Polynomial fits and the proton radius puzzle

E. Kraus,[1,*] K. E. Mesick,[1] A. White,[1] R. Gilman,[1] and S. Strauch[2]
[1]*Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08854, USA*
[2]*Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208, USA*
(Received 27 June 2014; revised manuscript received 17 September 2014; published 17 October 2014)

The proton radius puzzle refers to the $\approx 7\sigma$ discrepancy that exists between the proton charge radius determined from muonic hydrogen and that determined from electronic hydrogen spectroscopy and electron-proton scattering. One possible partial resolution to the puzzle includes errors in the extraction of the proton radius from *ep* elastic scattering data. This possibility is made plausible by certain fits that extract a smaller proton radius from the scattering data consistent with that determined from muonic hydrogen. The reliability of some of these fits that yield a smaller proton radius was studied. We found that fits of form factor data with a truncated polynomial fit are unreliable and systematically give values for the proton radius that are too small. Additionally, a polynomial fit with a $\chi^2_{\text{reduced}} \approx 1$ is not a sufficient indication for a reliable result.

DOI: 10.1103/PhysRevC.90.045206 PACS number(s): 25.30.Bf, 13.40.Gp

I. PHYSICS MOTIVATION

The *proton radius puzzle* pertains to the disagreement between the proton charge radius determined from muonic hydrogen and from electron-proton systems: atomic hydrogen and *ep* elastic scattering. The muonic hydrogen result [1,2] of $r_p = 0.840\,87 \pm 0.000\,39$ fm is about 13 times more precise and $\approx 7\sigma$ different than the recent CODATA 2010 [3] result of $r_p = 0.8775 \pm 0.0051$ fm. The CODATA analysis includes atomic hydrogen and the precise cross section measurements of Bernauer *et al.* [4,5], which give $r_p = 0.879 \pm 0.008$ fm, but not the more recent confirmation of Zhan *et al.* [6] which yields $r_p = 0.875 \pm 0.010$ fm. For a recent review, see Ref. [7].

Many possible explanations of the proton radius puzzle have been ruled out. There are, for example, no known issues with the atomic theory, or with the muonic hydrogen experiment. It appears that the most likely explanations are novel physics beyond the Standard Model that differentiates μp and ep interactions, novel two-photon exchange effects that differentiate μp and ep interactions, and errors in the *ep* experiments. It is therefore important to examine possible issues in the *ep* experiments before concluding that interesting physics is required.

While the extracted radius values given above have been confirmed by some analyses, other analyses of *ep* scattering data give a smaller radius consistent with the muonic hydrogen result. Examples of confirming analyses include the z expansion of [8] ($r_p = 0.871$ fm $\pm$ 0.009 fm $\pm$ 0.002 fm $\pm$ 0.002 fm), and the sum-of-Gaussians fit of [9–11] ($r_p = 0.886$ fm $\pm$ 0.008 fm). However, three recent analyses give smaller radii, consistent with the muonic hydrogen result. Griffioen and Carlson [12] observed that a truncated linear polynomial fit of the low Q^2 Bernauer data yields $r_p \approx 0.84$ fm, with good χ^2. The dispersion relation analysis of Lorenz *et al.* [13] yields $r_p = 0.84 \pm 0.01$ fm with a large $\chi^2_{\text{reduced}} \approx 2.2$, in a simultaneous fit of proton and neutron data. The fluctuating radius fit found in Ref. [14] yields $r_p = 0.8333 \pm 0.0004$ fm with $\chi^2_{\text{reduced}} \approx 4$—but note the criticism of the authors of Ref. [15]. A summary of some recent proton radius determinations can be seen in Fig. 1. The variation in the radius determined from scattering experiments calls into question the reliability of the proton radius determination from scattering experiments.

In this paper we study the reliability of proton radius determinations from the *ep* elastic scattering experiments. We note that there are a number of issues in extracting a radius from the experimental data, as discussed in Refs. [7] and [11]. In particular, we look at the radius extraction through the Taylor series expansion of the proton electric form factor: $G_E^p(Q^2) = 1 - Q^2 r_p^2/6 + Q^4 r_p^4/120 + \cdots$ such that $r_p^2 = -6 dG_E^p(Q^2)/dQ^2|_{Q^2=0}$. We use a polynomial fit[1] that has the same functional form as a truncated Taylor series expansion, and note that a polynomial fit exhibits unphysical behavior in extrapolations to large Q^2, as it necessarily diverges to infinity, and this might also affect a radius determination.

The basic result of this paper—that radius extractions with polynomial fits cannot be trusted to be reliable—has already been argued by Sick [16], who claimed that higher-order terms in the expansion prevent a precise determination of the proton radius for any Q^2 region. Determining the Q^2 term precisely requires a larger Q^2 range to determine the Q^4 term precisely, which requires an even larger Q^2 range to determine the Q^6 term precisely, and so on. The inefficiency and inconsistency of the truncated polynomial fit has also been demonstrated in unpublished numerical work by Distler [18]. Lastly, Borisyuk [19] has argued that there is a systematic error related to the deviation of a fitted radius and the true radius due to the inadequacy of the form factor parametrizations in describing the true form factor. In this paper, we find with the polynomial fits an offset between the real radius and the radius extracted with a fit that results from truncating the power-series expansion to fit a finite range of data. This is a systematic error that we call the *truncation offset*.

*Permanent address: Syracuse University, Syracuse, NY 13210, USA.

[1]Note that in similar analyses, the polynomial fit is commonly called the Taylor series expansion.

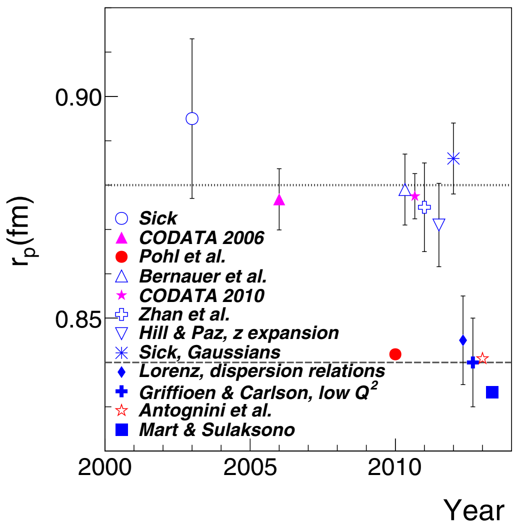

FIG. 1. (Color online) A summary of some recent proton charge radius determinations: Sick [16], CODATA 2006 [17], Pohl *et al.* [1], Bernauer *et al.* [4,5], CODATA 2010 [3], Zhan *et al.* [6], Hill & Paz [8], Sick Gaussians [9,10], Lorenz *et al.* [13], Griffioen & Carlson [12], Antognini *et al.* [2], and Mart & Sulaksono [14]. The dashed and dotted lines are drawn at 0.88 and 0.84 fm, respectively, for reference.

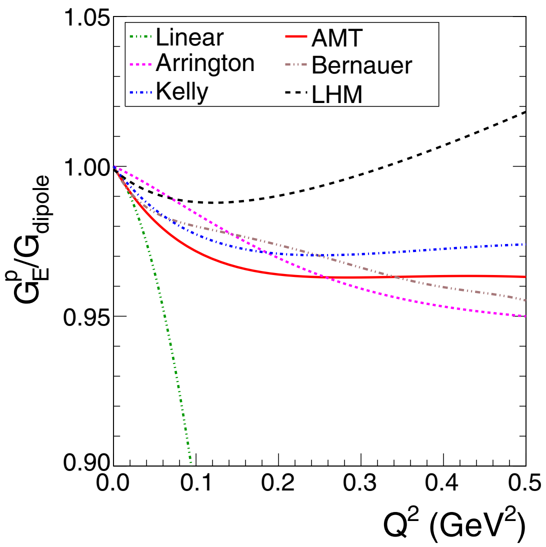

FIG. 2. (Color online) Parametrizations of the proton electric form factor used to generate pseudodata, relative to the dipole form factor, $G_{\text{dipole}} = (1 + Q^2/0.71\,\text{GeV}^2)^{-2}$.

II. METHOD

The most precise *ep* elastic scattering data come from Bernauer *et al.* [4,5], but for our purposes it is more useful to generate pseudodata for G_E^p from a parametrization with a known radius. To get data similar in shape to the actual proton form factor, and to study how sensitive the result is to the input, we generate the pseudodata from six parametrizations of the proton form factor data as follows:

(i) the Arrington, Melnitchouk, Tjon (AMT) fit [20], a Padé parametrization with $r_p \approx 0.878$ fm;
(ii) the Arrington fit [21], an inverse polynomial parametrization with $r_p \approx 0.829$ fm;
(iii) the Bernauer $n = 10$ polynomial fit [22], with $r_p \approx 0.887$ fm;
(iv) the standard dipole fit, with $r_p \approx 0.811$ fm;
(v) the Kelly fit [23], a Padé parametrization with $r_p \approx 0.863$ fm; and
(vi) the Lorenz, Hammer, and Meissner (LHM) fit [13], which combines dispersion relations with a vector meson dominance parameterization with $r_p \approx 0.84$ fm.

In addition, the numerical procedures were confirmed by generating pseudodata from a linear function with $r_p = 0.86$ fm. Figure 2 compares the parametrizations listed above.

For each form factor parametrization, we generate pseudodata points with 0.2% uncertainties (corresponding to 0.4% cross section uncertainties) spaced every 0.001 GeV^2 in Q^2 from $Q^2_{\text{min}} = 0.004$ GeV^2 to a variable Q^2_{max}. These pseudodata have roughly the uncertainties and data-point density of the Bernauer data, but reflect a known radius. We fit the data with polynomials in Q^2:

$$a_0\left[1 + \sum_{i=1}^{n} a_i (Q^2)^i\right], \tag{1}$$

with $n = 1, 2, 3,$ and 4, and where a_0 was statistically consistent with unity and $a_1 \propto r^2$. For each parametrization, polynomial order, and Q^2_{max}, the pseudodata generation and fitting is repeated 5000 times to generate distributions of r^2, $\sigma(r^2)$, and χ^2. From these distributions we extract the proton charge radius and its uncertainty, and the mean χ^2. Numerical work was done using CERN MINUIT and ROOT.

III. RESULTS

We find the results from all six form factor parametrizations are qualitatively similar. The AMT pseudodata fits are representative of the typical behavior and are shown here. Figure 3 shows the truncation offset versus Q^2_{max}. The lines shown indicate the truncation offset, while the width of the bands indicates the root mean square (r.m.s.) width of the distribution of proton radii from the 5000 fits done, corresponding to the statistical uncertainty of the radius extraction in the fit. Figure 4 shows how χ^2_{reduced} varies with Q^2_{max}. Lastly, Fig. 5 shows the truncation error in units of the fit uncertainty versus χ^2_{reduced}. In all plots, the four series of fits shown correspond to the polynomials of order 1 to 4 as defined in Eq. (1).

Some observations related to these figures include the following.

(i) Fit uncertainties decrease with increasing Q^2_{max} due to the greater number of data points and the greater "lever arm" of the data.
(ii) Low Q^2 data with the uncertainties and data-point density we have assumed do not by themselves determine a precise radius.

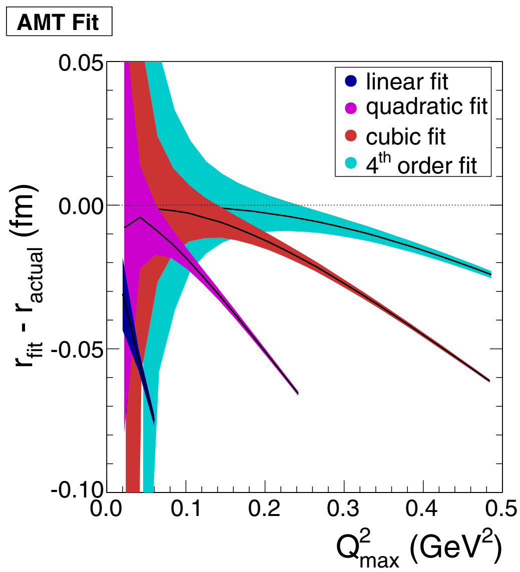

FIG. 3. (Color online) The truncation offset versus $Q^2_{\rm max}$ for the AMT parametrization. The lines indicate the size of the truncation offset and the bands the r.m.s. width of the radius distribution from the 5000 fits.

(iii) As there is more curvature in the generating functions than in the fit functions, the truncation offset generally grows with $Q^2_{\rm max}$, but decreases with increasing order of the fit.

(iv) The nature of the curvature in the proton electric form factor is such that the truncation offset using these parametrizations almost always leads to a fit radius that is smaller than the "real" radius.

(v) Comparing Figs. 3 and 4 shows that $\chi^2_{\rm reduced}$ *is not a reliable guide to the quality of the radius extracted.* There can already be a significant truncation offset before the $\chi^2_{\rm reduced}$ is obviously far from unity. For example, in the cubic AMT fit with $Q^2_{\rm max} = 0.24$ GeV^2, $\chi^2_{\rm reduced} = 1.018$, but the extracted radius of 0.859 ± 0.002 fm differs from the 0.878-fm radius of the AMT fit by 0.019 fm, about half of the proton-radius-puzzle discrepancy, and about ten times the fit uncertainty.

(vi) The above point is also demonstrated in Fig. 5, which shows that a small $\chi^2_{\rm reduced}$ does not guarantee an accurate determination of the radius; even with a small truncation error, the truncation offset of the fit is several times the fit uncertainty.

(vii) Even fits with $\chi^2_{\rm reduced} < 1.1$ can result in a truncation offset equal to the difference between the *ep* and μp proton radius determinations, $\Delta r \approx 0.037$ fm.

(viii) One can find combinations of $Q^2_{\rm max}$ and fit order for which there is no significant truncation offset and good statistical precision on the extracted radius (as done in Ref. [19], but with a different fitting parametrization). However, the combinations vary with form factor parametrization, and it is problematic in practice to ensure that a radius extracted with a polynomial fit from actual data is reliable.

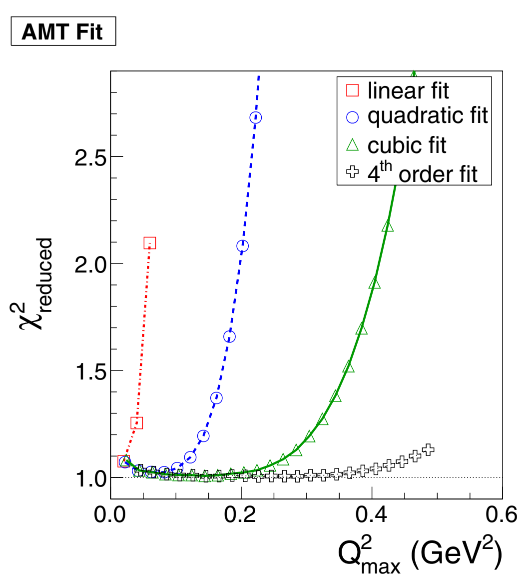

FIG. 4. (Color online) Reduced χ^2 for the fits of the pseudodata generated from the AMT form factor parametrization versus $Q^2_{\rm max}$.

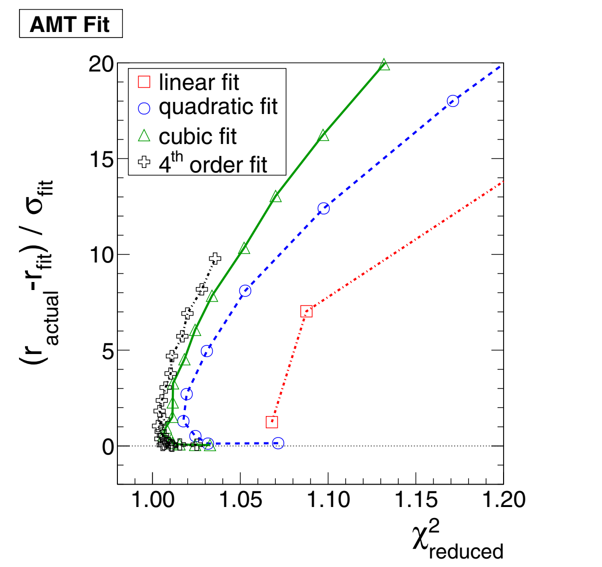

FIG. 5. (Color online) The truncation offset divided by the fit uncertainty as a function of the fit $\chi^2_{\rm reduced}$, for the AMT parametrization.

To summarize, a proton radius determination through a polynomial fit analysis is suspect. It is believed that other fit functions, such as the inverse polynomial or z expansion have smaller, but still significant, truncation offsets [24].

As mentioned, six different form factor parametrizations were studied. Figures 6 to 8 compare the third-order cubic fit results for all the form factor parametrizations. Fits to a form factor following the Arrington parametrization give the smallest truncation offset and is the least sensitive to fit order, while fits to a form factor following the Bernauer polynomial parametrization result in the largest truncation offset and sensitivity to fit order.

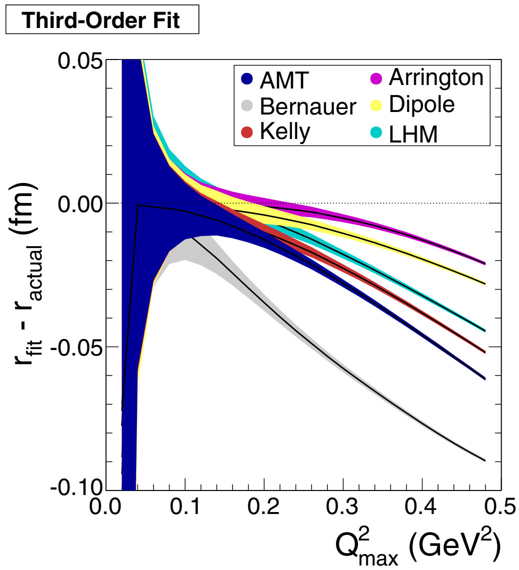

FIG. 6. (Color online) The truncation offset versus Q^2_{max} for the third-order fits of the pseudodata generated from the different form factor parametrizations.

Also of interest is how the truncation offset might affect upcoming experiments should a truncated polynomial fit be used, in particular low Q^2 measurements of the proton radius. In this case, the region of interest is $Q^2_{\text{max}} < 0.1$ GeV^2. The results fitting from $Q^2_{\text{min}} = 0.004$ GeV^2 up to $Q^2_{\text{max}} = 0.01$–0.1 GeV^2 are shown for fits of order 1 and 2 in Figs. 9 and 10 for the truncation offset and the χ^2_{reduced}, respectively, using the Arrington parametrization. Ultimately, the statistical fit uncertainty on extracting the radius depends on the final uncertainties the future experiments achieve, however, the truncation error depends on the Q^2 range.

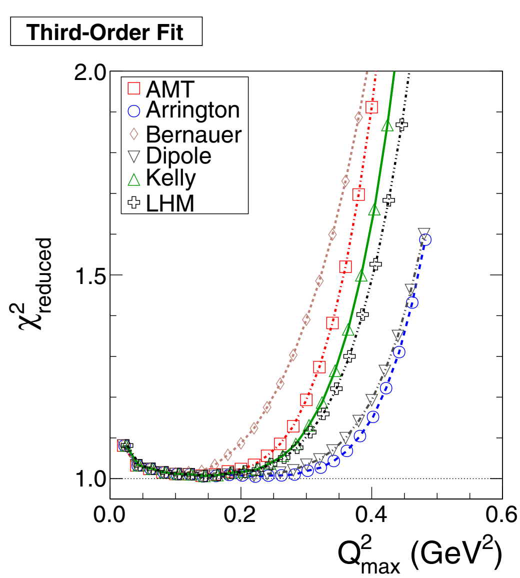

FIG. 7. (Color online) Reduced χ^2 versus Q^2_{max} for the third-order fits of the pseudodata generated from the different form factor parametrizations.

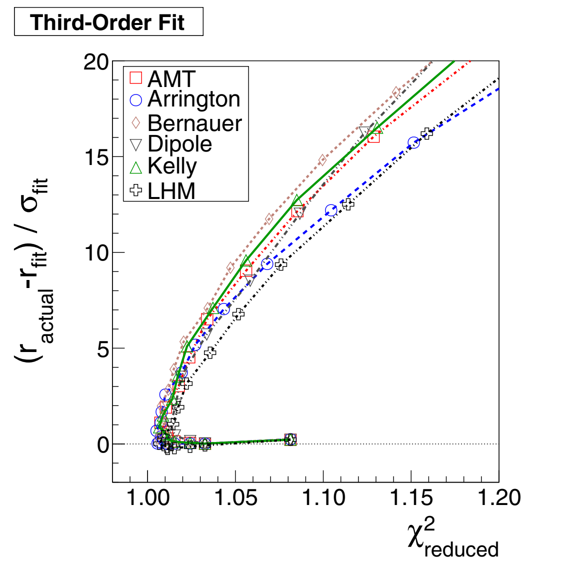

FIG. 8. (Color online) The truncation offset divided by the fit uncertainty as a function of the fit χ^2_{reduced}, for the six different parametrizations.

One upcoming experiment is Jefferson Lab E12-11-106 [25], which plans to measure elastic *ep* scattering in the range $Q^2 \approx 10^{-4}$–0.02 GeV^2. We simulate the experiment using 12 data points at the Q^2 values shown in Fig. 30 of the proposal—note that other estimates in the proposal rebin the data into more points. Under these assumptions, a linear fit to pseudodata yields a truncation offset ranging from 0.016 fm for the Arrington parametrization to 0.025 fm when the AMT parametrization is used. The χ^2_{reduced} for both fit examples is $\approx$1.1. A higher-order quadratic fit reduces the truncation offset by an order of magnitude, however, results

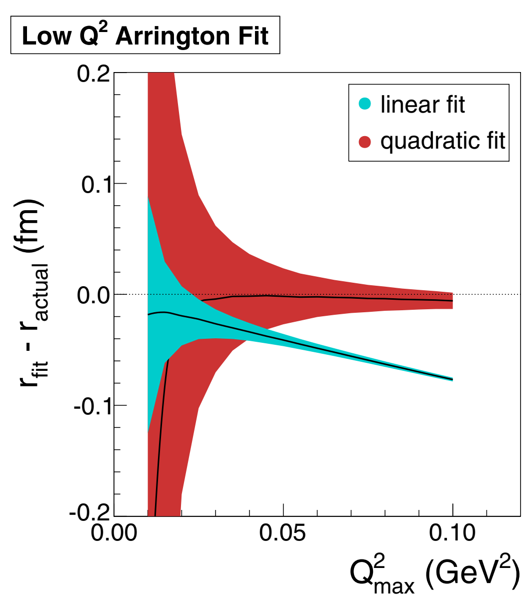

FIG. 9. (Color online) The truncation offset versus Q^2_{max} for liner and quadratic fits in the low Q^2 region.

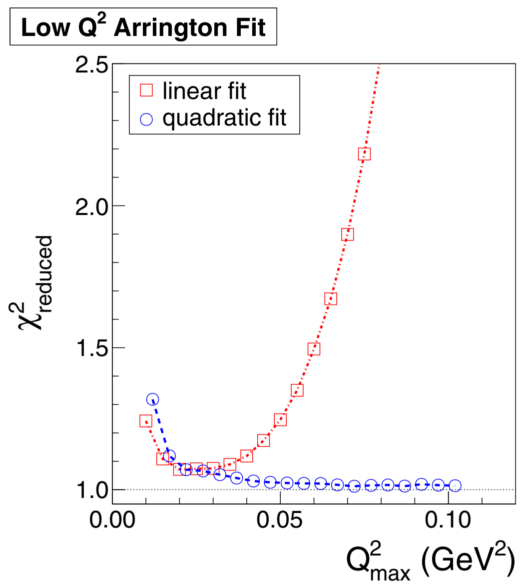

FIG. 10. (Color online) Reduced χ^2 versus Q^2_{max} for linear and quadratic fits in the low Q^2 region.

in a statistical fit uncertainty of 0.05 fm assuming 0.4% point-to-point cross section uncertainties. This demonstrates that a truncated polynomial fit of the E12-11-106 data alone is highly suspect as a technique to determine an accurate radius.

A second upcoming experiment is the MUSE measurement of $\mu^{\pm}p$ and $e^{\pm}p$ scattering at the Paul Scherrer Institute [26]. This experiment will have six independent datasets (three different beam momentum, two polarities) for each particle type, covering a Q^2 range of 0.0025–0.0775 GeV^2. MUSE will make a relative comparison of the ep and μp elastic scattering cross sections and form factors, largely canceling several systematic uncertainties including the truncation offset in the radius extraction. Doing so will allow for a ≈0.01 fm measurement of the difference between the proton charge radius as measured by electrons versus muons.

In summary, Sick [16] and Distler [18] have indicated that a precise proton radius could not reliably be extracted using a polynomial fit of the form factor. Using six form factor parametrizations for which the radius is known, we have confirmed that this is the case. In particular, we have shown that the condition $\chi^2_{reduced} \approx 1$ is not sufficient for the extracted radius to be reliable. Due to the higher-order terms in the polynomial fit, even an apparently good fit of the data can have a significant offset from the real radius. This truncation offset increases with fitting a wider range of data, but decreases with fitting with a higher-order expansion. Even for a fit with a small truncation offset, the offset can be large relative to the fit uncertainty. Finally, we have also observed that for the six form factor parametrizations used to generate pseudodata, the truncation offset generally results in an extracted radius that is smaller than the true radius.

ACKNOWLEDGMENTS

This work was supported by the U.S. National Science Foundation through Grant Nos. PHY 1263280 and PHY 1306126 to Rutgers University, and through Grant No. PHY 1205782 to the University of South Carolina. We thank Andrew Jiang for his assistance with the calculation in this paper.

[1] R. Pohl *et al.*, Nature (London) **466**, 213 (2010).
[2] A. Antognini *et al.*, Science **339**, 417 (2013).
[3] P. J. Mohr, B. N. Taylor, and D. B. Newell, Rev. Mod. Phys. **84**, 1527 (2012).
[4] J. Bernauer *et al.* (A1 Collaboration), Phys. Rev. Lett. **105**, 242001 (2010).
[5] J. Bernauer *et al.* (A1 Collaboration), Phys. Rev. C **90**, 015206 (2014).
[6] X. Zhan *et al.*, Phys. Lett. B **705**, 59 (2011).
[7] R. Pohl, R. Gilman, G. A. Miller, and K. Pachucki, Ann. Rev. Nucl. Part. Sci. **63**, 175 (2013).
[8] R. J. Hill and G. Paz, Phys. Rev. D **82**, 113005 (2010).
[9] I. Sick, Few-Body Syst. **50**, 367 (2011).
[10] I. Sick, Prog. Part. Nucl. Phys. **67**, 473 (2012).
[11] I. Sick and D. Trautmann, Phys. Rev. C **89**, 012201 (2014).
[12] K. Griffioen and C. E. Carlson (unpublished; see http://www.mpq.mpg.de/~rnp/wiki/stuff/Workshop/Talks/204_Griffioen.pdf).
[13] I. Lorenz, H.-W. Hammer, and U.-G. Meissner, Eur. Phys. J. A **48**, 151 (2012).
[14] T. Mart and A. Sulaksono, Phys. Rev. C **87**, 025807 (2013).
[15] E. J. Downie, W. J. Briscoe, R. Gilman, and G. Ron, Phys. Rev. C **88**, 059801 (2013).
[16] I. Sick, Phys. Lett. B **576**, 62 (2003).
[17] P. J. Mohr, B. N. Taylor, and D. B. Newell, Rev. Mod. Phys. **80**, 633 (2008).
[18] M. O. Distler (unpublished).
[19] D. Borisyuk, Nucl. Phys. A **843**, 59 (2010).
[20] J. Arrington, W. Melnitchouk, and J. A. Tjon, Phys. Rev. C **76**, 035205 (2007).
[21] J. Arrington, Phys. Rev. C **69**, 022201 (2004).
[22] J. C. Bernauer, Ph.D. thesis, Johannes Gutenberg-Universität, Mainz, 2010.
[23] J. J. Kelly, Phys. Rev. C **70**, 068202 (2004).
[24] J. Arrington (private communication).
[25] A. Gasparian *et al.* (unpublished; Jefferson Lab experiment 12-11-106).
[26] R. Gilman *et al.* (MUSE Collaboration), arXiv:1303.2160 [nucl-ex].