Nuclear clusters bound to doubly magic nuclei: The case of ^{212}Po

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An effective α-particle equation is derived for cases where an α particle is bound to a doubly magic nucleus. As an example, we consider ^{212}Po with the α on top of the ^{208}Pb core. We consider the core nucleus infinitely heavy, so that the α particle moves with respect to a fixed center; that is, recoil effects are neglected. The fully quantal solution of the problem is discussed. The approach is inspired by the Tohsaki-Horiuchi-Schuck-Röpke wave function concept that has been successfully applied to light nuclei. Shell-model calculations are improved by including four-particle (α-like) correlations that are of relevance when the matter density becomes low. In the region where the α-like cluster penetrates the core nucleus, the intrinsic bound-state wave function transforms at a critical density into an unbound four-nucleon shell-model state. Exploratory calculations for ^{212}Po are presented. Such preformed cluster states are very difficult to describe with shell-model calculations. Reasons for the different physics behavior of an α-like cluster with respect to a deuteron-like cluster are discussed.

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I. INTRODUCTION

The shell model of nuclei has been proven as a very successful concept, describing properly many features of the structure of nuclei [1,2]. A mean-field potential is introduced, defining single-nucleon states that are populated up to a maximum energy that is the chemical potential or the Fermi energy of the neutrons or protons, respectively. Pairing can be included in a mean-field approach using a Bogoliubov transformation among the single-particle orbits. In general, the treatment of correlations is a difficult problem in a single-nucleon mean-field approach beyond the two-particle case with pairing. However, cluster formation may occur in special situations, and the systematic treatment of correlations beyond the mean-field theory is a great challenge in the actual treatment of nuclear structure [3–15].

The problem of cluster formation in or on a nucleus is that, besides the deuteron cluster, heavier clusters like t, ^{3}He, and α are very difficult to handle technically if one wants to treat the relative motion of the cluster versus the core nucleus correctly. In principle, this is a very complicated three-, four-, etc., body problem. The solution should join two limiting cases, the situation where the cluster is well inside the core nucleus and a shell- model mean-field calculation can be performed (Hartree-Fock-Bogoliubov) and the limit of distant clusters. In the present work, we focus on four-particle (α-like) correlations. Because of spin-isospin degeneracy, such correlations are quite strong and of relevance in low-density nuclear systems [3–5].

The main ingredient is the introduction of a collective variable, describing the center-of-mass (c.m.) motion of the considered cluster, and variables that describe the intrinsic motion. A suitable choice consists of Jacobian coordinates; see Sec. III B for the four-particle case. The separation of an energy eigenstate Ψ of the few-particle cluster into a contribution $\Phi(\mathbf{R})$, where $\mathbf{R}$ denotes the c.m. coordinate, and an intrinsic part depending only on relative coordinates is strict for a homogeneous system because the total momentum $\mathbf{P}$ is conserved. This simple decomposition is not possible for finite systems such as nuclei considered here. As shown in Sec. II, in the general case of inhomogeneous systems, as, e.g., nuclei, the intrinsic wave function $\varphi^{\rm intr}(\mathbf{r}_i - \mathbf{r}_j, \mathbf{R})$ of the cluster (we focus on four-nucleon clusters) also depends on the c.m. position $\mathbf{R}$. This is mainly attributable to the Pauli blocking which depends on the local nucleon density near $\mathbf{R}$. The cluster (α-like) nucleonic wave function in momentum space is blocked out inside the Fermi sphere. Also, the global form of the four-particle wave function changes from a Gaussian-like shape at low densities, where α-particle-like bound states are formed, to a shape which corresponds to the wave function of four single nucleons found in shell-model states; see Sec. IV and [6].

The introduction of the c.m. coordinate $\mathbf{R}$ as a new dynamical collective degree of freedom simplifies the treatment of correlated nuclear systems beyond the single-quasiparticle approximation. Although the shell model gives a complete

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basis, the relevant coordinates are obtained only at the cost of very high configurational mixing. As an example, the Tohsaki-Horiuchi-Schuck-Röpke (THSR) ansatz [16] to describe the Hoyle state in ^{12}C is very successful because after separating the c.m. motion of the α-like clusters from their intrinsic motion, a simple form for the wave function can be given which describes the Hoyle state in excellent approximation. In contrast to the Brink ansatz, the c.m. motion should be treated dynamically [7] with a full freedom of the wave function, for instance, in what concerns its extension avoiding the superposition of many states with α clusters fixed at different positions.

The separation of the c.m. motion is crucial to simplify the problem in the case of α-cluster formation. It has been applied, besides in ^{8}Be, also to other systems, like ^{16}O or ^{20}Ne [8,9]. It is always of importance that not only the separation of the c.m. motion from the intrinsic motion is performed, but also that the Pauli blocking is respected by the full antisymmetrization of the nucleonic wave function. Once these ingredients are taken into account, the description is reasonable even in the case of the deuteron. One may consider $^{6}\mathrm{Li} = \alpha + d$, $^{18}\mathrm{F} = 16\mathrm{O} + d$, or $^{210}\mathrm{Bi} = {}^{208}\mathrm{Pb} + d$. However, as discussed in more detail below, there exists a crucial difference between a two-body cluster and the α-like cluster. Namely, as we have shown in previous works [6,17], a quartet (α-particle) dissolves very fast as a function of increasing baryonic density and around nuclear-matter density $n_{B,\mathrm{cluster}} \approx n_0/5$, with $n_0 \approx 0.15$ fm^{-3} the saturation density, the α particle as a well-formed cluster has disappeared. The deuteron is also dissolved as a bound state, but Cooper pairing remains also at high densities. As we know, standard pairing persists to much higher densities and even beyond n_0. Reasons for this difference between the pairing and quartetting cases are given below. We want to neglect the recoil of the core. Then a heavier nucleus like ^{212}Po is a better choice, and the separation of the c.m. motion refers only to the α-like cluster. Note that a similar problem to separate different degrees of freedom arises also in other fields such as the Born-Oppenheimer approximation in electron-ion systems [18].

The treatment of correlations in nuclei with one α on top of doubly magic nuclei such as in ^{212}Po has a long-standing tradition; see Refs. [10–15]. One α bound to the doubly magic nuclei ^{16}O to describe ^{20}Ne was considered using the generalized THSR wave function recently [8,9]. We use the implementation of correlations according to the THSR approach [16], which is able to unify clustering in nuclei with shell-model approaches, if the parameters of the variational approach are chosen correspondingly. It is a challenge to present nuclear structure calculations to give a general in-medium description which contains both the limit of cluster formation at low densities, i.e., outside the nucleus, as well as the quasiparticle (shell-model) approach, which is applicable at high densities, as already known from nuclear-matter calculations in homogeneous systems [19]; see also [6] for the four-nucleon case.

Shell-model calculations tend to underestimate the decay width of α emitting nuclei like ^{212}Po substantially [20]. Preformation of α-like correlations is indispensable [21] to explain the observed decay widths. Cluster states have been considered already some time ago; see Ref. [22]. Only recently have systematic approaches been considered which combine the shell model with cluster model calculations; see [10,23] and references given in there. The preformation amplitude obtained there is in reasonable agreement with the experimental data; the amount of $\{\mathrm{core} + \alpha\}$ clustering amplitude in the parent state of about 30% is found, which is much higher than former microscopic estimates. A calculation using a modified Woods-Saxon potential has been published recently [24]. In spite of the fact that the form of the single-particle potential is chosen *ad hoc*, the results are very reasonable. A microscopic approach leading to this empirical pocket-structure mean-field potential is, however, missing. Very recently [25] it was shown that also in a restricted Hartree-Fock calculation cluster formation can be described approximately; however, the separation of the c.m. motion has to be performed in a rigorous manner. For this, the single-particle approach must be improved, treating few-particle correlations responsible in forming bound states.

After explaining the separation of the c.m. motion in Sec. II, α-like correlations are treated in Sec. III. Exploratory calculations for ^{212}Po are presented in Sec. IV, showing the formation of a pocket in the effective α-cluster potential near the surface of the doubly magic ^{208}Pb core nucleus. Discussions and conclusions are drawn finally in Sec. V.

II. THE c.m. AND INTRINSIC SCHRÖDINGER EQUATIONS

We consider a few-body cluster, in particular A_c nucleons of mass m with two-body interaction $V_{ij}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}'_i,\mathbf{r}'_j)$. Further details, such as isospin dependence of the interaction and of the masses, are neglected so that $m_n \approx m_p = m$. More details of the interaction potential are discussed in the following sections.

To characterize the state of a system, we can introduce the positions $\mathbf{r}_i$ (coordinate space representation) or the momenta $\mathbf{p}_i$ (momentum space representation), whereas spin and isospin are not considered explicitly. If the interaction depends only on the relative positions $\mathbf{r}_i - \mathbf{r}_j$ and there is no external potential, the problem is homogeneous in space and the total momentum is conserved. It is advantageous to introduce new observables, the c.m. position $\mathbf{R} = \sum_i^{A_c} \mathbf{r}_i/A_c$, the relative coordinates $\mathbf{s}_j$, $j = 1 \cdots A_c - 1$, and, in particular, Jacobian coordinates. Canonically conjugate momenta are the total momentum $\mathbf{P} = \sum_i^{A_c} \mathbf{p}_i$ and the relative momenta $\mathbf{k}_j$, $j = 1 \cdots A_c - 1$. As an example, for $A_c = 4$ such transformations to Jacobi-Moshinsky coordinates are given in Sec. III B.

The introduction of the c.m. motion as a collective degree of freedom is also of general importance if we consider clusters (bound states) consisting of A_c particles. If the intrinsic interaction is strong compared with external influences from, e.g., core nuclei or homogeneous nuclear matter, such clusters can be considered as new elementary particles as it may happen at low density or when the cluster is quite far out in the surface of a nucleus. Then, the dynamical behavior is only given by the c.m. motion, whereas the intrinsic structure is nearly not changing.

In quantum theory, we try to subdivide the wave function $\Psi(\mathbf{R},\mathbf{s}_j)$ into two parts,

$$\Psi(\mathbf{R},\mathbf{s}_j) = \varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R})\,\Phi(\mathbf{R}). \tag{1}$$

This subdivision is unique [up to a phase factor $\Phi(\mathbf{R}) \to e^{i\alpha(\mathbf{R})}\Phi(\mathbf{R})$, $\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R}) \to e^{-i\alpha(\mathbf{R})}\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R})$] if, besides the normalization $\int dR\, ds_j\, |\Psi(\mathbf{R},\mathbf{s}_j)|^2 \equiv \int d^3R \int d^{3A-3}s_j\, |\Psi(\mathbf{R},\mathbf{s}_j)|^2 = 1$, one also imposes the individual normalizations (multiple integrals are not indicated explicitly within the present section)

$$\int dR\, |\Phi(\mathbf{R})|^2 = 1, \tag{2}$$

and for each $\mathbf{R}$

$$\int ds_j |\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R})|^2 = 1. \tag{3}$$

The Hamiltonian of a cluster may be written as

$$H = \left(-\frac{\hbar^2}{2Am}\nabla_R^2 + T\big[\nabla_{s_j}\big]\right)\delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{s}_j-\mathbf{s}'_j) + V(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j), \tag{4}$$

where the kinetic energy of the c.m. motion is explicitly given. The kinetic energy of the internal motion of the cluster, $T[\nabla_{s_j}]$, depends on the choice of the Jacobi coordinates (see Sec. III B for $A = 4$). The interaction $V(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)$ contains the mutual interaction $V_{ij}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}'_i,\mathbf{r}'_j)$ between the particles as well as the interaction with an external potential (for instance, the mean-field potential of the core nucleus) and is, in general, nonlocal in space. We specify the interaction $V(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)$ when considering the α particle on top of a doubly magic core nucleus in Secs. III and IV. At present, a local external potential may be considered to explain the separation of the c.m. motion.

To find stationary states, we take the expectation value of (4) with (1) and minimize

$$\delta\left\{\int dR\, ds_j\, dR'\, ds'_j\; \Psi^*(\mathbf{R},\mathbf{s}_j)H\Psi(\mathbf{R}',\mathbf{s}'_j) - E\int dR|\Phi(\mathbf{R})|^2 - \int dR\, F(\mathbf{R})\int ds_j|\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R})|^2\right\} = 0. \tag{5}$$

The variation of the wave function is not restricted after the boundary conditions (2) and (3) are taken into account by the Lagrange parameters E and $F(\mathbf{R})$.

The variation with respect to $\Phi^*(\mathbf{R})$ yields the wave equation for the c.m. motion

$$-\frac{\hbar^2}{2Am}\nabla_R^2\Phi(\mathbf{R}) - \frac{\hbar^2}{Am}\int ds_j \varphi^{\rm intr,*}(\mathbf{s}_j,\mathbf{R})[\nabla_R\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R})][\nabla_R\Phi(\mathbf{R})] - \frac{\hbar^2}{2Am}\int ds_j\varphi^{\rm intr,*}(\mathbf{s}_j,\mathbf{R})\big[\nabla_R^2\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R})\big]\Phi(\mathbf{R}) + \int dR'\, W(\mathbf{R},\mathbf{R}')\Phi(\mathbf{R}') = E\,\Phi(\mathbf{R}), \tag{6}$$

with the c.m. potential

$$W(\mathbf{R},\mathbf{R}') = \int ds_j\, ds'_j\, \varphi^{\rm intr,*}(\mathbf{s}_j,\mathbf{R})\big[T\big[\nabla_{s_j}\big]\delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{s}_j-\mathbf{s}'_j) + V(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)\big]\varphi^{\rm intr}(\mathbf{s}'_j,\mathbf{R}'). \tag{7}$$

The variation of $\varphi^{\rm intr,*}(\mathbf{s}_j,\mathbf{R})$ at fixed $\mathbf{R}$ yields the wave equation for the intrinsic motion

$$-\frac{\hbar^2}{Am}\Phi^*(\mathbf{R})[\nabla_R\Phi(\mathbf{R})][\nabla_R\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R})] - \frac{\hbar^2}{2Am}|\Phi(\mathbf{R})|^2\nabla_R^2\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R}) + \int dR'\, ds'_j\, \Phi^*(\mathbf{R})\big\{T\big[\nabla_{s_j}\big]\delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{s}_j-\mathbf{s}'_j) + V(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)\big\}\Phi(\mathbf{R}')\varphi^{\rm intr}(\mathbf{s}'_j,\mathbf{R}') = F(\mathbf{R})\varphi^{\rm intr}(\mathbf{s}_j,\mathbf{R}). \tag{8}$$

We emphasize that we should allow for nonlocal interactions. In particular, the Pauli blocking considered below is nonlocal. Also, the nucleon-nucleon interaction can be taken as a nonlocal potential. To simplify the calculations, often local approximations are used for the potentials. If in addition to the external potential also further conditions have to be implemented, further Lagrange multipliers are needed. For instance, the antisymmetrization with respect to the states of the core nucleus leads to a norm kernel N [26] to be considered in Sec. III.

III. THE α-PARTICLE CASE

A. Quasiparticle representation

We apply this formalism to the α-particle case, or, more generally, to the correlation of four nucleons moving in a nuclear system. The four nucleons are taken with different spin or isospin (not indicated explicitly in the following), which may form an α particle. The nucleon-nucleon interaction V_{N-N} is specified below; see Eq. (25). Concerning the nuclear system, we consider first nuclear matter (Sec. III B). This case is comparatively simple because it is homogeneous and the total momentum $\mathbf{P} = \sum_i^4 \mathbf{p}_i$ of the few-particle system is conserved. After that we consider finite nuclei. For reasons to be discussed below, the formation of an α particle bound to a doubly magic nucleus is of particular interest.

In principle, the theoretical formulation of an α cluster on top of a heavy doubly magic nucleus like ^{208}Pb, the case to be considered in this work, is rather straightforward. In the so-called Tamm-Dancoff approximation (TDA), we consider

the following Schrödinger equation:

$$\begin{aligned}&\left(\varepsilon_{n_1}+\varepsilon_{n_2}+\varepsilon_{n_3}+\varepsilon_{n_4}\right)\Psi^{\nu}_{n_1n_2n_3n_4}\\&+\frac{1}{2}\sum_{n_1'n_2'}\left[1-f\left(\varepsilon_{n_1}\right)\right]\left[1-f\left(\varepsilon_{n_2}\right)\right]\bar{v}_{n_1n_2n_1'n_2'}\Psi^{\nu}_{n_1'n_2'n_3n_4}\\&+\text{ permutations}=E_{\nu}N_{n_1n_2n_3n_4}\Psi^{\nu}_{n_1n_2n_3n_4}.\end{aligned}\tag{9}$$

The ε_{n_i} are the single-particle shell-model energies corresponding to the mean-field potential of the ^{208}Pb core, that is, $\hat{h}_i|n_i\rangle=\varepsilon_{n_i}|n_i\rangle$, where $\hat{h}$ is the single-particle Hamiltonian of nucleons moving in the mean field of the lead core and $|n_i\rangle$ are the corresponding eigenfunctions. In this basis the antisymmetrized matrix elements of the two-body force are given by $\bar{v}_{n_1n_2n_3n_4}$. Furthermore, the single-nucleon occupation ($\tau=n,p$) (Fermi-Dirac function at zero temperature) is defined as

$$f\left(\varepsilon_{n_\tau}\right)=\Theta\left(\mu_\tau-\varepsilon_{n_\tau}\right),\tag{10}$$

and the projector on single-particle states above the doubly magic core is given by the norm kernel

$$\begin{aligned}N_{n_1n_2n_3n_4}=\langle n_1n_2n_3n_4|&\Theta(\hat{h}_1-\mu_1)\Theta(\hat{h}_2-\mu_2)\\&\Theta(\hat{h}_3-\mu_3)\Theta(\hat{h}_4-\mu_4)|n_1n_2n_3n_4\rangle,\end{aligned}\tag{11}$$

where $\Theta(x)$ is the step function and the μ_i's are the chemical potentials of the valence nucleons. Of course, for the α-like cluster considered here the chemical potentials are pairwise equal.

The above four-particle Tamm-Dancoff equation is formally easy. However, in the case of an α particle, i.e., an asymptotically strongly bound cluster, the solution of this equation is absolutely nontrivial. The problem lies in the fact that one has to reproduce two limits correctly: On the one hand, for the α particle being at large distances from the Pb core, the solution should contain the correct asymptotic limit of a lead core interacting only via the Coulomb force with an otherwise unperturbed α. On the other hand, once the α-like four-nucleon cluster gets inside the Pb core, its cluster aspect gets dissolved and the four nucleons shall be described within the usual shell-model approach. To have a consistent incorporation of both limits is, as is well known, a very hard problem and has only been achieved so far within crude approximations [12,24]. A further very important aspect of the α-particle cluster, to be discussed in detail below in Sec. IV, is the fact that, in contrast to the case of the deuteron, the binding of the α particle gets lost quite abruptly once it enters the tail of the Pb core density. We have studied this effect in quite some detail in a series of earlier papers of α particles in low-density nuclear matter [6,17]. We think that the effect persists in finite systems. One could envisage to solve the above equation with a two-center shell model, one for the α particle and the other for the lead core. However, this procedure also is not free of problems concerning, for instance, spurious center-of-mass motion, etc. In this work we adopt a different strategy. Our focus is on how the α particle is modified entering from the outside into the region of finite density of the Pb core. We treat the c.m. motion in local density approximation (LDA). However, the intrinsic wave function of the α particle is considered fully quantal.

Within a quantum many-particle approach, the treatment of the interacting many-nucleon system needs some approximations which may be obtained in a consistent way from a Green's functions approach. In a first step, we can introduce the quasiparticle picture where the nucleons are moving independently in a mean field, described by a single-particle Hamiltonian $\hat{h}$ given above, with single-nucleon states $|n_i\rangle$ as the shell-model states of the ^{208}Pb core. In the next step we go beyond the quasiparticle picture and take the full interaction within the A_c-particle cluster into account. In the case of four nucleons considered here, we have for Eq. (9) in position space the representation

$$\begin{aligned}&[E_4-\hat{h}_1-\hat{h}_2-\hat{h}_3-\hat{h}_4]\Psi_4(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4)\\&=\int d^3\mathbf{r}_1'\,d^3\mathbf{r}_2'\langle\mathbf{r}_1\mathbf{r}_2|B\;V_{N-N}|\mathbf{r}_1'\mathbf{r}_2'\rangle\Psi_4(\mathbf{r}_1'\mathbf{r}_2'\mathbf{r}_3\mathbf{r}_4)\\&\quad+\int d^3\mathbf{r}_1'\;d^3\mathbf{r}_3'\langle\mathbf{r}_1\mathbf{r}_3|B\;V_{N-N}|\mathbf{r}_1'\mathbf{r}_3'\rangle\Psi_4(\mathbf{r}_1'\mathbf{r}_2\mathbf{r}_3'\mathbf{r}_4)\\&\quad+\text{four further permutations.}\end{aligned}\tag{12}$$

The six nucleon-nucleon interaction terms contain, besides the nucleon-nucleon potential V_{N-N}, also the blocking operator B, which can be given in quasiparticle state representation. For the first term on the right-hand side of Eq. (12), the expression

$$B(1,2)=[1-f_1(\hat{h}_1)-f_2(\hat{h}_2)]\tag{13}$$

results, which is the typical blocking factor of the so-called particle-particle random-phase approximation [2]. The phase-space occupation (we give the internal quantum state $\nu=\sigma,\tau$ explicitly)

$$f_\nu(\hat{h})=\sum_n^{\text{occ.}}|n,\nu\rangle\langle n,\nu|\tag{14}$$

indicates the phase space which, according to the Pauli principle, is not available for an interaction process of a nucleon with internal quantum state ν. Here we use the TDA expression $[1-f_1(\hat{h}_1)][1-f_2(\hat{h}_2)]$, which neglects the hole-hole contributions which are of relevance in deriving the gap equation if pairing is considered. In this way, our treatment is similar to the study of Cooper pairs by Cooper [27], which uses the TDA form of Eq. (13), only extended here to the case of quartets. We do not consider the Bogoliubov transformation introducing BCS quasiparticles so that we discuss in the following the TDA expression. The Pauli blocking factor can be given in the form of a projection operator $\mathcal{P}^{\text{Pauli}}=1-\sum_n^{\text{occ.}}|n,\nu\rangle\langle n,\nu|$ so that the quasiparticle subspace used to form the cluster is orthogonal to the subspace of the occupied shell-model states in the core nucleus. Then, the norm kernel N can be dropped. In homogeneous matter, the states below the Fermi energy are blocked out. In the LDA used in the present work, the reduction of the phase space owing to the Pauli principle is taken into account by the ansatz for the wave function; see Eq. (37) below. Note, however, that in the general case where the overlap between the occupied shell-model states in the core nucleus and the wave function of the α-like cluster remains finite, the norm kernel (9) and (11) cannot be dropped. This problem shall be investigated in future work. A systematic derivation of these expressions, also

for the general case of finite temperatures, can be given using the Matsubara Green's function method [4,19].

Considering homogeneous nuclear matter characterized by the nucleon densities n_τ with $\tau = (n,p)$ (we drop the spin variable σ), the quasiparticle states are momentum eigenstates so that the in-medium wave equation (12) becomes simpler in momentum representation. The single-particle Hamiltonian $\hat{h}_i$ as well as the Pauli blocking operator B are diagonal in momentum representation, and Eq. (12) reads for the α-like state (we mark Fourier transformed quantities with a tilde)

$$\begin{aligned}&\left[\varepsilon^{\rm mf}_{\tau_1}(\mathbf{p}_1)+\varepsilon^{\rm mf}_{\tau_2}(\mathbf{p}_2)+\varepsilon^{\rm mf}_{\tau_3}(\mathbf{p}_3)+\varepsilon^{\rm mf}_{\tau_4}(\mathbf{p}_4)\right]\tilde{\Psi}_4(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4)\\ &\quad+\int\frac{d^3\mathbf{p}_1'}{(2\pi)^3}\int\frac{d^3\mathbf{p}_2'}{(2\pi)^3}\left\{1-f_{\tau_1}\left[\varepsilon^{\rm mf}_{\tau_1}(\mathbf{p}_1)\right]\right\}\left\{1-f_{\tau_2}\left[\varepsilon^{\rm mf}_{\tau_2}(\mathbf{p}_2)\right]\right\}\\ &\quad\times\tilde{V}_{N-N}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')\tilde{\Psi}_4(\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3\mathbf{p}_4)\\ &\quad+\text{five permutations}=E_4(\mathbf{P})\tilde{\Psi}_4(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4).\end{aligned}\tag{15}$$

Here $\varepsilon^{\rm mf}_\tau(\mathbf{p})=\hbar^2p^2/2m+V^{\rm mf}_\tau(\mathbf{p})$ contains the quasiparticle mean-field shift $V^{\rm mf}_\tau(\mathbf{p})$, and the Fermi function $f_\tau(E)=\{\exp[(E-\mu_\tau)/(k_BT)]+1\}^{-1}$ becomes the step function $\Theta(E_{{\rm Fermi},\tau}-E)$ for zero temperature $T=0$, where $E_{{\rm Fermi},\tau}=\mu_\tau$ denotes the Fermi energy of the neutrons or protons; see Eq. (10). Note that Eq. (15) can be generalized for the case of finite temperatures T. Then, the energy eigenvalue E_4 as well as the wave function $\tilde{\Psi}_4$ will depend in addition to n_n, n_p also on T. The solution of this four-particle in-medium equation for homogeneous matter at arbitrary temperatures has been investigated extensively; see [4,6,17,28,29].

We discuss the in-medium wave equation (15) more in detail. The medium modifications originate from two effects.

(i) The self-energy shifts $V^{\rm mf}_\tau(\mathbf{p})$ contained in the single-particle Hamiltonian $\hat{h}_i$. We denote these contributions by the external part,

$$\begin{aligned}\tilde{V}^{(4),{\rm ext}}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4)=&\,V^{\rm mf}_{\tau_1}(\mathbf{p}_1)+V^{\rm mf}_{\tau_2}(\mathbf{p}_2)\\ &+V^{\rm mf}_{\tau_3}(\mathbf{p}_3)+V^{\rm mf}_{\tau_4}(\mathbf{p}_4).\end{aligned}\tag{16}$$

(ii) The Pauli blocking terms that modify the nucleon-nucleon interaction. We denote the interaction part including the Pauli blocking by the intrinsic part

$$\begin{aligned}&\tilde{V}^{(4),{\rm intr}}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4,\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\mathbf{p}_4')\\ &\quad=\left\{1-f_{\tau_1}\left[\varepsilon^{\rm mf}_{\tau_1}(\mathbf{p}_1)\right]\right\}\left\{1-f_{\tau_2}\left[\varepsilon^{\rm mf}_{\tau_2}(\mathbf{p}_2)\right]\right\}\\ &\qquad\times\tilde{V}_{N-N}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')\delta(\mathbf{p}_3-\mathbf{p}_3')\delta(\mathbf{p}_4-\mathbf{p}_4')\\ &\qquad+\text{five permutations},\end{aligned}\tag{17}$$

the integrals in Eq. (15) being modified correspondingly. In the account of the Pauli blocking in the effective wave equation (15), it is indispensable to have a conserving approximation which relates the approximation for the single-nucleon self-energy to the two-particle propagator [30]. Both the self-energy in mean-field approximation and the Pauli blocking given by the Fermi distribution are obtained in the approximation of an uncorrelated medium. Higher-order approximations to the in-medium few-particle Green's functions will improve the in-medium wave equation allowing for correlations in the medium as discussed in Sec. V.

B. α-like correlations in homogeneous nuclear matter

To solve the four-nucleon problem separating the c.m. motion as a collective degree of freedom, we introduce relative and c.m. Jacobi-Moshinsky coordinates (for details, see Ref. [28]):

$$\begin{aligned}\mathbf{r}_1&=\mathbf{R}+\mathbf{s}/2+\mathbf{s}_{12}/2,\quad \mathbf{r}_2=\mathbf{R}+\mathbf{s}/2-\mathbf{s}_{12}/2,\\ \mathbf{r}_3&=\mathbf{R}-\mathbf{s}/2+\mathbf{s}_{34}/2,\quad \mathbf{r}_4=\mathbf{R}-\mathbf{s}/2-\mathbf{s}_{34}/2.\end{aligned}\tag{18}$$

In momentum space we have the conjugate Jacobi momenta:

$$\begin{aligned}\mathbf{p}_1&=\mathbf{P}/4+\mathbf{k}/2+\mathbf{k}_{12},\quad \mathbf{p}_2=\mathbf{P}/4+\mathbf{k}/2-\mathbf{k}_{12},\\ \mathbf{p}_3&=\mathbf{P}/4-\mathbf{k}/2+\mathbf{k}_{34},\quad \mathbf{p}_4=\mathbf{P}/4-\mathbf{k}/2-\mathbf{k}_{34}.\end{aligned}\tag{19}$$

1. Zero-density limit: The free α particle

To be more transparent, we consider first the free α particle, i.e., the zero-density case $n_B=0$. The ansatz (1) reads now (we denote the free case by the index 0)

$$\Psi_0(\mathbf{R},\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34})=\varphi_0^{\rm intr}(\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34})\Phi_0(\mathbf{R}).\tag{20}$$

The Hamiltonian in position representation contains the intrinsic kinetic energy

$$T_4\left[\nabla_{s_j}\right]=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\mathbf{s}^2}-\frac{\hbar^2}{m}\frac{\partial^2}{\partial\mathbf{s}_{12}^2}-\frac{\hbar^2}{m}\frac{\partial^2}{\partial\mathbf{s}_{34}^2}\tag{21}$$

and the nucleon-nucleon interaction potential $V^{(4),{\rm intr}}(\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34};\mathbf{s}',\mathbf{s}_{12}',\mathbf{s}_{34}')$ depending on intrinsic coordinates only. The potential $V^{(4),{\rm intr}}$ contains six pair interaction terms; see Eq. (12), where the Cartesian coordinates are transformed to Jacobian coordinates according to Eq. (18). In the homogeneous system, there is no external force acting on the α particle. Mean-field self-energy shifts and Pauli blocking vanishes in the zero-density limit.

Because the interaction does not contain any dependence on $\mathbf{R}$, the intrinsic wave function $\varphi_0^{\rm intr}(\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34})$ is also not dependent on $\mathbf{R}$ [a trivial phase factor $e^{i\alpha(\mathbf{R})}$ can be eliminated, as discussed above, below Eq. (1)]. The system of wave equations (6) and (8) is considerably simplified. With respect to the application to homogeneous matter, it is convenient to use the momentum representation. For the free α particle (zero-density limit), Eq. (8) reads

$$\begin{aligned}&\frac{\hbar^2}{2m}\left[k^2+2k_{12}^2+2k_{34}^2\right]\tilde{\varphi}_0^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34})+\int\frac{d^3k'}{(2\pi)^3}\frac{d^3k_{12}'}{(2\pi)^3}\frac{d^3k_{34}'}{(2\pi)^3}\tilde{V}^{(4),{\rm intr}}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34};\mathbf{k}',\mathbf{k}_{12}',\mathbf{k}_{34}')\tilde{\varphi}_0^{\rm intr}(\mathbf{k}',\mathbf{k}_{12}',\mathbf{k}_{34}')\\ &\quad=E_\alpha^{(0)}\tilde{\varphi}_0^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34}).\end{aligned}\tag{22}$$

Here the four-nucleon interaction $V^{(4),\mathrm{intr}}$ contains the six pair interactions in the free α cluster. The new Lagrange parameter $E_\alpha^{(0)} = F(\mathbf{R})/|\Phi(\mathbf{R})|^2$ coincides with the intrinsic energy of the free α particle (of course, independent of $\mathbf{R}$),

$$E_\alpha^{(0)} = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\frac{\hbar^2}{2m}\left[k^2+2k_{12}^2+2k_{34}^2\right]\left|\tilde{\varphi}_0^{\mathrm{intr}}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34})\right|^2 + \int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3k'_{12}}{(2\pi)^3}\frac{d^3k'_{34}}{(2\pi)^3}\tilde{\varphi}_0^{\mathrm{intr},*}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34})V^{(4),\mathrm{intr}}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34};\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}'_{34})\tilde{\varphi}_0^{\mathrm{intr}}(\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}'_{34}). \quad (23)$$

The empirical value is $E_\alpha^{(0)} = -B_\alpha = -28.3$ MeV.

In the free α-particle case, the c.m. potential $W(\mathbf{R},\mathbf{R}')$, Eq. (7), is local. According to Eq. (23), it reads $W(\mathbf{R},\mathbf{R}') = E_\alpha^{(0)}\delta(\mathbf{R}-\mathbf{R}')$ so that, in the zero-density case, Eq. (6) reads

$$\frac{\hbar^2 P^2}{8m}\tilde{\Phi}_0(\mathbf{P}) + E_\alpha^{(0)}\tilde{\Phi}_0(\mathbf{P}) = E_0(\mathbf{P})\,\tilde{\Phi}_0(\mathbf{P}). \quad (24)$$

Equation (22) is the Schrödinger equation for the intrinsic motion of the α particle, and Eq. (24) is the Schrödinger equation for the c.m. motion. The Lagrange parameter $E_0(\mathbf{P}) \equiv \hbar^2/(8m)P^2 + E_\alpha^{(0)}$ has the meaning of the total energy of the α particle.

The wave functions $\Phi_0(\mathbf{R}), \varphi_0^{\mathrm{intr}}(\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34})$ or their Fourier transforms follow, solving the Schrödinger equations. The solution for the c.m. motion, Eq. (24), is trivial in the case of homogeneous matter. In position representation results a plain wave with wave vector $\mathbf{P}$. To solve the wave equation for the intrinsic motion (22) we have to define the interaction. We choose a separable interaction [6] with Gaussian form factor,

$$V_{N-N}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}'_1,\mathbf{p}'_2) = \lambda e^{-(\mathbf{p}_1-\mathbf{p}_2)^2/4\gamma^2}e^{-(\mathbf{p}'_1-\mathbf{p}'_2)^2/4\gamma^2}\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}'_1-\mathbf{p}'_2), \quad (25)$$

and find the approximate solution from a variational approach.

In particular, for Gaussian wave functions as a simple variational ansatz, the c.m. motion can be easily separated. For vanishing c.m. motion, $\mathbf{P}=0$, we have for the internal wave function

$$\tilde{\varphi}_0^{\mathrm{intr}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = \tfrac{1}{\mathrm{norm}}e^{-(p_1^2+p_2^2+p_3^3+p_4^2)/b^2}\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4), \quad (26)$$

with the normalization $\sum_{p_1,p_2,p_3,p_4}|\tilde{\varphi}_0^{\mathrm{intr}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)|^2 = 1$, or, explicitly,

$$\tilde{\varphi}_0^{\mathrm{intr}}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34}) = \frac{2^6(2\pi)^{9/4}}{b^{9/2}}e^{-2\mathbf{k}_{12}^2/b^2}e^{-2\mathbf{k}_{34}^2/b^2}e^{-\mathbf{k}^2/b^2}. \quad (27)$$

With potential parameters $\lambda = -1449.6$ MeV fm^3 and $\gamma = 1.152$ fm^{-1} in Eq. (25), the binding energy and rms radius of the free α particle are reproduced, using the Gaussian variational ansatz for the intrinsic motion. To show this, we calculate the intrinsic energy according to Eq. (23),

$$\hat{E}_\alpha^{(0)}(b) = \frac{9}{8}\frac{\hbar^2}{m}b^2 + 6\lambda\frac{\gamma^6 b^3}{\pi^{3/2}(b^2+2\gamma^2)^3}, \quad (28)$$

with a minimum $E_\alpha^{(0)} = -28.3$ MeV for the ground-state energy at $b = 1.034$ fm^{-1}. The parameter b reproduces the nucleonic point rms radius $\sqrt{\langle r^2\rangle} = 1.45$ fm as $b^2 = 9/(4\langle r^2\rangle) = 1.069$ fm^{-2}.

2. α-like correlations in homogeneous nuclear matter at finite densities

We continue to discuss the case of homogeneous nuclear matter, which is of relevance when we later introduce a local-density approach. In homogeneous systems, the separation of the c.m. motion is exact because the c.m. momentum is conserved. Because there the effective c.m. potential $W(\mathbf{R},\mathbf{R}')$ depends only on $\mathbf{R}-\mathbf{R}'$ and, thus, gradient terms like $\nabla_R\varphi^{\mathrm{intr}}$ can be dropped, we have from Eq. (6)

$$-\frac{\hbar^2}{8m}\nabla_R^2\Phi(\mathbf{R}) + \int d^3R'\,W(\mathbf{R}-\mathbf{R}')\Phi(\mathbf{R}') = E_4\,\Phi(\mathbf{R}). \quad (29)$$

After Fourier transformation (remember, the transformed quantities are marked with a tilde), we have

$$\left[-\frac{\hbar^2}{8m}\mathbf{P}^2 + \tilde{W}(\mathbf{P})\right]\tilde{\Phi}(\mathbf{P}) = E_4(\mathbf{P})\,\tilde{\Phi}(\mathbf{P}). \quad (30)$$

To identify different contributions to the effective c.m. potential $\tilde{W}(\mathbf{P})$, we consider the in-medium wave equation (15) given in momentum representation. The eigenvalue $E_4(\mathbf{P})$ will depend on the total momentum $\mathbf{P}$ not only owing to the kinetic energies of the single-nucleon states, $P^2/(8m)$ but also owing to the mean-field shifts $V_\tau^{\mathrm{mf}}(\mathbf{p})$ that may be absorbed into the chemical potential in the rigid shift approximation, as well as owing to the Pauli blocking terms $f_\tau[\varepsilon_\tau^{\mathrm{mf}}(\mathbf{p})]$ [approximately, this dependence of $E_4(\mathbf{P})$ on P can be described introducing an effective mass of the α-like cluster].

The wave equation for the intrinsic motion (8) becomes also simplified in homogeneous systems,

$$T_4\left[\nabla_{s_j}\right]\varphi^{\mathrm{intr}}(\mathbf{s}_j,\mathbf{R}) + \int d^3R'\,d^9s'_j\,V^{(4)}(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)\frac{\Phi(\mathbf{R}')}{\Phi(\mathbf{R})}\varphi^{\mathrm{intr}}(\mathbf{s}'_j,\mathbf{R}') = \frac{F(\mathbf{R})}{|\Phi(\mathbf{R})|^2}\varphi^{\mathrm{intr}}(\mathbf{s}_j,\mathbf{R}). \quad (31)$$

Whereas the intrinsic kinetic energy $T_4[\nabla_{s_j}]$ is given by (21), the interaction $V^{(4)}(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)$ contains in addition to the mutual interaction also the medium effects, in particular, the self-energy shifts and the Pauli blocking terms. For homogeneous systems

it is convenient to pass over to momentum representation. With the Jacobi-Moshinsky momenta (19), Eq. (15) reads now [cf. Eqs. (8) and (22)]

$$\frac{\hbar^2}{2m}\left[k^2+2k_{12}^2+2k_{34}^2\right]\tilde{\varphi}^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{P})+\int\frac{d^3k'}{(2\pi)^3}\frac{d^3k'_{12}}{(2\pi)^3}\frac{d^3k'_{34}}{(2\pi)^3}\tilde{V}^{(4)}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34};\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}'_{34};\mathbf{P})\tilde{\varphi}^{\rm intr}(\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}'_{34},\mathbf{P})$$
$$=\tilde{W}(\mathbf{P})\tilde{\varphi}^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{P}). \tag{32}$$

Here we have used that for homogeneous systems the interaction term $\tilde{V}^{(4)}$ is diagonal with respect to the total momentum $\mathbf{P}$. The new Lagrange parameter $\tilde{W}(\mathbf{P})$ is the Fourier transform of $F(\mathbf{R})/|\Phi(\mathbf{R})|^2$ and can be considered as the intrinsic energy of the four-nucleon system.

The effective in-medium interaction $\tilde{V}^{(4)}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34};\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}'_{34};\mathbf{P})$ contains the external part (16) as well as the intrinsic part (17) (to be transformed to Jacobi-Moshinsky momenta). In addition to the terms which describe the intrinsic motion of the free α particle, additional contributions arise from the single-nucleon self-energy shift $V_\tau^{\rm mf}$ and the Pauli blocking term $f_\tau[\varepsilon_\tau^{\rm mf}(\mathbf{p})]$. Accordingly, we decompose the effective c.m. potential,

$$\tilde{W}(\mathbf{P})=\tilde{W}^{\rm ext}(\mathbf{P})+\tilde{W}^{\rm intr}(\mathbf{P}), \tag{33}$$

into an external part $\tilde{W}^{\rm ext}(\mathbf{P})$, collecting the mean-field shifts $V_\tau^{\rm mf}$ of the surrounding matter, and an intrinsic part $\tilde{W}^{\rm intr}(\mathbf{P})$ which contains the intrinsic kinetic energy as well as the mutual interaction of the constituents including the Pauli blocking. As seen from Eqs. (15) and (16), the quasiparticle mean-field shift $V_\tau^{\rm mf}(\mathbf{p})$ gives the first contribution,

$$\tilde{W}^{\rm ext}(\mathbf{P})=\int\frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}|\tilde{\varphi}^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{P})|^2$$
$$\times\left[V_{\tau_1}^{\rm mf}\left(\frac{\mathbf{P}}{4}+\frac{\mathbf{k}}{2}+\mathbf{k}_{12}\right)+V_{\tau_1}^{\rm mf}\left(\frac{\mathbf{P}}{4}+\frac{\mathbf{k}}{2}-\mathbf{k}_{12}\right)\right.$$
$$\left.+V_{\tau_1}^{\rm mf}\left(\frac{\mathbf{P}}{4}-\frac{\mathbf{k}}{2}+\mathbf{k}_{34}\right)+V_{\tau_1}^{\rm mf}\left(\frac{\mathbf{P}}{4}-\frac{\mathbf{k}}{2}-\mathbf{k}_{34}\right)\right], \tag{34}$$

to the Fourier transform of the four-particle c.m. potential $W(\mathbf{R},\mathbf{R}')$, Eq. (7), which depends for homogeneous systems only on $\mathbf{R}-\mathbf{R}'$. This term acts on the free nucleons in quasiparticle states, as well as on the bound nucleons in the cluster. If the momentum dependence of the mean-field shift $V_\tau^{\rm mf}(\mathbf{p})$ can be neglected (rigid shift approximation), both the scattering states as well as the bound four-nucleon states are shifted by the same amount. Then, the contribution to the shift of the binding energy (the difference between scattering-state and bound-state energies) is canceled. Simple approximations for the mean-field shifts in homogeneous matter are, e.g., given by Skyrme forces or relativistic mean-field energy shifts and are not discussed here in detail. Let us mention, however, that the mean-field shifts are most of the time incorporated into a rigid shift not depending on $\mathbf{p}$ and an effective mass, which give only a small contribution; see [28] for further details. For finite nuclei, expressions for the mean-field shift like the Woods-Saxon potential are given in Sec. IV.

A second contribution to the influence of the surrounding matter on the four-nucleon system in Eq. (15) is attributable to Pauli blocking, given by the occupation $f_\tau[\varepsilon_\tau^{\rm mf}(\mathbf{p})]$ of single-quasiparticle nucleon states. As already given above Eq. (10) and below Eq. (15), in homogeneous matter (no dependence on $\mathbf{R}$), we adopt the single-nucleon occupation ($\tau=n,p$) as

$$f_{\tau,\mathbf{p}}=f_\tau\left[\varepsilon_\tau^{\rm mf}(\mathbf{p})\right]=f(\mathbf{p};\mu_\tau,T=0)=\Theta\left[\mu_\tau-\varepsilon_\tau^{\rm mf}(\mathbf{p})\right]. \tag{35}$$

The chemical potentials μ_τ coincide at zero temperature with the Fermi energy, $\mu_\tau=E_{{\rm Fermi},\tau}=(\hbar^2/2m)(3\pi^2n_\tau)^{2/3}$ and are determined by the respective densities.

The evaluation of the Pauli blocking term for arbitrary temperatures and arbitrary c.m. momenta $\mathbf{P}$ has been given in Ref. [29]. Some special results for the zero temperature case that are not discussed in Ref. [29] are given below. For the Pauli blocking, we consider the wave equation (15) for zero total momentum, $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4=0$. Note again that we replaced the RPA blocking term $[1-f_{\tau_1}(\varepsilon_{p_1})-f_{\tau_2}(\varepsilon_{p_2})]$ with the TDA term $[1-f_{\tau_1}(\varepsilon_{p_1})][1-f_{\tau_2}(\varepsilon_{p_2})]$, which excludes the participation of already occupied single-particle states (below the Fermi surface) from the propagation of the four-nucleon state. The medium is treated as uncorrelated, and also the formation of a BCS state is excluded.

3. *Energy of intrinsic motion in homogeneous matter at $P=0$*

We can expand $\tilde{W}(\mathbf{P})$ with respect to $\mathbf{P}$ but in this work we only evaluate the terms for $\mathbf{P}=0$. For the external part $\tilde{W}^{\rm ext}(\mathbf{P})$ the higher orders in $\mathbf{P}$ are zero if the mean-field potential is local. In general, as is well known, within a gradient expansion the next term can be absorbed introducing effective masses. In particular, the mean-field shift $V_\tau^{\rm mf}(\mathbf{p})$ in a homogeneous system can be treated this way, leading to a rigid shift $V_\tau^{\rm mf}(0)$ and to the introduction of an effective nucleon mass m^*. We discuss here only the lowest order of the expansion with respect to the single-nucleon momentum $\mathbf{p}$. The introduction of the effective nucleon mass is straightforward; see Ref. [28], where corresponding expressions for the homogeneous case are given.

The in-medium wave equation (15) can be given in a Hermitian form and can be solved with a variational approach. After a projected product ansatz, self-consistent equations to solve the single-nucleon wave function are considered in Ref. [17]. For simplicity, here we use a Gaussian ansatz [see Eq. (27)] that reads

$$\tilde{\varphi}^{\rm intr}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=\tfrac{1}{\rm norm}\varphi_{\tau_1}(\mathbf{p}_1)\varphi_{\tau_1}(\mathbf{p}_2)\varphi_{\tau_1}(\mathbf{p}_3)\varphi_{\tau_1}(\mathbf{p}_4)$$
$$\times\,\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4), \tag{36}$$

with

$$\varphi_\tau(\mathbf{p}) = e^{-\mathbf{p}^2/b^2}\Theta[p - p_{\mathrm{Fermi},\tau}], \tag{37}$$

so that the Fermi sphere $p_{\mathrm{Fermi},\tau} = (3\pi^2 n_\tau)^{1/3}$ is blocked out and b is a variational parameter. To simplify the calculations we average the Fermi energies with respect to the isospin $\tau = n, p$ (symmetric matter), so that we perform the calculations for an excluded Fermi sphere $p_{\mathrm{Fermi}} = (3\pi^2 n_B/2)^{1/3}$ with the total baryon density $n_B = n_n + n_p$.

Within the variational calculation, we have to evaluate the norm of the trial function (37) as well as the kinetic and potential energy. The Pauli blocking is already taken into account by the choice of the trial wave function and must not be considered anymore. After transforming to the internal Jacobian coordinates $\mathbf{k}, \mathbf{k}_{12}, \mathbf{k}_{34}$, one has to perform multiple integrals; see Appendix A.

The intrinsic motion of the four-nucleon system contains the kinetic energy and the interaction energy within the cluster, taking into account Pauli blocking. Besides the shift $\tilde{W}^{\mathrm{ext}}$, which acts on the nucleons both in the scattering (single-nucleon) states as well as in bound states, the dependence of the effective c.m. potential (7) $\tilde{W} = \tilde{W}^{\mathrm{ext}} + \tilde{W}^{\mathrm{intr}}$ on the c.m. momentum $\mathbf{P}$ and the baryon density is determined by the internal part $\tilde{W}^{\mathrm{intr}}$, which is sensitive to the formation of bound states. The dependence of $\tilde{W}^{\mathrm{intr}}(\mathbf{P})$ on $\mathbf{P}$ is attributable to the Pauli blocking term B and has been considered in detail in Ref. [29]. Here we restrict ourselves to the value $\tilde{W}^{\mathrm{intr}}$ at $\mathbf{P} = 0$. Using Eq. (32), we separate the mean-field shifts from $\tilde{V}^{(4)}(\mathbf{k}, \mathbf{k}_{12}, \mathbf{k}_{34}; \mathbf{k}', \mathbf{k}'_{12}, \mathbf{k}'_{34}; \mathbf{P})$, which give the contribution $\tilde{W}^{\mathrm{ext}}$. The in-medium equation for the intrinsic part of the α-particle wave function is given by

$$\left(\tilde{W}^{\mathrm{intr}} - \frac{\hbar^2}{2m}\left[k^2 + 2k_{12}^2 + 2k_{34}^2\right]\right)\tilde{\varphi}_4^{\mathrm{intr}}(\mathbf{k}, \mathbf{k}_{12}, \mathbf{k}_{34}) = \int \frac{d^3k'}{(2\pi)^3}\frac{d^3k'_{12}}{(2\pi)^3}\frac{d^3k'_{34}}{(2\pi)^3} V_4^{\mathrm{intr}}(\mathbf{k}, \mathbf{k}_{12}, \mathbf{k}_{34}, \mathbf{k}', \mathbf{k}'_{12}, \mathbf{k}'_{34}, \mathbf{P} = 0)\tilde{\varphi}_4^{\mathrm{intr}}(\mathbf{k}', \mathbf{k}'_{12}, \mathbf{k}'_{34}), \tag{38}$$

where the four-nucleon interaction term V_4^{intr} contains also the Pauli blocking terms for $\mathbf{P} = 0$; see Eqs. (12) and (13). The explicit form is obtained from

$$V_4^{\mathrm{intr}}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4) = [1 - f(\mathbf{p}_1)][1 - f(\mathbf{p}_2)]V_{N-N}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}'_1, \mathbf{p}'_2)\delta(\mathbf{p}_3 - \mathbf{p}'_3)\delta(\mathbf{p}_4 - \mathbf{p}'_4) + \text{five permutations} \tag{39}$$

after transforming to Jacobian momenta (19). A solution of this equation within a variational approach is described for the free α particle in Sec. III B 1. We do the same at finite density with the variational ansatz (36) and (37); see also Appendix A. In contrast to the expression (23) for the zero-density case, for arbitrary $\mathbf{P}$ the minimum of

$$\tilde{W}(\mathbf{P}) = \frac{\hbar^2}{2m}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\left[k^2 + 2k_{12}^2 + 2k_{34}^2\right]|\tilde{\varphi}^{\mathrm{intr}}(\mathbf{k}, \mathbf{k}_{12}, \mathbf{k}_{34}, \mathbf{P})|^2 + \tilde{W}^{\mathrm{ext}}(\mathbf{P})$$
$$+ \int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3k'_{12}}{(2\pi)^3}\frac{d^3k'_{34}}{(2\pi)^3}\tilde{\varphi}^{\mathrm{intr},*}(\mathbf{k}, \mathbf{k}_{12}, \mathbf{k}_{34}, \mathbf{P})V_4^{\mathrm{intr}}(\mathbf{k}, \mathbf{k}_{12}, \mathbf{k}_{34}, \mathbf{k}', \mathbf{k}'_{12}, \mathbf{k}'_{34}, \mathbf{P})\tilde{\varphi}^{\mathrm{intr}}(\mathbf{k}', \mathbf{k}'_{12}, \mathbf{k}'_{34}, \mathbf{P}) \tag{40}$$

has to be found with Eqs. (36) and (37) and b again is the single variational parameter. $\tilde{W}^{\mathrm{ext}}(\mathbf{P})$ is given by (34) and V_4^{intr} is given by (39) with arbitrary $\mathbf{P}$.

In contrast to the free α-particle case detailed in Sec. III B 1, the eigenvalue $\tilde{W}^{\mathrm{intr}}$ now becomes dependent on the density which enters the Pauli blocking via the Fermi momentum. The results are shown in Fig. 1. An interpolation formula which reproduces these results is given below in Eq. (45).

We discuss the result for $\tilde{W}^{\mathrm{intr}}$ in more detail. To add four nucleons (neutrons and protons, two spin orientations) to nuclear matter with density n_τ, we consider two cases which are based on the scenario which we described in the Introduction; that is, the α particle as a bound state only exists in the far surface. As soon as the α enters the region of higher density, its binding fades away and the four nucleons go over into shell-model states (eventually with pairing). Therefore, we see the following.

(i) At first, the four nucleons are treated as free, uncorrelated particles which correspond to the shell-model states. If uncorrelated, free nucleons are introduced in nuclear matter with density n_τ; this can happen only above the Fermi energy so that the minimum energy to add four uncorrelated nucleons is

$$\tilde{W}^{\mathrm{intr,free}}[n_\tau] = 2E_{\mathrm{Fermi}}(n_n) + 2E_{\mathrm{Fermi}}(n_p) = \frac{\hbar^2}{m}[(3\pi^2 n_n)^{2/3} + (3\pi^2 n_n)^{2/3}]. \tag{41}$$

The four nucleons are introduced at the corresponding Fermi energy with zero total momentum. Only the kinetic energy is needed to determine the edge of the continuum of scattering states. This four-particle energy for single-nucleon states is shown for symmetric matter ($n_n = n_p = n_B/2$) in Fig. 1 with the (blue) line starting at zero energy.

(ii) Below the continuum of scattering states, bound states may occur in the four-nucleon system at very low densities. In the zero-density limit, we have the formation of the α particle at the bound-state energy $E_\alpha^{(0)} = -28.3$ MeV for the internal motion, the energy of the c.m. motion vanishes at $\mathbf{P} = 0$. The energy of the four-nucleon bound state is shifted at finite density of the surrounding nuclear matter owing to Pauli blocking so that

$$\tilde{W}^{\mathrm{intr,bound}}[n_\tau] = E_\alpha^{(0)} + \tilde{W}^{\mathrm{Pauli}}(n_\tau) = -28.3\ \mathrm{MeV} + \tilde{W}^{\mathrm{Pauli}}(n_\tau). \tag{42}$$

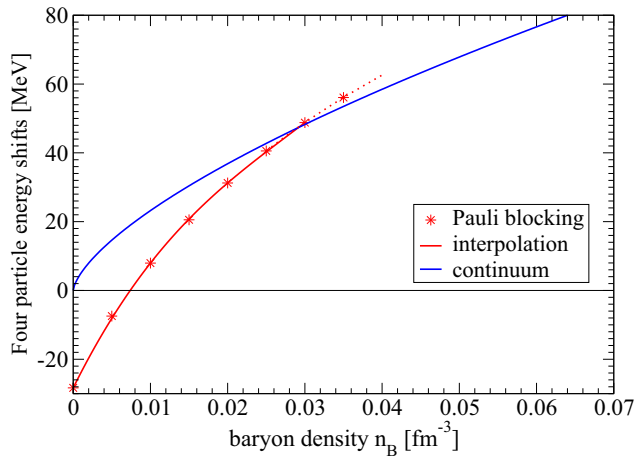

FIG. 1. (Color online) Internal four-nucleon energy (no c.m. motion) in a medium with nucleon density $n_B = n_n + n_p$. The continuum edge of free single-particle states is given by $4E_{\rm Fermi}$ [Eq. (41)]. At zero baryon density, the four-nucleon energy is given by the binding energy of the α particle, $E^0_\alpha = -B^0_\alpha = -28.3$ MeV. With increasing density, the binding energy B^0_α is decreasing owing to the Pauli blocking [Eq. (42)] (stars). The four-nucleon bound state disappears at $n_B \approx 0.03$ fm^{-3}. A fit to the calculated values, Eq. (45), is also shown.

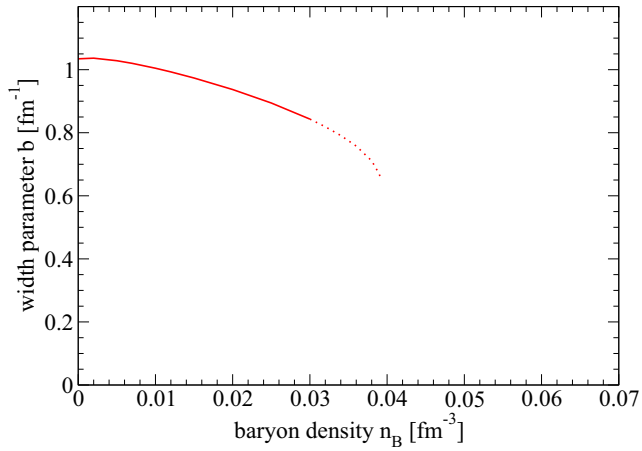

FIG. 2. (Color online) Width parameter b according to Eq. (37) as function of the baryon density. Below $n_B \approx 0.03$ fm^{-3} a bound state (α-like) arises (full line). Above $n_B \approx 0.03$ fm^{-3}, a resonance occurs; see Fig. 1 (dotted line).

> The Pauli blocking shift $\tilde{W}^{\rm Pauli}(n_\tau)$ for nuclear matter is caused by the terms containing the phase-space occupations $f_\tau(E)$.

The minimum of the energy leads, with increasing density, to a wave function $\tilde{\varphi}_\tau(\mathbf{p})$ which has near the Fermi momentum a sharp maximum for the distribution of the occupation of the single-nucleon states. It is expected for any added nucleons that, at minimum energy, it occupies the Fermi momentum if the interaction is neglected. We obtain a solution at the continuum edge of single-particle states at high densities, whereas below a critical value $n_{B,\rm cluster} \approx 0.03$ fm^{-3} a bound state is formed. The corresponding energies as a function of density are shown in Fig. 1 with the red asterisks. Note that the sharp appearance of a bound state at a critical density where blue and red lines cross is possibly a consequence of the simple variational ansatz that contains only one parameter b. Until now, there is no exact solution of the four-particle problem near the so-called Mott point [4] where, owing to Pauli blocking, the bound state is dissolved in the continuum of scattering states. The same applies also for finite temperatures, discussing, for instance, the disappearance of quartetting with increasing density [6], which seems to be a sharp transition to pairing. In principle, one cannot exclude, however, a fast but smooth merging of both solutions.

The variational approach with the ansatz (37) contains the width parameter b, which is shown as function of the baryon density in Fig. 2. Below $n_{B,\rm cluster} \approx 0.03$ fm^{-3}, where an α-like bound state exists, the change in the width parameter is small. At the critical density $n_{B,\rm cluster}$ the uncorrelated four-nucleon solution becomes favorable, and the solution of the variational approach jumps to the corresponding minimum of energy. Above $n_{B,\rm cluster}$, the local minimum of the energy as function of b is also shown in Fig. 2 which may be interpreted as a resonance. This particular behavior of the cluster wave function has also been observed in other approaches (see Ref. [17]), as well as in an effective four-body equation of the Alt-Grassberger-Sandhas (AGS) type that includes the dominant medium effects, i.e., self-energy corrections and Pauli blocking in a consistent way [31], so that it is assumed to be valid, also improving the variational approach.

As just discussed, in contrast to the two-nucleon case where the pairing solution exists also in the degenerate case, the α-like four-nucleon bound state may disappear abruptly at $n_{B,\rm cluster}$, which can be explained considering the density of states near the Fermi energy [6]. Let us discuss this difference in more detail. Supposing that the c.m. of the particles is at rest ($\mathbf{P} = 0$), we obtain for the two-particle case the level density

$$g_2(\omega = 2\mu) \propto \int d^3P \int d^3k \bar{n}_{\mathbf{P}/2-\mathbf{k}} \bar{n}_k \delta(2\mu - e_{\mathbf{P}/2-\mathbf{k}} - e_k)\delta(\mathbf{P}) = \propto \sqrt{\mu}, \tag{43}$$

where $\bar{n}_k = 1 - n_k$ with $n_k = \Theta(\mu - e_k)$ and $e_k = \frac{k^2}{2m}$.

Analogously, we obtain for the four-particle level density at the Fermi energy with total c.m. at rest

$$g_4(\omega = 4\mu) \propto \int d^3P d^3P' d^3k d^3k' \bar{n}_{\mathbf{P}/2-\mathbf{k}} \bar{n}_{\mathbf{P}2+\mathbf{k}} \bar{n}_{\mathbf{P}'2-\mathbf{k}'} \bar{n}_{\mathbf{P}'/2+\mathbf{k}'} \times \delta(4\mu - e_{\mathbf{P}/2-\mathbf{k}} - e_{\mathbf{P}/2+\mathbf{k}} - e_{\mathbf{P}'/2-\mathbf{k}'} - e_{\mathbf{P}'/2+\mathbf{k}'}) \times \delta(\mathbf{P} + \mathbf{P}') = 0. \tag{44}$$

We see that in the four-particle case, for positive μ, energy conservation and the phase-space constraint cannot be fulfilled simultaneously and, thus, no four-particle correlations can build up around the Fermi energy. This is a quite dramatic difference to the two-particle case, where the level density remains finite at the Fermi level. For negative μ, i.e., for the case where there is binding, the Fermi step n_k is zero and no qualitative difference between two- and four-particle cases exists. The two-particle case is, therefore, very exceptional

with respect to all heavier clusters. Therefore, when the α particle approaches the ^{208}Pb core, the internal structure of the α-like cluster remains relatively stable until it is dissolved quite abruptly at the critical density $n_{B,\mathrm{cluster}} = 0.03$ fm^{-3}, which is very low. In addition to the deformation by the Fermi momentum, described above, saying that no states in momentum space are occupied below the Fermi level, the change in the variational parameter b, which describes the width of the Gaussian wave function, is moderate. It changes from its value $b = 1.034$ fm^{-1} at $n_B = 0$ to $b = 0.84$ fm^{-1} at $n_{B,\mathrm{cluster}} = 0.03$ fm^{-3}, which means that the α-particle size increases by about 20%.

In conclusion, considering homogeneous nuclear matter, additional nucleons (two neutrons, two protons) can form an α-like cluster. In the zero-density limit the binding energy amounts to 28.3 MeV. As soon as the density takes a finite value, owing to the Pauli blocking, the binding energy is shifted. Bound states are possible for $n_B \leqslant 0.03$ fm^{-3}. To give a simple relation for the dependence on the baryon density, the fit formula derived within a variational approach to solve the in-medium four-nucleon wave equation,

$$\tilde{W}^{\mathrm{Pauli}}(n_B) = 4515.9\, n_B - 100\,935\, n_B^2 + 1\,202\,538\, n_B^3, \quad (45)$$

can be used, $n_n = n_p = n_B/2$. For $n_B \geqslant 0.03$ fm^{-3}, no bound state is formed, and the four nucleons added to nuclear matter are implemented on top of the Fermi energy μ; see Fig. 1.

The intrinsic wave function (36) and (37) is $\mathbf{R}$ dependent via the Fermi momentum if the inhomogeneous case is considered, for instance, an α particle on top of a heavy nucleus whose c.m. position is fixed at $\mathbf{R}_{\mathrm{core}} = 0$. Also, the intrinsic energy $W(\mathbf{R})$ introduced in Eq. (6) becomes dependent on $\mathbf{R}$ via $n_\tau(\mathbf{R})$. This is discussed in Sec. IV with the introduction of an effective potential for the α-like state near the lead core in ^{212}Po.

C. α-like correlations in a nucleus, Thomas-Fermi approximation

Now we discuss the formation of α-like correlations for a finite nuclear system, in particular, the nucleus ^{212}Po considered below. A mean-field potential $V_\tau^{\mathrm{mf}}(\mathbf{r})$ acts on the nucleons, taken as local and depending on isospin τ. As is well known from the shell model, a harmonic oscillator potential or a Woods-Saxon like potential can be used to determine single-nucleon orbits that are occupied up to the Fermi energy. Often this potential is considered as a local one, only depending on the nucleon coordinate $\mathbf{r}$. For comparison, in the homogeneous case considered before, any dependence on $\mathbf{r}$ disappears, and the mean-field contribution is a constant that can be added to the intrinsic energy.

The solution of the four-nucleon system using the c.m. coordinate $\mathbf{R}$ as a new degree of freedom as well as relative coordinates is not as simple as in the homogeneous case. We start from the general expressions given in Sec. II. In particular, we neglect the terms containing $\nabla_R \varphi_4^{\mathrm{intr}}(\mathbf{s}_j,\mathbf{R})$ so that Eqs. (6) and (8) reduce to

$$-\frac{\hbar^2}{8m}\nabla_R^2 \Phi(\mathbf{R}) + \int d^3R'\, W(\mathbf{R},\mathbf{R}')\ \Phi(\mathbf{R}') = E_4\, \Phi(\mathbf{R}), \quad (46)$$

with the effective c.m. potential

$$W(\mathbf{R},\mathbf{R}') = \int d^9 s_j\, d^9 s'_j\, \varphi_4^{\mathrm{intr},*}(\mathbf{s}_j,\mathbf{R})\big\{T_4\big[\nabla_{s_j}\big]\delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{s}_j-\mathbf{s}'_j) + V_4(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)\big\}\varphi_4^{\mathrm{intr}}(\mathbf{s}'_j,\mathbf{R}'). \quad (47)$$

The in-medium four-particle interaction $V_4(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)$ follows from Eq. (15). Besides the intrinsic nucleon-nucleon interaction V_{N-N}, it contains also two medium effects, the quasiparticle mean-field shift $V_\tau^{\mathrm{mf}}(\mathbf{r})$, which leads to the contribution $W^{\mathrm{ext}}(\mathbf{R},\mathbf{R}')$ [see Eqs. (34) and (50) below], and the Pauli blocking terms $\propto f_\tau(\varepsilon_\tau^{\mathrm{mf}})V_{N-N}$, which leads to the contribution $W^{\mathrm{Pauli}}(\mathbf{R},\mathbf{R}')$; see Eqs. (38) and (42). Both contributions $W^{\mathrm{ext}}(\mathbf{R},\mathbf{R}')$, $W^{\mathrm{Pauli}}(\mathbf{R},\mathbf{R}')$ depend on the density of the nuclear medium and vanish for the free α-particle case. In general, these medium contributions are nonlocal and depend on $\mathbf{R},\mathbf{R}'$.

The variation of the functional (5) with respect to $\varphi_4^{\mathrm{intr},*}(\mathbf{s}_j,\mathbf{R})$ at fixed $\mathbf{R}$ yields

$$\int d^3R'\, d^9 s'_j\, \big\{T_4\big[\nabla_{s_j}\big]\delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{s}_j-\mathbf{s}'_j) + V_4(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)\big\}\frac{\Phi(\mathbf{R}')}{|\Phi(\mathbf{R})|^2}\varphi_4^{\mathrm{intr}}(\mathbf{s}'_j,\mathbf{R}') = E_4^{\mathrm{intr}}(\mathbf{R})\varphi_4^{\mathrm{intr}}(\mathbf{s}_j,\mathbf{R}), \quad (48)$$

where we introduced the intrinsic energy $E_4^{\mathrm{intr}}(\mathbf{R}) = F(\mathbf{R})/|\Phi(\mathbf{R})|^2$ in analogy to Eqs. (23) and (32). In contrast to the free α-particle energy $E_\alpha^{(0)}$, the intrinsic energy contains in-medium effects and depends on the c.m. position $\mathbf{R}$. If the effective c.m. potential $W(\mathbf{R},\mathbf{R}')$ is taken in local approximation, we have $W(\mathbf{R},\mathbf{R}') = E_4^{\mathrm{intr}}(\mathbf{R})\delta(\mathbf{R}-\mathbf{R}')$.

In general, these equations are nonlocal in $\mathbf{R}$ space owing to the potential energy $V_4(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j)$, which contains the mean-field contribution V_4^{ext} defined below as well as the intrinsic interaction V_4^{intr} within the four-nucleon cluster [cf. also Eqs. (16) and (17)],

$$V_4(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j) = V_4^{\mathrm{ext}}(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j) + V_4^{\mathrm{intr}}(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j). \quad (49)$$

We discuss both contributions separately together with some approximations.

Usually, the mean field of the nucleus is taken as local in position space, neglecting momentum dependence which makes also $W^{\mathrm{ext}}(\mathbf{R},\mathbf{R}')$ local. Below we use the Woods-Saxon potential $V_\tau^{\mathrm{mf}}(\mathbf{r})$ that depends on the position $\mathbf{r}_i$ of the four nucleons, $\tau = n,p$. Transforming to Jacobi coordinates we have for the interaction with an external (mean-field) potential

$$V_4^{\mathrm{ext}}(\mathbf{R},\mathbf{s}_j;\mathbf{R}',\mathbf{s}'_j) = \big[V_{\tau_1}^{\mathrm{mf}}\big(\mathbf{R}+\tfrac{1}{2}\mathbf{s}+\tfrac{1}{2}\mathbf{s}_{12}\big) + V_{\tau_2}^{\mathrm{mf}}\big(\mathbf{R}+\tfrac{1}{2}\mathbf{s}-\tfrac{1}{2}\mathbf{s}_{12}\big) + V_{\tau_3}^{\mathrm{mf}}\big(\mathbf{R}-\tfrac{1}{2}\mathbf{s}+\tfrac{1}{2}\mathbf{s}_{34}\big) + V_{\tau_4}^{\mathrm{mf}}\big(\mathbf{R}-\tfrac{1}{2}\mathbf{s}-\tfrac{1}{2}\mathbf{s}_{34}\big)\big]$$
$$\times\, \delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{s}-\mathbf{s}')\delta(\mathbf{s}_{12}-\mathbf{s}'_{12})\delta(\mathbf{s}_{34}-\mathbf{s}'_{34}). \quad (50)$$

For the effective c.m. potential $W(\mathbf{R},\mathbf{R}') = W^{\rm ext}(\mathbf{R})\delta(\mathbf{R}-\mathbf{R}') + W^{\rm intr}(\mathbf{R},\mathbf{R}')$ we have the mean-field contribution

$$W^{\rm ext}(\mathbf{R}) = \int d^3s\, d^3s_{12}\, d^3s_{34}\, \left|\varphi_4^{\rm intr}(\mathbf{s}_j,\mathbf{R})\right|^2 V_4^{\rm ext}(\mathbf{R},\mathbf{s}_j;\mathbf{R},\mathbf{s}_j). \tag{51}$$

Similar to the introduction of a double-folding potential, the effective c.m. interaction term owing to the mean-field potential follows after averaging with the intrinsic density distribution.

As a further component to the effective c.m. potential energy, the Pauli blocking appears. The Pauli principle, as consequence of antisymmetrization, means that states below the Fermi energy are blocked if further nucleons are added to the lead core (we consider ^{212}Po). We denote this as the intrinsic four-particle energy $\tilde{W}^{\rm intr}[n_\tau(\mathbf{R})]$ [Eqs. (38) and (39), and, more explicitly, Eqs. (42) and (45)] of the intrinsic motion which is a functional of the nucleon density $n_\tau(\mathbf{R})$ of the surrounding medium. We obtain these local-density expressions within a more general approach which is able to go also beyond the LDA and makes the terms that are neglected in the LDA more transparent. In principle, the full quantal solution may be possible. Here we consider the Thomas-Fermi approximation as a simple approximation to the LDA. The reader not interested in the technical details of how to get to LDA and the approximations involved, can jump directly to Eq. (55), where the same expression for the intrinsic energy as in (40) is given, only in LDA.

The intrinsic interaction including blocking terms B reads in position space ($i = 1,\ldots,4$) [cf. Eq. (39) in momentum representation]

$$\begin{aligned} V_4^{\rm intr}(\mathbf{r}_i;\mathbf{r}_i') = \int d^3r_1''\, d^3r_2''\ \langle \mathbf{r}_1\mathbf{r}_2|\big[1-f_1\big(\varepsilon_{n_1}\big)\big]\big[1-f_2\big(\varepsilon_{n_2}\big)\big] \\ \times |\mathbf{r}_1''\mathbf{r}_2''\rangle\langle\mathbf{r}_1''\mathbf{r}_2''|V_{N-N}|\mathbf{r}_1'\mathbf{r}_2'\rangle\delta(\mathbf{r}_3'-\mathbf{r}_3)\delta(\mathbf{r}_4'-\mathbf{r}_4) \\ + \text{five permutations}, \end{aligned} \tag{52}$$

where $\langle\mathbf{r}_1|f_1(\varepsilon_{n_1})|\mathbf{r}'\rangle$ which is defined with the single-nucleon quasiparticle states $\psi_n(\mathbf{r})$, is given in a local approximation in the following.

We can introduce Jacobi coordinates to separate the c.m. motion and perform a Fourier transformation to momentum representation. As above, the nucleon-nucleon interaction can be taken in a separable form so that

$$\begin{aligned} &\langle\mathbf{r}_1''\mathbf{r}_2''|V_{N-N}|\mathbf{r}_1'\mathbf{r}_2'\rangle\delta(\mathbf{r}_3'-\mathbf{r}_3'')\delta(\mathbf{r}_4'-\mathbf{r}_4'') \\ &= \int \frac{d^3k_{12}'}{(2\pi)^3}\frac{d^3k_{12}''}{(2\pi)^3} e^{i\mathbf{s}_{12}''\cdot\mathbf{k}_{12}''-i\mathbf{s}_{12}'\cdot\mathbf{k}_{12}'} V_{N-N}(\mathbf{k}_{12}'';\mathbf{k}_{12}') \\ &\quad\times\delta(\mathbf{s}'-\mathbf{s}'')\delta(\mathbf{s}_{34}'-\mathbf{s}_{34}'') \\ &= \langle\mathbf{s}'',\mathbf{s}_{12}'',\mathbf{s}_{34}''|V_{N-N}|\mathbf{s}',\mathbf{s}_{12}',\mathbf{s}_{34}'\rangle. \end{aligned} \tag{53}$$

More difficult is the treatment of the Pauli blocking term B, which is an exchange term and nonlocal in position space. We delegate it to Appendix B, where the corresponding approximations are given. In future work we may eliminate some of the approximation made here.

We recover in Thomas-Fermi approximation the expression for the shift given in the homogeneous case, only with the parametric dependence on the c.m. position $\mathbf{R}$ via the baryon density $n_B(\mathbf{R})$. Though we gave here the whole series of approximations leading in the end to LDA or TF expressions, where, in principle, corrections can be evaluated, we give below general arguments in favor of such a local procedure for the c.m. motion of the α particle.

After the local approximation with respect to $\mathbf{R}$ was introduced, we solve Eq. (48) within a variational approach. With Eq. (51), which contains also the intrinsic wave function, the minimum of the functional

$$\begin{aligned} &\left[W_4^{\rm ext}(\mathbf{R}) + \int d^9s_j\, \varphi_4^{\rm intr,*}(\mathbf{s}_j,\mathbf{R}) T_4\big[\nabla_{s_j}\big]\varphi_4^{\rm intr}(\mathbf{s}_j,\mathbf{R}) \right. \\ &\left. + \int d^9s_j d^9s_j' d^9s_j''\, \varphi_4^{\rm intr,*}(\mathbf{s}_j,\mathbf{R}) B(\mathbf{R},\mathbf{s}_j,\mathbf{s}_j')\, V_{N-N}^{(4)}(\mathbf{s}_j',\mathbf{s}_j'')\varphi_4^{\rm intr}(\mathbf{s}_j'',\mathbf{R})\right]\left[\int d^9s_j\, \left|\varphi_4^{\rm intr}(\mathbf{s}_j,\mathbf{R})\right|^2\right]^{-1} = E_4^{\rm intr}(\mathbf{R}), \end{aligned} \tag{54}$$

within a given set of functions $\varphi_4^{\rm intr}(\mathbf{s}_j,\mathbf{R})$ gives an approximation for the intrinsic wave function and the intrinsic energy. The Pauli blocking term B depends on the position $\mathbf{R}$. In the approximation considered here, it is diagonal in momentum representation, and the dependence on $\mathbf{s}_j,\mathbf{s}_j'$ follows after Fourier transformation, as shown in Eq. (B9).

In the following section we perform exploratory calculations with the separable interaction given above. It is of advantage to use a mixed representation where the intrinsic part is given in momentum representation. Again we use the Fermi blocked Gaussian ansatz (37) for the intrinsic wave function with the width parameter as the only variational input (which becomes density dependent and, via the local density, also $\mathbf{R}$ dependent). The intrinsic interaction and the Pauli blocking give contributions to the potential owing to the interaction between the nucleons 1 and 2 (the other five follow from permutations and give rise to the factor six below). Explicitly, Eq. (54) reads

$$\begin{aligned} &\left[W^{\rm ext}(\mathbf{R}) + \frac{\hbar^2}{2m}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\left[k^2+2k_{12}^2+2k_{34}^2\right]\left|\tilde{\varphi}_4^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{R})\right|^2 \right. \\ &+6\int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{12}'}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\,\tilde{\varphi}_4^{\rm intr,*}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{R})\left[1-f_1\left(\mathbf{R},\frac{\mathbf{k}}{2}+\mathbf{k}_{12}\right)\right]\left[1-f_2\left(\mathbf{R},\frac{\mathbf{k}}{2}-\mathbf{k}_{12}\right)\right] \\ &\left.\times V_{N-N}(\mathbf{k}_{12},\mathbf{k}_{12}')\tilde{\varphi}_4^{\rm intr}(\mathbf{k},\mathbf{k}_{12}',\mathbf{k}_{34},\mathbf{R})\right]\left[\int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\left|\tilde{\varphi}_4^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{R})\right|^2\right]^{-1} = E_4^{\rm intr}(\mathbf{R}). \end{aligned} \tag{55}$$

This is now the LDA version of Eq. (40). We emphasize that the approach given here makes it possible to improve on LDA. In particular, the Pauli blocking is only approximately determined by the baryon density $n_B(\mathbf{R})$ at the c.m. position $\mathbf{R}$. As discussed in context with Eq. (B7), the baryon density and, correspondingly, the Fermi momentum $p_{\rm Fermi}(n_B)$ have to be averaged over the neighborhood of $\mathbf{R}$ corresponding to the spatial extension of the intrinsic wave function $\varphi_4^{\rm intr}(\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34},\mathbf{R})$. An improvement of the LDA is, e.g., given if for $f_1^{\rm Wigner}$ the actual position such as $\mathbf{R}+(\mathbf{s}+\mathbf{s}_{12})/2$ is taken.

The interaction term contains the Pauli blocking, which is simple in momentum representation because it is diagonal in that representation. For $\mathbf{P}=0$ we used within a variational approach a Gaussian internal wave function $\tilde{\varphi}_4^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34};\mathbf{R})$, where in the phase space $\{\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34}\}$ the volume $|\mathbf{k}+\mathbf{k}_{12}/2|\leqslant k_{\rm Fermi}(\mathbf{R})$, $|\mathbf{k}-\mathbf{k}_{12}/2|\leqslant k_{\rm Fermi}(\mathbf{R})$, $|\mathbf{k}+\mathbf{k}_{34}/2|\leqslant k_{\rm Fermi}(\mathbf{R})$, and $|\mathbf{k}-\mathbf{k}_{34}/2|\leqslant k_{\rm Fermi}(\mathbf{R})$ is excluded. Consequently, the variational ansatz for the internal wave function should vanish within that excluded volume. The blocking term is taken in local-density (Fermi gas) approximation. The nucleon-nucleon interaction V_{N-N} without blocking terms gives the bound-state energy of the α particle $E_\alpha^{(0)}=-28.3$ MeV. We separate this part so that $W_4^{\rm int}(\mathbf{R})=E_\alpha^{(0)}+W^{\rm Pauli}(\mathbf{R})$. The dependence of $W^{\rm Pauli}(\mathbf{R})$ on the surrounding baryon density $n_B(\mathbf{R})$ is given by Eq. (45). This nucleon density $n_B(\mathbf{R})$ is determined by the core nucleus, which may be described in shell-model approximation.

Note that the Thomas-Fermi approximation given here, i.e., the introduction of a "local momentum," is possible because the inverse width parameter b of the intrinsic wave function of the α-like bound state remains nearly unchanged; it is reduced only by 17% when it merges with the quasicontinuum of shell-model single quasiparticle states. This means that the α particle, even up to the breakup point, remains a rather compact entity with small extension, of the same order as the surface width of the core nucleus entailing that a local approach can be used at least as a first reasonable attempt. This is quite opposite to the pairing case, where the size of the Cooper pairs can be as large as the nucleus itself, invalidating a LDA approach. The derivation given here makes it possible to go beyond LDA if the corresponding approximations are improved. In principle, also a fully quantal solution can be envisaged.

IV. EXPLORATORY CALCULATIONS

For demonstration we consider ^{212}Po, i.e., an α particle on top of the doubly magic ^{208}Pb core nucleus [32]. We take Woods-Saxon mean-field potentials [24,33,34], which are used for the description of nuclei in the lead region. In particular, for the neutrons of the ^{208}Pb core we use

$$V_n^{\rm mf}(r)=-\frac{40.6}{1+e^{(r-R_n)/a}}, \tag{56}$$

with $R_n=1.347A^{1/3}=7.891$ fm and $a=0.7$ fm. For the protons we take

$$V_p^{\rm mf}(r)=-\frac{58.7}{1+e^{(r-R_p)/a}}+V^{\rm Coul}(r), \tag{57}$$

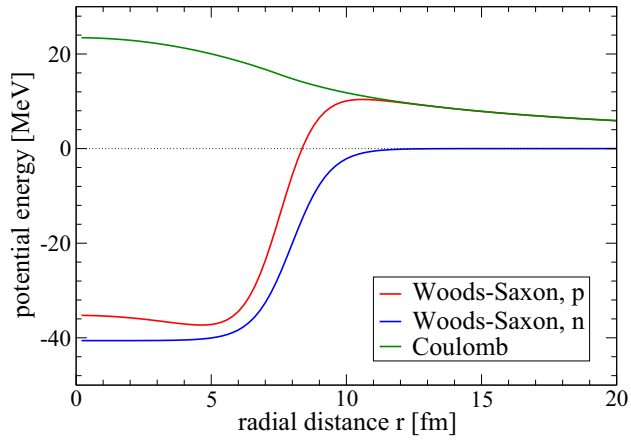

FIG. 3. (Color online) Coulomb potential and isospin-dependent Woods-Saxon potentials for the ^{208}Pb core.

with $R_p=1.275A^{1/3}=7.554$ fm and $a=0.7$ fm. The Coulomb potential produced by the lead core is taken for a homogeneously charged sphere as (units in MeV, fm)

$$V^{\rm Coul}(r)=82\frac{1.44}{r},\quad r>R_p;$$
$$V^{\rm Coul}(r)=82\frac{1.44}{R_p}\left[\frac{3}{2}-\frac{1}{2}\frac{r^2}{R_p^2}\right],\quad r<R_p. \tag{58}$$

These potentials are shown in Fig. 3; see also Fig. 2 of Ref. [24].

On the two-neutron, two-proton cluster (α-like cluster) acts the potential given by Eqs. (51) and (50). As a local approximation we take the mean-field potential at the c.m. position $\mathbf{R}$, i.e., $2V_n^{\rm mf}(R)+2V_p^{\rm mf}(R)+2V^{\rm Coul}(R)$, to simplify the calculations, but avoid performing the spatial average with the intrinsic wave function. The correction

$$\Delta V^{\rm ext}(\mathbf{R})=W^{\rm ext}(\mathbf{R})-\left[2V_n^{\rm mf}(\mathbf{R})+2V_p^{\rm mf}(\mathbf{R})+2V^{\rm Coul}(\mathbf{R})\right], \tag{59}$$

owing to the average over the intrinsic wave function in $W^{\rm ext}(\mathbf{R})$ is of interest in the low-density region $n_B\leqslant 0.03$ fm^{-3}, where α-like bound states can be formed, but it is assumed to be small because the potentials are smooth and the α particle is well localized in coordinate space so that this correction $\Delta V^{\rm ext}(R)$ can be neglected. The local approximation where the mean-field potential $W^{\rm ext}(\mathbf{R})$ is replaced by the sum of the mean-field potentials of the four constituents at the c.m. position $\mathbf{R}$ can be improved by taking into account the correction $\Delta V^{\rm ext}(\mathbf{R})$.

For the internal part $W_4^{\rm intr}(R)$ of the c.m. potential we have to estimate the baryon density $n_B(R)$ that is responsible for the Pauli blocking. To be consistent within the local-density approach given here, we use the Thomas-Fermi approximation in the average baryon potential $W_4^{\rm ext}(R)/4$,

$$n_B(R)=\frac{2}{3\pi^2}\left[\frac{2m}{\hbar^2}\left(\mu-\frac{1}{4}W_4^{\rm ext}(R)\right)\right]^{3/2}. \tag{60}$$

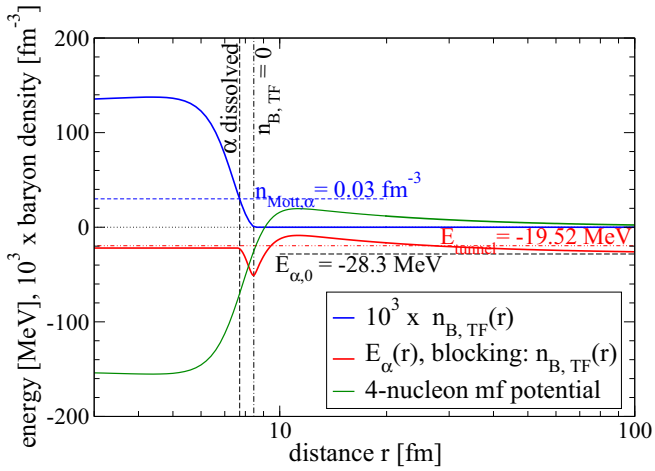

FIG. 4. (Color online) Local effective potential $W(\mathbf{R})$ (62) (red solid line) with respect to the lead ^{208}Pb core for the Woods-Saxon-like distribution (59), $A = 208$. The Thomas-Fermi density and the Fermi energy $4E_{\text{Fermi}}$ of the four added nucleons is shown, as is the measured energy (Q value) of the emitted α particle. The distance at which the density becomes the critical value $n_B = 0.0292$ fm^{-3} where the α particle is dissolved is indicated.

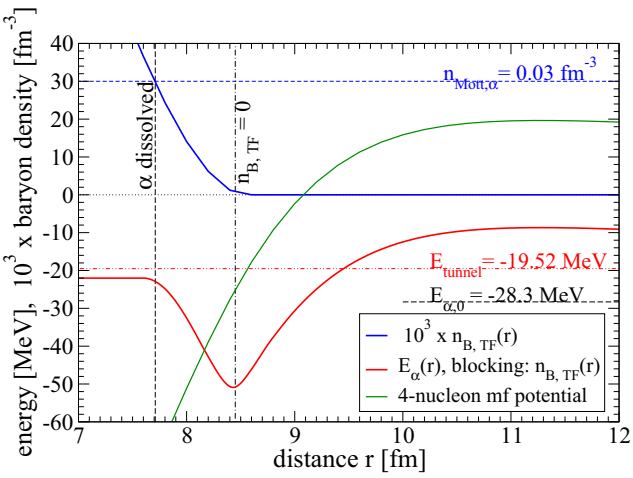

FIG. 5. (Color online) Insertion of Fig. 4.

From the mass number $A = \int n_B(R) d^3R = 208$ of the core nucleus, the value $\mu = -5.504$ MeV is obtained for the chemical potential (Fermi energy).

We consider the case of inhomogeneous nuclear matter where, compared with the homogeneous case, the c.m. motion is not trivial. Instead of Eq. (30) for the homogeneous case, we have now from Eq. (6)

$$\left[-\frac{\hbar^2}{8m}\frac{\partial^2}{\partial \mathbf{R}^2} + W(\mathbf{R})\right]\Phi(\mathbf{R}) = E_4\Phi(\mathbf{R}), \tag{61}$$

with

$$\begin{aligned} W(\mathbf{R}) = E_4^{\text{intr}}(\mathbf{R}) &= W^{\text{ext}}(\mathbf{R}) + W^{\text{intr}}(\mathbf{R}) \\ &= W^{\text{ext}}(\mathbf{R}) + E_\alpha^{(0)} + W^{\text{Pauli}}(\mathbf{R}). \end{aligned} \tag{62}$$

Note that, in general, the effective c.m. potential $W(\mathbf{R})$ is not local in space but depends on two variables $\mathbf{R}$ and $\mathbf{R}'$.

The effective c.m. potential $W(\mathbf{R})$ is shown in Fig. 4. At large distances, only the bound-state energy of the free α particle remains, $\lim_{R\to\infty} W(R) = E_\alpha^{(0)} = -B_\alpha^{(0)} = -28.3$ MeV. For finite distances $R > 14$ fm, the Coulomb repulsion between the α particle and the lead core dominates the effective potential. Below $R \approx 14$ fm, the mean-field $[4V^{\text{mf}}(\mathbf{R})]$ of the lead core becomes relevant, tempting to attract the α particle. At distances inside the Coulomb barrier, the intrinsic four-nucleon energy shifts strongly downward. As soon as the core nucleons have a finite density (within the Thomas-Fermi model at $R \approx 8.46$ fm), the blocking of the α particle acts. The shift $W^{\text{Pauli}}(\mathbf{R})$ reduces the binding energy at distances R where the densities of the α particle and the core nucleus overlap. The bound state disappears if the baryon density n_B approaches the value $n_{B,\text{cluster}} = 0.0292$ fm^{-3}, which happens at $R_{\text{cluster}} \approx 7.72$ fm. At this point, the four-nucleon system has the (local) Fermi energy [Eq. (41)], with $4\mu \approx -22.016$ MeV, which is the edge of the quasiparticle continuum. At higher densities, the solution of the four-nucleon problem is given by the single-nucleon shell states, and the empty states above the Fermi energy $E_{\text{Fermi},\tau} = \mu_\tau$ are occupied by the added four nucleons on top of the ^{208}Pb core. An interesting result is the occurrence of a "pocket" near $R \approx 8.5$ fm in the effective α potential $W(\mathbf{R})$. Details are shown in the insertion in Fig. 5.

The calculations can be improved using a more detailed nucleon-nucleon interaction V_{N-N} for the α particle such as the Volkov force. Furthermore, the intrinsic wave function can be improved within the variational approach similar to the treatment given in Ref. [6], so that the α-like cluster becomes more stable and the transition to the continuum states becomes smoother. The Pauli blocking is overestimated using $P = 0$. A more appropriate expression for the Pauli blocking should also consider finite c.m. momenta for the bound, α-like cluster state; see Ref. [29].

The wave function $\Phi(\mathbf{R})$ is calculated solving the corresponding Schrödinger equation (61). The pocket of $W(R)$ shown in Fig. 4 is quite deep (-51.3 MeV at $R = 8.46$ fm) and a bound state at -32.47 MeV appears. The reason for the sharp minimum is the sharp disappearance of the nucleon density in the Thomas-Fermi model at the distance $R = 8.46$ fm where the mean-field potential $V^{\text{mf}}(R)$ coincides with the chemical potential. More realistic nucleon densities of heavy nuclei show longer tails so that the Pauli blocking acts already at larger values of R. Nevertheless, we used the Thomas-Fermi model for our exploratory calculations because the physical background for the appearance of the potential pocket becomes more transparent. Future calculations have to improve this approximation so that the density distribution in the tail that is of relevance in our approach is treated quantum mechanically. See Appendix C for further discussions.

V. DISCUSSION AND CONCLUSIONS

The physics of cluster formation in homogeneous matter is reasonably well understood; however, the numerical treatment is quite complex (see Refs. [6,17]). There, the c.m. momentum is a good quantum number so that the separation into the c.m.

motion and the relative motion is simple. In addition to the formation of clusters which are modified by the surrounding matter, we have also the formation of quantum condensates such as pairing and quartetting.

In the present work, we consider cluster formation in inhomogeneous nuclear systems, in particular, α-like clustering in heavy nuclei. We treated the particular situation where only a single α particle sits on top of a doubly magic nucleus which, thus, can be treated as a shell-model core (Fermi gas). In particular, we considered ^{212}Po, that is, one α on top of the ^{208}Pb core. The α particle as a cluster plays a rather particular role among possible clusters. The physics of the deuteron is very different, as explained in the main text. Heavier clusters may be treated with the fission-fusion scenario. At which mass number of the cluster the transition from our present description to the latter one occurs is not very clear. In any case, the α particle is by itself a doubly magic nucleus (the lightest) and, therefore, very stable with its first excited state at $\sim$20 MeV. However, as we have shown in earlier works [29], light clusters including the α particle are extremely sensitive to Pauli blocking from surrounding matter. Already at a fifth of saturation density, the α particle more or less suddenly becomes dissolved and gets mixed up with the surrounding Fermi gas. Translated to our $\alpha + {}^{208}$Pb case, this means that an α approaching the Pb core stays a compact almost elementary particle until it feels the tail of the Pb density at around $\rho_0/5$. There it quite suddenly dissolves and its four nucleons go over into single-particle shell-model states with, eventually, pair correlations in the open shells on top of the ^{208}Pb core. However, before its dissolution, the α particle already feels the attraction of the mean field of the core, so that one can understand the formation of a potential pocket at the surface of the Pb core.

As we know, the description of a well-defined cluster on top of a core nucleus is extremely difficult in a one-center shell-model description. Therefore, the main ingredient of this work is the introduction of the c.m. motion as a collective degree of freedom and an intrinsic motion that characterizes the cluster. To go beyond the single-quasiparticle approach, four-nucleon correlations are then described by an in-medium Schrödinger equation. Besides the mean field, the crucial effect of the surrounding nuclear system is Pauli blocking as a consequence of antisymmetrization. As just explained, an α-like bound state can exist in nuclear matter only at low densities, $n_B \leqslant n_{B,\mathrm{cluster}} \approx 0.03$ fm^{-3} and will be dissolved at higher densities into nearly free single-quasiparticle states forming the continuum of scattering states. It is clear that in a heavy nucleus only states near the Fermi energy can form an α-like cluster because only these single-particle states extend to the low-density regions at the surface of the nucleus. Deeper mean-field energy levels are situated in the region of higher densities. There, the role of cluster formation becomes irrelevant because of strong Pauli blocking.

The introduction of the c.m. motion and the intrinsic motion for clusters in nuclei, with full antisymmetrization of the nucleon wave function, was investigated within the THSR approach for light, low-density nuclei [7,16]. This gives a simple and adequate description of the properties of nuclei with cluster structure such as the Hoyle state. We reconsidered the preformation of α-like correlations within a generalized THSR approach which considers a fully antisymmetrized state of an α-like cluster and the core nucleus. The c.m. motion of both constituents has to be treated in a consistent way. In contrast to our recent calculation for ^{20}Ne [8,9], we here replaced the wave function of the doubly magic core nucleus with a shell-model wave function. Furthermore, we neglected the c.m. motion of the core nucleus because we treat a heavy system. However, for the nonlocalized α particle the c.m. motion is taken into account. After separation of the intrinsic motion within the α cluster, an effective potential has been derived which describes the c.m. motion of the α cluster under the influence of Pauli blocking with the surrounding medium.

The approach presented in this work to include few-nucleon correlations, in particular, bound states, is based on a first-principles approach to nuclear many-body systems. However, several approximations have been performed to make the approach practicable and to make the physical content transparent. In particular, derivatives of the intrinsic wave function $\varphi_4^{\mathrm{intr}}(\mathbf{s}_j,\mathbf{R})$ with respect to the c.m. coordinate $\mathbf{R}$ have been neglected. For the nucleon-nucleon interaction V_{N-N} a simple separable potential was taken, and Woods-Saxon-like expressions have been used for the mean-field potential $V_\tau^{\mathrm{mf}}(r)$. Furthermore, the effective c.m. potential $V_4^{\mathrm{c.m.}}(\mathbf{R})$ is taken in local approximation, and instead of the correct self-consistent mean-field single-particle states for a nucleus, the Thomas-Fermi (TF) model as a LDA was used. In general, the Pauli blocking as an exchange term leads to a nonlocal single-particle potential. These approximations can be improved in more sophisticated future calculations. The TF approximation for the c.m. motion of the α particle can be justified from the fact that, before its abrupt dissolution, the α particle is still quite compact in extension, its radius having increased by only about 20%. Therefore, the extension of the α particle is never much larger than the surface width of the Pb core qualifying the TF approximation as a reasonable lowest-order approach.

The intrinsic energy, called $W(R)$, of the α particle, thus becomes a function of the distance R of the center of the core nucleus. It has two contributions. The effect of $W^{\mathrm{Pauli}}(\mathbf{R})$ is to reduce the attractive shift $W^{\mathrm{ext}}(\mathbf{R}) + E_\alpha^{(0)}$ of the four-nucleon cluster at distances R where the densities of the α particle and the core nucleus overlap. It compensates the binding energy if the nucleon density $n_B(R)$ exceeds about 1/5 of the saturation density. This gives a microscopic derivation for the potential inferred by Delion and Liotta [24]. The approach [24] considers a fixed position of the α particle as described by the pocket at a fixed position. This resembles the adiabatic approach in describing fission of ^{212}Po into two daughter nuclei. The approach presented here considers the nonlocalized α particle where the c.m. motion is expressed by the wave function $\Phi(\mathbf{R})$. The corresponding in-medium Schrödinger equations for the c.m. motion and the intrinsic motion are derived within a quantum statistical approach. This may serve also to further elaborate on recent approaches using constrained Hartree-Fock calculations that have been performed for dilute nuclei showing a fragmentation of the mean field and correspondingly the appearance of fragments [25], in particular, to implement the c.m. motion.

Let us discuss the relation of our present study of ^{212}Po with respect to the similar situation of ^{20}Ne, which has been treated extensively already 40 y ago with the resonating group method (RGM) (see Matsuse *et al.* [35]) and also recently with the THSR wave function [7–9]. In both cases one considers an α particle on top of a doubly magic core. In the case of ^{20}Ne the core ^{16}O is light and its c.m. motion must be treated correctly. This is done with the RGM as well as with THSR approaches. However, in the case of ^{212}Po the ^{208}Pb core is too massive for an application of those methods for technical reasons. This, however, makes it possible to treat the ^{208}Pb core as infinitely heavy and then the corresponding treatment boils down to a four-nucleon TDA equation as discussed earlier in the text. It is interesting to see that the effective α particle-core potentials for ^{20}Ne and ^{212}Po show some similarity. In both cases they become strongly attractive inside the Coulomb barrier, see, e.g., Fig. 5 in Ref. [35] and Figs. 3 and 4 in present work. It would be interesting to also analyze the THSR approach in this respect. A rigorous separation of the c.m. motion and the antisymmetrization can be made using Gaussian functions for the internal cluster wave functions as well as for the relative c.m. motion. This has been shown in several papers related to the THSR approach [36–38]. In particular, let us outline the relation of our present treatment with the case of ^{20}Ne consisting of ^{16}O and an α cluster [8]. Contrary to the latter case, we here supposed that the big cluster is infinitely heavy, so that we can represent it as a shell-model nucleus with a fixed c.m. position at $R_{\text{c.m.}} = 0$, from where all coordinates are measured. The antisymmetrization of the total wave function which we had in the case of ^{20}Ne is then here replaced with the Pauli blocking factors. This means that the THSR approach has the advantage that the α particle is treated in a correlated medium in contrast to the single-particle, uncorrelated Pauli blocking term (Θ function in momentum space) considered in this work. A cluster-mean-field approach [4] would improve that. The extension of the original THSR approach to heavy nuclei is numerically not feasible at present. However, in the doubly magic ^{208}Pb core nucleus the α-like correlations are not strong so that a shell-model approach is reasonable. Nevertheless, a comparison of the results obtained using the THSR ansatz with the approach given in our work if applied to light nuclei such as ^{20}Ne would be of interest (as well as with former RGM calculations [26] and recent investigations [39]). The intrinsic wave function of the α particle in ^{212}Po has the same meaning as in ^{20}Ne case. The c.m. wave function $\Phi(\mathbf{R})$ plays the role of the relative wave function in the ^{20}Ne case. The difficult point is the Pauli blocking factor which is a very nonlocal operator.

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APPENDIX A: EVALUATION OF THE VARIATIONAL FUNCTIONAL EQ. (40)

We look for the minimum of the intrinsic energy, see Eq. (40), of an α-like cluster,

$$\tilde{W}^{\text{intr}}(\mathbf{P}) = \frac{\hbar^2}{2m}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\left[k^2+2k_{12}^2+2k_{34}^2\right]|\tilde{\varphi}^{\text{intr}}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{P})|^2$$
$$+\int \frac{d^3k}{(2\pi)^3}\frac{d^3k_{12}}{(2\pi)^3}\frac{d^3k_{34}}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3k'_{12}}{(2\pi)^3}\frac{d^3k'_{34}}{(2\pi)^3}\tilde{\varphi}^{\text{intr},*}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{P})V_4^{\text{intr}}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}'_{34},\mathbf{P})\tilde{\varphi}^{\text{intr}}(\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}'_{34},\mathbf{P}). \tag{A1}$$

The evaluation has been done for a special ansatz for the wave function, Eqs. (36) and (37), which contain a unique variational parameter b. The in-medium four-particle interaction V_4^{intr} is given by (39) with arbitrary $\mathbf{P}$. The Pauli blocking is fulfilled by the ansatz (37) for the wave function so that it must not considered any more. For simplicity, we consider only the c.m. momentum $\mathbf{P} = 0$. (To discuss finite $\mathbf{P}$, a series expansion with respect to powers of P can be performed.) We have to transform from the single-nucleon momenta $\mathbf{p}_i$ to Jacobi-Moshinsky momenta $\mathbf{k}_i$ [Eq. (19)].

To simplify the calculations we average the Fermi energies with respect to the isospin $\tau = n,p$ (symmetric matter), so that we perform the calculations for an excluded Fermi sphere $p_{\text{Fermi}} = k_F = (3\pi^2 n_B/2)^{1/3}$ with the total baryon density $n_B = n_n + n_p$.

The kinetic energy gives a 9-fold integral; the potential energy (after exploiting the δ functions) gives a 12-fold integral. We use spherical coordinates where the integrals over the angles can be performed. By reason of isotropy, we can fix the direction of $\mathbf{k}$ and denote the $\cos\theta$ of the directions of $\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{k}'_{12}$ relatively to $\mathbf{k}$ with z_{12},z_{34},z'_{12}, respectively; i.e., $z_{12} = \cos(\mathbf{k}_{12},\mathbf{k})$, etc. In Jacobi momenta, the expressions $F(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34})$ which have to be integrated have the form

$$F(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34}) \equiv F(k,k_{12},z_{12},k_{34},z_{34}), \tag{A2}$$

occurring for the norm or the kinetic energy and with additional variables k'_{12}, z'_{12} for the potential energy. For the functions F considered here, the integral over k is divided into two parts:

$$\int d^3k\, d^3k_{12}\, d^3k_{34} F(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34}) = 4\pi \int_0^{2k_F} dk G^<(k,\mathbf{k}_{12},\mathbf{k}_{34}) + 4\pi \int_{2k_F}^{\infty} dk G^>(k,\mathbf{k}_{12},\mathbf{k}_{34}). \tag{A3}$$

Next we consider the integral over $\mathbf{k}_{12}$. The excluded region in momentum space which is occupied by the Fermi sphere leads to a restriction of the limits of the integrals over $z_{12} = \cos(\mathbf{k}_{12},\mathbf{k})$. Geometrical considerations give for $k \leqslant 2k_F$ the limits where the Fermi sphere is touched,

$$G^<(k,\mathbf{k}_{12},\mathbf{k}_{34}) = \int d^3k_{12} H^<(k,k_{12},z_{12},\mathbf{k}_{34}) = 2\pi \left[\int_{\sqrt{k_F^2-k^2/4}}^{k_F+k/2} k_{12}^2 dk_{12} 2 \int_{(k_F^2-k^2/4-k_{12}^2)/kk_{12}}^{0} dz_{12} H^<(k,k_{12},z_{12},\mathbf{k}_{34}) \right.$$
$$\left. + \int_{k_F+k/2}^{\infty} k_{12}^2 dk_{12} \int_{-1}^{1} dz_{12} H^<(k,k_{12},z_{12},\mathbf{k}_{34}) \right], \tag{A4}$$

and for $k \geqslant 2k_F$,

$$G^>(k,\mathbf{k}_{12},\mathbf{k}_{34}) = \int d^3k_{12} H^>(k,k_{12},z_{12},\mathbf{k}_{34})$$
$$= 2\pi \left[\int_0^{k/2-k_F} k_{12}^2 dk_{12} \int_{-1}^{1} dz_{12} H^>(k,k_{12},z_{12},\mathbf{k}_{34}) + \int_{k/2+k_F}^{\infty} k_{12}^2 dk_{12} \int_{-1}^{1} dz_{12} H^>(k,k_{12},z_{12},\mathbf{k}_{34}) \right.$$
$$\left. + \int_{k/2-k_F}^{k/2+k_F} k_{12}^2 dk_{12} 2 \int_{(k_F^2-k^2/4-k_{12}^2)/kk_{12}}^{0} dz_{12} H^>(k,k_{12},z_{12},\mathbf{k}_{34}) \right]. \tag{A5}$$

The remaining integrals are performed in the same way. For the special trial function (37), the integral over the angular part z_{12}, etc., can be performed analytically. The norm, the kinetic energy, and the potential energy are calculated as integrals over k after the relative momenta k_{12}, k_{34}, k'_{12} have been integrated over. Thus, the 9- or 12-fold integrals are reduced to 3- or 4-fold integrals, respectively, that can be handled. For a given density, which also determines the blocked phase space for the four-particle wave function, the trial wave function (37) contains the parameter b, which describes how fast the wave function is decreasing with increasing single-particle momentum. For a similar evaluation of multiple integrals, see also [17].

With this variational ansatz, the minimum of the energy is determined for the optimal b parameter for each density. Results are given in Sec. III B 3. To improve the variational solution of the wave equation (8) for the intrinsic motion, the class of functions (37) can be extended.

APPENDIX B: LOCAL APPROXIMATION FOR THE PAULI BLOCKING TERM

As an example, we consider the term $\langle \mathbf{r}_1\mathbf{r}_2| f_1(\varepsilon_{n_1})|\mathbf{r}''_1\mathbf{r}''_2\rangle = \langle \mathbf{r}_1| f_1(\varepsilon_{n_1})|\mathbf{r}''_1\rangle \delta(\mathbf{r}''_2 - \mathbf{r}_2)$ occurring in B. We transform into a "mixed" (Wigner) representation,

$$\langle \mathbf{r}_1| f_1(E_{n_1})|\mathbf{r}''_1\rangle = \int \frac{d^3p_1}{(2\pi)^3} e^{i\mathbf{p}_1\cdot(\mathbf{r}_1-\mathbf{r}''_1)} f_1^{\text{Wigner}}\left(\frac{\mathbf{r}_1+\mathbf{r}''_1}{2},\mathbf{p}_1\right). \tag{B1}$$

The occupation of the phase space is given by the quasiparticle wave functions $\psi_n(\mathbf{r})$ (we take $\frac{\mathbf{r}_1+\mathbf{r}''_1}{2} = \mathbf{R}_1$),

$$f_1^{\text{Wigner}}(\mathbf{R}_1,\mathbf{p}_1) = \int d^3s_1 e^{-i\mathbf{p}_1\cdot\mathbf{s}_1} \sum_n^{\text{occupied}} \psi_n^*\left(\mathbf{R}_1 - \frac{\mathbf{s}_1}{2}\right) \times \psi_n\left(\mathbf{R}_1 + \frac{\mathbf{s}_1}{2}\right). \tag{B2}$$

Within the Thomas-Fermi model which corresponds to a LDA (or rather "local momentum approximation"), we have

$$f_1^{\text{Wigner}}(\mathbf{R}_1,\mathbf{p}_1) \approx \Theta\left\{E_{\text{Fermi}}[n_B(\mathbf{R}_1)] - \frac{p_1^2}{2m}\right\}; \tag{B3}$$

see also Eq. (35). The phase-space occupation is determined by the Fermi energy $E_{\text{Fermi}}(n_B) = (\hbar^2/2m)(3\pi^2 n_B/2)^{2/3}$, where we consider for simplicity the symmetric case $n_n = n_p = n_B/2$ as in Sec. III B. Now the baryon density $n_B(\mathbf{R}_1)$ depends on the position $\mathbf{R}_1$.

Again we introduce Jacobi coordinates to extract the c.m. motion as collective degree of freedom so that

$$\langle \mathbf{R},\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34}| f_1(\varepsilon_{n_1})|\mathbf{R}'',\mathbf{s}'',\mathbf{s}''_{12},\mathbf{s}''_{34}\rangle = \int \frac{d^3p_1}{(2\pi)^3} e^{i\mathbf{p}_1\cdot(\mathbf{s}_{12}-\mathbf{s}''_{12})} f_1^{\text{Wigner}}\left(\mathbf{R}+\frac{\mathbf{s}+\mathbf{s}''_{12}}{2},\mathbf{p}_1\right)$$
$$\times \delta(\mathbf{s}''_{34}-\mathbf{s}_{34})\delta(\mathbf{s}''-\mathbf{s}-2\mathbf{R}''+2\mathbf{R})\delta(\mathbf{s}''_{12}-\mathbf{s}_{12}-4\mathbf{R}''+4\mathbf{R}). \tag{B4}$$

To evaluate the contribution of $V_4^{\rm intr}$ to the intrinsic energy we use a "mixed" representation where the intrinsic motion is given in momentum representation,

$$\tilde{\varphi}_4^{\rm intr}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{R}) = \int d^3s\, d^3s_{12}\, d^3s_{34} e^{-i\mathbf{k}\cdot\mathbf{s}-i\mathbf{k}_{12}\cdot\mathbf{s}_{12}-i\mathbf{k}_{34}\cdot\mathbf{s}_{34}} \varphi_4^{\rm intr}(\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34},\mathbf{R}). \tag{B5}$$

To evaluate the Pauli blocking contribution to the effective c.m. potential $W(\mathbf{R},\mathbf{R}')$ [Eqs. (47), (49), and (52)], we have to average over the intrinsic motion. We consider here only one of the different terms (the others follow analogously). We give the general expression, which is complicated but will immediately be reduced to a more tractable form below in (B9):

$$\begin{aligned} F_1(\mathbf{R},\mathbf{R}') = &\int d^9s_j\, d^9s'_j\, d^9s''_j\, d^3R''\, \varphi_4^{\rm intr,*}(\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34},\mathbf{R}) \langle \mathbf{R},\mathbf{s},\mathbf{s}_{12},\mathbf{s}_{34}|f_1\left(\varepsilon_{n_1}\right)|\mathbf{R}'',\mathbf{s}'',\mathbf{s}''_{12},\mathbf{s}''_{34}\rangle \\ &\times \langle \mathbf{s}'',\mathbf{s}''_{12},\mathbf{s}''_{34}|V_{N-N}|\mathbf{s}',\mathbf{s}'_{12},\mathbf{s}'_{34}\rangle \delta(\mathbf{R}''-\mathbf{R}') \varphi_4^{\rm intr}(\mathbf{s}',\mathbf{s}'_{12},\mathbf{s}'_{34},\mathbf{R}'), \end{aligned} \tag{B6}$$

$$\begin{aligned} F_1(\mathbf{R},\mathbf{R}') = &\int d^3s\, d^3s_{12}\, d^3s''_{12}\, d^3k\, d^3k_{12}\, d^3k_{34} d^3k'\, d^3k'_{12}\, d^3k''_{12} \frac{d^3p_1}{(2\pi)^{21}} \tilde{\varphi}_4^{\rm intr,*}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{s}+i\mathbf{k}_{12}\cdot\mathbf{s}_{12}} \\ &\times e^{i\mathbf{p}_1\cdot(\mathbf{s}_{12}-\mathbf{s}''_{12})} f_1^{\rm Wigner}\left(\mathbf{R}+\frac{\mathbf{s}+\mathbf{s}''_{12}}{2},\mathbf{p}_1\right) e^{i\mathbf{k}'\cdot(\mathbf{s}-\mathbf{s}_{12}/2+\mathbf{s}''_{12}/2)+i\mathbf{k}''_{12}\cdot\mathbf{s}''_{12}} V_{N-N}(\mathbf{k}''_{12};\mathbf{k}'_{12}) \\ &\times \varphi^{\rm intr}\left(\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}_{34},\mathbf{R}-\frac{\mathbf{s}_{12}-\mathbf{s}''_{12}}{4}\right) \delta\left(\mathbf{R}'-\mathbf{R}+\frac{\mathbf{s}_{12}-\mathbf{s}''_{12}}{4}\right). \end{aligned} \tag{B7}$$

As expected, expression (B6) is not local in $\mathbf{R}$. The Wigner function limits the $\mathbf{p}_1$ integral as $\int_0^{p_{\rm Fermi}[n_B(\mathbf{R}+\frac{\mathbf{s}+\mathbf{s}''_{12}}{2})]} \frac{d^3p_1}{(2\pi)^3}$. We can expand near $\mathbf{R}$ so that additional terms near the Fermi surface are neglected. Also, the wave function $\tilde{\varphi}^{\rm intr}(\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}_{34},\mathbf{R}-\frac{\mathbf{s}_{12}-\mathbf{s}''_{12}}{2})$ can be expanded near $\mathbf{R}$. Neglecting higher-order contributions, we have

$$\begin{aligned} F_1(\mathbf{R},\mathbf{R}') \approx &\int d^3s\, d^3s_{12}\, d^3s''_{12}\, d^3k\, d^3k_{12}\, d^3k_{34} d^3k'\, d^3k'_{12}\, d^3k''_{12} \frac{d^3p_1}{(2\pi)^{21}} \tilde{\varphi}_4^{\rm intr,*}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{s}+i\mathbf{k}_{12}\cdot\mathbf{s}_{12}} \\ &\times e^{i\mathbf{p}_1\cdot(\mathbf{s}_{12}-\mathbf{s}''_{12})} f_1^{\rm Wigner}(\mathbf{R},\mathbf{p}_1)\, e^{i\mathbf{k}'\cdot(\mathbf{s}-\mathbf{s}_{12}/2+\mathbf{s}''_{12}/2)+i\mathbf{k}''_{12}\cdot\mathbf{s}''_{12}} V_{N-N}(\mathbf{k}''_{12};\mathbf{k}'_{12}) \tilde{\varphi}_4^{\rm intr}(\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}_{34},\mathbf{R}) \delta(\mathbf{R}'-\mathbf{R}), \end{aligned} \tag{B8}$$

which is diagonal in $\mathbf{R}$ space. Higher-order terms are connected with intrinsic coordinates $\mathbf{s}_j$ and are averaged out with the intrinsic wave function. The local approximation contains leading terms but can be improved in a systematic way.

Now we can integrate over the intrinsic coordinates $\mathbf{s}_j$ and obtain

$$F_1(\mathbf{R},\mathbf{R}') = \int d^3k\, d^3k_{12}\, d^3k_{34} \frac{d^3k'_{12}}{(2\pi)^{12}} \tilde{\varphi}_4^{\rm intr,*}(\mathbf{k},\mathbf{k}_{12},\mathbf{k}_{34},\mathbf{R}) f_1^{\rm Wigner}(\mathbf{R},\mathbf{k}+\mathbf{k}_{12})\, V_{N-N}(\mathbf{k}_{12};\mathbf{k}'_{12}) \varphi_4^{\rm intr}(\mathbf{k}',\mathbf{k}'_{12},\mathbf{k}_{34},\mathbf{R}) \delta(\mathbf{R}'-\mathbf{R}). \tag{B9}$$

The integral over $\mathbf{k}_1 = \mathbf{k}+\mathbf{k}_{12}$ is restricted to the Fermi sphere, $k_1 \leqslant k_{{\rm Fermi},\tau}[n(\mathbf{R}+\frac{\mathbf{s}_1+\mathbf{s}'_1}{4})]$. We can expand with respect to $\frac{\mathbf{s}_1+\mathbf{s}'_1}{4}$ so that the integral over the Fermi sphere gets additional contributions at $E_{{\rm Fermi},\tau}$. In zeroth order, we have only the Fermi energy at the c.m. position $\mathbf{R}$, but this can be improved taking the terms $\frac{\mathbf{s}_1+\mathbf{s}'_1}{4}$ into account.

APPENDIX C: IMPROVED DENSITY PROFILE

The Thomas-Fermi model gives a rather sharp pocket so that the α particle is well formed at the surface of the core nucleus. However, then the core nucleus may form further clusters so that the single-nucleon Thomas-Fermi approximation is not consistent. The Thomas-Fermi model is a quasiclassical approach which cannot describe the behavior of the tails of the density distribution, which are of relevance for the cluster formation. Thus, the nucleon density abruptly disappears at the value of the radius where the Fermi energy coincides with the potential energy. Real density distributions are more smooth and show long tails also in the region where the potential energy is larger than the Fermi energy owing to quantum tunneling. A shell-model calculation would give a better description of that region.

Because we are interested in the region where the nucleon density is low, we discuss here the consequences of long-range tails of the density. We use the nucleon density profile according to Shlomo [40], $n_B(R)$=0.17{1+ exp[$-(R-6.4914)/0.54$]}$^{-1}$ (units in fm), for the lead core. This density profile is shown in Fig. 6. The region of finite density, where Pauli blocking occurs, is extended to higher distances, above the value 8.46 fm obtained in the Thomas-Fermi model. The critical density where the α particle is dissolved occurs at the distance $R = 7.3416$ fm, which is smaller than the value given by the Thomas-Fermi model. These considerations are based on the given density profile, and instead of the theoretical estimates also experimental density distributions can be used. In a more detailed approach, different density profiles for neutrons and protons can be considered.

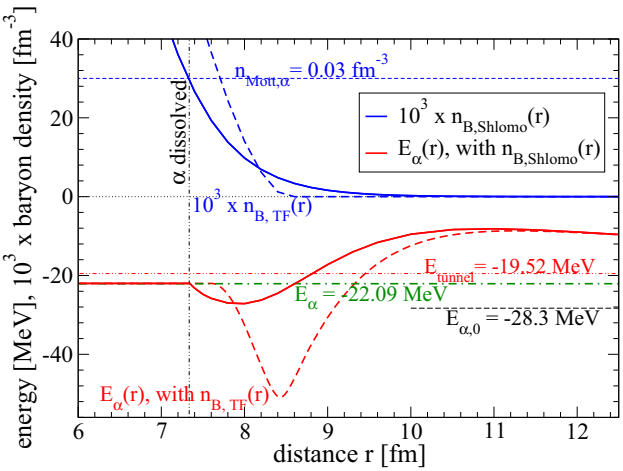

FIG. 6. (Color online) Nucleon density $n_B(r)$ (blue solid lines) according to Shlomo [40] compared with the Thomas-Fermi approach (blue dashed lines). The local effective potential $W(\mathbf{R})$ (62) (red dashed lines) according to the Thomas-Fermi approach is compared with an effective potential (red solid lines), which describes the Pauli blocking of the long-ranged density tails at the surface of the lead core. The ground-state energy level (green dash-dotted line) is also shown.

Now we have to introduce the mean-field potential of the core nucleons. If the position of the Fermi energy (−22.014 MeV) remains unchanged, the Woods-Saxon potential yields a too-big density at $R = 7.3416$ fm. Because now the density profile is inferred owing to the Shlomo approach, we can adapt the Woods-Saxon potential correspondingly so that the value of the critical density is reproduced within the Thomas-Fermi approximation. For our present estimation, the contribution of the neutron potential (56) to the four-nucleon potential was reduced by the factor 0.585. Then the condition for the disappearance of the α particle at the critical radius $R_{\rm crit}$ is correctly implemented. If we solve the c.m. Schrödinger equation with the effective c.m. potential we find the bound-state energy at −22.088 MeV.

For illustration, in Fig. 6 the nucleon density for the ^{208}Pb core according to Shlomo [40] is shown. It is clearly seen that the tail of the nucleon density n_B extends to larger values of R. Therefore, the pocket becomes shallow, and the bound-state energy of the four-nucleon (α-like) bound state becomes less negative. In an exploratory calculation where the Pauli blocking is calculated with the nucleon density profile according to Shlomo [40], the minimum of the pocket is −27.21 MeV at $R = 7.93$ fm. The corresponding solution of the c.m. Schrödinger equation (61) yields a bound-state energy at −22.088 MeV in better agreement with the empirical value −19.52 MeV. More systematic calculations based on shell-model states instead of the Thomas-Fermi model will provide us with more accurate results solving the c.m. motion of the α-like cluster on top of a heavy core nucleus.

The possibility to describe four-particle correlations and preformed α-like clusters near the surface can provide us with a theoretical tool to attack the cluster structure of nuclei like ^{212}Po, where, until now, only semiempirical approaches were known to determine the α decay. α-like correlations can survive in nuclear matter only up to densities $n_B \leqslant 0.03$ fm^{-3}, i.e., in the outer region of the nucleus. Preformation of α clusters and the α-like content of the four-nucleon wave function can be treated within the approach given here. For this, we consider the four-nucleon wave function $\Psi(\mathbf{R},\mathbf{s}_j)$ (1). The intrinsic part $\varphi^{\rm intr}$ is normalized for each $\mathbf{R}$, and the c.m. part $\Phi(\mathbf{R})$, which follows from the solution of a wave equation with the effective c.m. potential W, is normalized as well. For an estimation, we assume that in the region where the α-like cluster may exist, the overlap of the intrinsic wave function with the free α intrinsic wave function is equal to one, and it is zero in the remaining part where intrinsic wave function of the four-nucleon system is given as product of single-nucleon states. We integrate over the space $R \geqslant R_{\rm cluster}$ to find the amount of α clustering,

$$S = \int_0^\infty d^3R |\Phi(\mathbf{R})|^2 \Theta\left[n_B^{\rm critical} - n_B(\mathbf{R})\right] \approx 0.371, \quad \text{(C1)}$$

where $R_{\rm cluster}$ denotes the radius where the baryon density has the critical value where α-like clusters are destroyed because of Pauli blocking, $n_B(R_{\rm cluster}) = n_B^{\rm critical} = 0.03$ fm^{-3}. This result is in reasonable agreement with other estimations; see Refs. [10,23]. Note that the intrinsic state remains α-like as far as the change of the width parameter b is small.

The results given here should be improved by systematic shell-model calculations. We emphasize that our treatment, worked out with some approximations to allow for exploratory calculations, gives the possibility to improve the approximations, in particular, the LDA, the neglect of the gradient terms of the intrinsic wave function, and the introduction of nonlocal interaction potentials. Furthermore, a systematic improvement of the Green's functions approach allows also to include correlations in the nuclear matter, which is treated here as an uncorrelated medium, in contrast to the THSR approach, which treats four-particle correlations coherently.

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