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Improved Jänecke mass formula

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In this paper we improve an empirical mass formula constructed by Jänecke and collaborators. This formula is enlightened by the Garvey-Kelson mass relations. The new version of the Jänecke formula reproduces 2275 atomic masses with neutron number $N \geqslant 10$ and proton number $Z \geqslant 6$, at an average accuracy of 128 keV, by employing 576 parameters. The predictive power of our formula is exemplified by comparison with predicted results of other mass models.

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I. INTRODUCTION

Nuclear mass (or binding energy) and nucleon-separation energies are fundamental quantities in both nuclear physics and astrophysics [1]. Generally speaking, the approaches toward describing and understanding nuclear masses are classified into two types, one of which is usually called global, and the other local. Examples of global formulas are the Duflo-Zuker (D-Z) model [2], Skyrme-Hartree-Fock-Bogoliubov theory [3–5], finite-range droplet model (FRDM) [6], and improved Weizsäcker mass formula [7]. The local mass formulas predict the mass of a nucleus by using available experimental data of its neighboring nuclei, i.e., by extrapolation. We mention three methods of local mass relations: the Audi-Wasptra extrapolation [8–11], mass relations connected with neutron-proton (n-p) interactions [12–16], and the Garvey-Kelson mass relations (GKs) [17–34].

The purpose of this paper is to suggest an improved version of the mass formula developed by Jänecke and collaborators [19–24]. This formula was enlightened by the compact form of the Garvey-Kelson mass relations. If one chooses different subsets of experimental masses (called "skeletons" in Ref. [25]) to set up an ensemble, the mass of a given nucleus is predicted to be the ensemble average of different subsets [28–31]. In another context, the Garvey-Kelson mass relations were treated as partial difference equations, and their general solutions were obtained from the least-squares fit to all experimental masses [18–24]. In this paper, we go one step forward along the latter line.

This paper is organized as follows. In Sec. II we present a short introduction to the Jänecke formula and derive its new version. In Sec. III we investigate the interpretive and predictive power of our formula for nuclear masses and nucleon-separation energies, with the refinement of the generalized hybrid method. Our summary and conclusion are given in Sec. IV.

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II. MASS FORMULAS

In this section we shall review the efforts made by Jänecke and collaborators in construction of mass formulas which are applicable to the whole nuclear chart but with many parameters. Then we shall present the improved version of the Jänecke mass formula. The main difference is inclusion of the pairing interaction and an isospin-dependent term. These two terms are found to be very useful in reducing the root-mean-squared deviation of the formula from experimental data.

A. Previous efforts

Let us begin with a short historical survey of previous efforts along this line. Garvey and Kelson [18] proposed two simple relations of masses for six neighboring nuclei,

$$M(N,Z+1)+M(N-1,Z-1)+M(N+1,Z)$$
$$-M(N,Z-1)-M(N-1,Z)-M(N+1,Z+1)=0, \quad (1)$$

$$M(N,Z-1)+M(N-1,Z+1)+M(N+1,Z)$$
$$-M(N,Z+1)-M(N-1,Z)-M(N+1,Z-1)=0, \quad (2)$$

where $M(N,Z)$ denotes the mass of a nucleus with neutron number N and proton number Z. Equation (1) is called the longitudinal Garvey-Kelson relation (GKL), and Eq. (2) the transverse (GKT). For the GKL and GKT, odd-odd nuclei with $N=Z$ should be excluded, and $N \geqslant Z$ is additionally required for the GKT. Numerical experiment by using the Atomic Mass Evaluation 2012 (AME2012) [11] yields the root-mean-squared deviation (RMSD) ~200 keV [33]. The Garvey-Kelson relations can be rewritten in terms of mass number $A=N+Z$ and neutron excess $E=N-Z$ as below:

$$M(A+1,E-1)+M(A-2,E)+M(A+1,E+1)$$
$$-M(A-1,E+1)-M(A-1,E-1)$$
$$-M(A+2,E)=0, \quad (3)$$

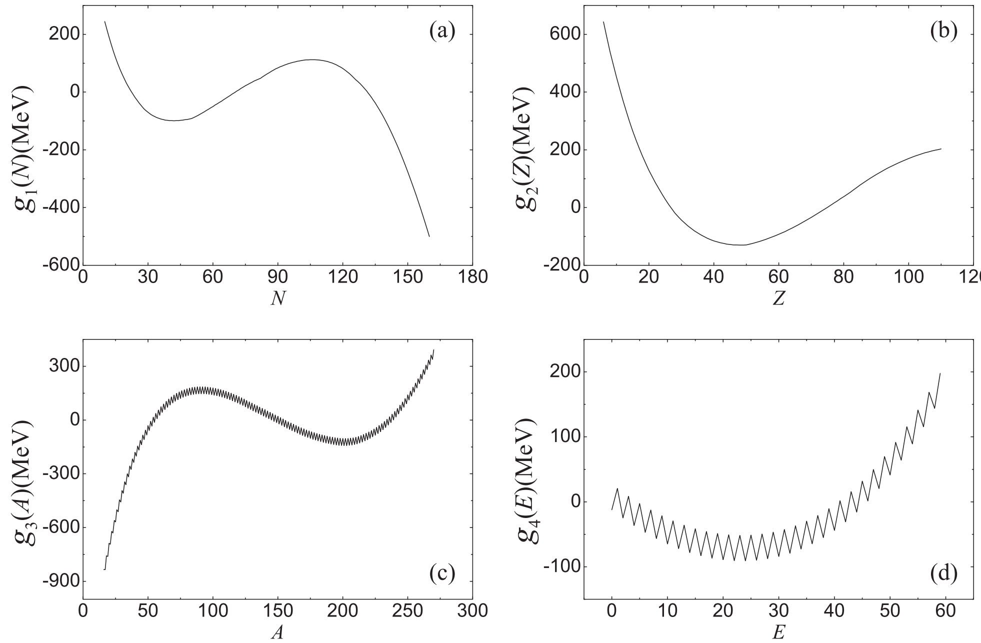

FIG. 1. Values of $g_1(N)$, $g_2(Z)$, $g_3(A)$, and $g_4(E)$ with $g_5(N,Z) = 0$ and $g_6(N,Z) = E_{\rm p}$. One sees that $g_1(N)$ and $g_2(Z)$ are smooth and that $g_3(A)$ and $g_4(E)$ exhibit odd-even staggerings.

$$\begin{aligned} &M(A-1,E+1) + M(A,E-2) + M(A+1,E+1) \\ &\quad - M(A+1,E-1) - M(A-1,E-1) \\ &\quad - M(A,E+2) = 0. \end{aligned} \tag{4}$$

Neutron excess E is equivalent to the z projection of the isospin $T_z = \frac{1}{2}(N-Z) = \frac{1}{2}E$.

According to Eqs. (1) and (3), the GKL connects three pairs of nuclear masses which have the same N, Z, or E; and according to Eqs. (2) and (4), the GKT connects three pairs of nuclear masses which have the same N, Z, or A. As suggested by Garvey, Kelson, and their collaborators in their review paper [18], one would conjecture that nuclear masses take the following forms:

$$M_{\rm L}(N,Z) = f_1(N) + f_2(Z) + f_3(E), \tag{5}$$

$$M_{\rm T}(N,Z) = g_1(N) + g_2(Z) + g_3(A), \tag{6}$$

corresponding to the GKL and GKT, respectively. Here $f_i(k)$ and $g_i(k)$ are arbitrary point functions. Garvey *et al.* further constructed a formula which satisfied both the GKL and GKT as follows:

$$M(N,Z) = h_1(N) + h_2(Z) + \lambda NZ + \tfrac{1}{2}\mu[1-(-1)^{NZ}], \tag{7}$$

where $h_i(k)$ were arbitrary point functions and λ and μ were constants.

Unfortunately, Eq. (7) does not well describe experimental data [18], and thus two improvements have been developed. The first is to take Eqs. (5) and (6) separately, even though they are based on the same physical consideration [18–20]. Garvey *et al.* adopted this idea and presented all values of $f_i(k)$ and $g_i(k)$ in tabular form which were obtained by minimizing the deviations from measured masses [18]. In Ref. [19], the authors pointed out that Eq. (6) was more accurate than Eq. (5). In Ref. [20], the experimental data of neutron-rich and proton-rich nuclei were treated separately to reduce systematic errors in long-range extrapolations. In doing so, there are 928 adjustable parameters for about 1550 mass values by using Eq. (6) [20].

The second improvement is to consider a so-called inhomogeneous term into the GKL and GKT [21–24], and this can be done in two approaches. The first approach is to

TABLE I. $\sigma_M/\sigma_{S_n}/\sigma_{S_p}$ values, i.e., the RMSD values of masses, one-neutron separation energies, and one-proton separation energies. These RMSD values are calculated by using Eqs. (6), (14), and (15), with $g_6(N,Z)$ taking the form of Eq. (13). Here we also present these RMSD values separately for even-N and even-Z (ee), even-N and odd-Z (eo), odd-N and even-Z (oe), odd-N and odd-Z (oo) nuclei. The last row enumerates the numbers of the experimentally known masses assumed in our calculations.

	Total	ee	eo	oe	oo
Eq. (6)	323/238/271	347/247/279	303/254/270	315/236/267	325/210/267
Eq. (14)	221/195/226	245/190/232	194/203/211	222/196/233	218/193/230
Eq. (15)	128/161/161	152/149/158	107/150/161	117/184/160	128/157/164
Numbers	2275/2120/2074	605/549/544	563/500/529	579/564/503	528/507/498

take general solutions of so-called homogeneous equations as corrections. In Ref. [22], the GKL and GKT were derived from the neutron-proton interaction between the last neutron and proton, viz.,

$$\begin{aligned}\delta V_{1n\text{-}1p}(A,E) &= B(A,E)+B(A-2,E)\\ &\quad -B(A-1,E-1)-B(A-1,E+1)\\ &= -M(A,E)-M(A-2,E)\\ &\quad +M(A-1,E-1)+M(A-1,E+1). \qquad (8)\end{aligned}$$

In terms of $\delta V_{1n\text{-}1p}$, the GKL and GKT are rewritten as

$$\delta V_{1n\text{-}1p}(A+2,E)-\delta V_{1n\text{-}1p}(A,E)=0, \qquad (9)$$

$$\delta V_{1n\text{-}1p}(A+1,E-1)-\delta V_{1n\text{-}1p}(A+1,E+1)=0. \qquad (10)$$

If $\delta V_{1n\text{-}1p}(A,E)$ is E dependent only, one obtains the GKL and its general solution as given in Eq. (5); if $\delta V_{1n\text{-}1p}(A,E)$ is A dependent only, one obtains the GKT and its general solution Eq. (6). In Ref. [22], the authors assumed $\delta V_{1n\text{-}1p}(A,E)$ to be constant (separately for even-A and odd-A nuclei) and obtained a general solution $M^*(N,Z)$, a correction term of theoretically calculated mass [denoted as $M^{\text{th}}(N,Z)$]. The final predicted mass is $M^{\text{th}}+M^*$. The second approach is to evaluate corrections based on the general solutions of so-called inhomogeneous equations. For example, Ref. [24] showed that an inhomogeneous term is required in the GKT, and that the inclusion of the T_z^3 term led to remarkable improvement in long-range extrapolations. The disadvantage, however, is that the parameters of this approach are very unstable [24].

B. Improved Jänecke mass formula

Because Eq. (6) is more accurate than Eq. (5), Eq. (6) has been more widely applied in previous studies [20–25]. Equation (6) assumes the A dependence in nuclear mass, while Eq. (5) assumes the isospin dependence. In this paper, we take a more general form which includes both Eqs. (5) and (6), viz.,

$$M_{\text{LT}}(N,Z)=g_1(N)+g_2(Z)+g_3(A)+g_4(E). \qquad (11)$$

Substituting Eq. (11) into Eq. (8), one obtains

$$\begin{aligned}\delta V_{1n\text{-}1p}(A,E) &= -[g_3(A)+g_3(A-2)-2g_3(A-1)]\\ &\quad +[g_4(E+1)+g_4(E-1)-2g_4(E)]\\ &= G_3(A)+G_4(E). \qquad (12)\end{aligned}$$

References [14,20] studied the A and E dependence of $\delta V_{1n\text{-}1p}(A,E)$, indicating that corrections are still needed for Eq. (11).

In Refs. [32,33] it was pointed out that to the third order the smooth dependence of N and Z was canceled out in the GKL and GKT. Recently, an odd-even feature of the GKL and GKT was observed and explained in terms of a refined form of pairing interaction [33]. We include a smooth function $g_5(N,Z)$ and a nonsmooth function [35,36] $g_6(N,Z)$ in Eq. (11). In this paper we take $g_5(N,Z)$ in the form of the Coulomb energy E_{C} and symmetry energy (including both volume-symmetry energy E_{sv} and surface-symmetry energy E_{ss}) as in the Weizsäcker mass formula, and $g_6(N,Z)$ the form of the pairing interaction [37] as below.

$$E_{\text{p}}=a_{\text{p}}A^{-1/3}\times\begin{cases}2-|I|, & N \text{ and } Z \text{ even},\\ |I|, & N \text{ and } Z \text{ odd},\\ 1-|I|, & N \text{ even, } Z \text{ odd, } N>Z,\\ 1-|I|, & N \text{ odd, } Z \text{ even, } N<Z,\\ 1, & N \text{ even, } Z \text{ odd, } N<Z,\\ 1, & N \text{ odd, } Z \text{ even, } N>Z.\end{cases} \qquad (13)$$

Here $I=(N-Z)/A$ is the charge-asymmetry parameter. In recent years this form of the pairing interaction has been adopted by many authors [7,14,33,38–40]. Our mass formula takes the following form:

$$\begin{aligned}M(N,Z) &= g_1(N)+g_2(Z)+g_3(A)+g_4(E)\\ &\quad +g_5(N,Z)+g_6(N,Z). \qquad (14)\end{aligned}$$

Although we adopt g_5 and g_6 as the form of the Coulomb energy, symmetry energy, and the pairing interaction in the Weizsäcker mass formula, they should not take the same magnitudes as in the Weizsäcker mass formula, as the essential parts have already been considered in Eq. (11), namely, the $g_1(N)$, $g_2(Z)$, $g_3(A)$, and $g_4(E)$ terms. We shall see in the next section that the inclusion of g_5 does not improve our

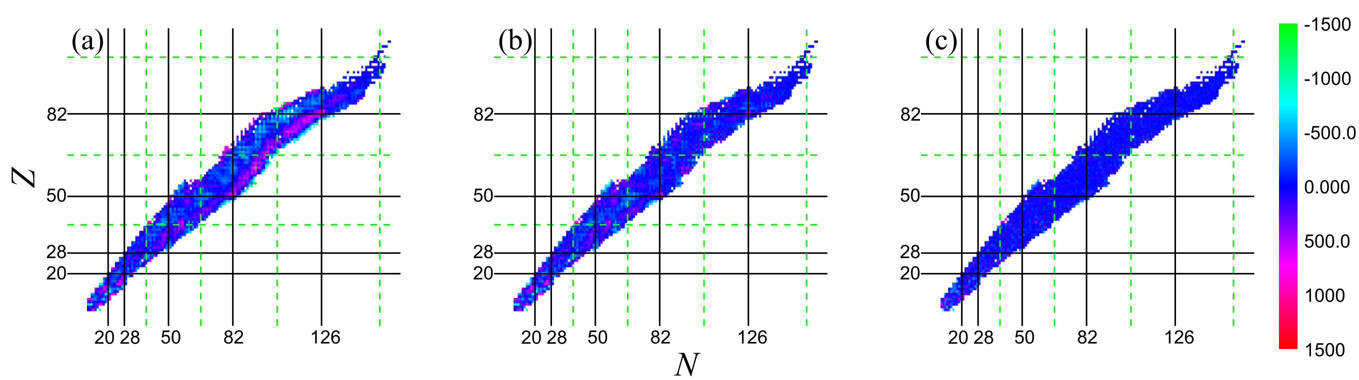

FIG. 2. (Color online) Deviations $D(N,Z)=M^{\text{th}}(N,Z)-M^{\text{expt}}(N,Z)$ (in keV) of calculated masses from experimental data. Panels (a), (b), (c) correspond to results obtained by using Eqs. (6), (14), (15), respectively.

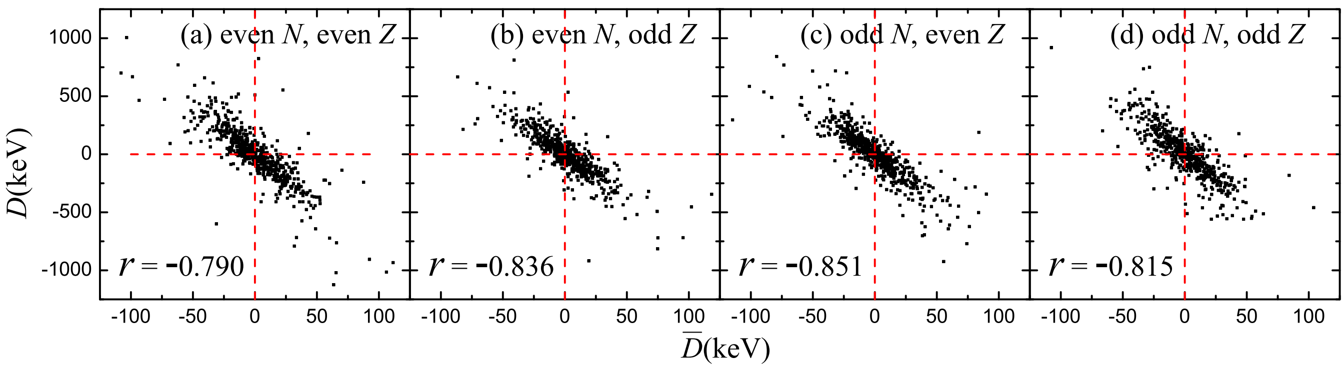

FIG. 3. (Color online) Correlation between deviations $D(N,Z) = M^{\rm th}(N,Z) - M^{\rm expt}(N,Z)$ and $\overline{D}$ [see Eq. (15)] for different parity of (N,Z). r represents the linear correlation coefficients and is around -0.8 here. This means a strong correlation between $D(N,Z)$ and $\overline{D}$.

mass formula [Eq. (14)], and that the inclusion of g_6 in the refined form [Eq. (13)] is very useful to reduce the deviations of calculated results from experimental data.

III. INTERPRETIVE AND PREDICTIVE POWER OF THE NEW FORMULA

In this section we investigate the accuracy of our mass formula, Eq. (14), by using the AME2012 database [11], as well as comparisons of the RMSDs with other approaches. We shall also make use of the so-called generalized hybrid method to improve our approach.

A. Description of available experimental data

Calculation of $g_i(k)$ for each integer k was performed in Ref. [18]. We follow the same procedure to optimize the coefficients in Eq. (14), by the least-squares fitting of the AME2012 database [11] for $10 \leqslant N \leqslant 160, 6 \leqslant Z \leqslant 110$ and $N > Z$ or $N = Z =$ even (mass number $16 \leqslant A \leqslant 270$, and neutron excess $0 \leqslant E \leqslant 59$), in total 2275 nuclei. Besides the very few coefficients in g_5 and g_6, there are $P = 571$ parameters here. For the very few cases without known masses (such as $Z = 107$), we take their $g_i(k)$ values by using a parabolic fit to their neighbors.

According to our calculation, the inclusion of $g_6(N,Z)$ is very useful in improving our formula in both description of the known masses and evaluation of the unknown ones. The resultant $a_{\rm p} = -4.9644$ MeV is slightly different from that in Ref. [7] where $a_{\rm p} = -5.5108$ MeV and that in Ref. [41] where $a_{\rm p} = -5.4423$ MeV. As will be discussed later, the pairing interaction is partly taken into account in $g_3(A)$ and $g_4(E)$. The usefulness of $g_6(N,Z)$ is not surprising, because the pairing interaction [Eq. (13)] reproduces subtle odd-even features of the Garvey-Kelson mass relations [33] and these features are not reflected in the g_1, g_2, g_3, and g_4 functions. The inclusion of $g_5(N,Z)$ does not considerably improve the agreement between Eq. (14) and experimental data, and furthermore the values of parameters in $g_5(N,Z)$ are unstable in different regions. This is not surprising, as the Coulomb energy is well considered in the $g_3(Z)$ term and the symmetry energy in the $g_4(E)$ term. The motivation of including the g_5 term is to further improve the description of the smooth part of our mass formula, and this part is now well considered by the g_1, g_2, g_3, and g_4 functions.

In Fig. 1 we plot $g_1(N)$, $g_2(Z)$, $g_3(A)$, and $g_4(E)$. Their values are also listed in Table IV in the Appendix. In Fig. 1, one sees that $g_1(N)$ and $g_2(Z)$ are smooth and that $g_3(A)$ and $g_4(E)$ exhibit odd-even staggerings. According to Ref. [18], the nuclear masses are generally parabolic along lines of constant N, Z, and A. Here, however, $g_1(N)$, $g_2(Z)$, and $g_3(A)$ behave in a more complicated manner. According to the isobaric multiplet mass equation (IMME) [42], the masses of

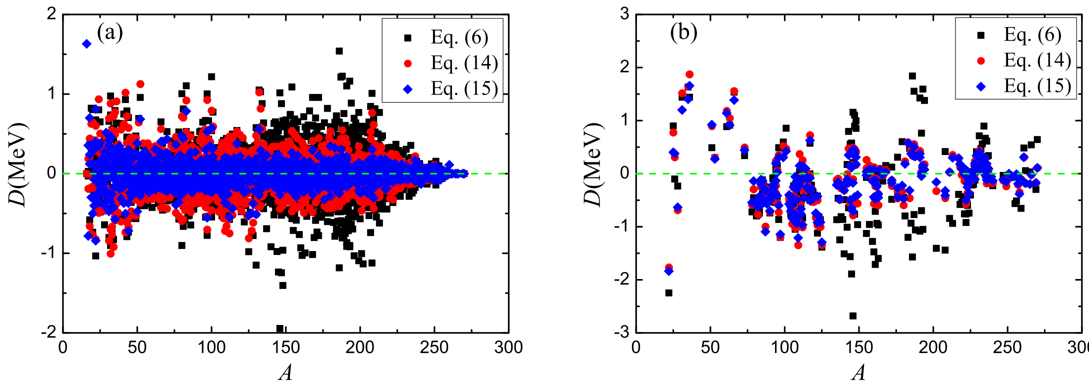

FIG. 4. (Color online) Deviations $D(N,Z)$ (in MeV) for Eqs. (6), (14), and (15) versus mass number A. The results in panel (a) correspond to the 2275 nuclei considered in this paper, and those in panel (b) correspond to deviations of theoretical masses based on the AME2003 database [10] with respect to the AME2012 database (without those known nuclei in the AME2003 database).

TABLE II. σ_M and σ_{S_n} for a few global models with respect to the AME2003 database [10]. Here HFB-14, HFB-17, FRDM, WS, WS*, and WS* + Δ_T represent the Hartree-Fock-Bogoliubov (HFB) calculations in Ref. [45], the HFB in Ref. [46], the FRDM calculations by Möller *et al.* in Ref. [47], the modified Weizsäcker mass formula in Ref. [7], the mass formula in Ref. [41], and the mass formula in Ref. [41].

	HFB-14	HFB-17	FRDM	WS	WS*	WS* + Δ_T	Eq. (6)	Eq. (14)	Eq. (15)
σ_M	729	581	656	516	441	417	292	211	137
σ_{S_n}	598	506	399	346	332	330	240	203	184

isobaric nuclei are given by $M(A,T_z) = aT_z^2 + bT_z + c$. Here indeed g_4 is approximately parabolic (separately for even-E and odd-E) with E. The odd-even staggerings in $g_3(A)$ and $g_4(E)$ are reflections of the pairing interaction.

In the first and second rows of Table I we present the RMSDs of our calculated masses (denoted by σ_M) by using Eqs. (6) and (14) with respect to experimental data [11]. One sees that the σ_M are different for different parity of (N,Z). The results of one-neutron separation energies and one-proton separation energies (denoted by S_n and S_p, respectively) are shown. The RMSDs for M, S_n, and S_p obtained by using Eq. (14) are smaller than those obtained by using Eq. (6) by 32%, 18% and 17%, respectively. In Fig. 2 we plot the deviations $D(N,Z) = M^{\rm th}(N,Z) - M^{\rm expt}(N,Z)$ for the 2275 nuclei considered in this paper. One sees substantial improvements by including the g_4 and g_6.

Now we discuss the $g_4(E)$ term. In Ref. [24], the authors suggested that higher-order corrections in isospin proportional to T_z^3 might have important influence in long-range extrapolations. However, the calculated results are very sensitive to the form of this term and the shells [24]. In this paper $g_4(E)$ is close to parabolic (different for even-E and odd-E) which is much simpler than that of Ref. [24].

It is also useful to investigate the possible correlations among the deviations of Eq. (14) from experimental data. Enlightened by the radial basis function (RBF) approach [43] and the hybrid method for α-decay energies [44], we investigate this correlations in a procedure as follows. We denote the predicted mass of a given nucleus with neutron number N and proton number Z based on Eq. (14) by using $M^{\rm th1}(N,Z)$. We calculate the average deviation, denoted as $\overline{D}(N,Z)$, of its neighboring nuclei (N',Z'), with the requirement $|A - A'| \leqslant 2$, and $N \neq N'$, $Z \neq Z'$. From Fig. 3 one easily sees a statistically linear correlation between D and $\overline{D}$. Here the linear correlation coefficient r values are about -0.80 which corresponds to strong linear correlation. Therefore our new predicted mass [denoted by using $M^{\rm th2}(N,Z)$] can be written in the form

$$M^{\rm th2}(N,Z) = M^{\rm th1}(N,Z) + \overline{D}(N,Z)\phi,$$
$$\overline{D}(N,Z) = \frac{\sum_{i=1}^{\mathcal{N}}[M^{\rm expt}(N',Z') - M^{\rm th1}(N',Z')]}{\mathcal{N}}, \quad (15)$$

where the optimal ϕ equals -6.77, -6.97, -7.24, and -7.42 for even-even, even-odd, odd-even, and odd-odd nuclei, respectively.

The σ of M, S_n, and S_p for Eq. (15) are reduced by about 60%, 32%, and 41%, respectively, in comparison with the results of Eq. (6). The deviations of masses for Eq. (15) from experimental data are plotted in Figs. 2(c) and 4(a). In Table II we present σ_M and σ_{S_n} by using Eqs. (6), (14), and (15) and those by a few other popular models with respect to the AME2003 [10]. One sees that the RMSDs of Eqs. (14) and (15) are considerably smaller.

B. Extrapolation and predictive power

In this section we investigate the predictive power of our formulas. We perform two numerical experiments. In the first numerical experiment, we take the AME2003 database [10] and predict new masses compiled in the AME2012 database [11]. Namely, we predict unknown masses in the AME2003 database and compared our predicted results with experimental data which were available afterward. We focus on the RMSD values of our predicted results. We also study the behavior

TABLE III. $\sigma_M/\sigma_{S_n}/\sigma_{S_p}$ of predicted results based on the AME2003 database [10]. The $\sigma_M/\sigma_{S_n}/\sigma_{S_p}$ values are calculated only for those which are unknown in the AME2003 database and are compiled in the AME2012 database [11], by using different formulas Eqs. (6), (11), (14), and (15), and models [7,10,40,41,45–47]. The $\sigma_M/\sigma_{S_n}/\sigma_{S_p}$ results of our formulas are smaller than other global models and are competitive with local Audi-Wapstra extrapolation [10].

	HFB-14 $M/S_n/S_p$	Eq. (6) $M/S_n/S_p$	WS $M/S_n/S_p$	FRDM $M/S_n/S_p$	HFB-17 $M/S_n/S_p$	WS* + Δ_T $M/S_n/S_p$
$A \geqslant 16$	925/541/616	777/371/427	750/348/408	761/333/487	646/520/602	602/332/367
$A \geqslant 60$	878/486/464	752/353/394	704/325/381	658/311/406	611/453/483	580/320/355
$A \geqslant 120$	868/434/336	845/374/452	806/279/319	627/258/313	607/341/351	620/271/286

	WS* $M/S_n/S_p$	Eq. (11) $M/S_n/S_p$	Bao $M/S_n/S_p$	Eq. (14) $M/S_n/S_p$	Eq. (15) $M/S_n/S_p$	Audi-Wapstra $M/S_n/S_p$
$A \geqslant 16$	582/331/367	545/334/367		515/287/350	475/271/334	396/235/306
$A \geqslant 60$	562/318/355	488/308/317	502/304/371	457/260/281	424/253/277	391/210/265
$A \geqslant 120$	600/265/286	395/281/326	387/241/270	361/225/271	326/207/267	270/171/224

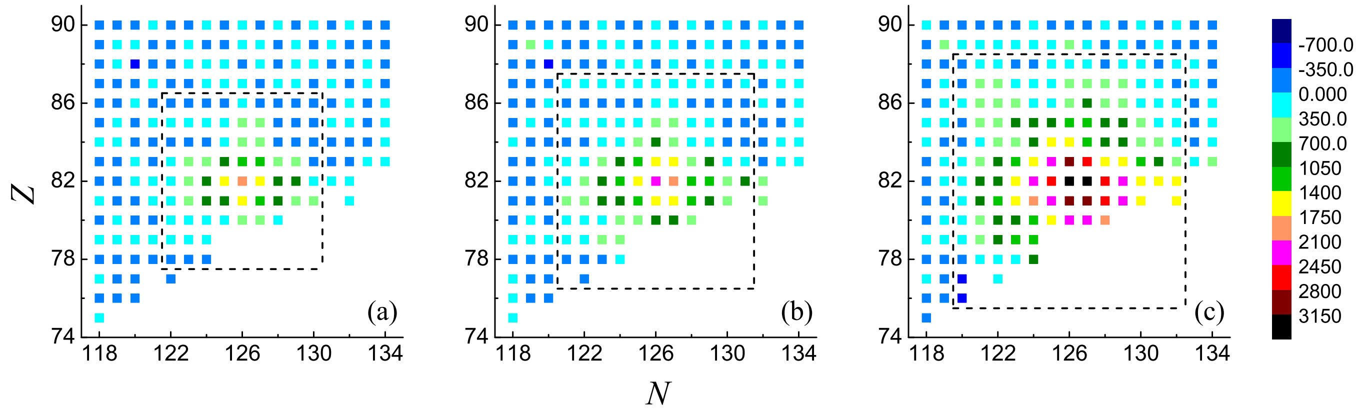

FIG. 5. (Color online) Contours of differences (in keV) between the extrapolated and experimental values of the excluding nuclei satisfying $|N-126| \leqslant d$ and $|Z-82| \leqslant d$, where panels (a), (b), and (c) correspond to $d = 4$,5, and 6, respectively.

of our formula when we are close to a shell gap. In the second numerical experiment, we investigate the difficulty of our predictions when the unknown subsets include a doubly magic nucleus.

In Table III we present the RMSDs of several global formulas in this paper as well as those from previous studies [7,10,40,41,45–47]. One sees that Eq. (15) works better than other global formulas except for the local Audi-Wasptra extrapolation [10]. Based on this table, we note the improvements in our formulas. The first improvement is the isospin-dependence correction, the $g_4(E)$ term. This correction is seen to be useful by comparing the results of Eqs. (6) and (11). The second is the pairing term, i.e., $g_6(N,Z)$. The usefulness of this term is clearly seen by comparing the results of Eqs. (11) and (14). The third is a correlation term in Eq. (15). This correction is useful for extrapolation close to the known border. One sees its role by comparing the RMSDs of Eqs. (14) and (15). In Fig. 4(b), we present the deviations D versus A in the above investigation. The deviations are within 500 keV for $A \geqslant 150$.

The second experiment concerns the predictive power of our improved formula in the case of the unknown subset with doubly magic nuclei. For this purpose, we remove the experimental data for nuclei with $|N-126| \leqslant d$ and $|Z-82| \leqslant d$ from the AME2003 database, with $d = 4, 5$, and 6, and calculate the deviations from the AME2012 database. The deviations by using such a database are plotted in Fig. 5. One concludes that the extrapolation deteriorates with d. One sees also that the extrapolation is reasonable if the doubly magic nucleus is not very far from the border of known masses in the nuclear chart.

IV. SUMMARY

To summarize, in this paper we revisit the Jänecke mass formula. This formula are given in *explicit form* of proton number Z and neutron number N. We improve the Jänecke formula by inclusion of the pairing interaction and an isospin-dependent term. Although the number of parameters $[g_1(N), g_2(Z), g_3(A), g_4(E)]$ are very large (576 in total), these parameters do not change quickly in local regions; they change in a smooth manner, instead (see Fig. 1). For the current experimental database, our formulas provide us with a description in a very good accuracy.

We investigate the predictive power of these new formulas by numerical experiments. They are competitive with local mass relations for unknown masses which are close to the known borders of the current database in the nuclear chart. Without details we note that the predicted results of the formulas in this paper are reasonably consistent with those of other models for nuclei far from the known borders. The deviation of predicted results from experimental values is large, if the unknown subset includes doubly magic nuclei. As our formulas are global, we present our predictions based on our formulas in the Supplemental Material [48].

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APPENDIX: VALUES OF POINT FUNCTIONS $g_i(k)$

In Table IV we present the values of point functions $g_1(N)$, $g_2(Z)$, $g_3(A)$, and $g_4(E)$ in Eq. (14),

$$M(N,Z) = g_1(N) + g_2(Z) + g_3(A) + g_4(E) + g_5(N,Z) + g_6(N,Z).$$

Here $g_5(N,Z) = 0$ and $g_6(N,Z)$ takes the form of Eq. (13) with $a_p = -4.9644$ MeV. The values are obtained from the least-squares adjustment to 2275 experimental masses in the AME2012 database [11] for nuclei with $10 \leqslant N \leqslant 160$, $6 \leqslant Z \leqslant 110$ and $N > Z$ or $N = Z =$ even. There are 571 parameters for $g_1(N)$, $g_2(Z)$, $g_3(A)$, and $g_4(E)$. There are in addition four parameters for ϕ in Eq. (15) for different parity of (N,Z). In total there are 576 parameters for Eq. (15).

TABLE IV. Values of point functions $g_1(N)$, $g_2(Z)$, $g_3(A)$, and $g_4(E)$ (in keV). The first line gives $g_1(1), g_1(2), \ldots, g_1(10)$; the second line gives $g_1(11), g_1(12), \ldots, g_1(20)$, etc. As $g_4(E)$ starts from $E = 0$, for consistency g_4 are given as $g_4(E-1)$ with $E = 1, 2 \ldots, 60$. The last row is the accumulated k of the $g_i(k)$ function in each column.

				$g_1(N)$						
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	244524.6	10
216756.5	189994.9	165321.1	141002.3	118870.4	98548.6	80016.0	62743.8	46856.7	31205.3	20
18019.3	4937.5	−6810.1	−18621.8	−28965.4	−39042.9	−48026.4	−56924.7	−64057.8	−70214.5	30
−75756.1	−80656.9	−84475.8	−88068.0	−90690.5	−93481.7	−95127.1	−97089.2	−97893.7	−99021.0	40
−99009.9	−99433.3	−98774.4	−98764.9	−97675.4	−97161.7	−95813.2	−94953.5	−93271.9	−92154.9	50
−88718.4	−84941.9	−80985.9	−76981.2	−72599.8	−68443.6	−63680.1	−59246.6	−54285.1	−49963.2	60
−45024.3	−40686.7	−35653.2	−31158.3	−26001.2	−21461.0	−16190.9	−11683.3	−6375.1	−1870.1	70
3371.3	7772.2	12930.6	17072.7	21923.6	25794.3	30221.4	33858.9	37972.2	41210.8	80
44786.6	47996.6	53149.5	58044.1	62879.9	67359.1	71997.7	75944.8	80246.0	83511.3	90
87227.1	90059.1	93211.9	95753.8	98623.5	100730.1	103132.1	104812.5	106776.9	107986.6	100
109501.5	110292.4	111315.2	111650.1	112200.5	112036.9	112093.0	111387.1	111057.2	109854.9	110
108941.8	107101.8	105567.6	103041.4	100753.2	97456.9	94321.5	90301.0	86256.1	81315.4	120
76236.5	70385.3	64258.9	57405.8	50091.8	42657.6	36172.9	28857.3	21170.4	12563.8	130
3741.8	−6077.3	−16048.6	−26946.8	−38067.5	−50084.0	−62300.0	−75525.0	−88851.4	−103202.8	140
−117819.9	−133270.1	−149018.5	−165526.9	−182485.3	−200031.6	−218053.3	−236812.0	−256130.6	−275880.5	150
−296197.9	−317083.3	−338168.0	−359843.6	−381970.5	−404758.0	−428015.8	−451620.2	−475776.6	−500421.3	160
				$g_2(Z)$						
0.0	0.0	0.0	0.0	0.0	643274.2	591241.8	540240.4	496067.4	452328.7	10
411463.7	371262.5	334480.6	298326.1	265510.1	234570.1	205746.8	178559.3	153184.2	128636.8	20
107282.1	86045.0	66428.2	47141.5	29051.7	11940.4	−3749.3	−18936.1	−31472.6	−42969.4	30
−53481.8	−63369.2	−71984.4	−80418.9	−87485.9	−94559.7	−100421.5	−106341.2	−110992.1	−115329.8	40
−118389.1	−121683.5	−123898.7	−126422.3	−127816.2	−129150.1	−129523.7	−130051.8	−129645.6	−129534.5	50
−126814.2	−124039.6	−120766.5	−117598.0	−113765.0	−110204.4	−105745.4	−101727.2	−96995.6	−92612.4	60
−87431.1	−82483.7	−76902.5	−71538.1	−65464.3	−59653.4	−53170.4	−47073.3	−40306.2	−33901.6	70
−26928.9	−20233.4	−13106.2	−6317.4	986.5	8061.6	15687.3	22987.0	30544.2	37726.2	80
45221.5	52689.8	61162.3	69243.7	77615.8	85277.9	93110.0	100231.6	107506.6	114186.4	90
120960.1	127247.6	133449.6	139285.9	144862.8	150229.2	155426.8	160325.3	165223.0	169775.6	100
174215.5	178566.3	182460.0	186173.6	189529.4	192653.4	195634.4	198311.2	200737.4	202913.2	110
				$g_3(A)$						
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	10
0.0	0.0	0.0	0.0	0.0	−834146.8	−834648.8	−757734.3	−761094.0	−687407.5	20
−693250.4	−621470.7	−629858.7	−560051.3	−569106.1	−500473.1	−511477.0	−445008.3	−457863.6	−393186.4	30
−407899.3	−344824.6	−360996.9	−299692.5	−317169.5	−256832.6	−275506.5	−216690.7	−236821.3	−179487.2	40
−200949.1	−144785.6	−167582.7	−112957.5	−137014.9	−83596.3	−109032.0	−56603.2	−82703.7	−30917.4	50
−57883.9	−6749.5	−34253.6	16019.3	−12521.5	36901.8	7688.2	56221.1	26083.6	73817.2	60
42828.2	89665.1	57985.4	104135.3	71707.9	117046.2	83957.6	128808.3	95168.6	139289.5	70
105031.9	148619.4	113633.8	156476.3	120874.2	163222.5	127131.7	169154.8	132567.9	174042.8	80
137187.3	178321.6	140849.9	181474.5	143541.9	183582.6	145218.5	184844.8	146073.4	185490.5	90
146438.0	185516.9	146149.3	184918.2	145219.8	183721.3	143818.0	182005.9	141776.2	179523.7	100
138999.6	176598.3	135809.4	173199.7	132181.2	169314.5	128128.4	165073.2	123745.4	160411.3	110
118806.9	155321.5	113501.0	149898.0	107816.7	143969.6	101759.8	137758.6	95368.5	131092.6	120
88517.1	124102.6	81347.4	116794.3	74008.9	109307.7	66497.3	101855.4	58952.5	94384.5	130
51493.3	86874.2	44094.9	79581.4	36891.5	72404.6	29745.8	65285.0	22590.0	58040.3	140
15331.1	50718.4	7906.3	43281.5	491.4	35903.0	−6904.8	28481.1	−14335.3	20962.7	150
−21931.9	13311.2	−29588.3	5602.7	−37264.7	−1995.6	−44805.0	−9458.8	−52162.9	−16660.3	160
−59275.4	−23744.4	−66216.1	−30558.2	−72938.0	−37055.4	−79336.6	−43396.3	−85469.7	−49407.6	170
−91341.2	−55092.1	−96951.1	−60549.5	−102182.5	−65559.2	−107128.2	−70410.0	−111765.9	−74969.3	180
−116256.6	−79215.9	−120397.5	−83131.9	−124084.9	−86774.3	−127545.2	−90106.4	−130741.5	−93236.1	190
−133774.8	−96051.6	−136464.9	−98594.6	−138715.6	−100614.5	−140521.5	−102162.2	−141722.1	−103046.3	200
−142298.9	−103225.6	−142178.9	−102762.5	−141281.2	−101421.9	−139515.9	−99097.2	−136725.9	−95853.0	210
−132918.4	−91535.0	−128175.3	−86346.7	−122521.9	−80360.6	−116269.0	−73841.9	−109411.6	−66635.6	220
−101843.1	−58769.6	−93625.2	−50150.1	−84678.6	−40903.4	−75023.8	−30882.4	−64630.4	−20037.3	230
−53431.0	−8312.9	−41287.8	4268.4	−28125.0	17933.6	−14084.3	32467.3	989.6	48044.3	240
17045.8	64628.4	34202.1	82200.2	52229.5	100605.0	71073.1	119981.8	90894.9	140205.9	250
111546.8	161282.7	133052.7	183282.1	155558.0	206227.4	178942.4	230051.8	203250.8	254784.0	260
228414.2	280536.0	254704.5	307051.2	281693.0	334506.1	309751.5	362911.0	338782.3	392242.2	270

TABLE IV. (*Continued.*)

$g_4(E-1)$										
–12187.6	20605.1	−24895.0	8531.6	−36510.1	−2609.4	−47019.5	−12604.3	−56434.9	−21602.8	10
–64848.5	−29354.2	−72016.5	−35946.1	−78041.8	−41465.6	−82980.8	−45751.8	−86636.0	−48874.7	20
–89145.4	−50865.8	−90538.6	−51663.2	−90743.3	−51278.7	−89734.2	−49664.1	−87490.0	−46779.0	30
–83983.1	−42639.1	−79147.0	−37035.7	−72670.5	−29784.7	−64681.6	−21013.2	−55120.9	−10616.0	40
–43837.6	1569.9	−30736.9	15645.6	−15737.5	31695.7	1355.3	49698.0	20366.4	69685.9	50
41339.1	91636.0	64274.1	115526.5	89095.6	141221.8	115636.1	168639.5	143905.5	197630.8	60

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