonian is a scalar. As a result, one only needs to examine the conditions under which the Hamiltonian may commute (approximately) with the other generators of the Eu(5). Specifically, one can prove the following commutation relations by using the standard angular momentum coupling techniques [12,37]:

$$[\hat{T}_u^{(3)}, \hat{n}_d] = 0, \tag{30}$$

$$[\hat{T}_u^{(3)}, (d^\dagger s + s^\dagger \tilde{d})_v^{(2)}] = -\frac{\sqrt{14}}{5}\langle 3u2v|2u+v\rangle$$
$$\times (d^\dagger s + s^\dagger \tilde{d})_{u+v}^{(2)}, \tag{31}$$

$$[\hat{T}_u^{(3)}, (d^\dagger \tilde{d})_v^{(2)}] = 2\sqrt{70}\sum_{k=2,4}\langle 3u2v|ku+v\rangle$$
$$\times \begin{Bmatrix} 2 & 3 & k \\ 2 & 2 & 2 \end{Bmatrix}(d^\dagger\tilde{d})_{u+v}^{(k)}, \tag{32}$$

$$[\hat{Q}_u^{(2)}, \hat{n}_d] = \frac{\sqrt{2}}{2}(\tilde{d} + d^\dagger)_u^{(2)}, \tag{33}$$

$$[\hat{Q}_u^{(2)}, (d^\dagger s + s^\dagger \tilde{d})_v^{(2)}] = (-)^u \delta_{u,-v}(s + s^\dagger), \tag{34}$$

$$[\hat{Q}_u^{(2)}, (d^\dagger \tilde{d})_v^{(2)}] = \frac{\sqrt{2}}{2}\langle 2u2v|2u+v\rangle(\tilde{d} + d^\dagger)_{u+v}^{(2)}. \tag{35}$$

By using the above relations, one can derive

$$[\hat{T}_q^{(3)}, \hat{H}(\eta, \chi)]$$
$$= \frac{3\sqrt{5}\varepsilon\eta\chi}{28N}\{\sqrt{10}[(\hat{B}^{(2)}\hat{A}^{(2)})_q^{(3)} - (\hat{A}^{(2)}\hat{B}^{(2)})_q^{(3)}]$$
$$-2[(\hat{B}^{(4)}\hat{A}^{(2)})_q^{(3)} - (\hat{A}^{(2)}\hat{B}^{(4)})_q^{(3)}] - 2\chi[(\hat{B}^{(4)}\hat{B}^{(2)})_q^{(3)}$$
$$-(\hat{B}^{(2)}\hat{B}^{(4)})_q^{(3)}]\}, \tag{36}$$

where $\hat{A}_q^{(2)} = (s^\dagger\tilde{d} + d^\dagger s)_q^{(2)}$ and $\hat{B}_q^{(k)} = (d^\dagger\tilde{d})_q^{(k)}$ with $k = 2, 4$. Furthermore, by implementing the matrix elements related to the s-boson operators under the U(6) ⊃ U(5) ⊃ SO(5) ⊃

SO(3) basis vectors $\{|Nn_d\tau\Delta L\rangle\}$, where N, n_d, τ, and L are the quantum numbers of U(6), U(5), SO(5), and SO(3), respectively, and Δ is the additional quantum number to characterize the multiplicity of L in τ, one gets the replacements $s^\dagger \rightarrow \sqrt{\hat{n}_s + 1}$ and $s \rightarrow \sqrt{\hat{n}_s}$ with $\hat{n}_s = N - \hat{n}_d$. Then, in the $n_d/N \ll 1$ limit, one can derive that

$$[\hat{Q}_q^{(2)}, \hat{H}(\eta, \chi)] = \frac{\sqrt{2}\varepsilon}{2}(1 - 2\eta)\hat{C}_q^{(2)}$$
$$-\frac{\sqrt{2}\varepsilon\eta\chi}{8N}[(\hat{A}^{(2)}\hat{C}^{(2)})_q^{(2)} + (\hat{C}^{(2)}\hat{A}^{(2)})_q^{(2)}$$
$$+2\hat{B}_q^{(2)} + \chi(\hat{C}^{(2)}\hat{B}^{(2)})_q^{(2)} + \chi(\hat{B}^{(2)}\hat{C}^{(2)})_q^{(2)}], \tag{37}$$

where $\hat{C}_q^{(2)} = (\tilde{d} + d^\dagger)_q^{(2)}$. In order to make the commutators given in (36) and (37) vanish at the same time, it is uniquely required that ($\eta = 0.5, \chi = 0$), under which the IBM Hamiltonian just locates at the single triple point mentioned above. The result clearly shows that the Hamiltonian at the triple point is approximately invariant under the Eu(5) transformations in the $n_d/N \ll 1$ limit. It should be noted that the approximation condition $n_d/N \ll 1$ is well satisfied for low-lying states generated from the IBM Hamiltonian (27) with $\eta \in [0, 0.5]$ and $\chi = 0$ in large-N cases [35,38,39]. In fact, if implementing the matrix elements related to s-boson operators under the basis vectors $\{|Nn_d\tau\Delta L\rangle\}$, one can write the Hamiltonian (27) at the triple point ($\eta = 0.5, \chi = 0$) as

$$\hat{H}_{\text{tri}} = \frac{\varepsilon}{8}\Big\{4\hat{n}_d - \frac{1}{N}[\hat{n}_d(N - \hat{n}_d + 1)$$
$$+ (N - \hat{n}_d)(\hat{n}_d + 5) + d^\dagger \cdot d^\dagger\sqrt{(N - \hat{n}_d)(N - \hat{n}_d - 1)}$$
$$+ \sqrt{(N - \hat{n}_d + 1)(N - \hat{n}_d + 2)}\tilde{d} \cdot \tilde{d}]\Big\}. \tag{38}$$

In the $n_d/N \ll 1$ limit, Eq. (38) can be further approximated as [8]

$$\hat{H}_{\text{tri}} \simeq \frac{\varepsilon}{4}\Big[\hat{n}_d - \frac{5}{2} - \frac{1}{2}(P_d^\dagger + P_d)\Big]$$
$$= \frac{\varepsilon}{4}[\hat{C}_2[\text{Eu}(5)] - 5], \tag{39}$$

which is explicitly given as the Casimir operator of the Eu(5) up to a constant and a scale factor. It is thus confirmed that the Eu(5) DS indeed occurs in the $n_d/N \ll 1$ limit at the triple point. Rigorously speaking, the Eu(5) DS occurring at the triple point is only almost exact for the ground state in the large-N limit, but is approximate for the excited states because the condition $n_d/N \ll 1$ for the excited states at the triple point becomes weaker with the increasing of the excitation energies. In addition, it should be emphasized that the Eu(5) DS is the (approximate) DS associated with a β-soft potential since the scale potential surface deduced from (29) at the triple point is soft in β, in contrast to other DSs in the IBM, of which the potential surfaces are all relatively rigid in β.

V. POSSIBLE EU(5) CANDIDATES

Based on the concept of dynamical symmetry, the Hamiltonian with the Eu(5) DS can generally be written as

$$\hat{H}_{\text{Eu(5)}} = a\hat{C}_2[\text{Eu(5)}] + b\hat{C}_2[\text{SO(5)}] + c\hat{C}_2[\text{SO(3)}], \quad (40)$$

where a, b, and c are adjustable parameters, and $\hat{C}_2[\text{SO(5)}]$ and $\hat{C}_2[\text{SO(3)}]$ are the Casimir operators of SO(5) and SO(3) defined as

$$\hat{C}_2[\text{SO(5)}] = \hat{T}^3 \cdot \hat{T}^3 + \hat{T}^1 \cdot \hat{T}^1, \quad (41)$$

$$\hat{C}_2[\text{SO(3)}] = 5\hat{T}^1 \cdot \hat{T}^1, \quad (42)$$

with $\hat{T}^1$ and $\hat{T}^3$ being those given in (3). One can construct eigenstates of (40) from the SO(5) basis vectors $\{|\tau \Delta L\rangle \equiv |n_d = \tau, \tau \Delta L\rangle\}$, which is well defined in the IBM, since SO(5) is the subalgebra of the Eu(5) as shown in (6). Specifically, the eigenstates of the Hamiltonian in the d-boson system with the seniority τ and angular momentum L being good quantum numbers can be expressed as

$$|\xi \tau \Delta L\rangle = \sum_{k=0}^{m} C_k^\xi (\hat{P}_d^\dagger)^k |\tau \Delta L\rangle, \quad (43)$$

where $m + 1$ represents the dimension of the Hilbert subspace, and C_k^ξ is the expansion coefficient with ξ being the additional quantum number to distinguish the states with the same τ, Δ, and L. The expansion coefficients $\{C_k^\xi\}$ are determined by the eigenequation

$$\hat{H}_{\text{Eu(5)}}|\xi \tau \Delta L\rangle = E_{\tau L}^\xi |\xi \tau \Delta L\rangle. \quad (44)$$

Concretely, one can diagonalize the Hamiltonian (40) in the $m + 1$ dimensional Hilbert subspace to get the eigenvalues $E_{\tau L}^\xi$ and the corresponding expansion coefficients $\{C_k^\xi\}$. Generally, for the IBM Hamiltonian such as that of the O(6) DS, $m = [\frac{N-\tau}{2}]$ is required, where $[x]$ is the integer part of x [1], and the s-boson part $|n_s\rangle$ should also be involved in (43) for the IBM with $n_s = N - 2k - \tau$. In the present case, the s-boson part is irrelevant, while m should, in principle, be taken as infinite

since the dimension of the Hilbert subspace is infinite due to the noncompactness of the Eu(5) algebra [24]. However, the analysis [21] shows that the dynamical structure of the Eu(5) DS may be well kept in the finite-m cases, which indicates that one can diagonalize the Eu(5) Hamiltonian within a finite subspace with sufficient large-m truncation. In our calculation, the parameter a in (40) is reset as $a = \alpha m$ for convenience since the energy levels generated by $\hat{C}_2[\text{Eu(5)}]$ may scale with m^{-1} as shown in [21]. In contrast, the energy levels generated by H_{tri} shown in (38) may scale with $m^{-1/3}$ [40]. Besides the diagonalization scheme, it should be mentioned that the eigenstates corresponding to a DS in the IBM may be built through the so called spectrum algebra method. For example, the eigenstates in the O(6) DS can be analytically constructed by acting the generalized boson pairing operators on the O(6) basis vector [1]. Similarly, as shown in (43), eigenstates of the Eu(5) DS are constructed by acting the generalized boson pairing operator $\sum_{k=0}^{m} C_k^\xi (\hat{P}_d^\dagger)^k$ on the SO(5) basis vectors with m being infinite. But the coefficients C_k^ξ can be only calculated in a numerical way at present.

In the previous work [21], it was shown that the simplest version of Eu(5) DS involving only the first term in (40) provides an algebraic description of the E(5) CPS [22], in which the E(5) CPS has been widely confirmed [41–48]. It is thus suggested that the E(5) nuclei may be the candidate of the Eu(5) DS in experiments. Here, we choose ^{108}Pd [49], ^{134}Ba [50], ^{64}Zn [51], and ^{114}Cd [52], which were previously identified as the candidates of the E(5) CPS [41,42,46,47], as examples to show the possible Eu(5) patterns in experiments. In our calculation, the E2 transition operator is taken as

$$\hat{T}_u = e(d^\dagger + \tilde{d})_u^{(2)} \quad (45)$$

with the effective boson charge e determined by the corresponding experiment value of $B(E2; 2_1 \rightarrow 0_1)$ (in W.u.). The low-lying patterns of these E(5) nuclei and the corresponding results obtained from the Eu(5) DS are shown in Figs. 2–5. It can be clearly observed from Fig. 2 that the low-lying
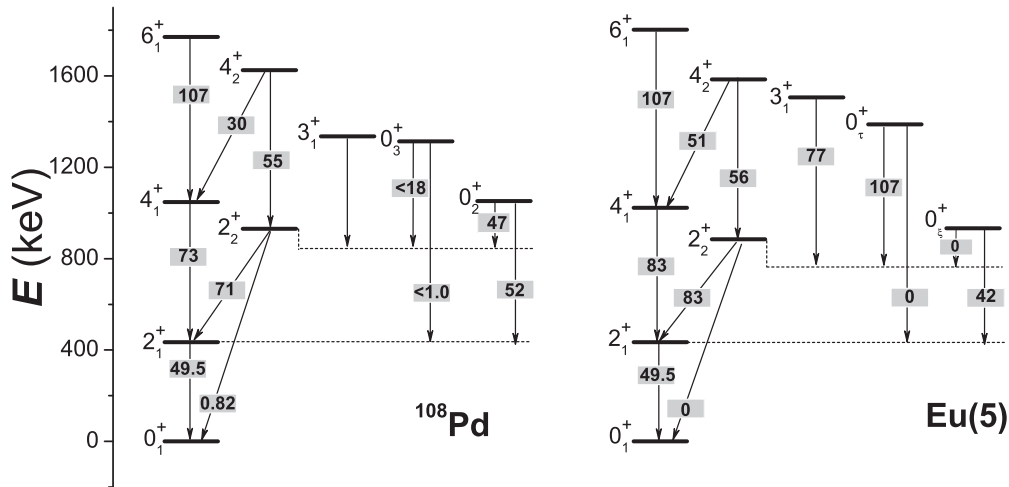

FIG. 2. The low-lying structure of ^{108}Pd [49] and the results calculated for the Eu(5) Hamiltonian (40) with the $m = 100$ truncation, where 0_τ^+ and 0_ξ^+ represent the excited 0^+ state with $\tau = 3$ in the $\xi = 1$ family and that with $\tau = 0$ in the $\xi = 2$ family, respectively, as those in the E(5) CPS [22]. The parameters involved in the Eu(5) Hamiltonian are set as $\alpha = 49.4$ keV, $b = 14.8$ keV, and $c = 9.9$ keV.
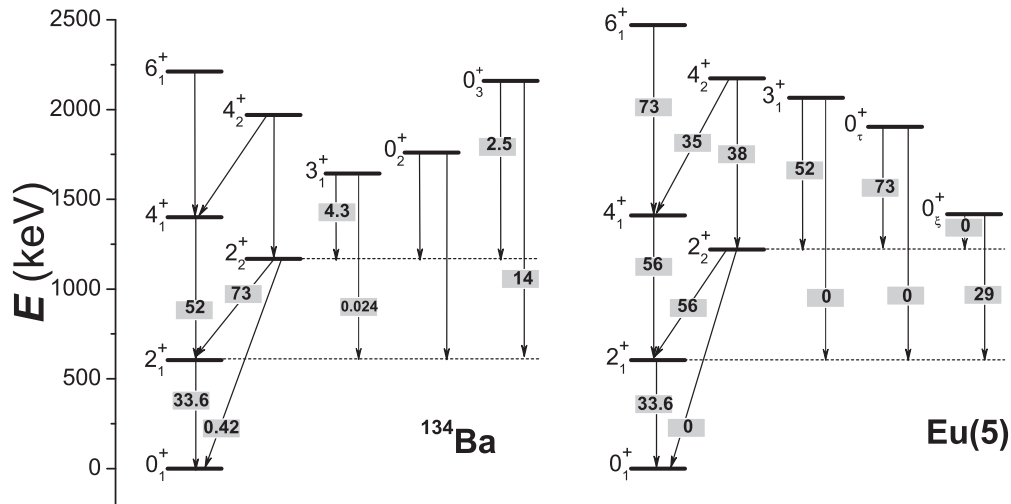
FIG. 3. The same as in Fig. 2 but for ^{134}Ba [50] and the corresponding Eu(5) pattern, for which the parameters are set as $\alpha = 74.9$ keV, $b = 11.2$ keV and $c = 13.5$ keV.
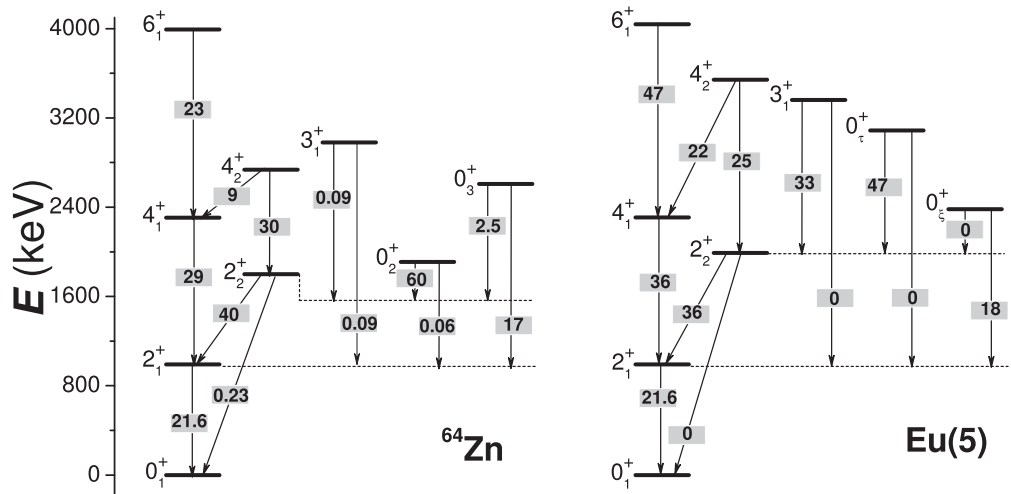

FIG. 4. The same as in Fig. 2 but for ^{64}Zn [51] and the corresponding Eu(5) pattern, for which the parameters are set as $\alpha = 126$ keV, $b = 12.6$ keV and $c = 22.7$ keV.

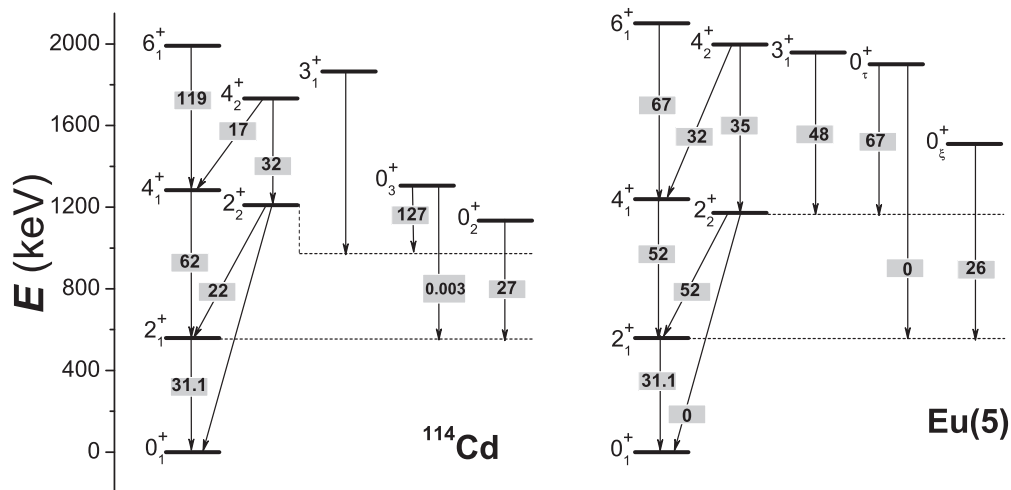FIG. 5. The same as in Fig. 2 but for ^{114}Cd [52] and the corresponding Eu(5) pattern, for which the parameters are set as $\alpha = 79.9$ keV, $b = 4.8$ keV and $c = 4.8$ keV.

TABLE I. Typical energy and $B(E2)$ ratios of ^{108}Pd [49], ^{134}Ba [50], ^{64}Zn [51], and ^{114}Cd [52] calculated from the Eu(5) DS extracted from Figs. 2–5, the IBM at the triple point with the boson number N taken as the number of valence nucleon (or hole) pairs for each nucleus, and the E(5) CPS [22] in comparison with the corresponding experimental data.

	$\frac{E(4_1^+)}{E(2_1^+)}$	$\frac{E(2_2^+)}{E(4_1^+)}$	$\frac{E(0_\xi^+)}{E(2_1^+)}$	$\frac{B(E2;4_1^+\to2_1^+)}{B(E2;2_1^+\to0_1^+)}$	$\frac{B(E2;0_2^+\to2_1^+)}{B(E2;2_1^+\to0_1^+)}$
^{108}Pd	2.42	0.89	2.43	1.48	1.05
Eu(5)	2.36	0.86	2.15	1.67	0.85
Tri($N=8$)	2.14	1.00	3.00	1.55	0.81
E(5)	2.20	1.00	3.03	1.68	0.86
^{134}Ba	2.32	0.83	3.57	1.55	0.42
Eu(5)	2.33	0.87	2.34	1.67	0.85
Tri($N=5$)	2.15	1.00	3.18	1.41	0.66
E(5)	2.20	1.00	3.03	1.68	0.86
^{64}Zn	2.33	0.78	2.63	1.34	0.79
Eu(5)	2.33	0.86	2.40	1.67	0.85
Tri($N=4$)	2.16	1.00	3.29	1.32	0.58
E(5)	2.20	1.00	3.03	1.68	0.86
^{114}Cd	2.30	0.94	2.03	1.99	0.87
Eu(5)	2.22	0.95	2.70	1.67	0.85
Tri($N=9$)	2.14	1.00	2.96	1.57	0.84
E(5)	2.20	1.00	3.03	1.68	0.86

spectrum of ^{108}Pd can be well described by the Eu(5) pattern determined by (40). Particularly, the relative $B(E2)$ strengths in the Eu(5) DS are independent of the parameters, while the data of ^{108}Pd seem to be well reproduced by those of the Eu(5) DS. In addition, $B(E2;0_{3,2}^+ \to 2_2^+)$ evidently deviates from $B(E2;0_\tau^+ \to 2_2^+)$ in the Eu(5) DS, which indicates that the 0_ξ^+ and 0_τ^+ components may be mixing in the excited 0^+ states of ^{108}Pd. As shown in Figs. 3–5, the low-lying pattern of ^{134}Ba, ^{64}Zn, and ^{114}Cd can also be globally reproduced well in the Eu(5) DS, which further confirms that the Eu(5) DS emerges in these nuclei. Meanwhile, deviations from the experimental results still exist. For example, $B(E2;3_1^+ \to 2_2^+)$ calculated from the Eu(5) DS description seems too large, and the ordering of the first two excited 0^+ levels in ^{134}Ba and ^{64}Zn is altered in the Eu(5) DS as show in Figs. 3 and 4, which show that the Eu(5) DS is still an approximate symmetry. As analyzed in [53], the order of the first two excited 0^+ states can be altered by using a γ-independent displaced infinite well β potential in the Bohr Hamiltonian, in contrast to the one used in the E(5) CPS. It would be very interesting to investigate whether such an improvement in the E(5) CPS will be achieved in the Eu(5) DS, which may be discussed elsewhere.

For comparison of the Eu(5) DS results with those of the E(5) and the IBM consistent-Q Hamiltonian at the triple

point, some typical level energies and $B(E2)$ ratios calculated from these models in comparison with the corresponding experimental data are shown in Table I. As clearly shown in Table I, the results of these models are similar and are in accord with the experimental data with quantitative differences. In addition, one may notice from Table I that the $B(E2)$ ratios obtained from the E(5) CPS are almost the same as those calculated from the Eu(5) DS with $m = 100$ up to the second decimal place. Actually, the minor differences at the second decimal place in the $B(E2)$ ratios shown in Table I can also be removed if these quantities are calculated with larger m truncation, which indicates that the Eu(5) Hamiltonian (40) is just an algebraic equivalent description of the E(5) CPS with SO(5) and SO(3) invariants being involved, with which an intimate relation between the E(5) CPS and the IBM beyond the mean-field approximation [54–56] is thus revealed. It should be mentioned that one can also eliminate the differences of the energy ratios in the Eu(5) DS from the E(5) description shown in Table I by adding a linear combination of the SO(5) and SO(3) Casimir operators in the E(5) CPS Hamiltonian [22,23].

VI. SUMMARY

In summary, the boson realization of the Eu(5) algebras has been presented, based on which the relation between the Eu(5) dynamical symmetry description and the IBM is discussed. Specifically, it is shown that the Eu(5) dynamical symmetry may emerge at the triple point of the IBM phase diagram in the $n_d/N \ll 1$ limit, which thus provides an alternative insight into the dynamical structure at or around this isolated point and the experimental data associated with it. On the other hand, this work also shows that the results of the E(5) CPS can be realized in a fully DS way within the algebraic frame, which thus reveals a more intimate relationship between the E(5) CPS and the IBM beyond the mean-field approximation. Finally, a preliminary examination of the Eu(5) DS in ^{108}Pd, ^{134}Ba, ^{64}Zn, and ^{114}Cd is made. The results indicate that the low-lying dynamics of these nuclei are indeed dominated by the Eu(5) DS.

[1] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University, Cambridge, UK, 1987).

[2] F. Iachello and R. D. Levine, *Algebraic Theory of Molecules* (Oxford University, Oxford, UK, 1995).

[3] H. Yépez-Martínez, J. Cseh, and P. O. Hess, Phys. Rev. C **74**, 024319 (2006).

[4] J. Jolie, R. F. Casten, P. von Brentano, and V. Werner, Phys. Rev. Lett. **87**, 162501 (2001).

[5] A. Leviatan, Phys. Rev. Lett. **77**, 818 (1996).

[6] A. Leviatan, Phys. Rev. Lett. **98**, 242502 (2007).

[7] J. E. Garcìa-Ramos, A. Leviatan, and P. Van Isacker, Phys. Rev. Lett. **102**, 112502 (2009).

[8] D. J. Rowe, Phys. Rev. Lett. **93**, 122502 (2004).

[9] D. J. Rowe, P. S. Turner, and G. Rosensteel, Phys. Rev. Lett. **93**, 232502 (2004).

[10] D. J. Rowe, Nucl. Phys. A **745**, 47 (2004).

[11] D. Bonatsos, E. A. McCutchan, and R. F. Casten, Phys. Rev. Lett. **104**, 022502 (2010).

[12] D. Bonatsos, S. Karampagia, and R. F. Casten, Phys. Rev. C **83**, 054313 (2011).

[13] Y. Alhassid and N. Whelan, Phys. Rev. Lett. **67**, 816 (1991).

[14] J. Jolie, R. F. Casten, P. Cejnar, S. Heinze, E. A. McCutchan, and N. V. Zamfir, Phys. Rev. Lett. **93**, 132501 (2004).

[15] C. Kremer, J. Beller, A. Leviatan, N. Pietralla, G. Rainovski, R. Trippel, and P. Van Isacker, Phys. Rev. C **89**, 041302(R) (2014).

[16] P. Cejnar, J. Jolie, and R. F. Casten, Rev. Mod. Phys. **82**, 2155 (2010).

[17] P. Cejnar and J. Jolie, Prog. Part. Nucl. Phys. **62**, 210 (2009).

[18] R. F. Casten and E. A. McCutchan, J. Phys. G **34**, R285 (2007).

[19] J. Jolie, P. Cejnar, R. F. Casten, S. Heinze, A. Linnemann, and V. Werner, Phys. Rev. Lett. **89**, 182502 (2002).

[20] D. Warner, Nature (London) **420**, 614 (2002).

[21] Y. Zhang, Y. X. Liu, F. Pan, Y. Sun, and J. P. Draayer, Phys. Lett. B **732**, 55 (2014).

[22] F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000).

[23] M. A. Caprio and F. Iachello, Nucl. Phys. A **781**, 26 (2007).

[24] B. G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).

[25] P. Feinsilver, Acta Appl. Math. **43**, 289 (1996).

[26] D. Bonatsos, E. A. McCutchan, and R. F. Casten, Phys. Rev. Lett. **101**, 022501 (2008).

[27] H. Ui, Prog. Theor. Phys. **44**, 153 (1970).

[28] D. J. Rowe, Nucl. Phys. A **735**, 372 (2004).

[29] D. J. Rowe and P. S. Turner, Nucl. Phys. A **753**, 94 (2005).

[30] D. J. Rowe, J. Phys. A **38**, 10181 (2005).

[31] D. J. Rowe, T. A. Welsh, and M. A. Caprio, Phys. Rev. C **79**, 054304 (2009).

[32] F. Pan and J. P. Draayer, Nucl. Phys. A **636**, 156 (1998).

[33] F. Pan, X. Zhang, and J. P. Draayer, J. Phys. A **35**, 7173 (2002).

[34] D. D. Warner and R. F. Casten, Phys. Rev. C **28**, 1798 (1983).

[35] F. Iachello and N. V. Zamfir, Phys. Rev. Lett. **92**, 212501 (2004).

[36] F. Iachello, N. V. Zamfir, and R. F. Casten, Phys. Rev. Lett. **81**, 1191 (1998).

[37] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).

[38] J. M. Arias, J. Dukelsky, and J. E. Garcìa-Ramos, Phys. Rev. Lett. **91**, 162502 (2003).

[39] F. Pan, Y. Zhang, and J. P. Draayer, J. Phys. G **31**, 1039 (2005).

[40] S. Dusuel, J. Vidal, J. M. Arias, J. Dukelsky, and J. E. Garcìa-Ramos, Phys. Rev. C **72**, 011301(R) (2005).

[41] D. L. Zhang and Y. X. Liu, Phys. Rev. C **65**, 057301 (2002).

[42] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **85**, 3584 (2000).

[43] A. Frank, C. E. Alonso, and J. M. Arias, Phys. Rev. C **65**, 014301 (2001).

[44] N. V. Zamfir *et al.*, Phys. Rev. C **65**, 044325 (2002).

[45] R. M. Clark *et al.*, Phys. Rev. C **69**, 064322 (2004).

[46] C. Mihai *et al.*, Phys. Rev. C **75**, 044302 (2007).

[47] J. F. Zhang *et al.*, Chin. Phys. Lett. **20**, 1231 (2003).

[48] L. Coquard *et al.*, Phys. Rev. C **80**, 061304(R) (2009)

[49] J. Blachot, Nucl. Data Sheets **81**, 599 (1997).

[50] A. A. Sonzogni, Nucl. Data Sheets **103**, 1 (2004).

[51] B. Singh, Nucl. Data Sheets **78**, 395 (1996).

[52] J. Blachot, Nucl. Data Sheets **97**, 593 (2002).

[53] D. Bonatsos, D. Lenis, N. Pietralla, and P. A. Terziev, Phys. Rev. C **74**, 044306 (2006).

[54] J. M. Arias, C. E. Alonso, A. Vitturi, J. E. Garcìa-Ramos, J. Dukelsky, and A. Frank, Phys. Rev. C **68**, 041302(R) (2003).

[55] J. E. Garcìa-Ramos and J. M. Arias, Phys. Rev. C **77**, 054307 (2008).

[56] Y. Zhang, F. Z. Hou, and Y. X. Liu, and Y. Sun, Sci. China Phys. Mech. Astron. **54**, 227 (2011).