Relativistic three-body bound state in a 3D formulation

M. R. Hadizadeh, Ch. Elster

Institute of Nuclear and Particle Physics and Department of Physics and Astronomy, Ohio University, Athens, Ohio 45701, USA

W. N. Polyzou

Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA

Background: The relativistic three-body problem has a long tradition in few-nucleon physics. Calculations of the triton binding energy based on the solution of the relativistic Faddeev equation, in general, lead to a weaker binding than the corresponding nonrelativistic calculation.

Purpose: In this work we solve for the three-body binding energy as well as the wave function and its momentum distribution. The effect of the different relativistic ingredients is studied in detail.

Method: Relativistic invariance is incorporated within the framework of Poincaré invariant quantum mechanics. The relativistic momentum-space Faddeev equation is formulated and directly solved in terms of momentum vectors without employing a partial-wave decomposition.

Results: The relativistic calculation gives a three-body binding energy which is about 3% smaller than its nonrelativistic counterpart. In the wave function, relativistic effects are manifested in the Fermi motion of the spectator particle.

Conclusions: Our calculations show that though the overall relativistic effects in the three-body bound state are small, individual effects by themselves are not necessarily small and must be taken into account consistently.

PACS number(s): 21.45.−v, 21.10.Dr, 21.30.Fe

I. INTRODUCTION

We solve the relativistic three-nucleon bound-state problem and compare the resulting wave functions to the corresponding nonrelativistic bound-state wave functions. While the wave functions themselves are not observable, the difference between the relativistic and nonrelativistic wave functions provides useful information about which observables might be sensitive to the difference. Before going into detail we need to define what we consider the relativistic three-nucleon problem, define what we mean by relativistic effects, and summarize what has been learned from previous work on this problem.

In discussing the three-nucleon problem we limit our considerations to an idealized system modeled on a three-nucleon Hilbert space. This limitation allows us to make meaningful comparisons with the nonrelativistic problem. The difference between a relativistic and a nonrelativistic model is the underlying symmetry group of the theory. For relativistic models the symmetry group is the Poincaré group and for nonrelativistic models it is the Galilean group. Symmetries of a quantum theory preserve observables, i.e., probabilities, expectation values, and ensemble averages. This ensures the invariance of these observables in all inertial reference frames. In the relativistic case the inertial frames are related by Poincaré transformations, while in the nonrelativistic case inertial frames are related by Galilean transformations. Symmetries in a quantum theory are implemented by unitary or antiunitary transformations. In the relativistic case the dynamics is implemented by a unitary projective representation of the Poincaré group [1]. In the nonrelativistic case the dynamics is given by unitary projective representation of the central extension [2] of the Galilean group. Neither of these symmetries impose strong constraints on the dynamics.

Normally, the dynamics is formulated in a particular frame, e.g., the laboratory frame, the center-of-momentum (c.m.) frame, etc. The symmetry only ensures that the results are consistent in frames related to this particular frame by the symmetry transformations.

A second related constraint is cluster separability. In the three-body system, cluster separability means that isolated one- and two-body subsystems should exhibit the same symmetries as the system itself. This is needed to ensure that either special or Galilean relativity can be tested on isolated subsystems. To understand the role of this condition, assume, for example, that the two- and three-body dynamics are formulated in the two- and three-body rest frames, respectively. The two-body subsystems in the three-body rest frame are not generally in the two-body rest frame. However, if the model satisfies cluster properties, then the two-body symmetry transformation determines how to transform the two-body subsystem from its rest frame to the three-nucleon rest frame. This embedding is different for the Galilean and Poincaré symmetry groups and is the source of the relativistic effects that are studied in this work.

One feature of realistic nucleon-nucleon (NN) interactions is that while they are formally motivated by, e.g., meson-exchange models, when cast in a nonrelativistic two-nucleon Hamiltonian, the model is adjusted so that the predicted NN observables agree with experiment with a χ^2 per degree of freedom close to 1 [3,4]. The experimental data are consistent with special relativity, while the nonrelativistic calculation is consistent with a Galilean symmetry. There is an immediate question about what is being compared to the data to obtain the quoted χ^2. The answer depends on how the analysis is performed. Normally, laboratory frame data are correctly transformed to the c.m. frame using a Lorentz transformation.

The correctly transformed scattering data are then compared to the nonrelativistic c.m. scattering solutions. Parameters of the interaction are fine tuned to achieve agreement with the data. In this case, phase shifts are identified as functions of the relativistic and nonrelativistic relative momenta. This means that in any other frame the experimental and computed cross sections will no longer be identical functions of the laboratory energy.

The important consequence of this is that in the preferred c.m. frame the nonrelativistic two-body calculation gives the experimental result. It is also possible to introduce relativistic NN interactions that are consistent with the same scattering data. If one were to take the nonrelativistic limit of the relativistic model, the scattering observables would change as a result of the approximation and would not agree with the results of a nonrelativistic model that is fit to the same experimental data. This aspect of realistic NN interactions must be taken into account when interpreting relativistic corrections in the three-body problem.

In this work we construct the relativistic NN interaction so that the rest-frame relativistic and nonrelativistic NN wave functions and phase shifts are identical. Cluster properties determine how these two-body models should be embedded in the three-nucleon Hilbert space. Relativistic effects are attributable entirely to the different ways that these two-body interactions appear in the three-nucleon problem to satisfy cluster properties.

The resulting formulation of the relativistic three-nucleon problem has the property that in the limit that the momenta are all small compared to the nucleon masses, the relativistic Faddeev equation reduces to the nonrelativistic one, which justifies our interpretation of the difference being attributed to "relativistic effects." It is appropriate to think of the relativistic effects being attributable to the difference between relativistic and nonrelativistic treatments of Fermi motion, which involve Galilean or Poincaré boosts of the subsystems.

In this initial work the nonrelativistic NN potential is a Malfliet-Tjon V (MT-V) [5] type interaction. The formal definition of the phase and wave-function equivalent relativistic NN interactions that we use was given by Coester, Pieper, and Serduke [6]. The construction of the corresponding relativistic NN transition operators was given in Ref. [7] and successfully implemented in Refs. [8–10]. The two-body unitary representation of the Poincaré group is formulated using a construction given by Bakamjian and Thomas [11]. The corresponding three-body unitary representation of the Poincaré group that satisfies S-matrix cluster properties was introduced by Coester [12]. Note that while it is possible to realize cluster properties of the unitary representation of the Poincaré group [13–15], for the three-body problem it is sufficient and far simpler to formulate a model where the S matrix clusters properly. The difference is that cluster properties of the unitary representation of the Poincaré group requires three-body forces that are generated by the two-body forces. In the formulation where only the S matrix clusters, the required three body-forces are replaced by two-body interactions that depend on the spectator momentum. The important relation is that both formulations of the relativistic three-body problem give identical three-body scattering observables. They are related by an S-matrix preserving unitary transformation that becomes the identity in the three-body rest frame [14].

While many of the methods mentioned in the previous paragraph are formulated using one of Dirac's forms of dynamics [16], in our calculations the form of dynamics is only relevant if we choose to transform our results from the three-body rest frame to an arbitrary frame. This can be done consistently in any form of dynamics [17].

In this work we solve the relativistic Faddeev equation using direct integration [18–20]. This method gives the three-body wave function directly, so it does not have to be reconstructed using partial-wave methods. Part of our motivation for using direct integration in the bound-state problem is to provide test problems that can be compared to previous relativistic partial-wave calculations. Ultimately, these calculations need to be extended to treat spin-dependent interactions and scattering at relativistic energies, where direct-integration methods are essential.

The first relativistic three-body calculations using the formalism that we use in this paper were performed by Glöckle, Coester, and Lee [21]. Their calculations used the same MT-V interaction. However, they approximated the relativistic interaction so the phase equivalence with the nonrelativistic interaction was only approximate and they used a partial-wave expansion that was truncated to s waves. They found a small decrease in the binding energy due to relativistic effects. Relativistic bound-state calculations with a realistic interaction were performed by Kamada *et al.* [22]. These were fully converged partial-wave calculations. The resulting binding energy corrections were comparable to the corrections obtained by Glöckle, Coester, and Lee. Three-body scattering calculations using a realistic interaction and including a three-nucleon force were also performed using partial-wave methods for energies up to 250 MeV [23]. These calculations exhibited relativistic effects in the breakup observables and large-angle elastic scattering showed evidence of missing degrees of freedom. Direct-integration scattering calculations were successfully performed using the MT-V interaction for energies up to 2 GeV [8–10]. These also exhibited strong relativistic effects in certain breakup observables and demonstrated the value of direct-integration methods at higher energies. Our long-term goal is to perform few-GeV-scale scattering calculations with realistic interactions. This requires using direct integration with realistic interactions. Because there are no such calculations, the first step is to establish that the method works for the bound-state problem, where the results can be compared to partial-wave calculations. This work is a precursor to including a relativistic treatment with realistic interactions using direct-integration methods.

This work addresses two omissions of the body of work discussed above. First there are no relativistic bound-state calculations that have utilized direct-integration methods. Second, while the three-body binding energy has been computed, there is no published work comparing the relativistic and nonrelativistic bound-state wave functions. While wave functions are not directly observable, observables are sensitive to the structure of the wave functions and differences in the relativistic and nonrelativistic wave functions are responsible for relativistic effects in three-body observables.

II. THE POINCARÉ INVARIANT FADDEEV EQUATION

A. Kinematic variables

In both the relativistic and nonrelativistic three-body problems, Poincaré or Galilean symmetry relates the state of the system in its rest frame to its state in a general frame. The dynamics is usually formulated in the rest frame of the system. Thus, it is useful to formulate the three-body problem using variables that describe the momenta of particles in the three-body rest frame. The two-body subsystem, however, uses variables that describe the momenta of particles in the rest frame of the subsystem. The relevant variables are defined by boosting the single-particle momenta to the three-body rest frame and then boosting two-body subsystem momenta to the rest frame of the subsystem. Using Galilean boosts results in the Jacobi momenta that are used in nonrelativistic three-body calculations [24]. Replacing the Galilean boosts by Lorentz boosts leads to relativistic Jacobi momenta. In both cases these are changes of variables from single-particle momenta to variables that are more convenient in three-body applications.

In what follows we assume that all nucleons have the same mass, m. We denote the single-particle four-momenta by p_i^μ. We define the total four-momentum of the noninteracting three-body system $P^\mu := \sum_{i=1}^3 p_i^\mu$ and its invariant mass $M_0^2 = -P^\mu P_\mu$. Relativistic Jacobi momenta are constructed by first boosting the p_i^μ to the three-body rest frame with a rotationless boost $\Lambda^{-1}(\mathbf{P}/M_0)^\mu{}_\nu$,

$$k_i^\mu := \Lambda^{-1}(\mathbf{P}/M_0)^\mu{}_\nu p_i^\nu. \tag{1}$$

The vector components of k_i^μ are

$$\mathbf{k}_i = \mathbf{p}_i + \frac{\mathbf{P}}{M_0} \left[\frac{\mathbf{P} \cdot \mathbf{p}_i}{M_0 + \sqrt{M_0^2 + \mathbf{P}^2}} - \omega_m(p_i) \right], \tag{2}$$

where $\mathbf{P}$ is the three-vector part of P^μ and $\omega_m(p_i) = p_i^0 = \sqrt{m^2 + p_i^2}$ is the energy of the ith particle.

The vectors $\mathbf{k}_i$ are not independent and satisfy

$$\sum_{i=1}^3 \mathbf{k}_i = \mathbf{0}, \quad M_0 := \sum_{i=1}^3 \omega_m(k_i). \tag{3}$$

The other relativistic Jacobi momentum variables are obtained by boosting k_j^μ to the rest frame of the pair (jk). Following Eq. (2) we denote the four-momentum of the pair (jk) in the three-body rest frame by $k_{jk}^\mu = k_j^\mu + k_k^\mu$ and the two-body invariant mass of the subsystem (jk) by $m_{0jk}^2 = -k_{jk}^\mu k_{\mu jk}$. The other relativistic Jacobi momenta are defined by

$$p_{jk}^\mu := \Lambda^{-1}(\mathbf{k}_{jk}/m_{0jk})^\mu{}_\nu k_j^\nu. \tag{4}$$

The vector components of p_{jk}^μ are

$$\mathbf{p}_{jk} = \mathbf{k}_j + \frac{\mathbf{k}_{jk}}{m_{0jk}} \left\{ \frac{\mathbf{k}_{jk} \cdot \mathbf{k}_j}{m_{0jk} + \sqrt{m_{0jk}^2 + k_{jk}^2}} - \omega_m(k_j) \right\}. \tag{5}$$

The inverse of Eq. (5) is given as

$$\mathbf{k}_j = \mathbf{p}_{jk} + \frac{\mathbf{k}_{jk}}{m_{0jk}} \left\{ \frac{\mathbf{k}_{jk} \cdot \mathbf{p}_{jk}}{m_{0jk} + \sqrt{m_{0jk} + k_{jk}^2}} + \omega_m(p_{jk}) \right\}. \tag{6}$$

The pairs $(\mathbf{p}_{jk}, \mathbf{k}_i)$ are the relativistic analogs of the usual Jacobi momenta. If the Lorentz boosts $\Lambda^{-1}(\cdot)^\mu{}_\nu$ are replaced by Galilean boosts these become the nonrelativistic Jacobi momenta [24].

The different choices of independent momentum variables are the single-particle momenta $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, the total momentum plus the momenta of any two particles in the three-body rest frame, $\{\mathbf{P}, \mathbf{k}_j, \mathbf{k}_k\}$, and the relativistic Jacobi momenta for the pair (jk), $\{\mathbf{P}, \mathbf{p}_{jk}, \mathbf{k}_i\}$. The Jacobian of the variable change $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \leftrightarrow \{\mathbf{P}, \mathbf{k}_j, \mathbf{k}_k\}$ is one when $\mathbf{P} = 0$, while the Jacobian of the variable change $\{\mathbf{P}, \mathbf{k}_j, \mathbf{k}_k\} \leftrightarrow \{\mathbf{P}, \mathbf{p}_{jk}, \mathbf{k}_i\}$ is given by

$$\begin{aligned} \mathcal{N}^2(\mathbf{k}_j, \mathbf{k}_k) &:= \left| \frac{\partial(\mathbf{k}_j, \mathbf{k}_k)}{\partial(\mathbf{p}_{jk}, \mathbf{k}_{jk})} \right| \\ &= \frac{\omega_m(p_{jk}) + \omega_m(p_{jk})}{\omega_m(k_j) + \omega_m(k_k)} \frac{\omega_m(k_j)\omega_m(k_k)}{\omega_m(p_{jk})\omega_m(p_{jk})}. \end{aligned} \tag{7}$$

In the limit that the momenta are much smaller than the masses the relativistic Jacobi momenta become identical to the nonrelativistic Jacobi momenta and the Jacobian becomes 1.

B. Two-body interactions embedded in the three-body space

Realistic two-body interactions may be, e.g., motivated by meson exchange or other effective field theories, but the parameters of the interaction must be fine tuned to be consistent with experimental two-body scattering observables. This means that the nonrelativistic interactions are already consistent with data and at the two-body level should not be considered as approximations to a relativistic two-body model. Instead, a realistic relativistic two-body model should be consistent with the same data. Once the two-body model is defined, cluster properties dictate how the two-body interactions appear in the three-body problem. This is different in the relativistic and nonrelativistic formalisms.

Given a nonrelativistic two-body model fit to scattering data, we define a relativistic interaction fit to the same data by requiring that the relativistic wave functions as a function of $\mathbf{p}_{jk}$ are identical to the nonrelativistic wave functions as a function of the corresponding nonrelativistic Jacobi momentum. Because the phase shifts can be extracted from asymptotic properties of the wave functions, this ensures that both interactions give the same phase shifts as a function of the $\mathbf{p}_{jk}$. This can be proved using the invariance principle [15,25].

We begin by defining the interacting two-body invariant mass operator (relativistic rest-frame Hamiltonian) for the pair (jk) in terms of the nonrelativistic two-body interaction by

$$m_{jk}^2 = 4m \left(\frac{p_{jk}^2}{m} + v_{jk}^{\mathrm{nr}} + m \right), \tag{8}$$

where v_{jk}^{nr} is the nonrelativistic NN interaction between particles j and k. Because m_{jk}^2 is a function of the nonrelativistic

rest-frame Hamiltonian, $\frac{p_{jk}^2}{m} + v_{jk}^{\text{nr}}$, it has the same eigen-functions. The phase-equivalent relativistic interaction, v_{jk}^{r}, is defined in terms of m_{jk} by

$$v_{jk}^{\text{r}} = m_{jk} - m_{0jk}. \tag{9}$$

While it is possible to formally solve the nonlinear relation needed to express v_{jk}^r in terms of v_{jk}^{nr} [26], this is not needed to formulate the relativistic Faddeev equation.

The input to the Faddeev equation is the two-body transition operators properly embedded in the three-body Hilbert space. As in the two-body case we define interactions as the difference between the three-body mass operator with and without the two-body interaction. This will satisfy S-matrix cluster properties if this $2 + 1$-body mass operator leads to the same two-body scattering operator as the two-body mass operator m_{jk}^2. This will be true if we can write the three-body interacting mass operator with pair (jk) interacting as a function of m_{jk}. This can be achieved by defining

$$M_{jk} = \sqrt{k_i^2 + m_{jk}^2} + \omega_m(k_i)$$
$$= \sqrt{k_i^2 + 4m^2 + 4p_{jk}^2 + 4mv_{jk}^{\text{nr}}} + \omega_m(k_i). \tag{10}$$

The two-body interactions embedded in the three-body Hilbert space are

$$V_{jk} = M_{jk} - M_0 \tag{11}$$
$$= \sqrt{k_i^2 + 4m^2 + 4p_{jk}^2 + 4mv_{jk}^{\text{nr}}} - \sqrt{k_i^2 + 4m^2 + 4p_{jk}^2}. \tag{12}$$

The three-body bound states are eigenstates of the three-body mass operator

$$M_t = M_0 + V_{12} + V_{23} + V_{31}. \tag{13}$$

The Faddeev kernel involves the two-body transition operators $T_{jk}(z)$ that act in the three-particle Hilbert space. They are

defined by

$$T_{jk}(z) := V_{jk} + V_{jk}(z - M_{jk})^{-1}V_{jk}. \tag{14}$$

This is a function of the nonrelativistic two-body interaction between particles j and k. Because of this relation, matrix elements of $T_{jk}(z)$ can be obtained directly from the nonrelativistic two-body transition matrix elements using a two-step process. This method is exact and avoids the problem of computing the relativistic two-body interaction.

The first step is to use the general relation between the interaction, scattering wave functions and half-shell transition operators

$$V_{jk}|(\mathbf{p}_{jk}, \mathbf{k}_i)^+\rangle = T_{jk}(z_0)|\mathbf{p}_{jk}, \mathbf{k}_i\rangle, \tag{15}$$

where $z_0 = \sqrt{m_{0jk}^2(p_{jk}) + k_i^2} + i0^+$ is the on-shell energy. Using the relativistic and nonrelativistic versions of this relation leads to the identity

$$T_{jk}\left(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; \sqrt{m_{0jk}^2(\mathbf{p}'_{jk}) + k_i^2} + i0^+\right)$$
$$= F(p_{jk}, p'_{jk}, k_i)\, t_{\text{nr}}\left(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; \frac{p_{jk}'^2}{m} + i0^+\right), \tag{16}$$

where the ratio of the half-shell transition matrix elements is

$$F(p_{jk}, p'_{jk}, k_i) = \frac{4m}{\sqrt{m_{0jk}^2(p_{jk}) + k_i^2} + \sqrt{m_{0jk}^2(p'_{jk}) + k_i^2}}, \tag{17}$$

with $m_{0jk}(p_{jk}) = 2\omega_m(p_{jk})$ and

$$\langle \mathbf{p}_{jk}, \mathbf{k}_i|T_{jk}(z)|\mathbf{p}'_{jk}, \mathbf{k}'_i\rangle = \delta(\mathbf{k}_i - \mathbf{k}'_i)T_{jk}(\mathbf{p}_{jk}, \mathbf{p}'_{jk}, z - \omega_m(k_i)). \tag{18}$$

Equation (16) can be used to express $T_{jk}(\mathbf{p}_{jk}, \mathbf{p}'_{jk};$ $\sqrt{m_{0jk}^2(\mathbf{p}'_{jk}) + k_i^2} + i0^+)$ in terms of $t_{\text{nr}}(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; \frac{p_{jk}'^2}{m} + i0^+)$. However, this relation is only valid for half on-shell transition matrices. In the Faddeev equation $T_{jk}(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; z)$ is needed for off-shell values of z. Those can be obtained by solving an integral equation for $T_{jk}(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; z)$ that uses Eq. (16) as input,

$$T_{jk}(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; z - \omega_m(k_i)) = T_{jk}\left(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; \sqrt{m_{0jk}^2(p'_{jk}) + k_i^2} + i0^+\right)$$
$$+ \int d\mathbf{p}''_{jk}\left[\frac{1}{z - \sqrt{m_{0jk}^2(p''_{jk}) + k_i^2}} - \frac{1}{\sqrt{m_{0jk}^2(p'_{jk}) + k_i^2} - \sqrt{m_{0jk}^2(p''_{jk}) + k_i^2} + i0^+}\right]$$
$$\times T_{jk}(\mathbf{p}_{jk}, \mathbf{p}''_{jk}; z - \omega_m(k_i))T_{jk}\left(\mathbf{p}''_{jk}, \mathbf{p}'_{jk}; \sqrt{m_{0jk}^2(p'_{jk}) + k_i^2} + i0^+\right). \tag{19}$$

This relation, which follows from first resolvent equations, is derived in the Appendix. To compute the relativistic Faddeev kernel, one only needs to solve the integral equation, Eq. (19), for $T_{jk}(\mathbf{p}_{jk}, \mathbf{p}'_{jk}; z - \omega_m(k_i))$, which uses the relativistic half-shell transition matrix elements as input.

C. Faddeev equations

The relativistic three-body bound state is a discrete eigen-state of the relativistic three-body mass operator M defined in Eq. (13). The eigenvalue problem can be reformulated as a system of coupled integral equations for the Faddeev components of the wave functions. For identical particles

this reduces to a single equation for one of the Faddeev components,

$$|\psi_i\rangle = (M_t - M_0)^{-1} \, T_{jk}(M_t) \, P \, |\psi_i\rangle, \tag{20}$$

where $M_t = E_t + 3m$ is three-body mass eigenvalue, and $P = P_{12}P_{23} + P_{13}P_{23}$ is the permutation operator for three identical particles. This equation has nonzero solutions when

M_t is an eigenvalue of Eq. (20). The bound-state wave function constructed from the solution of (20) is

$$|\Psi\rangle = |\psi_i\rangle + P|\psi_i\rangle. \tag{21}$$

For the explicit calculation we use the basis $|\mathbf{p}_{jk},\mathbf{k}_i\rangle$, where i is fixed. In this basis Eq. (20) becomes

$$\langle\mathbf{p}_{jk},\mathbf{k}_i|\psi_i\rangle = \int d\mathbf{p}'_{jk}\,d\mathbf{k}'_i\,d\mathbf{p}''_{jk}\,d\mathbf{k}''_i\,\frac{\delta(\mathbf{k}_i - \mathbf{k}'_i)\,T_{jk}(\mathbf{p}_{jk},\mathbf{p}'_{jk};M_t - \omega_m(k_i))}{M_t - M_0(p_{jk},k_i)}\langle\mathbf{p}'_{jk},\mathbf{k}'_i|P|\mathbf{p}''_{jk},\mathbf{k}''_i\rangle\langle\mathbf{p}''_{jk},\mathbf{k}''_i|\psi_i\rangle$$

$$= \frac{1}{M_t - M_0(p_{jk},k_i)} \int d\mathbf{p}'_{jk}\,d\mathbf{k}'_i\,d\mathbf{p}''_{jk}\,d\mathbf{k}''_i\,\delta(\mathbf{k}_i - \mathbf{k}'_i)\,T_{jk}(\mathbf{p}_{jk},\mathbf{p}'_{jk};M_t - \omega_m(k_i))$$

$$\times(\langle\mathbf{p}'_{jk},\mathbf{k}'_i|\mathbf{p}''_{ki},\mathbf{k}''_j\rangle + \langle\mathbf{p}'_{jk},\mathbf{k}'_i|\mathbf{p}''_{ij},\mathbf{k}''_k\rangle)\,\langle\mathbf{p}''_{jk},\mathbf{k}''_i|\psi_i\rangle. \tag{22}$$

Here the permutation operators contain two delta functions which eliminate two of the integrals. We use them to eliminate the integrals over $\mathbf{p}'_{jk}$ and $\mathbf{p}''_{jk}$. The matrix elements of the permutation operators have the form

$$\langle\mathbf{p}'_{jk},\mathbf{k}'_i|P|\mathbf{p}''_{jk},\mathbf{k}''_i\rangle = \langle\mathbf{p}'_{jk},\mathbf{k}'_i|\mathbf{p}''_{ki},\mathbf{k}''_j\rangle + \langle\mathbf{p}'_{jk},\mathbf{k}'_i|\mathbf{p}''_{ij},\mathbf{k}''_k\rangle$$

$$= \frac{\delta^3(\mathbf{p}'_{jk} - \mathbf{p}_{jk}(\mathbf{k}''_i, -\mathbf{k}'_i - \mathbf{k}''_i))\,\delta^3(\mathbf{p}''_{jk} - \mathbf{p}_{jk}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}'_i))}{\mathcal{N}(\mathbf{k}''_i, -\mathbf{k}'_i - \mathbf{k}''_i)\,\mathcal{N}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}'_i)}$$

$$+ \frac{\delta^3(\mathbf{p}'_{jk} - \mathbf{p}_{jk}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}''_i))\,\delta^3(\mathbf{p}''_{jk} - \mathbf{p}_{jk}(\mathbf{k}'_i, -\mathbf{k}'_i - \mathbf{k}''_i))}{\mathcal{N}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}''_i)\,\mathcal{N}(\mathbf{k}'_i, -\mathbf{k}'_i - \mathbf{k}''_i)}, \tag{23}$$

where $\mathcal{N}(\mathbf{k}_i,\mathbf{k}_j)$ is the square root of the Jacobian of the variable change defined in Eq. (7).

Using the symmetry property $\mathcal{N}(\mathbf{k}_j,\mathbf{k}_k) = \mathcal{N}(\mathbf{k}_k,\mathbf{k}_j)$ of the Jacobian, the matrix elements of the permutation operator P, Eq. (23), can be written as

$$\langle\mathbf{p}'_{jk},\mathbf{k}'_i|P|\mathbf{p}''_{jk},\mathbf{k}''_i\rangle = N(\mathbf{k}'_i,\mathbf{k}''_i)\{\delta^3(\mathbf{p}'_{jk} - \mathbf{p}_{jk}(\mathbf{k}''_i, -\mathbf{k}'_i - \mathbf{k}''_i))\,\delta^3(\mathbf{p}''_{jk} - \mathbf{p}_{jk}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}'_i))$$

$$+ \delta^3(\mathbf{p}'_{jk} - \mathbf{p}_{jk}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}''_i))\,\delta^3(\mathbf{p}''_{jk} - \mathbf{p}_{jk}(\mathbf{k}'_i, -\mathbf{k}'_i - \mathbf{k}''_i))\}, \tag{24}$$

where

$$N(\mathbf{k}'_i,\mathbf{k}''_i) = \mathcal{N}^{-1}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}''_i)\,\mathcal{N}^{-1}(-\mathbf{k}'_i - \mathbf{k}''_i,\mathbf{k}'_i). \tag{25}$$

Inserting Eq. (24) into Eq. (22) and integrating over the δ functions leads to the relativistic Faddeev integral equation

$$\langle\mathbf{p}_{jk},\mathbf{k}_i|\psi_i\rangle = \frac{1}{M_t - M_0(p_{jk},k_i)} \int d\mathbf{k}'_i\,N(\mathbf{k}_i,\mathbf{k}'_i)\,T^{\text{sym}}_{jk}(\mathbf{p}_{jk},\tilde{\boldsymbol{\pi}};M_t - \omega_m(k_i))\,\langle\boldsymbol{\pi},\mathbf{k}'_i|\psi_i\rangle, \tag{26}$$

where T^{sym}_{jk} is the symmetrized boosted two-body T matrix, defined by

$$T^{\text{sym}}_{jk}(\mathbf{p}_{jk},\mathbf{p}'_{jk};\epsilon) = T_{jk}(\mathbf{p}_{jk},\mathbf{p}'_{jk};\epsilon) + T_{jk}(-\mathbf{p}_{jk},\mathbf{p}'_{jk};\epsilon) \tag{27}$$

and

$$\tilde{\boldsymbol{\pi}} = \mathbf{p}_{jk}(\mathbf{k}'_i, -\mathbf{k}_i - \mathbf{k}'_i) = \mathbf{k}'_i + \tfrac{1}{2}C(\mathbf{k}_i,\mathbf{k}'_i)\,\mathbf{k}_i,$$

$$\boldsymbol{\pi} = \mathbf{p}_{jk}(\mathbf{k}_i + \mathbf{k}'_i, -\mathbf{k}'_i) = \mathbf{k}_i + \tfrac{1}{2}C(\mathbf{k}'_i,\mathbf{k}_i)\,\mathbf{k}'_i. \tag{28}$$

The coefficient $C(\mathbf{k}_i,\mathbf{k}'_i)$ is defined as [8]

$$C(\mathbf{k}_i,\mathbf{k}'_i) \equiv 1 + \frac{\omega_m(k'_i) - \omega_m(|\mathbf{k}_i + \mathbf{k}'_i|)}{\omega_m(k'_i) + \omega_m(|\mathbf{k}_i + \mathbf{k}'_i|) + \sqrt{[\omega_m(k'_i) + \omega_m(|\mathbf{k}_i + \mathbf{k}'_i|)]^2 - k_i^2}}. \tag{29}$$

In deriving Eq. (26) we used the property

$$\langle\mathbf{p}_{jk},\mathbf{k}_i|\psi_i\rangle = \langle-\mathbf{p}_{jk},\mathbf{k}_i|\psi_i,\rangle. \tag{30}$$

The relativistic momenta $\tilde{\boldsymbol{\pi}}$ and $\boldsymbol{\pi}$ defined in Eq. (28) become the corresponding nonrelativistic ones if the coefficient $C(\mathbf{k}_i,\mathbf{k}'_i)$ is set equal to 1, which is the case if the momenta are small with respect to the masses.

To solve the integral equation Eq. (26), we follow Ref. [18] and choose a coordinate system where $\mathbf{k}_i$ is parallel to the z axis and $\mathbf{p}_{jk}$ is in the $x-z$ plane. Then the variables in the Faddeev integral equation are the magnitudes of the vectors and the angle

between them,

$$
\begin{aligned}
x &\equiv x_{p_{jk}} = \hat{\mathbf{k}}_i \cdot \hat{\mathbf{p}}_{jk}, \\
x' &\equiv x_{k'_i} = \hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}'_i, \\
y &\equiv x_{p_{jk}k'_i} = \hat{\mathbf{p}}_{jk} \cdot \hat{\mathbf{k}}'_i = xx' + \sqrt{1-x^2}\sqrt{1-x'^2}\cos(\phi_{k'_i}), \\
\tilde{\pi} &= \sqrt{\frac{1}{4}C^2(k_i,k'_i,x_{q'})k_i^2 + k_i'^2 + C(k_i,k'_i,x_{k'_i})k_i k'_i x_{k'_i}}, \\
\pi &= \sqrt{k_i^2 + \frac{1}{4}C^2(k'_i,k_i,x_{q'})k_i'^2 + C(k'_i,k_i,x_{k'_i})k_i k'_i x_{k'_i}}, \\
x_{p_{jk}\tilde{\pi}} &= \frac{\frac{1}{2}C(k_i,k'_i,x_{k'_i})k_i x_{p_{jk}} + k'_i y}{\tilde{\pi}}, \\
x_{\pi k'_i} &= \frac{k_i x_{k'_i} + \frac{1}{2}C(k'_i,k_i,x_{k'_i})k'_i}{\pi}.
\end{aligned}
\tag{31}
$$

Using these variables, Eq. (26) takes the explicit form

$$
\langle p_{jk},k_i,x_{p_{jk}}|\psi_i\rangle = \frac{1}{M_t - \omega_m(k_i) - \sqrt{m_{0jk}^2(p_{jk})+k_i^2}} \int_0^\infty dk'_i k_i'^2 \int_{-1}^{+1} dx_{k'_i} \int_0^{2\pi} d\phi_{k'_i}
$$

$$
\times N(k_i,k'_i,x_{k'_i})\, T_{jk}^{\text{sym}}(p_{jk},\tilde{\pi},x_{p_{jk}\tilde{\pi}};M_t-\omega_m(k_i))\langle\pi,k'_i,x_{\pi k'_i}|\psi_i\rangle.
\tag{32}
$$

To solve Eq. (32), which has the form $|\psi_i\rangle = K(M_t)|\psi_i\rangle$, we treat it as an eigenvalue problem of the form $\lambda|\psi_i\rangle = K(M_t)|\psi_i\rangle$ and vary M_t until λ is 1 to a given precision.

D. Three-body wave function

Once the Faddeev component $\langle\mathbf{p}_{jk},\mathbf{k}_i|\psi_i\rangle$ is calculated, the three-body wave function can be obtained from Eq. (21) as

$$
\begin{aligned}
\langle\mathbf{p}_{jk},\mathbf{k}_i|\Psi\rangle &= \langle\mathbf{p}_{jk},\mathbf{k}_i|\psi_i\rangle \\
&+ \left|\frac{\partial(\mathbf{p}_{ki},\mathbf{k}_j)}{\partial(\mathbf{p}_{jk},\mathbf{k}_i)}\right|^{1/2}\langle\mathbf{p}_{ki},\mathbf{k}_j|\psi_i\rangle \\
&+ \left|\frac{\partial(\mathbf{p}_{ij},\mathbf{k}_k)}{\partial(\mathbf{p}_{jk},\mathbf{k}_i)}\right|^{1/2}\langle\mathbf{p}_{ij},\mathbf{k}_k|\psi_i\rangle,
\end{aligned}
\tag{33}
$$

where the Jacobi momenta in systems (ki,j) and (ij,k) are connected to the ones in system (jk,i), i.e., $\mathbf{p}_{jk},\mathbf{k}_i$, by

$$
\mathbf{k}_j = \mathbf{p}_{jk} + \frac{\mathbf{k}_i}{m_{0jk}}\left[\frac{\mathbf{k}_i \cdot \mathbf{p}_{jk}}{m_{0jk} + \sqrt{m_{0jk}^2 + k_i^2}} - \omega_m(p_{jk})\right],
$$

$$
\begin{aligned}
\mathbf{k}_k &= -\mathbf{k}_i - \mathbf{k}_j \\
&= -\mathbf{k}_i - \mathbf{p}_{jk} - \frac{\mathbf{k}_i}{m_{0jk}}\left[\frac{\mathbf{k}_i \cdot \mathbf{p}_{jk}}{m_{0jk} + \sqrt{m_{0jk}^2 + k_i^2}} - \omega_m(p_{jk})\right],
\end{aligned}
$$

$$
\mathbf{p}_{ij} = \mathbf{k}_i + \frac{\mathbf{k}_k}{m_{0ij}}\left[\frac{\mathbf{k}_k \cdot \mathbf{k}_i}{m_{0ij} + \sqrt{m_{0ij}^2 + k_k^2}} + \omega_m(k_i)\right],
$$

$$
\mathbf{p}_{ki} = \mathbf{k}_k + \frac{\mathbf{k}_j}{m_{0ki}}\left[\frac{\mathbf{k}_j \cdot \mathbf{k}_k}{m_{0ki} + \sqrt{m_{0ki}^2 + k_j^2}} + \omega_m(k_k)\right],
\tag{34}
$$

with

$$
\begin{aligned}
m_{0ij} &= \sqrt{[\omega_m(k_i)+\omega_m(k_j)]^2 - k_k^2}, \\
m_{0ki} &= \sqrt{[\omega_m(k_k)+\omega_m(k_i)]^2 - k_j^2}.
\end{aligned}
\tag{35}
$$

The Jacobians for changing the basis states from system (jk,i) to (ki,j) and (ij,k) are

$$
\begin{aligned}
\left|\frac{\partial(\mathbf{p}_{jk},\mathbf{k}_i)}{\partial(\mathbf{p}_{ki},\mathbf{k}_j)}\right| &= \left|\frac{\partial(\mathbf{p}_{jk},\mathbf{k}_i)}{\partial(\mathbf{k}_j,\mathbf{k}_k)}\right|\left|\frac{\partial(\mathbf{k}_j,\mathbf{k}_k)}{\partial(\mathbf{k}_i,\mathbf{k}_k)}\right|\left|\frac{\partial(\mathbf{k}_i,\mathbf{k}_k)}{\partial(\mathbf{p}_{ki},\mathbf{k}_j)}\right| \\
&= \frac{\mathcal{N}^2(\mathbf{k}_i,\mathbf{k}_k)}{\mathcal{N}^2(\mathbf{k}_j,\mathbf{k}_k)},
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{\partial(\mathbf{p}_{jk},\mathbf{k}_i)}{\partial(\mathbf{p}_{ij},\mathbf{k}_k)}\right| &= \left|\frac{\partial(\mathbf{p}_{jk},\mathbf{k}_i)}{\partial(\mathbf{k}_j,\mathbf{k}_k)}\right|\left|\frac{\partial(\mathbf{k}_j,\mathbf{k}_k)}{\partial(\mathbf{k}_i,\mathbf{k}_j)}\right|\left|\frac{\partial(\mathbf{k}_i,\mathbf{k}_j)}{\partial(\mathbf{p}_{ij},\mathbf{k}_k)}\right| \\
&= \frac{\mathcal{N}^2(\mathbf{k}_i,\mathbf{k}_j)}{\mathcal{N}^2(\mathbf{k}_j,\mathbf{k}_k)}.
\end{aligned}
\tag{36}
$$

The Jacobi momenta in Eq. (34) can be explicitly given as a function of the Jacobi momenta of system (jk,i),

$$
\begin{aligned}
\mathbf{k}_j &= \mathbf{p}_{jk} + \alpha\mathbf{k}_i, \\
\mathbf{k}_k &= -\mathbf{p}_{jk} - \beta\mathbf{k}_i, \\
\mathbf{p}_{ij} &= \gamma_p\mathbf{p}_{jk} + \gamma_k\mathbf{k}_i, \\
\mathbf{p}_{ki} &= \xi_p\mathbf{p}_{jk} + \xi_k\mathbf{k}_i,
\end{aligned}
\tag{37}
$$

where

$$
\alpha = \frac{1}{m_{0jk}}\left(\frac{p_{jk}k_i x_{p_{jk}}}{m_{0jk}+\sqrt{m_{0jk}^2+k_i^2}} - \frac{1}{2}m_{0jk}\right),
$$

$$
\beta = 1 + \alpha,
$$

$$\gamma_p = \frac{1}{m_{0ij}}\left(\frac{p_{jk}k_i x_{p_{jk}} + \beta k_i^2}{m_{0ij} + \sqrt{m_{0ij}^2 + k_k^2}} + m_{0i}\right),$$

$$\gamma_k = 1 + \gamma_p\beta,$$

$$\xi_p = -1 - \frac{1}{m_{0ki}}\left[\frac{p_{jk}^2 + \alpha\beta k_i^2 + (\alpha+\beta)p_{jk}k_i x_{p_{jk}}}{m_{0ki} + \sqrt{m_{0ki}^2 + k_j^2}} - m_{0k}\right],$$

$$\xi_k = \alpha(\xi_p + 1) - \beta. \tag{38}$$

In the coordinate system defined by Eq. (31), the relativistic three-body wave function of Eq. (33) has the form

$$\langle p_{jk}, k_i, x_{p_{jk}}|\Psi\rangle = \langle p_{jk}, k_i, x_{p_{jk}}|\psi_i\rangle$$
$$+ \frac{\mathcal{N}(\mathbf{k}_j, \mathbf{k}_k)}{\mathcal{N}(\mathbf{k}_i, \mathbf{k}_k)}\langle p_{ki}, k_j, x_{p_{ki}k_j}|\psi_i\rangle$$
$$+ \frac{\mathcal{N}(\mathbf{k}_j, \mathbf{k}_k)}{\mathcal{N}(\mathbf{k}_i, \mathbf{k}_j)}\langle p_{ij}, k_k, x_{p_{ij}k_k}|\psi_i\rangle, \tag{39}$$

with

$$p_{ki} = |\mathbf{p}_{ki}| = |\xi_p\mathbf{p}_{jk} + \xi_k\mathbf{k}_i|$$
$$= \sqrt{\xi_p^2 p_{jk}^2 + \xi_k^2 k_i^2 + 2\xi_p\xi_k p_{jk}k_i x_{p_{jk}}},$$

$$k_j = |\mathbf{k}_j| = |\mathbf{p}_{jk} + \alpha\mathbf{k}_i|$$
$$= \sqrt{p_{jk}^2 + \alpha^2 k_i^2 + 2\alpha p_{jk}k_i x_{pjk}},$$

$$x_{p_{ki}k_j} \equiv \hat{\mathbf{p}}_{ki}\cdot\hat{\mathbf{k}}_j = \frac{\xi_p p_{jk}^2 + \alpha\xi_k k_i^2 + (\alpha\xi_p + \xi_k)p_{jk}k_i x_{pij}}{p_{ki}k_j},$$

$$p_{ij} = |\mathbf{p}_{ki}| = |\gamma_p\mathbf{p}_{jk} + \gamma_k\mathbf{k}_i|$$
$$= \sqrt{\gamma_p^2 p_{jk}^2 + \gamma_k^2 k_i^2 + 2\gamma_p\gamma_k p_{jk}k_i x_{p_{jk}}},$$

$$k_k = |\mathbf{k}_k| = |-\mathbf{p}_{jk} - \beta\mathbf{k}_i|$$
$$= \sqrt{p_{jk}^2 + \beta^2 k_i^2 + 2\beta p_{jk}k_i x_{pjk}},$$

$$x_{p_{ij}k_k} \equiv \hat{\mathbf{p}}_{ki}\cdot\hat{\mathbf{k}}_k$$
$$= -\frac{\gamma_p p_{jk}^2 + \beta\gamma_k k_i^2 + (\beta\gamma_p + \gamma_k)p_{jk}k_i x_{pjk}}{p_{ij}k_k}. \tag{40}$$

III. RESULTS AND DISCUSSION

A. Binding energy

To evaluate the relativistic effects in the three-body binding energy, we use the MT-V [5] potential,

$$\langle\mathbf{p}_{jk}|v^{\mathrm{nr}}|\mathbf{p}'_{jk}\rangle$$
$$= \frac{1}{2\pi^2}\left[\frac{V_R}{(\mathbf{p}_{jk} - \mathbf{p}'_{jk})^2 + \mu_R^2} - \frac{V_A}{(\mathbf{p}_{jk} - \mathbf{p}'_{jk})^2 + \mu_A^2}\right], \tag{41}$$

with parameters, V_R, V_A, μ_R, and μ_A given in Ref. [18]. In addition, we use $\hbar^2/m = 41.470\,\mathrm{MeV\,fm^2}$ and $\hbar c = 197.3286$ MeV fm. To numerically solve Eq. (32), the integrals over the continuous momenta and angle variables are replaced with sums over discrete quadrature points. To reach five significant
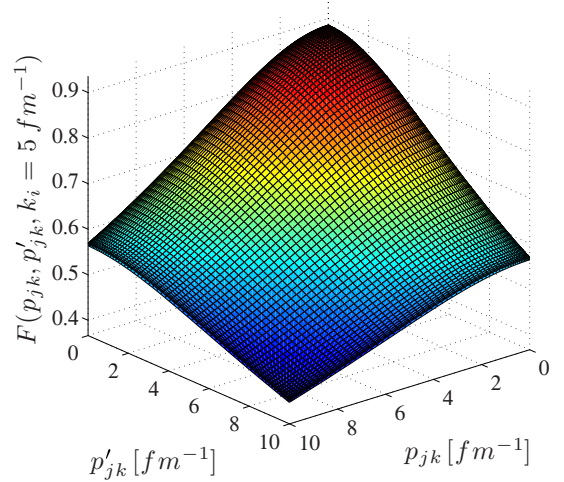

FIG. 1. (Color online) The analytical term $F(p_{jk}, p'_{jk}, k_i)$ connecting the relativistic and the nonrelativistic right-half-shell T matrices, Eq. (17), as function of the momenta p_{jk} and p'_{jk} in the two-body subsystem, for a fixed third-particle momentum $k_i = 5\,\mathrm{fm^{-1}}$.

digit convergence in the binding energy we use 100 Gaussian quadrature points for the Jacobi momentum p_{jk} in the interval $[0, 60\,\mathrm{fm^{-1}}]$, 60 quadrature points for the Jacobi momentum k_i in the interval $[0, 20\,\mathrm{fm^{-1}}]$, and 40 quadrature points for the angle variables. The Faddeev integral equation, Eq. (32), is solved by iteration using a Lanczos algorithm [27]. The iteration of this integral equation requires a large number of two-dimensional interpolations on the Faddeev component and the symmetrized two-body t matrix. We perform the interpolation using the cubic-Hermite splines of Ref. [28]. Seven iterations are, in general, sufficient to obtain a mass eigenvalue with a relative error of 10^{-6}.

The off-shell T matrix needed as input for the Faddeev integral equation, is computed by solving Eq. (A10). The input is the right-half-shell T matrix embedded in the three-body Hilbert space, and is obtained from the nonrelativistic t matrix by Eq. (16). In Fig. 1 we plot the ratio $F(p_{jk}, p'_{jk}, k_i)$ defined in Eq. (17) as a function of momenta p_{jk} and p'_{jk} for a third particle momentum $k_i = 5\,\mathrm{fm^{-1}}$. The slope of this function decreases as the value of k_i increases.

As a numerical test of the solution of first resolvent integral equation for negative energies, we reproduce the nonrelativistic three-body binding energy obtained from the direct solution of the Lippmann-Schwinger (LS) equation for the off-shell t matrix within four significant figures. The imaginary part of the transition matrix calculated from the first resolvent equation is of the order of $10^{-11}\,\mathrm{MeV^{-2}}$, which is 10^4 times smaller than the real part, which gives an additional measure of the accuracy of the calculation.

To visualize the Jacobian function $N(k_i, k'_i, x_{k'_i})$, which appears in the kernel of Faddeev integral equation, we parametrize it as $N(k\cos(\theta), k\sin(\theta), x \equiv x_{k'_i})$ and plot it in Fig. 2 as a function of θ and x for $k = 1, 5, 10$, and $20\,\mathrm{fm^{-1}}$. We use the same representation for the matrix elements of permutation coefficient $C(k\cos(\theta), k\sin(\theta), x \equiv x_{k'_i})$ displayed in Fig. 3.
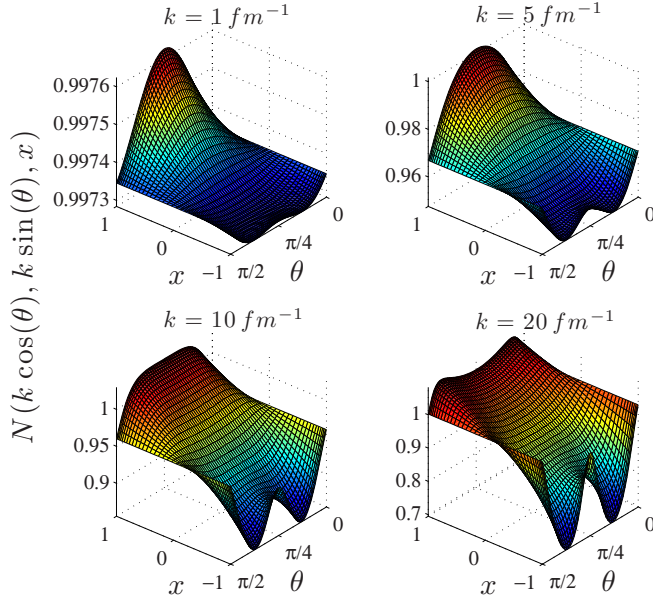
FIG. 2. (Color online) The matrix elements of Jacobian function $N(k\cos(\theta), k\sin(\theta), x)$, Eq. (25), as function of the angles x and θ calculated for different values of the momentum k.

The solution of the relativistic Faddeev equation leads to the three-body binding energy, $E_t^r = -7.4825$ MeV, which is slightly less than the nonrelativistic binding energy of $E_t^{nr} = -7.7382$ MeV. Thus, the relativistic effect is small, about 3.3%. This is consistent with a reduction of 2.7% for the s-wave calculation of Glöckle *et al.* [21].

The difference between the relativistic and nonrelativistic calculations arise from (1) the Jacobian function N, (2) the permutation coefficient C, (3) the relation between the
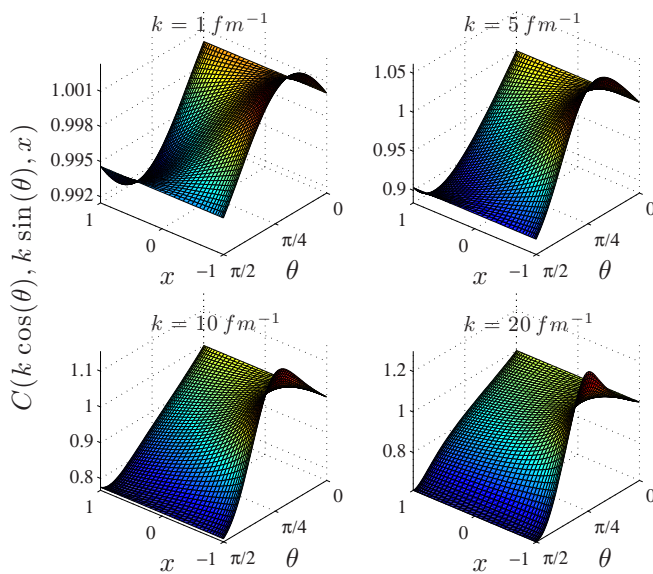

FIG. 3. (Color online) The matrix elements of the permutation coefficient $C(k\cos(\theta), k\sin(\theta), x)$, Eq. (29), as function of the angles x and θ calculated for different values of the momentum k.

TABLE I. The relativistic (r) and nonrelativistic (nr) [18] three-body binding energies calculated with the MT-V potential [5]. E_{approx} indicates the relativistic three-body binding energies calculated for Jacobian function $N(k_i, k_i', x_{k_i'}) = 1$, permutation coefficient $C(k_i', k_i, x_{k_i'}) = 1$, and the analytical term $F(p_{jk}, p_{jk}', k_i) = 1$ and using the nonrelativistic free propagator, as well as different combinations thereof.

	E_{nr} (MeV)	E_r (MeV)	$\frac{E_r - E_{nr}}{E_{nr}}$ (%)
	-7.7382	-7.4825	-3.30
	E_r (MeV)	E_{approx} (MeV)	$\frac{E_{approx} - E_r}{E_r}$ (%)
$N = 1$	-7.4825	-7.5412	$+0.78$
$C = 1$	-7.4825	-7.4361	-0.62
$N = C = 1$	-7.4825	-7.4934	$+0.15$
$F = 1$	-7.4825	-7.6606	$+2.38$
G_0^{nr}	-7.4825	-7.3446	-1.84
$F = 1, G_0^{nr}$	-7.4825	-7.4993	$+0.22$

relativistic and nonrelativistic transition matrices $T_{ij}(z)$, and (4) the relations between the relativistic and nonrelativistic free Green's functions. If we keep leading nonzero terms in the limit that the masses are larger than the momenta, all four of these factors reduce to the corresponding nonrelativistic quantities. For the off-shell transition matrix, the kernel of the first resolvent equation reduces to the nonrelativistic kernel in the same limit, which implies that both the half-shell and off-shell two-body transition matrix elements approach their nonrelativistic counterparts. The fact that all four corrections become small in that limit suggests that momentum/mass expansions are valid approximations. However, the corrections that relate the relativistic and nonrelativistic Faddeev equation are only attributable to relativistic effects associated with a relativistic treatment of the Fermi motion with respect to the spectator particle. It would be incorrect to apply these expansions to the two-body dynamics, which is fit to the same data in both the relativistic and the nonrelativistic cases.

While our calculations suggest that relativistic effects are small, it is important to remember that only the combination of all four ingredients leads to a small correction, while individually the corrections do not have to be small. In Table I the contributions of the Jacobian function N and the permutation coefficient C to the relativistic three-body binding energy are shown. Both of these functions become 1 in the nonrelativistic limit. By setting the Jacobian function N to 1 in our relativistic calculations, the binding energy increases about 0.8%. Setting the permutation coefficient C to 1 leads to a small decrease, about 0.6% of the binding energy. Finally, by setting both, the Jacobian function N and the permutation coefficient C to 1, the three-body binding energy has a small increase of about 0.15%. This means that ignoring the Jacobian function N and the permutation coefficient C in the relativistic formalism, leads to less than 0.2% change. The combined effect of the Jacobian and permutation operators is a factor of 4–5 smaller than the effect of each one individually. The main contribution of relativistic effects in the three-body binding energy comes from the relativistic transition operator and free propagator. To evaluate the contribution of relativistic T matrix to the

three-body binding energy, we replace the relativistic T matrix with a nonrelativistic one in the kernel of relativistic Faddeev integral equation. This substitution results in an increase of about 2.4% in the energy. When replacing the relativistic free propagator with the nonrelativistic one, a decrease of 1.8% in relativistic 3B binding energy can be observed. These numerical results imply that the main contribution of the relativistic effects in the three-body binding energy results from the relativistic T matrix. The remaining contribution stems from the free propagator, the Jacobian function, and the permutation coefficient.

B. Three-body wave function and momentum distribution

Using the Faddeev component from Eq. (32) the total wave function $\Psi(p_{jk}, k_i, x_{p_{jk}})$ can be obtained by three-dimensional interpolations on momentum and angle variables, as shown in Eq. (39). The wave function is normalized as

$$\langle \Psi | \Psi \rangle = 8\pi^2 \int_0^\infty dp_{jk}\, p_{jk}^2 \int_0^\infty dk_i\, k_i^2$$
$$\times \int_{-1}^{+1} dx_{p_{jk}}\, \Psi^2(p_{jk}, k_i, x_{p_{jk}}) = 1. \quad (42)$$

The left panel of Fig. 4 shows contour plots of the logarithm of the absolute value of the relativistic total wave functions for fixed angles $x_{p_{jk}} = 0$ (top left) and $x_{p_{jk}} = +1$ (bottom left). The right panel shows the absolute value of the difference between the relativistic and corresponding nonrelativistic wave function. These figures indicate that the largest relativistic effect appears at large values of the momentum of the third particle k_i. This is not surprising because the primary
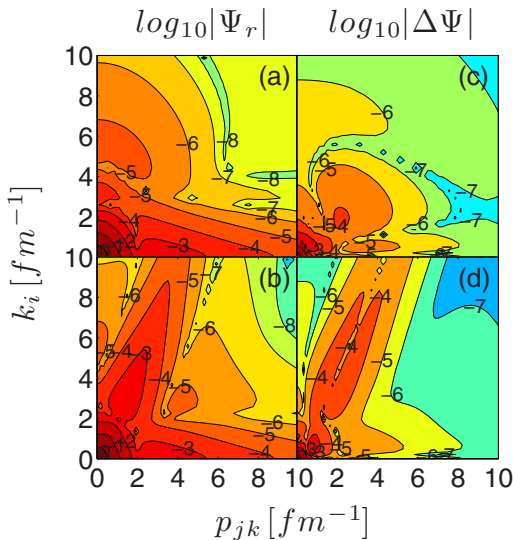
$$log_{10}|\Psi_r| \qquad log_{10}|\Delta\Psi|$$

FIG. 4. (Color online) The magnitude of the relativistic three-body bound-state wave function $\Psi(p_{jk}, k_i, x_{p_{jk}})$ as a function of the pair momentum p_{jk} and the spectator momentum k_i for fixed values $x_{p_{jk}} = 0$ (a) and $x_{p_{jk}} = +1$ (b) obtained with the MT-V potential [5,18]. The differences between the relativistic and nonrelativistic wave functions are shown for fixed values $x_{p_{jk}} = 0$ in panel (c) and $x_{p_{jk}} = +1$ in (d).
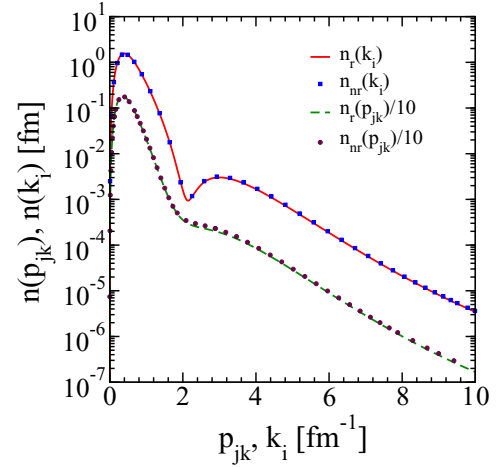
FIG. 5. (Color online) The relativistic and nonrelativistic momentum distribution functions $n(k_i)$ (solid line and solid squares) and $n(p_{jk})$ (dashed line and solid circles) obtained from the MT-V potential [5,18]. The function $n(p_{jk})$ is scaled with a factor 0.1.

relativistic effects are expected to be attributable to the Fermi motion of the spectator nucleon.

To provide further insight into the structure of the wave function, we calculate the momentum distribution functions $n(k_i)$, the probability to find a particle with momentum k_i in the nucleus, and $n(p_{jk})$, the probability to find a pair with momentum p_{jk} in the nucleus, which are defined as

$$n(k_i) = 8\pi^2 k_i^2 \int_0^\infty dp_{jk}\, p_{jk}^2 \int_{-1}^{+1} dx_{p_{jk}}\, \Psi^2(p_{jk}, k_i, x_{p_{jk}}),$$

$$n(p_{jk}) = 8\pi^2 p_{jk}^2 \int_0^\infty dk_i\, k_i^2 \int_{-1}^{+1} dx_{p_{jk}}\, \Psi^2(p_{jk}, k_i, x_{p_{jk}}).$$
$$(43)$$

Electron scattering is sensitive to the quantity $n(k_i)$. By considering the normalization of total wave function, given in Eq. (42), both momentum distribution functions are also normalized to 1; i.e., $\int_0^\infty n(p_{jk}) dp_{jk} = 1$ and $\int_0^\infty n(k_i) dk_i = 1$. The momentum distribution functions $n(p_{jk})$ and $n(k_i)$ calculated from relativistic and nonrelativistic wave functions are presented in Fig. 5. The difference between the relativistic and nonrelativistic momentum distribution functions is relatively small and is shown in Fig. 6. Both differences peak at $p_{jk} \sim 0.2$ fm^{-1} and $k_i \sim 0.2$ fm^{-1} and have a dip at $p_{jk} \sim 0.7$ fm^{-1} and $k_i \sim 0.7$ fm^{-1}. Setting the permutation coefficient $C = 1$ increases the difference, while setting the Jacobian function $N = 1$ decrease it and setting both of them $C = N = 1$ leads to a small decrease in the difference.

IV. SUMMARY AND OUTLOOK

In this work we solved the relativistic momentum-space Faddeev equation for three nucleons interacting with a spinless Malfliet-Tjon type potential without partial-wave decomposition for the three-body binding energy and calculated the corresponding bound-state wave function. To identify relativistic effects, the relativistic two-body interaction was
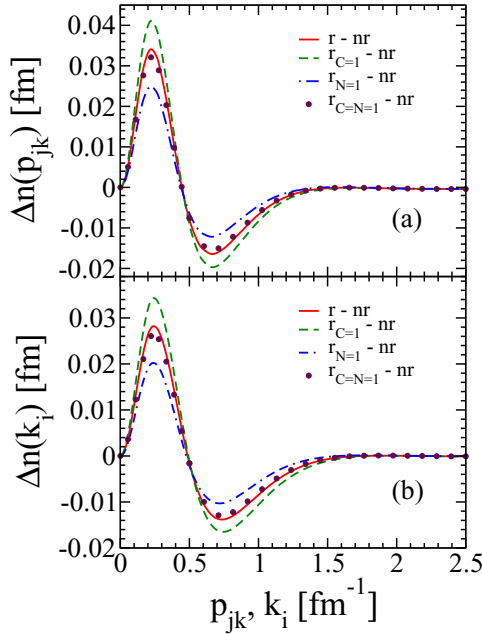
A comparison of the relativistic and nonrelativistic equations show four essential differences. Two are related to the Jacobi momenta, which leads to different treatments of the permutation operator in the relativistic and nonrelativistic Faddeev kernels. While these are choices of variables, the relativistic mass operator (13) that has an S matrix that clusters is naturally expressed in terms of the relativistic Jacobi momenta. The difference between these variables and the corresponding nonrelativistic Jacobi momenta appear in the coefficients C defined in Eq. (29) and the nontrivial Jacobians [factors N of Eq. (25)] in the relativistic case. For $C \to 1$ and $N \to 1$ these become the nonrelativistic Jacobi momenta. The combination of these two factors is associated with the kinematics of cluster properties. An important observation is that in our calculations these two quantities have opposite effects on the value of the binding energy and that the combination of those two quantities is essential to have a net effect, which is about a factor of 4–5 smaller than each effect separately.

The other two areas where relativity plays a role is in the part of the Faddeev kernel involving the two-body transition operator and the free three-body Green's functions. The difference between the relativistic and nonrelativistic half-shell t matrix is contained in the ratio factor F, Eq. (17), which becomes 1 in the limit that the masses are much larger than the momentum scales. This half-shell T appears in the kernel of the first resolvent equation, so setting $F = 1$ in the kernel of first resolvent equation and replacing the relativistic free Green's function with the nonrelativistic free Green's function leads to the nonrelativistic fully off-shell $T(z)$. The combination $T(z)g_0(z)$ is dimensionless, in both the relativistic and the nonrelativistic cases. Again, setting F to 1 without making a corresponding change in the free Green's function results in a significant increase in the binding energy. Replacing the relativistic free Green's function with the nonrelativistic one and keeping the function F leads to a decrease in binding energy by roughly the same amount. The combination of those two effects leads to a relatively small increase in the binding energy.

We calculated the relativistic three-body wave function and compared it to its nonrelativistic counterpart. The largest difference can be seen in the dependence on the spectator momentum k_i. This is not surprising because the k_i dependence is dictated by the different ways that the relativistic and nonrelativistic calculations treat the Fermi motion. Finally, we calculate that relativistic effects decrease the binding energy by about 3.3%. This is consistent with the findings in Refs. [21,22].

This work demonstrates that direct-integration techniques can be used to achieve the same results that can be obtained using partial-wave methods. Because our long-term interest is to first replace the Malfliet-Tjon interaction by a realistic interaction that has a more complicated spin-isospin dependence, and then to extend the calculations to treat scattering at the few GeV scale, it is important to test these methods in successive steps.

FIG. 6. (Color online) The difference between the relativistic and nonrelativistic momentum distribution functions (a) for $n(p_{jk})$ and (b) for $n(k_i)$ calculated from the MT-V potential (solid line). The dashed line shows the difference for the case when $C = 1$ in the relativistic calculation, while for the dash-dotted line $N = 1$. The solid circles represent a calculation in which both $C = N = 1$.

defined so that it gives in the two-body rest frame the same phase shifts and wave functions as the nonrelativistic interaction. Relativistic effects arise because the interacting two-body subsystems are not at rest in the three-body rest frame. Lorentz boosts associated with each subsystem are used to transform the two-body interactions from the two-body rest frames to the three-body rest frame. This transformation is determined by both the relativistic symmetry and the cluster properties.

The requirement that the relativistic and nonrelativistic two-body interactions be phase equivalent is attributable to the fact that realistic nonrelativistic NN interactions are already designed to be consistent with experiment and thus are consistent with special relativity. This means that all of the observable effects of special relativity are related to the different ways that the relativistic and nonrelativistic problems treat the Fermi motion. The relativistic Faddeev equation is simply a reformulation of the eigenvalue problem for the relativistic mass operator. However, it has the important advantage that in the limit that the mass scales are large compared to the momentum scales, both the variables and the kernel of the integral equation approach the corresponding quantities that appear in the nonrelativistic Faddeev equation. This relation is not as transparent when one compares the relativistic mass operator and the nonrelativistic rest Hamiltonian. This also suggests that as long as the Fermi momentum scales are small compared to the mass scales relativistic corrections are expected to be small. Of course, this naive picture is impacted by exchange symmetry.

APPENDIX: THE BOOSTED OFF-SHELL T MATRIX OBTAINED VIA RESOLVENT EQUATIONS

1. Resolvent equations

Starting from the resolvent of M, $g(z_i) = (z_i - M)^{-1}$, where $z_i = E_i + i\epsilon$ one obtains the first resolvent equation that relates the resolvent at two different values of z_i as

$$g(z_j) = g(z_i) + [g(z_j) - g(z_i)]$$
$$= g(z_i) + g(z_j)[g^{-1}(z_i) - g^{-1}(z_j)]g(z_i). \quad (A1)$$

Multiplying Eq. (A1) from the left and right by an interaction operator V and adding V to both sides of the equation leads to

$$V + Vg(z_j)V = V + Vg(z_i)V + V[g(z_j) - g(z_i)]V \quad (A2)$$

or

$$T(z_j) = T(z_i) + Vg(z_j)[z_i - z_j]g(z_i)V$$
$$= T(z_i) + T(z_j)g_0(z_j)[z_i - z_j]g_0(z_i)T(z_i)$$
$$= T(z_i) + T(z_j)[g_0(z_j) - g_0(z_i)]T(z_i), \quad (A3)$$

where we used the identity $\frac{1}{A} - \frac{1}{B} = \frac{1}{A}(B - A)\frac{1}{B}$, the definition of the transition operator $T(z_i) = V + Vg(z_i)V$, as well as the identity $g(z_i)V \equiv g_0(z_i)T(z_i)$, with $g_0(z_i) = (z_i - M_0)^{-1}$ being the resolvent of M_0. We now obtain an integral relation, which connects the transition operator at a given energy argument z_i with the transition operator at a different energy z_j.

Next we take matrix elements, $\langle \mathbf{p}|T(z_i)|\mathbf{p}'\rangle \equiv T(\mathbf{p},\mathbf{p}';z_i)$. Because the relativistic T matrix is only known for half-shell momentum variables, i.e., $T(\mathbf{p},\mathbf{p_i};z_i)$, we need to take matrix elements of Eq. (A3) leading to the inhomogeneous integral equation

$$\langle \mathbf{p}|T(z_j)|\mathbf{p_i}\rangle = \langle \mathbf{p}|T(z_i)|\mathbf{p_i}\rangle + \int d^3\mathbf{p}'' \langle \mathbf{p}|T(z_j)|\mathbf{p}''\rangle$$
$$\times [g_0(z_j) - g_0(z_i)]\langle \mathbf{p}''|T(z_i)|\mathbf{p_i}\rangle. \quad (A4)$$

Here the inhomogeneous term is given by the half-shell T-matrix elements $\langle \mathbf{p}|T(z_i)|\mathbf{p_i}\rangle = T(\mathbf{p},\mathbf{p_i};z_i)$, and we solve for the off-shell matrix elements $\langle \mathbf{p}|T(z_j)|\mathbf{p_i}\rangle = T(\mathbf{p},\mathbf{p_i};z_j)$.

2. Numerical realization

Writing Eq. (A4) explicitly using $z_i \equiv E(p_i)$ leads to

$$T(\mathbf{p},\mathbf{p_i};E(p_j))$$
$$= T(\mathbf{p},\mathbf{p_i};E(p_i)) + \int d^3\mathbf{p}'' T(\mathbf{p},\mathbf{p}'',E(p_j))$$

$$\times \left[\frac{1}{E(p_j) - E(p'') + i\epsilon} - \frac{1}{E(p_i) - E(p'') + i\epsilon} \right]$$
$$\times T(\mathbf{p}'',\mathbf{p_i};E(p_i)). \quad (A5)$$

Choosing the vector $\mathbf{p}$ parallel to the z axis and the vector $\mathbf{p_i}$ in the x-z plane leads to the following angle variables

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}_i \equiv x_i,$$
$$\hat{\mathbf{p}}'' \cdot \hat{\mathbf{p}}_i \equiv \hat{x}_i = x_i x'' - \sqrt{1 - x_i^2}\sqrt{1 - x''^2}\cos\phi_{p''p_i}$$
$$\equiv \hat{x}_i(x_i, x'', \cos\phi_{p''p_i}),$$
$$\hat{p} \cdot \hat{p}'' \equiv x''. \quad (A6)$$

Inserting the above variables into Eq. (A6) gives

$$T(p, p_i, x_i; E(p_j))$$
$$= T(p, p_i, x_i; E(p_i)) + \int_0^\infty dp\, p'' \int_{-1}^{+1} dx''$$
$$\times \left[\frac{1}{E(p_j) - E(p'') + i\epsilon} - \frac{1}{E(p_i) - E(p'') + i\epsilon} \right]$$
$$\times T(p, p'', x''; ; E(p_j)) \int_0^{2\pi} d\phi'' T$$
$$\times (p'', p_i, \hat{x}_i(x_i, x'', \cos\phi_{p''p_i}); E(p_i)). \quad (A7)$$

Here we note that the ϕ'' integration only affects one term in the integral equation and we can carry it out separately. For convenience, let us define

$$T(p'', p_i, x_i, x''; E(p_i))$$
$$\equiv \int_0^{2\pi} d\phi'' T(p'', p_i, \hat{x}_i(x_i, x'', \cos\phi_{p''p_i}); E(p_i)). \quad (A8)$$

The structure of Eq. (A7) is identical to the two-body LS equation [29] and can be solved in a similar fashion. However, one needs to carefully look at its singularities. For the calculation of the relativistic three-body bound-state equation, we need the off-shell T matrix at negative energies $E(p_j)$. Thus, the first propagator in Eq. (A7) is nonsingular, and its numerical value is always negative. However, the second propagator exhibits a singularity at $E(p_i) = E(p'')$ for each fixed momentum p_i. This singular point on the momentum grid can be numerically treated with a subtraction technique. In the actual calculation, we work on a momentum grid for p_i, and we use the same momentum grid for the integration over p''. In this case, when setting up the matrix equation to solve Eq. (A7), the singular point is located on the diagonal of this matrix, and all terms resulting from the analytic treatment of the singularity must be located on the diagonal.

To calculate the off-shell boosted T matrix $T_{jk}(p_{jk}, p'_{jk}, x_{p'_{jk}}; M_t - m_{0i}(k_i))$, which appears in the kernel of the Faddeev integral equation of the right-half-shell boosted T matrix, $T_{jk}(p_{jk}, p'_{jk}, x_{p'_{jk}}; \sqrt{m_{0jk}^2(pjk') + k_i^2})$, we solve Eq. (A7) for $E(p_j) = M_t - m_{0i}(k_i)$ and $E(p_i) = \sqrt{m_{0jk}^2(p'_{jk}) + k_i^2}$.

After calculating the singularity of the second integral of Eq. (A7) via subtraction, the explicit form of the first resolvent integral equation is

$$
T_{jk}(p_{jk},p'_{jk},x_{p'_{jk}};M_t - m_{0i}(k_i))
$$

$$
= T_{jk}\big(p_{jk},p'_{jk},x_{p'_{jk}};\sqrt{m_{0jk}^2(p'_{jk})+k_i^2}\big) + \int_0^\infty dp''_{jk}\, p''^2_{jk} \int_{-1}^{+1} dx_{p''_{jk}} \frac{1}{M_t - m_{0i}(\mathbf{k}_i) - \sqrt{m_{0jk}^2(p''_{jk})+k_i^2}}
$$

$$
\times T_{jk}\big(p''_{jk},p'_{jk},x_{p'_{jk}},x_{p''_{jk}};\sqrt{m_{0jk}^2(p'_{jk})+k_i^2}\big) T_{jk}\big(p_{jk},p''_{jk},x_{p''_{jk}};M_t - m_{0i}(\mathbf{k}_i)\big)
$$

$$
- \Bigg\{ \frac{1}{4} \int_0^\infty dp''_{jk}\, p''^2_{jk} \int_{-1}^{+1} dx_{p''_{jk}} \frac{\sqrt{m_{0jk}^2(p'_{jk})+k_i^2} - \sqrt{m_{0jk}^2(p''_{jk})+k_i^2}}{p'^2_{jk} - p''^2_{jk}} T_{jk}\big(p''_{jk},p'_{jk},x_{p'_{jk}},x_{p''_{jk}};
$$

$$
\sqrt{m_{0jk}^2(p'_{jk})+k_i^2}\big) T_{jk}\big(p_{jk},p''_{jk},x_{p''_{jk}};M_t - m_{0i}(\mathbf{k}_i)\big) - \frac{1}{2} \int_0^\infty dp''_{jk} \int_{-1}^{+1} dx_{p''_{jk}} \frac{p'^2_{jk}\sqrt{m_{0jk}^2(p'_{jk})+k_i^2}}{p'^2_{jk} - p''^2_{jk}}
$$

$$
\times T_{jk}\big(p'_{jk},p'_{jk},x_{p'_{jk}},x_{p''_{jk}};\sqrt{m_{0jk}^2(p'_{jk})+k_i^2}\big) T_{jk}\big(p_{jk},p'_{jk},x_{p''_{jk}};M_t - m_{0i}(k_i)\big)
$$

$$
- \frac{1}{4} \int_{-1}^{+1} dx_{p''_{jk}}\, p'_{jk}\sqrt{m_{0jk}^2(p'_{jk})+k_i^2} \left[i\pi + \ln\left(\frac{p''^{\max}_{jk} + p'_{jk}}{p''^{\max}_{jk} - p'_{jk}} \right) \right] T_{jk}\big(p'_{jk},p'_{jk},x_{p'_{jk}},x_{p''_{jk}};
$$

$$
\sqrt{m_{0jk}^2(p'_{jk})+k_i^2}\big) T_{jk}\big(p_{jk},p'_{jk},x_{p''_{jk}};M_t - m_{0i}(\mathbf{k}_i)\big) \Bigg\}.
\tag{A9}
$$

This integral equation is solved for a given value of momentum k_i, equal to the boosted momentum k_{jk}, and for all values of the left momentum p_{jk}.

[1] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[2] V. Bargmann, Ann. Math. **59**, 1 (1954).

[3] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C **51**, 38 (1995).

[4] R. Machleidt, Phys. Rev. C **63**, 024001 (2001).

[5] R. A. Malfliet and J. A. Tjon, Nucl. Phys. A **127**, 161 (1969).

[6] F. Coester, S. C. Pieper, and F. J. D. Serduke, Phys. Rev. C **11**, 1 (1975).

[7] B. D. Keister and W. N. Polyzou, Phys. Rev. C **73**, 014005 (2006).

[8] T. Lin, Ch. Elster, W. N. Polyzou, and W. Glöckle, Phys. Rev. C **76**, 014010 (2007).

[9] T. Lin, Ch. Elster, W. N. Polyzou, H. Witała, and W. Glöckle, Phys. Rev. C **78**, 024002 (2008).

[10] T. Lin, Ch. Elster, W. N. Polyzou, and W. Glöckle, Phys. Lett. B **660**, 345 (2008).

[11] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).

[12] F. Coester, Helv. Phys. Acta **38**, 7 (1965).

[13] S. N. Sokolov, Dokl. Akad. Nauk SSSR **233**, 575 (1977).

[14] F. Coester and W. N. Polyzou, Phys. Rev. D **26**, 1348 (1982).

[15] B. D. Keister and W. N. Polyzou, Adv. Nucl. Phys. **20**, 225 (1991).

[16] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[17] W. N. Polyzou, Phys. Rev. C **82**, 064001 (2010).

[18] Ch. Elster, W. Schadow, A. Nogga, and W. Glöckle, Few-Body Syst. **27**, 83 (1999).

[19] I. Fachruddin, Ch. Elster, and W. Glöckle, Phys. Rev. C **62**, 044002 (2000).

[20] H. Liu, Ch. Elster, and W. Glöckle, Phys. Rev. C **72**, 054003 (2005).

[21] W. Glöckle, T.-S. H. Lee, and F. Coester, Phys. Rev. C **33**, 709 (1986).

[22] H. Kamada et al., Mod. Phys. Lett. A **24**, 804 (2009).

[23] H. Witała, J. Golak, R. Skibinski, W. Glöckle, W. Polyzou et al., Mod. Phys. Lett. A **24**, 871 (2009).

[24] W. Glöckle, *The Quantum Mechanical Few-Body Problem*, Texts and Monographs in Physics (Springer Verlag, Berlin, 1983).

[25] T. Kato, *Perturbation Theory for Linear Operators* (Spinger-Verlag, Berlin, 1966).

[26] H. Kamada and W. Glöckle, Phys. Lett. B **655**, 119 (2007).

[27] A. Stadler, W. Glöckle, and P. U. Sauer, Phys. Rev. C **44**, 2319 (1991).

[28] D. Hüber, H. Witała, A. Nogga, W. Glöckle, and H. Kamada, Few-Body Syst. **22**, 107 (1997).

[29] Ch. Elster, J. H. Thomas, and W. Glöckle, Few-Body Syst. **24**, 55 (1998).