${}^{6}_{\Lambda\Lambda}$He in cluster effective field theory

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The hypernucleus ${}^{6}_{\Lambda\Lambda}$He is studied as a three-body ($\Lambda\Lambda\alpha$) cluster system in cluster effective field theory at leading order. We find that the three-body contact interaction exhibits the limit cycle when the cutoff in the integral equations is sent to the asymptotic limit and thus it should be promoted to leading order. We also derive a determination equation of the limit cycle which reproduces the numerically obtained limit cycle. We then study the correlations between the double Λ separation energy $B_{\Lambda\Lambda}$ of ${}^{6}_{\Lambda\Lambda}$He and the scattering length $a_{\Lambda\Lambda}$ of the S-wave $\Lambda\Lambda$ scattering. The role of the scale in this approach is also discussed.

Although the first observation of ${}^{6}_{\Lambda\Lambda}$He was reported in 1960s [1], there have been only a few reports on this light hypernucleus [2,3]. Among them, a track of ${}^{6}_{\Lambda\Lambda}$He was clearly caught in an emulsion experiment of the KEK-E373 Collaboration [3], now known as the "NAGARA" event, and the two-Λ separation energy $B_{\Lambda\Lambda}$ of ${}^{6}_{\Lambda\Lambda}$He is estimated as $B_{\Lambda\Lambda} = 6.93 \pm 0.16$ MeV after being averaged with that from the "MIKAGE" event [4,5]. This would be essential information to study the $\Lambda\Lambda$ interaction.

On the other hand, theoretical studies for double Λ hypernuclei mainly aim at extracting information on baryon-baryon interactions in the strangeness sector and searching for new exotic systems for which the value of $B_{\Lambda\Lambda}$ of ${}^{6}_{\Lambda\Lambda}$He plays an important role [6,7]. Theoretical studies on ${}^{6}_{\Lambda\Lambda}$He have been reported with various issues [8–12], primarily employing the three-body ($\Lambda\Lambda\alpha$) cluster model. One of those issues is the role of the mixing of the ΞN channel in the $\Lambda\Lambda$ interaction which is triggered by the small mass difference, about 23 MeV, between ΞN and $\Lambda\Lambda$ [11].

Effective field theories at very low energies are expected to provide a model-independent and systematic perturbative method where one introduces a high momentum separation scale Λ_H between relevant degrees of freedom in low energy and irrelevant degrees of freedom in high energy for the system in question. Then one constructs an effective Lagrangian expanded in terms of the number of derivatives order by order. Coupling constants appearing in the effective Lagrangian should be determined from available experimental or empirical data. For a review, see, e.g., Refs. [13,14] and references therein. In the previous publication [15], we studied ${}^{4}_{\Lambda\Lambda}$H, a bound state of a light double Λ hypernucleus, and the S-wave scattering of Λ and ${}^{3}_{\Lambda}$H below the hypertriton breakup threshold by treating ${}^{4}_{\Lambda\Lambda}$H as a three-body (Λ-Λ-deuteron) system in cluster effective field theory (EFT) at leading order (LO).

In this work, we apply this approach to study the structure of ${}^{6}_{\Lambda\Lambda}$He as a three-body ($\Lambda\Lambda\alpha$) cluster system. For this purpose, we treat the α particle field as an elementary field. The binding energy of the α particle is $B_4 \simeq 28.3$ MeV and its first excited state has the quantum numbers ($J^\pi = 0^+, I = 0$) and the excitation energy of $E_1 \simeq 20.0$ MeV, which is between the energy gap of ^{3}H-p (19.8 MeV) from the ground state energy and that of ^{3}He-n (20.6 MeV). Thus the large momentum scale of the α-cluster theory is $\Lambda_H \simeq \sqrt{2\mu E_1} \sim 170$ MeV where μ is the reduced mass of the (^{3}H, p) system or the (^{3}He, n) system so that $\mu \simeq \frac{3}{4}m_N$ with m_N being the nucleon mass. Therefore, the mixing of the ΞN channel in the $\Lambda\Lambda$ interaction becomes irrelevant because the mass difference $\sim$23 MeV of the two channels is larger than E_1, the large energy scale of this approach. On the other hand, we choose the binding momentum of ${}^{5}_{\Lambda}$He as the typical momentum scale Q of the theory. The Λ separation energy of ${}^{5}_{\Lambda}$He is $B_\Lambda \simeq 3.12$ MeV and thus the binding momentum of ${}^{5}_{\Lambda}$He as the ($\Lambda\alpha$) cluster system is $\gamma_{\Lambda\alpha} = \sqrt{2\mu_{\Lambda\alpha}B_\Lambda}$, where $\mu_{\Lambda\alpha}$ is the reduced mass of the Λ-α system. This leads to $\gamma_{\Lambda\alpha} \simeq 73.2$ MeV and thus our expansion parameter is $Q/\Lambda_H \sim \gamma_{\Lambda\alpha}/\Lambda_H \simeq 0.43$.

In addition, a modification of the counting rules is reported by Bedaque, Hammer, and van Kolck, which states that the three-body contact interaction should be promoted to LO because of the appearance of the "limit cycle" in its coupling in the S-wave neutron-deuteron (nd) scattering for spin doublet channel in the pionless EFT [16]. The limit cycle in a renormalization group analysis was suggested by Wilson [17] and it is also known that the limit cycle is associated with the Efimov states [18] in the unitary limit, where the scattering length in the NN interaction becomes infinity. Furthermore, a "determination equation" of the limit cycle, as an expression of the homogeneous part of the integral equation in the asymptotic limit, was obtained earlier by Danilov [19].

In this work, we investigate the bound state of the ($\Lambda\Lambda\alpha$) cluster system in cluster EFT at LO in order to describe ${}^{6}_{\Lambda\Lambda}$He. We find that the three-body contact interaction exhibits the limit cycle behavior when the coupled integral equations with a sharp cutoff are numerically solved. Thus the contact interaction should be promoted to LO. We also derive a determination equation of the limit cycle for the $\Lambda\Lambda\alpha$ system and find that the solution of the equation reproduces remarkably well the numerically obtained limit cycle. In addition, we investigate

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the correlation between $B_{\Lambda\Lambda}$ and the scattering length $a_{\Lambda\Lambda}$ of the S-wave $\Lambda\Lambda$ interaction including the three-body contact interaction with different cutoff values. The case without the three-body contact interaction will be studied as well.

The LO effective Lagrangian relevant to our study reads

$$\mathcal{L} = \mathcal{L}_\Lambda + \mathcal{L}_\alpha + \mathcal{L}_s + \mathcal{L}_t + \mathcal{L}_{\Lambda t}. \tag{1}$$

Here, $\mathcal{L}_\Lambda$ and $\mathcal{L}_\alpha$ are one-body Lagrangians for the spin-1/2 Λ and spin-0 α-cluster field in the heavy-baryon formalism [20,21], respectively,

$$\mathcal{L}_\Lambda = \mathcal{B}_\Lambda^\dagger \left[iv \cdot D + \frac{(v \cdot D)^2 - D^2}{2m_\Lambda} \right] \mathcal{B}_\Lambda + \cdots, \tag{2}$$

$$\mathcal{L}_\alpha = \phi_\alpha^\dagger \left[iv \cdot D + \frac{(v \cdot D)^2 - D^2}{2m_\alpha} \right] \phi_\alpha + \cdots, \tag{3}$$

where v^μ is the velocity vector $v^\mu = (1, \mathbf{0})$ and m_Λ and m_α are the Λ and α masses, respectively. The dots denote higher order terms. The Lagrangian of the auxiliary fields s and t are given by $\mathcal{L}_s$ and $\mathcal{L}_t$, respectively, where s is the dibaryon field of two Λ particles in the $^1S_0$ channel and t is the composite field of the $(\Lambda\alpha)$ system in the $^5_\Lambda\text{He}$ ($S = 1/2$) channel [15,22,23],

$$\mathcal{L}_s = \sigma_s s^\dagger \left[iv \cdot \partial + \frac{(v \cdot \partial)^2 - \partial^2}{4m_\Lambda} + \Delta_s \right] s$$
$$- y_s \left[s^\dagger (\mathcal{B}_\Lambda^T P^{(^1S_0)} \mathcal{B}_\Lambda) + \text{H.c.} \right] + \cdots, \tag{4}$$

$$\mathcal{L}_t = \sigma_t t^\dagger \left[iv \cdot \partial + \frac{(v \cdot \partial)^2 - \partial^2}{2(m_\Lambda + m_\alpha)} + \Delta_t \right] t$$
$$- y_t [t^\dagger \mathcal{B}_\Lambda \phi_\alpha + \text{H.c.}] + \cdots, \tag{5}$$

where $\sigma_{s,t}$ are sign factors. The mass differences between $\Lambda\Lambda$ and the s dibaryon state and between $\Lambda\alpha$ and the composite t state ($^5_\Lambda\text{He}$) are represented by $\Delta_{s,t}$, respectively. $P^{(^1S_0)} = -i\frac{1}{2}\sigma_2$ is the spin projection operator to the $^1S_0$ state. The dibaryon s state is coupled to two Λ in $^1S_0$ state and the composite t state to the S-wave $\Lambda\alpha$ state with the coupling constants $y_{s,t}$, respectively. The Lagrangian for the contact interaction of Λ and t reads

$$\mathcal{L}_{\Lambda t} = -2m_\alpha y_t^2 \frac{g(\Lambda_c)}{\Lambda_c^2} (\mathcal{B}_\Lambda^T P^{(^1S_0)} t)^\dagger (\mathcal{B}_\Lambda^T P^{(^1S_0)} t) + \cdots, \tag{6}$$

where the coupling $g(\Lambda_c)$ is a function of the cutoff Λ_c, which is defined in the coupled integral equations below.

In the present work we consider two composite states in the two-body part, namely, the s field and t field. The dibaryon s state was investigated in our previous publication [15], where the Feynman diagrams for the dressed dibaryon propagator can be found. The renormalized dressed dibaryon propagator is obtained as

$$D_s(p) = \frac{4\pi}{y_s^2 m_\Lambda} \frac{1}{\frac{1}{a_{\Lambda\Lambda}} - \sqrt{-m_\Lambda p_0 + \frac{1}{4}\mathbf{p}^2 - i\epsilon}}, \tag{7}$$

where $y_s = -\frac{2}{m_\Lambda}\sqrt{\frac{2\pi}{r_{\Lambda\Lambda}}}$. The scattering length and the effective range of S-wave $\Lambda\Lambda$ scattering are represented by $a_{\Lambda\Lambda}$ and

$r_{\Lambda\Lambda}$, respectively. We note that the expression of the dressed dibaryon propagator in Eq. (7) is for the large value of $a_{\Lambda\Lambda}$. In the case of a small value of $a_{\Lambda\Lambda}$ one can expand it in terms of the kinetic square root term [24]. The diagrams for the dressed $t(\Lambda\alpha)$ propagator can be found, e.g., in Ref. [15], which lead to the renormalized dressed $t(\Lambda\alpha)$ propagator as

$$D_t(p) = \frac{2\pi}{y_t^2 \mu_{\Lambda\alpha}} \frac{1}{\gamma_{\Lambda\alpha} - \sqrt{-2\mu_{\Lambda\alpha}\left(p_0 - \frac{1}{2(m_\alpha + m_\Lambda)}\mathbf{p}^2 + i\epsilon\right)}}, \tag{8}$$

where $y_t = -\frac{1}{\mu_{\Lambda\alpha}}\sqrt{\frac{2\pi}{r_{\Lambda\alpha}}}$. We also note that the dependence of $y_{s,t}$ on the effective ranges $r_{\Lambda\Lambda}$ and $r_{\Lambda\alpha}$ disappears in the final expression of the three-body coupled integral equations at LO [15].

The amplitude for S-wave elastic Λ-$^5_\Lambda\text{He}$ scattering in the center-of-mass (CM) frame can be described by the coupled integral equations at LO as

$$a(p,k;E) = K_{(a)}(p,k;E) - m_\alpha y_t^2 \frac{g(\Lambda_c)}{\Lambda_c^2}$$
$$- \frac{1}{2\pi^2} \int_0^{\Lambda_c} dl\, l^2 \left[K_{(a)}(p,l;E) - m_\alpha y_t^2 \frac{g(\Lambda_c)}{\Lambda_c^2} \right]$$
$$\times D_t\left(E - \frac{l^2}{2m_\Lambda}, \mathbf{l}\right) a(l,k;E)$$
$$- \frac{1}{2\pi^2} \int_0^{\Lambda_c} dl\, l^2 K_{(b1)}(p,l;E) \tag{9}$$
$$\times D_s\left(E - \frac{l^2}{2m_\alpha}, \mathbf{l}\right) b(l,k;E),$$

$$b(p,k;E) = K_{(b2)}(p,k;E) - \frac{1}{2\pi^2} \int_0^{\Lambda_c} dl\, l^2 K_{(b2)}(p,l;E)$$
$$\times D_t\left(E - \frac{l^2}{2m_\Lambda}, \mathbf{l}\right) a(l,k;E),$$

where the amplitudes $a(p,k;E)$ and $b(p,k;E)$ are half-off shell amplitudes for the elastic Λt channel and the inelastic Λt to αs channel, respectively. Here, $p = |\mathbf{p}|$ and $k = |\mathbf{k}|$ where $\mathbf{p}$ ($\mathbf{k}$) is the off-shell final (on-shell initial) relative momentum in the CM frame. Thus the total energy E is determined as $E = \frac{1}{2\mu_{\Lambda(\Lambda\alpha)}}k^2 - B_\Lambda$ where $\mu_{\Lambda(\Lambda\alpha)} = m_\Lambda(m_\Lambda + m_\alpha)/(2m_\Lambda + m_\alpha)$. A sharp cutoff Λ_c is introduced in the loop integrals of Eq. (9). The one-α and one-Λ exchange interactions are given by $K_{(a)}(l,k;E)$ and $K_{(b1,b2)}(l,k;E)$, respectively, where

$$K_{(a)}(p,l;E) = \frac{m_\alpha y_t^2}{2pl} \ln\left[\frac{\frac{m_\alpha}{2\mu_{\Lambda\alpha}}(p^2 + l^2) + pl - m_\alpha E}{\frac{m_\alpha}{2\mu_{\Lambda\alpha}}(p^2 + l^2) - pl - m_\alpha E} \right],$$

$$K_{(b1)}(p,l;E) = \frac{\sqrt{2}m_\Lambda y_s y_t}{2pl} \ln\left[\frac{p^2 + \frac{m_\Lambda}{2\mu_{\Lambda\alpha}}l^2 + pl - m_\Lambda E}{p^2 + \frac{m_\Lambda}{2\mu_{\Lambda\alpha}}l^2 - pl - m_\Lambda E} \right],$$

$$K_{(b2)}(p,l;E) = \frac{\sqrt{2}m_\Lambda y_s y_t}{2pl} \ln\left[\frac{\frac{m_\Lambda}{2\mu_{\Lambda\alpha}}p^2 + l^2 + pl - m_\Lambda E}{\frac{m_\Lambda}{2\mu_{\Lambda\alpha}}p^2 + l^2 - pl - m_\Lambda E} \right]. \tag{10}$$
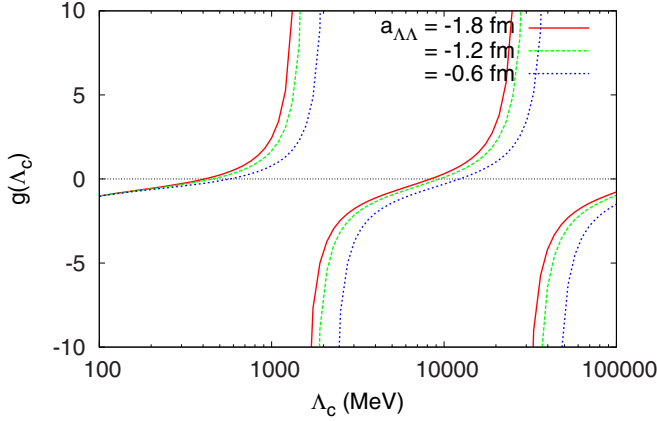
FIG. 1. (Color online) The coupling $g(\Lambda_c)$ as a function of Λ_c for $a_{\Lambda\Lambda} = -0.6, -1.2, -1.8$ fm where the values of $g(\Lambda_c)$ are fitted by $B_{\Lambda\Lambda} = 6.93$ MeV of $_{\Lambda\Lambda}^{6}$He.

In Fig. 1, we plot curves of $g(\Lambda_c)$ as a function of Λ_c with $a_{\Lambda\Lambda} = -0.6, -1.2, -1.8$ fm. The curves are numerically obtained from the homogeneous part of the coupled integral equations in Eq. (9) so as to reproduce the three-body bound state with $B_{\Lambda\Lambda} = 6.93$ MeV. One can see that the curves exhibit the limit cycle and the first divergence appears at $\Lambda_c \sim 1$ GeV. In addition, a larger value of $|a_{\Lambda\Lambda}|$ behaves as giving a larger attractive force and shifts the curves of $g(\Lambda_c)$ to the left in Fig. 1.

As pointed out in Ref. [25], one can check if the system exhibits the limit cycle behavior by studying the homogeneous part of the integral equation in the asymptotic limit. From Eq. (9), assuming the form of the amplitude in the asymptotic limit $p \gg k$ as $a(p,k) \sim p^{-1-s}$, we have

$$1 = C_1 I_1(s) + C_2 I_2(s) I_3(s), \qquad (11)$$

where

$$C_1 = \frac{1}{2\pi} \frac{m_\alpha}{\mu_{\Lambda\alpha}} \sqrt{\frac{\mu_{\Lambda(\Lambda\alpha)}}{\mu_{\Lambda\alpha}}}, \quad C_2 = \frac{\sqrt{2m_\Lambda \mu_{\Lambda(\Lambda\alpha)} \mu_{\alpha(\Lambda\Lambda)}}}{\pi^2 \mu_{\Lambda\alpha}^{3/2}},$$
$$(12)$$

and $\mu_{\alpha(\Lambda\Lambda)} = 2m_\Lambda m_\alpha/(2m_\Lambda + m_\alpha)$. The functions $I_{1,2,3}(s)$ are obtained by the Mellin transformation [26] and their explicit expressions are given in the Appendix. The imaginary solution $s = \pm i s_0$ indicates the limit cycle solution and we have

$$s_0 = 1.0496\ldots. \qquad (13)$$

On the other hand, the value of s_0 can be obtained from the curves of the limit cycle of $g(\Lambda_c)$ in Fig. 1. The $(n+1)$th values of Λ_n at which $g(\Lambda_c)$ vanishes can be parametrized as $\Lambda_n = \Lambda_0 \exp(n\pi/s_0)$. By using the second and third values of Λ_n for the three values of $a_{\Lambda\Lambda}$, we have $s_0 = \pi/\ln(\Lambda_2/\Lambda_1) \simeq 1.05$, which is in very good agreement with the value of Eq. (13). Furthermore, the value of s_0 may be checked by using Fig. 52 in Ref. [14] which is a plot of $\exp(\pi/s_0)$ versus m_1/m_3 for the mass-imbalanced system where $m_1 = m_2 \neq m_3$. In our case, $m_1/m_3 = m_\Lambda/m_\alpha \simeq 0.3$, which leads to $s_0 \simeq 1.05$ by
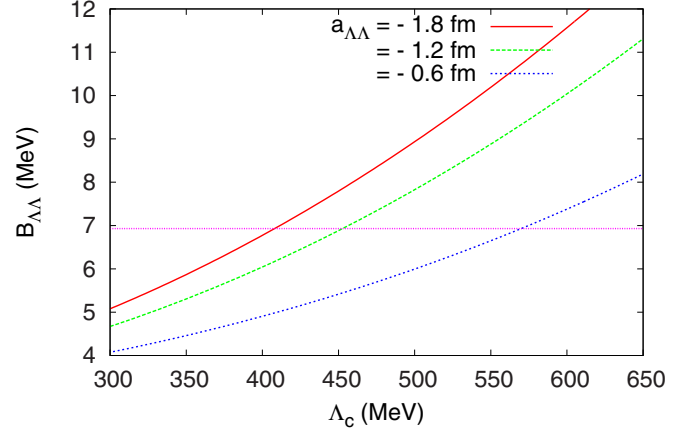

FIG. 2. (Color online) The two-Λ separation energy $B_{\Lambda\Lambda}$ as a function of the cutoff Λ_c for $a_{\Lambda\Lambda} = -0.6, -1.2, -1.8$ fm without the three-body contact interaction. The experimental data $B_{\Lambda\Lambda} = 6.93$ MeV is included as a reference line.

the result of Ref. [14]. This is in very good agreement with what we find in Eq. (13).

One may also reproduce the experimental value of $B_{\Lambda\Lambda}$ by adjusting the value of Λ_c without introducing the three-body contact interaction. In this case, the bound state of $_{\Lambda\Lambda}^{6}$He with $B_{\Lambda\Lambda} = 6.93$ MeV is found to appear only when the cutoff parameter Λ_c is larger than the critical value $\Lambda_{cr} \approx 300$ MeV, which is even larger than Λ_H of the theory. We found that $\Lambda_c \approx 300$ MeV leads to $a_{\Lambda\Lambda} \approx -3.4 \times 10^3$ fm. When we use $a_{\Lambda\Lambda} = -0.6$ to -1.8 fm as obtained from the $^{12}C(K^-, K^+\Lambda\Lambda X)$ data in Ref. [27], we should have $\Lambda_c = 570$ to 408 MeV. In Fig. 2, we plot $B_{\Lambda\Lambda}$ as a function of the cutoff Λ_c for $a_{\Lambda\Lambda} = -0.6, -1.2, -1.8$ fm. One can find that $B_{\Lambda\Lambda}$ is quite sensitive to both Λ_c and $a_{\Lambda\Lambda}$ and it becomes larger as Λ_c or $|a_{\Lambda\Lambda}|$ increases.

Figure 3 shows the two-Λ separation energy $B_{\Lambda\Lambda}$ as a function of $1/a_{\Lambda\Lambda}$ while $g(\Lambda_c)$ is renormalized at the point
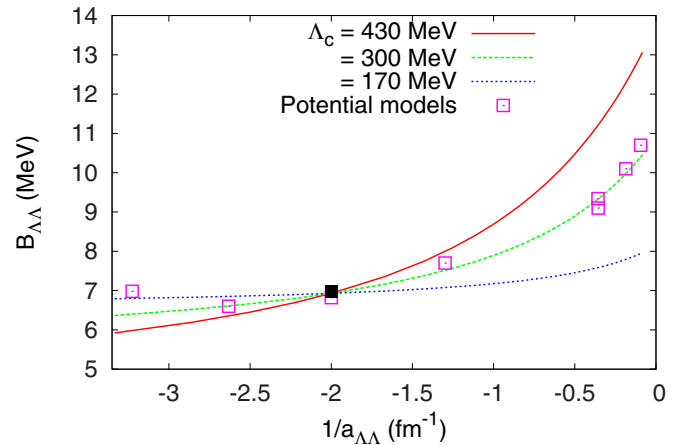

FIG. 3. (Color online) The two-Λ separation energy $B_{\Lambda\Lambda}$ as a function of $1/a_{\Lambda\Lambda}$ for $\Lambda_c = 170, 300, 430$ MeV, where $g(\Lambda_c)$ is renormalized at the point of $B_{\Lambda\Lambda} = 6.93$ MeV and $1/a_{\Lambda\Lambda} = -2.0$ fm^{-1} that is marked by a filled square. Open squares are the results from the potential models in Table 5 of Ref. [12].

marked by a filled square, i.e., $B_{\Lambda\Lambda} = 6.93$ MeV and $1/a_{\Lambda\Lambda} = -2.0$ fm^{-1}. This leads to $g(\Lambda_c) \simeq -0.715, -0.447, -0.254$ for $\Lambda_c = 170, 300, 430$ MeV, respectively. Open squares are the estimated values from the potential models given in Table 5 of Ref. [12]. We find that the curves are sensitive to the cutoff value and the results from the potential models are remarkably well reproduced by the curve with $\Lambda_c = 300$ MeV.

In summary, we have studied the hypernucleus $^{6}_{\Lambda\Lambda}$He as a three-body ($\Lambda\Lambda\alpha$) system in cluster EFT at LO. We found that the three-body contact interaction exhibits the limit-cycle and it is needed to be promoted to LO to make the result independent of the cutoff. The determination equation of the limit cycle for the bound state of $^{6}_{\Lambda\Lambda}$He is derived and its solutions remarkably well reproduce the numerically obtained results for the limit cycle. We here note that the determination equation depends on the masses and the spin-isospin quantum numbers of the state but not on the details of dynamics, and that the imaginary solution of the determination equation implies the Efimov states in the unitary limit [16]. Even though the system is not close to the unitary limit, the imaginary solution could imply the presence of a bound state as seen in this study. Therefore, the determination equation in three-body cluster systems may be useful to search for an exotic state.

We also found that $B_{\Lambda\Lambda}$ of $^{6}_{\Lambda\Lambda}$He can be reproduced even without introducing the three-body contact interaction, which, however, requires $\Lambda_c = 570$ to 410 MeV for $a_{\Lambda\Lambda} = -0.6$ to -1.8 fm. This range of the cutoff Λ_c may be converted to the length scale $r_c = \Lambda_c^{-1} = 0.35$ to 0.48 fm, which overlaps the range of a hard core potential in the early calculations of Ref. [9]. However, the $a_{\Lambda\Lambda}$ dependence is significant and it is unlikely to narrow the range of $a_{\Lambda\Lambda}$. More precise and diverse experimental data are thus required.

Finally, the correlation between $B_{\Lambda\Lambda}$ and $a_{\Lambda\Lambda}$ was investigated by introducing the three-body contact interaction and changing the value of Λ_c. We find that the results of the potential models can be reasonably reproduced by choosing $\Lambda_c = 300$ MeV, which may be understood to show the role of the two-pion exchange as the long range mechanism of the $\Lambda\Lambda$ interaction. Meanwhile, choosing $\Lambda_c > \Lambda_H$ ($\Lambda_H \simeq 170$ MeV) is inconsistent within our cluster theory because such a large cutoff probes the short range (or high momentum)

degrees of freedom such as the first excitation state of α, which is beyond the scope of the present calculation.

APPENDIX

The functions $I_{1,2,3}(s)$ in Eq. (11) are obtained by the Mellin transformation [26] as

$$I_1(s) = \int_0^\infty dx \ln\left(\frac{x^2 + ax + 1}{x^2 - ax + 1}\right) x^{s-1}$$
$$= \frac{2\pi}{s} \frac{\sin\left[s\sin^{-1}\left(\frac{1}{2}a\right)\right]}{\cos\left(\frac{\pi}{2}s\right)}, \qquad (A1)$$

$$I_2(s) = \int_0^\infty dx \ln\left(\frac{bx^2 + x + 1}{bx^2 - x + 1}\right) x^{s-1}$$
$$= \frac{2\pi}{s} \frac{1}{b^{s/2}} \frac{\sin[s\cot^{-1}(\sqrt{4b-1})]}{\cos\left(\frac{\pi}{2}s\right)}, \qquad (A2)$$

$$I_3(s) = \int_0^\infty dx \ln\left(\frac{x^2 + x + b}{x^2 - x + b}\right) x^{s-1}$$
$$= \frac{2\pi}{s} b^{s/2} \frac{\sin\left[s\cot^{-1}\left(\sqrt{4b-1}\right)\right]}{\cos\left(\frac{\pi}{2}s\right)}, \qquad (A3)$$

where $a = 2\mu_{\Lambda\alpha}/m_\alpha$ and $b = m_\Lambda/(2\mu_{\Lambda\alpha})$. When $a = b = 1$, they reproduce

$$I(s) = \int_0^\infty dx \ln\left(\frac{x^2 + x + 1}{x^2 - x + 1}\right) x^{s-1} = \frac{2\pi}{s} \frac{\sin\left(\frac{\pi}{6}s\right)}{\cos\left(\frac{\pi}{2}s\right)}. \quad (A4)$$

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