PHYSICAL REVIEW C **90**, 044908 (2014)

Transport coefficients for bulk viscous evolution in the relaxation-time approximation

Amaresh Jaiswal
Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

Radoslaw Ryblewski
Department of Physics, Kent State University, Kent, Ohio 44242, USA
and The H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, PL-31342 Kraków, Poland

Michael Strickland
Department of Physics, Kent State University, Kent, Ohio 44242, USA
(Received 4 August 2014; revised manuscript received 8 September 2014; published 23 October 2014)

We derive the form of the viscous corrections to the phase-space distribution function due to bulk viscous pressure and shear stress tensor using the iterative Chapman-Enskog method. We then calculate the transport coefficients necessary for the second-order hydrodynamic evolution of the bulk viscous pressure and the shear stress tensor. We demonstrate that the transport coefficients obtained using the Chapman-Enskog method are different than those obtained previously using the 14-moment approximation for a finite particle mass. Specializing to the case of boost-invariant and transversally homogeneous longitudinal expansion, we show that the transport coefficients obtained using the Chapman-Enskog method result in better agreement with the exact solution of the Boltzmann equation in the relaxation-time approximation compared to results obtained in the 14-moment approximation. Finally, we explicitly confirm that the time evolution of the bulk viscous pressure is significantly affected by its coupling to the shear stress tensor.

DOI: 10.1103/PhysRevC.90.044908 PACS number(s): 25.75.−q, 24.10.Nz, 47.75.+f

I. INTRODUCTION

Relativistic hydrodynamics has been quite successful in explaining a wide range of collective phenomena observed in astrophysics, cosmology, and the physics of high-energy heavy-ion collisions. The theory of relativistic hydrodynamics is formulated as a gradient expansion where ideal hydrodynamics is the zeroth order. The first-order relativistic Navier-Stokes (NS) theory [1,2] leads to acausal signal propagation, which is rectified in the second-order Israel-Stewart (IS) theory [3]. The derivation of IS equations proceeds in a variety of ways [4]. For instance, in the derivations based on the second law of thermodynamics ($\partial_\mu S^\mu \geqslant 0$), where S^μ is the generalized entropy four-current, the transport coefficients related to relaxation times for shear and bulk viscous pressures remain undetermined and have to be obtained from kinetic theory [3,5]. On the other hand, the derivations based on kinetic theory require the nonequilibrium phase-space distribution function, $f(x,p)$, to be specified. Consistent and accurate determination of the form of the dissipative equations and the associated transport coefficients is currently an active research area [6–24].

The existence of thermodynamic gradients in a nonequilibrium system gives rise to thermodynamic forces, which, in turn, results in various transport phenomena. In order to calculate the associated transport coefficients, it is convenient to first specify the nonequilibrium single particle phase-space distribution function $f(x,p)$. The two most commonly used methods to determine the form of $f(x,p)$ when the system is close to local thermodynamic equilibrium are (1) Grad's 14-moment approximation [25] and (2) the Chapman-Enskog method [26]. While Grad's moment method has been widely used in the formulation of causal relativistic dissipative hydrodynamics from kinetic theory [3–12], the Chapman-Enskog method remains less explored [13–15]. Although both methods involve expanding $f(x,p)$ around the equilibrium distribution function $f_0(x,p)$, in Refs. [14,15] it was demonstrated that the Chapman-Enskog method in the relaxation-time approximation (RTA) gives better agreement with both microscopic Boltzmann simulations and exact solutions of the RTA Boltzmann equation. This seems to stem from the fact that the Chapman-Enskog method does not require a fixed-order Grad's-moment expansion.

Relativistic viscous hydrodynamics has been used extensively to study and understand the evolution of the strongly interacting, hot and dense matter created in high-energy heavy-ion collisions; see Ref. [27] for a recent review. While much of the research on this topic is devoted to the extraction of the shear viscosity to entropy density ratio η/s from the analysis of the flow data [28–30], a systematic and self-consistent study of the effect of bulk viscosity in numerical simulations of heavy-ion collisions has not been performed. The relative lack of effort in this direction may be attributed to the fact that the bulk viscosity of hot QCD matter is estimated to be much smaller compared to the shear viscosity. However, it is important to note that, for the range of temperature probed experimentally in heavy-ion collisions, the magnitude and temperature dependence of bulk viscosity is unknown [31,32] and could be large enough to affect the spatio-temporal evolution of the QCD matter. Moreover, since QCD is a nonconformal field theory, bulk viscous corrections to the energy momentum tensor should not be neglected in order to correctly understand the dynamics of a QCD system.

From a theoretical perspective, the second-order transport coefficients that appear in the evolution equation for the bulk viscous pressure are less understood compared to those of the shear stress tensor. In Refs. [10,11], it was shown that the relaxation time for bulk viscous evolution can be obtained by employing the second law of thermodynamics in a kinetic theory setup. While, for finite masses, the transport coefficients corresponding to bulk viscous pressure and shear stress tensor have been explicitly obtained by employing the 14-moment approximation [12,33], they still remain to be determined using the Chapman-Enskog method. In this paper, we calculate the transport coefficients appearing in the second-order viscous evolution equations for nonvanishing masses using the method of Chapman-Enskog expansion. We compare the mass dependence of these coefficients with those obtained using the 14-moment approximation. In the case of one-dimensional scaling expansion of the viscous medium, we demonstrate that our results are in better agreement with the exact solution of the massive $(0+1)$-dimensional Boltzmann equation in the relaxation-time approximation [24] than those obtained using the 14-moment approximation. We also confirm that generation of bulk viscous pressure is affected more by its coupling to the shear stress tensor than to the first-order expansion rate of the system, in agreement with Ref. [33].

II. RELATIVISTIC HYDRODYNAMICS

The hydrodynamic evolution of a system having no net conserved charges (vanishing chemical potential) is governed by the local conservation of energy and momentum, $\partial_\mu T^{\mu\nu}=0$. The energy-momentum tensor, $T^{\mu\nu}$, characterizing the macroscopic state of a system, can be expressed in terms of a single-particle phase-space distribution function and tensor decomposed into hydrodynamic degrees of freedom [34],

$$T^{\mu\nu}=\int dP\, p^\mu p^\nu f(x,p)=\epsilon u^\mu u^\nu-(P+\Pi)\Delta^{\mu\nu}+\pi^{\mu\nu}. \tag{1}$$

Here $dP\equiv g d^3p/[(2\pi)^3 p^0]$ is the invariant momentum-space integration measure, where g is the degeneracy factor, p^μ is the particle four-momentum, and $f(x,p)$ is the phase-space distribution function. In the tensor decomposition, ϵ, P, Π, and $\pi^{\mu\nu}$ are energy density, thermodynamic pressure, bulk viscous pressure, and shear stress tensor, respectively. The projection operator $\Delta^{\mu\nu}\equiv g^{\mu\nu}-u^\mu u^\nu$ is orthogonal to the hydrodynamic four-velocity u^μ defined in the Landau frame: $T^{\mu\nu}u_\nu=\epsilon u^\mu$. The metric tensor is Minkowskian, $g^{\mu\nu}\equiv\mathrm{diag}(+1,-1,-1,-1)$.

The projection of $\partial_\mu T^{\mu\nu}=0$ along and orthogonal to u^μ leads to the evolution equations for ϵ and u^μ,

$$\dot\epsilon+(\epsilon+P+\Pi)\theta-\pi^{\mu\nu}\sigma_{\mu\nu}=0, \tag{2}$$

$$(\epsilon+P+\Pi)\dot u^\alpha-\nabla^\alpha(P+\Pi)+\Delta^\alpha_\nu\partial_\mu\pi^{\mu\nu}=0. \tag{3}$$

Here we have used the standard notation $\dot A\equiv u^\mu\partial_\mu A$ for the co-moving derivative, $\theta\equiv\partial_\mu u^\mu$ for the expansion scalar, $\sigma^{\mu\nu}\equiv\frac{1}{2}(\nabla^\mu u^\nu+\nabla^\nu u^\mu)-\frac{1}{3}\theta\Delta^{\mu\nu}$ for the velocity stress tensor, and $\nabla^\alpha\equiv\Delta^{\mu\alpha}\partial_\mu$ for spacelike derivatives. The inverse temperature, $\beta\equiv 1/T$, is determined by the matching condition $\epsilon=\epsilon_0$, where ϵ_0 is the equilibrium energy density. In terms of the equilibrium distribution function f_0, the energy density and the thermodynamic pressure can be written as

$$\epsilon_0=u_\mu u_\nu\int dP\, p^\mu p^\nu f_0, \tag{4}$$

$$P_0=-\frac{1}{3}\Delta_{\mu\nu}\int dP\, p^\mu p^\nu f_0, \tag{5}$$

respectively. For a classical Boltzmann gas with vanishing chemical potential, the equilibrium distribution function is given by $f_0=\exp(-\beta\, u\cdot p)$, where $u\cdot p\equiv u_\mu p^\mu$.

From Eqs. (4) and (5) one obtains $\dot\epsilon$ and $\nabla^\alpha P$ in terms of derivatives of β as

$$\dot\epsilon=-I_{30}^{(0)}\dot\beta,\quad \nabla^\alpha P=I_{31}^{(0)}\nabla^\alpha\beta, \tag{6}$$

where

$$I_{nq}^{(r)}\equiv\frac{1}{(2q+1)!!}\int dP\,(u\cdot p)^{n-2q-r}\,(\Delta_{\mu\nu}p^\mu p^\nu)^q f_0. \tag{7}$$

Here we readily identify $I_{20}^{(0)}=\epsilon$ and $I_{21}^{(0)}=-P$. The integrals $I_{nq}^{(r)}$ satisfy the following relations:

$$I_{nq}^{(r)}=I_{n-1,q}^{(r-1)}\quad\text{for}\quad n>2q, \tag{8}$$

$$I_{nq}^{(r)}=\frac{1}{(2q+1)}\left[m^2 I_{n-2,q-1}^{(r)}-I_{n,q-1}^{(r)}\right], \tag{9}$$

$$I_{nq}^{(0)}=\frac{1}{\beta}\left[-I_{n-1,q-1}^{(0)}+(n-2q)I_{n-1,q}^{(0)}\right]. \tag{10}$$

The above relations lead to the following identities:

$$I_{31}^{(0)}=-\frac{1}{\beta}(\epsilon+P), \tag{11}$$

$$I_{30}^{(0)}=\frac{1}{\beta}[3\epsilon+(3+z^2)P], \tag{12}$$

where $z\equiv\beta m$ with m being the mass of the particle. Substituting the expressions for $\dot\epsilon$ and $\nabla^\alpha P$ from Eq. (6) in Eq. (2), one obtains

$$\dot\beta=\frac{\beta(\epsilon+P)}{3\epsilon+(3+z^2)P}\theta+\frac{\beta(\Pi\theta-\pi^{\rho\gamma}\sigma_{\rho\gamma})}{3\epsilon+(3+z^2)P}, \tag{13}$$

$$\nabla^\alpha\beta=-\beta\dot u^\alpha-\frac{\beta}{\epsilon+P}\left(\Pi\dot u^\alpha-\nabla^\alpha\Pi+\Delta^\alpha_\nu\partial_\mu\pi^{\mu\nu}\right). \tag{14}$$

The above identities are used later to obtain the form of viscous corrections to the distribution function and derive evolution equations for shear and bulk viscous pressures.

Close to local thermodynamic equilibrium, the phase-space distribution function can be written as $f=f_0+\delta f$, where $\delta f\ll f$. From Eq. (1), the bulk viscous pressure Π and the shear stress tensor $\pi^{\mu\nu}$ can be expressed in terms of the nonequilibrium part of the distribution function δf as [34]

$$\Pi=-\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\,\delta f, \tag{15}$$

$$\pi^{\mu\nu}=\Delta^{\mu\nu}_{\alpha\beta}\int dP\, p^\alpha p^\beta\,\delta f, \tag{16}$$

where $\Delta^{\mu\nu}_{\alpha\beta}\equiv\frac{1}{2}(\Delta^\mu_\alpha\Delta^\nu_\beta+\Delta^\mu_\beta\Delta^\nu_\alpha)-\frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$ is a traceless symmetric projection operator orthogonal to u^μ. In the following, we iteratively solve the RTA Boltzmann equation to obtain δf up to first order.

III. VISCOUS EVOLUTION EQUATIONS

The relativistic Boltzmann equation in the RTA is given by [35]

$$p^\mu \partial_\mu f = -(u\cdot p)\frac{\delta f}{\tau_{\rm eq}}, \tag{17}$$

where $\tau_{\rm eq}$ is the relaxation time. To ensure the straightforward conservation of particle current and energy-momentum tensor, $\tau_{\rm eq}$ should be independent of momenta and u^μ should be defined in the Landau frame [35]. Rewriting Eq. (17) in the form $f = f_0 - (\tau_{\rm eq}/u\cdot p)\, p^\mu\partial_\mu f$ and solving iteratively, one obtains [14,36]

$$f_1 = f_0 - \frac{\tau_{\rm eq}}{u\cdot p}\, p^\mu\partial_\mu f_0, \tag{18}$$

$$f_2 = f_0 - \frac{\tau_{\rm eq}}{u\cdot p}\, p^\mu\partial_\mu f_1,$$

$$\vdots\ , \tag{19}$$

where $f_n = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots + \delta f^{(n)}$. To first order in derivatives, we have

$$\delta f^{(1)} = -\frac{\tau_{\rm eq}}{u\cdot p}\, p^\mu\partial_\mu f_0. \tag{20}$$

Using Eqs. (13) and (14) and consistently ignoring higher order gradient correction terms, one obtains [36]

$$\delta f = \frac{\beta\tau_{\rm eq}}{u\cdot p}\left\{\frac{1}{3}\left[m^2 - \left(1-3c_s^2\right)(u\cdot p)^2\right]\theta + p^\mu p^\nu\sigma_{\mu\nu}\right\} f_0. \tag{21}$$

Here, the velocity of sound squared, $c_s^2 \equiv dP/d\epsilon$, can be expressed as

$$c_s^2 = \frac{\epsilon + P}{3\epsilon + (3+z^2)P}. \tag{22}$$

We observe that the above expression reduces to $c_s^2 = 1/3$ in the ultrarelativistic ($z\to 0$) limit.

Substituting Eq. (20) in Eqs. (15) and (16), one obtains

$$\Pi = -\tau_{\rm eq}\beta_\Pi\theta, \tag{23}$$

$$\pi^{\mu\nu} = 2\tau_{\rm eq}\beta_\pi\sigma^{\mu\nu}, \tag{24}$$

where

$$\beta_\Pi = \tfrac{5}{3}\beta\, I_{42}^{(1)} - (\epsilon+P)c_s^2, \tag{25}$$

$$\beta_\pi = \beta\, I_{42}^{(1)}. \tag{26}$$

Replacing the velocity gradients appearing in Eq. (21) with viscous pressures using Eqs. (23) and (24), one obtains

$$\delta f = -\frac{\beta f_0}{3(u\cdot p)\beta_\Pi}\left[m^2 - \left(1-3c_s^2\right)(u\cdot p)^2\right]\Pi + \frac{\beta f_0}{2(u\cdot p)\beta_\pi}\, p^\mu p^\nu \pi_{\mu\nu}. \tag{27}$$

The above form of δf is analogous to the 14-moment approximation and can be used in the Cooper-Frye prescription for particle production [37].

To obtain second-order evolution equations for the bulk viscous pressure and the shear stress tensor, we follow the methodology discussed in Ref. [8]. We express the evolution of bulk viscous pressure and shear stress tensor given in Eqs. (15) and (16) as

$$\dot\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta \delta\dot f, \tag{28}$$

$$\dot\pi^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\int dP\, p^\alpha p^\beta \delta\dot f, \tag{29}$$

respectively. The co-moving derivative $\delta\dot f$ can be obtained by rewriting Eq. (17) in the form

$$\delta\dot f = -\dot f_0 - \frac{1}{u\cdot p}p^\gamma\nabla_\gamma f - \frac{\delta f}{\tau_{\rm eq}}. \tag{30}$$

Using the above expression for $\delta\dot f$ in Eqs. (28) and (29), one obtains

$$\dot\Pi = -\frac{\Pi}{\tau_{\rm eq}} + \frac{\Delta_{\alpha\beta}}{3}\int dP\, p^\alpha p^\beta\left(\dot f_0 + \frac{1}{u\cdot p}\, p^\gamma\nabla_\gamma f\right), \tag{31}$$

$$\dot\pi^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_{\rm eq}} - \Delta^{\mu\nu}_{\alpha\beta}\int dP\, p^\alpha p^\beta\left(\dot f_0 + \frac{1}{u\cdot p}\, p^\gamma\nabla_\gamma f\right). \tag{32}$$

It is clear from Eqs. (31) and (32) that there is only one time scale to describe the relaxation of the viscous evolution equations, i.e., $\tau_{\rm eq} = \tau_\Pi = \tau_\pi$. This stems from the fact that the RTA collision term in the Boltzmann equation (17) does not entirely capture the microscopic interactions. However, comparing the first-order equations, Eqs. (23) and (24), with the relativistic Navier-Stokes equations for bulk and shear pressures, $\Pi = -\zeta\theta$ and $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$, we obtain $\tau_\Pi = \zeta/\beta_\Pi$ and $\tau_\pi = \eta/\beta_\pi$. The first-order transport coefficients ζ and η can be calculated independently, by taking into account the full microscopic behavior of the system.

Substituting δf from Eq. (27) in Eqs. (31) and (32) and performing the integrations, one obtains the second-order evolution equations for the bulk viscous pressure and shear stress tensor,

$$\dot\Pi = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \tag{33}$$

$$\dot\pi^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}, \tag{34}$$

where $\omega^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu)$ is the vorticity tensor. The transport coefficients appearing above are

$$\delta_{\Pi\Pi} = -\frac{5}{9}\chi - c_s^2, \tag{35}$$

$$\lambda_{\Pi\pi} = \frac{\beta}{3\beta_\pi}\left(7I_{63}^{(3)} + 2I_{42}^{(1)}\right) - c_s^2, \tag{36}$$

$$\tau_{\pi\pi} = 2 + \frac{4\beta}{\beta_\pi}\, I_{63}^{(3)}, \tag{37}$$

$$\delta_{\pi\pi} = \frac{5}{3} + \frac{7\beta}{3\beta_\pi} I_{63}^{(3)}, \tag{38}$$

$$\lambda_{\pi\Pi} = -\frac{2}{3}\chi, \tag{39}$$

where

$$\chi = \frac{\beta}{\beta_\Pi}\left[\left(1-3c_s^2\right)\left(I_{42}^{(1)}+I_{31}^{(0)}\right) - m^2\left(I_{42}^{(3)}+I_{31}^{(2)}\right)\right]. \tag{40}$$

Apart from $I_{31}^{(0)} = -(\epsilon+P)/\beta$ [see Eq. (11)], we need to determine the integrals $I_{63}^{(3)}$, $I_{42}^{(1)}$, $I_{42}^{(3)}$, and $I_{31}^{(2)}$. In the following, we obtain expressions for these quantities in terms of modified Bessel functions of the second kind.

IV. TRANSPORT COEFFICIENTS

The transport coefficients obtained in the previous section can be expressed in terms of modified Bessel functions of the second kind. We start from the integral representation of the corresponding Bessel function,

$$K_n(z) = \int_0^\infty d\theta \cosh(n\theta)\, \exp(-z\cosh\theta). \tag{41}$$

Using the above form of the Bessel function, one obtains the following identities:

$$\int_0^\infty d\theta \cosh^5\theta\, \exp(-z\cosh\theta) = \frac{1}{16}[K_5 + 5K_3 + 10K_1], \tag{42}$$

$$\int_0^\infty d\theta \cosh^3\theta\, \exp(-z\cosh\theta) = \frac{1}{4}[K_3 + 3K_1], \tag{43}$$

where the z dependence of K_n is implicitly understood.

The thermodynamic integrals $I_{nq}^{(r)}$ can be cast in a similar form,

$$I_{nq}^{(r)} = \frac{g\,T^{n+2-r}z^{n+2-r}}{2\pi^2(2q+1)!!}(-1)^q \int_0^\infty d\theta\,(\cosh\theta)^{n-2q-r} \times (\sinh\theta)^{2q+2}\, \exp(-z\cosh\theta). \tag{44}$$

By using the identity $\cosh^2\theta - \sinh^2\theta = 1$, the integral in $I_{nq}^{(r)}$ can be expressed in terms of $\cosh\theta$ only. Employing Eqs. (42) and (43), one obtains

$$I_{63}^{(3)} = -\frac{gT^5z^5}{210\pi^2}\left[\frac{1}{16}(K_5 - 11K_3 + 58K_1) - 4K_{i,1} + K_{i,3}\right], \tag{45}$$

$$I_{42}^{(1)} = \frac{gT^5z^5}{30\pi^2}\left[\frac{1}{16}(K_5 - 7K_3 + 22K_1) - K_{i,1}\right], \tag{46}$$

$$I_{42}^{(3)} = \frac{gT^3z^3}{30\pi^2}\left[\frac{1}{4}(K_3 - 9K_1) + 3K_{i,1} - K_{i,3}\right], \tag{47}$$

$$I_{31}^{(2)} = -\frac{gT^3z^3}{6\pi^2}\left[\frac{1}{4}(K_3 - 5K_1) + K_{i,1}\right]. \tag{48}$$

Here the function $K_{i,n}$ is defined by the integral

$$K_{i,n}(z) = \int_0^\infty \frac{d\theta}{(\cosh\theta)^n}\, \exp(-z\cosh\theta), \tag{49}$$

which has the following property:

$$\frac{d}{dz}K_{i,n}(z) = -K_{i,n-1}(z). \tag{50}$$

This identity can also be written in integral form as

$$K_{i,n}(z) = K_{i,n}(0) - \int_0^z K_{i,n-1}(z')dz'. \tag{51}$$

We observe that, by using the series expansion of $K_{i,0}(z) = K_0(z)$, the above recursion relation can be employed to evaluate $K_{i,n}(z)$ up to any given order in z.

In the results section, we will use the exact expressions for the various transport coefficients. However, before proceeding to the numerical results it is possible to compare the analytic small-mass expansions of the transport coefficients with the results obtained using the 14-moment approximation. With this in mind, we now present small-mass expansions of the kinetic coefficients obtained in Eqs. (25), (26), and (35)–(39). We begin by noting that the quantity χ that appears in the transport coefficients (35)–(39) has the following small-mass expansion:

$$\chi = -\frac{9}{5} - \frac{9\pi z}{50} + \mathcal{O}(z^2\ln z). \tag{52}$$

The small-mass expansions of the transport coefficients entering the bulk evolution equation are

$$\begin{aligned}\frac{\beta_\Pi}{\epsilon+P} &= \frac{5z^4}{432} + \mathcal{O}(z^5),\\ \delta_{\Pi\Pi} &= \frac{2}{3} + \frac{\pi z}{10} + \mathcal{O}(z^2\ln z),\\ \lambda_{\Pi\pi} &= \frac{z^2}{18} - \frac{5z^4}{144} + \mathcal{O}(z^5).\end{aligned} \tag{53}$$

Similarly, the small-mass expansions of the transport coefficients entering the shear tensor evolution equation are

$$\begin{aligned}\frac{\beta_\pi}{\epsilon+P} &= \frac{1}{5} - \frac{z^2}{60} + \frac{z^4}{96} + \mathcal{O}(z^5),\\ \delta_{\pi\pi} &= \frac{4}{3} + \frac{z^2}{36} - \frac{25z^4}{864} + \mathcal{O}(z^5),\\ \tau_{\pi\pi} &= \frac{10}{7} + \frac{z^2}{21} - \frac{25z^4}{504} + \mathcal{O}(z^5),\\ \lambda_{\pi\Pi} &= \frac{6}{5} + \frac{3\pi z}{25} + \mathcal{O}(z^2\ln z).\end{aligned} \tag{54}$$

We observe that, while the expressions for β_Π and β_π in Eqs. (53) and (54) are identical to those obtained using the 14-moment method [12,33], the other coefficients agree only up to the constant term in their respective Taylor expansions in powers of z.

Having established that the Chapman-Enskog transport coefficients are different than the 14-moment transport coefficients even for small masses, we now turn to the exact numerical evaluation of the transport coefficients for arbitrary mass. In Fig. 1 we compare the exact transport coefficients obtained herein using the Chapman-Enskog method (blue dashed line) with those calculated using the 14-moment approximation (brown dotted line). Figures 1(a) and 1(b) show the transport

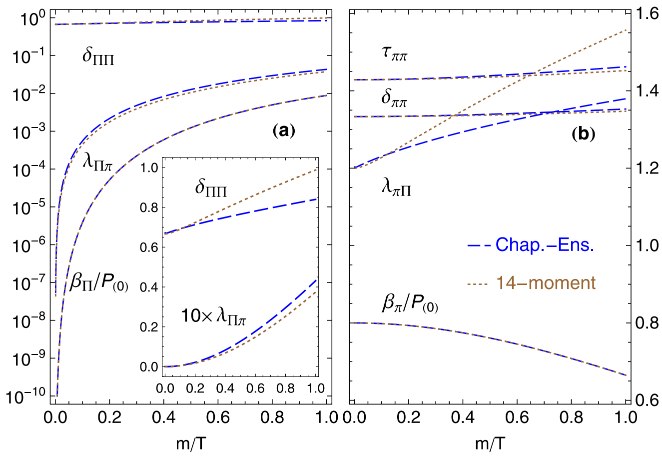

FIG. 1. (Color online) Comparison of the exact transport coefficients obtained herein using the Chapman-Enskog method (blue dashed line) with those calculated using the 14-moment approximation (brown dotted line). The two panels correspond to the transport coefficients which enter (a) the bulk viscous pressure and (b) the shear stress tensor evolution equations, as a function of the ratio of mass and temperature. The inset in panel (a) shows the m/T dependence of the transport coefficients $\delta_{\Pi\Pi}$ and $\lambda_{\Pi\pi}$ obtained by using the two methods on a linear scale. Here $P_{(0)}$ is the pressure at vanishing mass; i.e., $P_{(0)} \equiv P(m=0,T)$.

coefficients entering the evolution equations for the bulk viscous pressure and the shear stress tensor, respectively. In the inset of Fig. 1(a), we show the m/T dependence of the transport coefficients $\delta_{\Pi\Pi}$ and $\lambda_{\Pi\pi}$ (multiplied by a factor of 10) obtained by using the two methods on a linear scale. We observe that the two methods lead to very similar values of the transport coefficients for small values of $z=m/T$. For large values of z, the differences are significant for some transport coefficients. For example, at $z=1$, the values of $\lambda_{\pi\Pi}$, $\delta_{\Pi\Pi}$, and $\lambda_{\Pi\pi}$ in the two cases differ by approximately 15%, 20%, and 25%, respectively.

Another quantity of interest is the square of the sound velocity in the medium, c_s^2, which for small masses is approximately

$$\frac{1}{3}-c_s^2=\frac{z^2}{36}-\frac{5z^4}{864}+\mathcal{O}(z^6\ln z). \tag{55}$$

In the RTA, by comparing the relativistic NS equations, $\Pi = -\zeta\theta$ and $\pi^{\mu\nu}=2\eta\sigma^{\mu\nu}$, with Eqs. (23) and (24), one obtains $\zeta/\eta=\beta_\Pi/\beta_\pi$. Using the series expansion in z, one obtains

$$\frac{\zeta}{\eta}=75\left(\frac{1}{3}-c_s^2\right)^2+\mathcal{O}(z^5). \tag{56}$$

The relation in Eq. (56) can also be obtained by using the expressions for ζ and η presented in Ref. [24].[1] It is interesting to note that the form of the above expression is similar to the well-known relation $\zeta/\eta=15(1/3-c_s^2)^2$ derived by Weinberg [38]. However, we find the proportionality constant to be exactly five times larger than that obtained by Weinberg.

V. BOOST-INVARIANT (0 + 1)-DIMENSIONAL CASE

In the case of a transversely homogeneous and purely longitudinal boost-invariant expansion [39], all scalar functions of space and time depend only on the longitudinal proper time $\tau=\sqrt{t^2-z^2}$. In terms of Milne coordinates, (τ,x,y,η), the hydrodynamic four-velocity becomes $u^\mu=(1,0,0,0)$. The energy-momentum conservation equation together with equations (33) and (34) reduce to

$$\dot\epsilon=-\frac{1}{\tau}(\epsilon+P+\Pi-\pi), \tag{57}$$

$$\dot\Pi+\frac{\Pi}{\tau_\Pi}=-\frac{\beta_\Pi}{\tau}-\delta_{\Pi\Pi}\frac{\Pi}{\tau}+\lambda_{\Pi\pi}\frac{\pi}{\tau}, \tag{58}$$

$$\dot\pi+\frac{\pi}{\tau_\pi}=\frac{4}{3}\frac{\beta_\pi}{\tau}-\left(\frac{1}{3}\tau_{\pi\pi}+\delta_{\pi\pi}\right)\frac{\pi}{\tau}+\frac{2}{3}\lambda_{\pi\Pi}\frac{\Pi}{\tau}, \tag{59}$$

where $\pi\equiv-\tau^2\pi^{\eta\eta}$. We note that in this case the term involving the vorticity tensor, $2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}$, vanishes and hence has no effect on the dynamics of the fluid. We also note that the first terms on the right-hand side of Eqs. (58) and (59) are the first-order terms $\beta_\Pi\theta$ and $2\beta_\pi\sigma^{\mu\nu}$, respectively, whereas the rest are of second order.

We solve Eqs. (57)–(59) simultaneously assuming an initial temperature of $T_0=600$ MeV at the initial proper time $\tau_0=0.5$ fm/c, with relaxation times $\tau_{\rm eq}=\tau_\Pi=\tau_\pi=0.5$ fm/c corresponding to $(\eta/s)_{\tau=\tau_0}=3/4\pi$. We solve the equations for two different initial pressure configurations, $\xi_0=0$, corresponding to an isotropic pressure configuration $\pi_0=\Pi_0=0$, and $\xi_0=100$, corresponding to a highly oblate anisotropic configuration. Here ξ is the anisotropy parameter, which is related to the average transverse and longitudinal momentum in the local rest frame via $\xi=\frac{1}{2}\langle p_T^2\rangle/\langle p_L^2\rangle-1$. We consider two different masses, $m=300$ MeV, roughly corresponding to the constituent quark mass, and $m=1$ GeV, representing the approximate thermal mass of a gluon or quark. For comparison, we also solve Eqs. (57)–(59) with transport coefficients obtained by using the 14-moment method [12,33].

In Figs. 2–5 we show the proper-time evolution of the pressure anisotropy $\mathcal{P}_L/\mathcal{P}_T\equiv(P+\Pi-\pi)/(P+\Pi+\pi/2)$ (top) and the bulk viscous pressure times the proper time (bottom) for three different calculations: the exact solution of the RTA Boltzmann equation [24] (red solid line), second-order viscous hydrodynamics using the 14-moment method [12] (brown dotted line), and the Chapman-Enskog method used herein (blue dashed line). Figures 2 and 3 show the case for which $m=300$ MeV, while Figs. 4 and 5 show the case for which $m=1$ GeV. Figures 2 and 4 correspond to an isotropic initial condition ($\xi_0=0$), while Figs. 3 and 5 correspond to a highly oblate anisotropic initial condition ($\xi_0=100$).

From Figs. 2–5, we see that $\mathcal{P}_L/\mathcal{P}_T$ is quite insensitive to whether one uses the 14-moment or Chapman-Enskog transport coefficients obtained herein. However, the result for $\tau\Pi$ using the Chapman-Enskog method is in better agreement with the exact solution of the RTA Boltzmann equation than that using the 14-moment method.

[1]We note that the factor 75 is different than the value obtained in Ref. [12], which was 72.75.

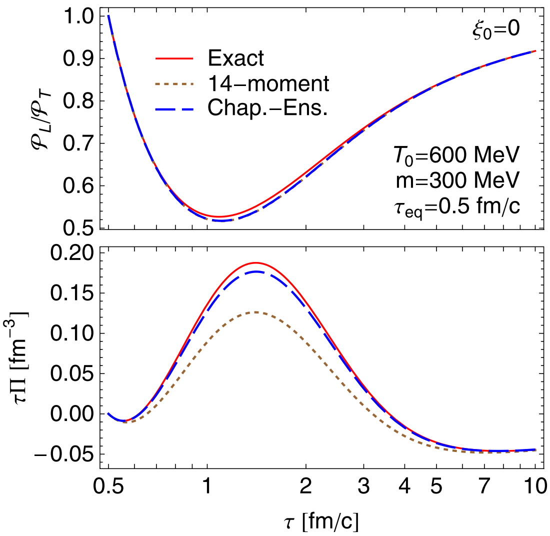

FIG. 2. (Color online) Time evolution of the pressure anisotropy $\mathcal{P}_L/\mathcal{P}_T$ (top) and the bulk viscous pressure times τ (bottom) for three different calculations: the exact solution of the RTA Boltzmann equation [24] (red solid line), second-order viscous hydrodynamics using the 14-moment method [12] (brown dotted line), and the Chapman-Enskog method used herein (blue dashed line). For both panels we use $T_0 = 600$ MeV at $\tau_0 = 0.5$ fm/c, $m = 300$ MeV, and $\tau_{\rm eq} = \tau_\pi = \tau_\Pi = 0.5$ fm/c. The initial spheroidal anisotropy in the distribution function, $\xi_0 = 0$, corresponds to isotropic initial pressures with $\pi_0 = 0$ and $\Pi_0 = 0$.

In Fig. 6 we plot the proper-time evolution of the second-order terms scaled by the first-order term in the evolution equation for bulk viscous pressure, Eq. (58). We observe that, for $m = 300$ MeV (top panel), the relative magnitude of the shear-bulk coupling term is greater than unity for the proper-time interval $0.6 \lesssim \tau \lesssim 3$ fm/c, indicating that

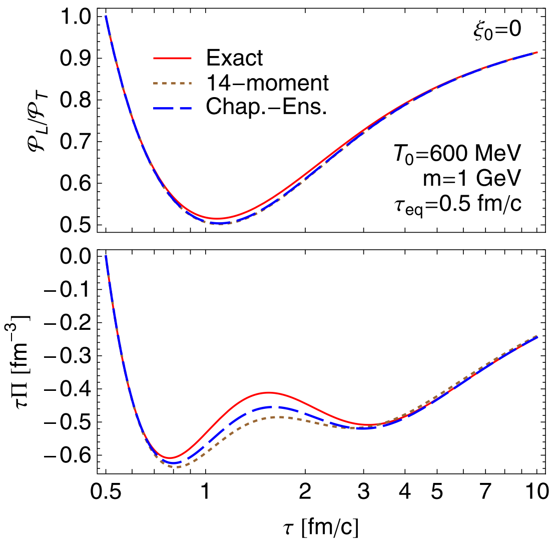

FIG. 4. (Color online) Same as Fig. 2 except here we take $m = 1$ GeV.

the evolution of bulk viscous pressure is dominated by its coupling to the shear for a long time on the time scales relevant to hydrodynamic evolution in relativistic heavy-ion collisions. For the case of $m = 1$ GeV (bottom panel), although the effect is not as prominent, the shear-bulk coupling term is still almost as important as the first-order expansion scalar.

VI. CONCLUSIONS AND OUTLOOK

In this paper we applied the iterative Chapman-Enskog method to the derive second-order viscous hydrodynamical equations and the associated transport coefficients for a massive gas in the relaxation-time approximation. The resulting dynamical equations (33) and (34) have precisely the same

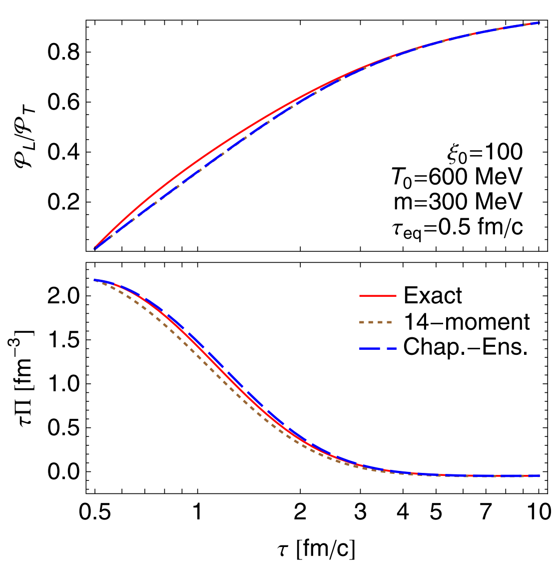

FIG. 3. (Color online) Same as Fig. 2 except here we take $\xi_0 = 100$ corresponding to $\pi_0 = 51.11$ GeV/fm^3 and $\Pi_0 = 0.85$ GeV/fm^3.

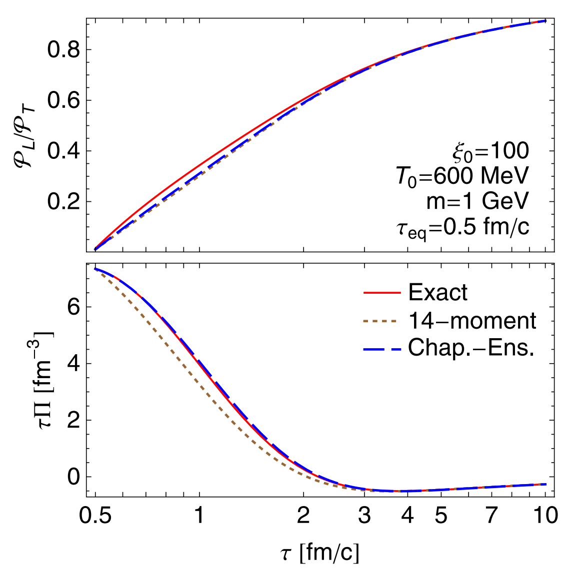

FIG. 5. (Color online) Same as Fig. 3 except here we take $m = 1$ GeV, which for $\xi_0 = 100$ implies $\pi_0 = 35.12$ GeV/fm^3 and $\Pi_0 = 3.08$ GeV/fm^3.

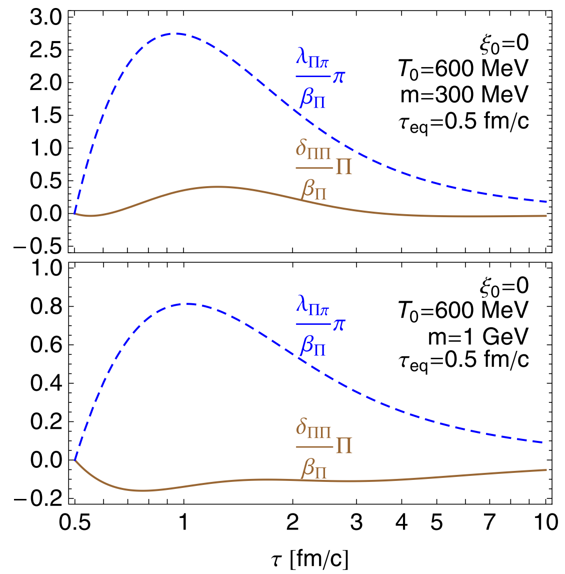

FIG. 6. (Color online) Proper time evolution of the second-order terms scaled by the first-order term in the evolution equation for bulk viscous pressure, Eq. (58). For both panels we use $T_0 = 600$ MeV at $\tau_0 = 0.5$ fm/c and $\tau_{\rm eq} = \tau_\pi = \tau_\Pi = 0.5$ fm/c. The initial spheroidal anisotropy in the distribution function, $\xi_0 = 0$, corresponds to an isotropic pressure configuration $\pi_0 = 0$ and $\Pi_0 = 0$. For the top panel, we show results for $m = 300$ MeV whereas the bottom panel corresponds to $m = 1$ GeV.

form as those obtained using the 14-moment approximation [12]; however, some of the transport coefficients are different than those obtained in the 14-moment approximation when $m > 0$. The equivalence or inequivalence of the various transport coefficients was established analytically by using Taylor expansions in m/T and also by direct numerical evaluation of the necessary integrals.

Having obtained the full set of dynamical equations necessary to self-consistently evolve both the bulk pressure and shear tensor, we then specialized to the case of a transversally homogeneous and longitudinally boost-invariant system. In this specific case it is possible to solve the RTA Boltzmann equation exactly [24]. Using this solution as a benchmark, we computed the pressure anisotropy and bulk pressure evolution using both the Chapman-Enskog method presented herein and the 14-moment method used in Ref. [12]. We demonstrated that the Chapman-Enskog method is able to reproduce the exact solution better than the 14-moment method. For the pressure anisotropy both methods give very similar results, but for the bulk pressure evolution the Chapman-Enskog method better reproduces the exact solution.

Finally, we presented a comparison of the magnitude of the shear-bulk coupling term in the dynamical equations for the bulk pressure to the term proportional to the first-order expansion scalar. We showed that, on the time scales relevant for relativistic heavy-ion collisions, the shear-bulk coupling in the bulk pressure evolution equation is equally as important as the term involving the expansion scalar, in agreement with previous findings [33]. We therefore conclude that, once the second-order terms for the bulk pressure are taken into account, at least in the relaxation-time approximation, we obtain very good agreement with the exact solution of the RTA Boltzmann equation. Since the latter does not rely on order-by-order expansion of the distribution function about equilibrium, this can be taken as evidence that in the RTA the second-order terms capture the most important nonequilibrium corrections.

At this point, we would like to clarify that we are using the exact solution of the RTA Boltzmann equation as a benchmark to compare different hydrodynamic formulations and that our minimal requirement for a viable nonconformal hydrodynamic theory is that it should be able to describe the dynamics in this simple case. It is true that the dynamics becomes more complicated when realistic scattering kernels are considered. These could, in fact, lead to a completely different parametric behavior fore bulk viscosity [40,41]. Looking forward, since the shear-bulk coupling term is as important as the first-order term, we believe it would be interesting to determine its impact in higher dimensional simulations. Moreover, from a phenomenological perspective, a large negative bulk viscous correction might lead to early onset of cavitation. It would therefore be instructive to see how the second-order transport coefficients obtained here influence cavitation. In addition, it would also be interesting to see whether the second-order results derived herein could be extended to third order. We leave these questions for a future work.

ACKNOWLEDGMENTS

We thank G. Denicol for useful discussions. R.R. was supported by Polish National Science Center Grant No. DEC-2012/07/D/ST2/02125 and US DOE Grant No. DE-SC0004104. M.S. was supported in part by US DOE Grant No. DE-SC0004104.

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