Asymptotics of the $3j$ and $9j$ coefficients

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We present the details of calculations we previously performed for the large j behavior of certain $3j$ and $9j$ symbols.

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In this Brief Report we focus on Eqs. (11) and (13) and Eqs. (23) and (24) of the work of Kleszyk and Zamick [1]. In particular we consider the case when the total angular momentum I is equal to $I_{\max} - 2n$ where $I_{\max} \equiv 4j - 2$ and $n = 0,1,2,\ldots$. We take the limit of large j where n becomes much smaller than j. For convenience, we also define $J = 2j$, where j is the total angular momentum of a single particle.

We first address the $3j$ coefficient, using the formula Eq. (13) of [1], a derivation of which is contained in the work of Racah [2]. The $3j$ in question is

$$\begin{pmatrix} 2j & 2j-2 & I \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

We express the total angular momentum I using a new variable m such that $I = 4j - 2m$, where this time $m = 1,2,3,\ldots$. We can separate parts of the $3j$, which now becomes

$$3j = \frac{\sqrt{(2m-1)!}}{(m-1)!}(-1)^m \sqrt{\frac{N_1!N_2!}{N_3!}\frac{N_4!}{N_5!N_6!}}, \tag{2a}$$

where the six factors N_i are

$$\begin{aligned} &N_1 = 2J - 2 - 2m \quad N_2 = 2J + 2 - 2m \\ &N_3 = 4J - 1 - 2m, \\ &N_4 = 2J - 1 - m \quad N_5 = J - 1 - m \\ &N_6 = J + 1 - m. \end{aligned} \tag{2b}$$

We use the Stirling approximation

$$\ln x! \approx x \ln x - x + \ln \sqrt{2\pi x} \tag{3}$$

and it should be noted that the approximation approaches the true value asymptotically. We present the results in Table I.

We can write $N_i = (\alpha_i + \beta_i m + \gamma_i J)$ with differing constant coefficients. In Eq. (2b) we give the contribuition of $-N, \ln\sqrt{2\pi N}, \alpha \ln N, m\beta \ln N$, and $\gamma J \ln N$. For the last of these we break things up into (a) "extreme" and (b) "next order." This is necessary because "next order" has contributions comparable to those in "$-N$."

First notice that the "$\gamma J \ln N$" result is 1/2, which cancels the +1/2 from "$-N$." Adding up all the totals we get

$$-m\ln(2) + \ln\left(\frac{2}{\pi J}\right)^{1/4} + \ln\left(\frac{1}{\sqrt{J}}\right) \tag{4}$$

$$= -m\ln(2) + \ln\left(\frac{2}{\pi J^3}\right)^{1/4}. \tag{5}$$

Taking the antilog we get

$$3j \approx e^{-m\ln(2)}\left(\frac{2}{\pi J^3}\right)^{1/4} \tag{6}$$

and note that $e^{-m\ln 2} = \frac{1}{2^m}$.

Putting everything together and putting things in terms of j and n we obtain

$$3j \to \frac{\sqrt{(2n)!}}{n!2^n}(-1)^n\left(\frac{1}{64\pi j^3}\right)^{1/4}. \tag{7}$$

We see that in the limit $n \ll j$, $3j$ goes as $\frac{1}{j^{3/4}}$. Alternatively, the Clebsch-Gordan (CG) has an asymptotic value

$$\mathrm{CG} \to \frac{\sqrt{(2n)!}}{n!2^n}(-1)^n\left(\frac{1}{\pi j}\right)^{1/4}. \tag{8}$$

We next consider the unitary $9j$ coefficient $\langle (jj)^{2j}(jj)^{2j} | (jj)^{2j}(jj)^{2j-2}\rangle^I$. This time we write $I = 4j - 2m$, where $m = 1,2,3,\ldots$. In Eq. (11) from [1], we have a factor $(2J + I + 1)!$ which becomes $(4J + 1 - 2m)!$. This can be written as $(4J + 1)! \times \mathrm{PROD}$ where $\mathrm{PROD} = (4J + 1)(4J),\ldots,(4J + 2 - 2m)$. For convenience we break this equation into several parts as follows:

$$U(9j) = \frac{\mathrm{FAC}}{\sqrt{\mathrm{PROD}}}\sqrt{\frac{(2J+1)(2J-3)}{2}} \times 3j, \tag{9}$$

where

$$\mathrm{FAC} = \frac{(C_1!)^2}{C_2!}\sqrt{\frac{C_3!}{C_4!C_5!}} \tag{10}$$

with

$$\begin{aligned} &C_1 = J \quad C_2 = 2J \quad C_3 = 4J + 1 \\ &C_4 = 2J + 1 \quad C_5 = 2J - 1. \end{aligned}$$

There are $2m$ terms in PROD. We use the fact that $(4J + 1 - 2m)! = (4J + 1)! \times \mathrm{PROD}$, and asymptotically we obtain

$$\sqrt{\frac{(2J+1)(2J-3)}{2}} \to J\sqrt{2}, \tag{11}$$

$$\mathrm{PROD} \to (4J)^{2m} = (8j)^{2m}. \tag{12}$$

Hence we have

$$\frac{1}{\sqrt{\mathrm{PROD}}} \to \frac{1}{(8j)^m}. \tag{13}$$

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TABLE I. Asymptotic contributions to the $3j$ coefficients.

	$-N_i$	$\ln\sqrt{2\pi N_i}$	$\alpha_i \ln N_i$	$\beta_i m \ln N_i$	$\gamma_i J \ln N_i$	$\gamma_i J \ln N_i$
(1)	$-\frac{1}{2}(2J-2-2m)$	$\frac{1}{2}\ln\sqrt{4\pi J}$	$-\ln(2J)$	$-m\ln(2J)$	$J\ln(2J)$	$-1-m$
(2)	$-\frac{1}{2}(2J+2-2m)$	$-\frac{1}{2}\ln\sqrt{8\pi J}$	$\ln(2J)$	$-m\ln(2J)$	$J\ln(2J)$	$1-m$
(3)	$\frac{1}{2}(4J-1-2m)$	$\frac{1}{2}\ln\sqrt{4\pi J}$	$\frac{1}{2}\ln(4J)$	$m\ln(4J)$	$-2J\ln(4J)$	$\frac{1}{2}+m$
(4)	$-(2J-1-m)$	$\frac{1}{2}\ln\sqrt{4\pi J}$	$-\ln(2J)$	$-m\ln(2J)$	$2J\ln(2J)$	$-1-m$
(5)	$(J-1-m)$	$-\frac{1}{2}\ln\sqrt{2\pi J}$	$\ln(J)$	$m\ln J$	$-J\ln(J)$	$1+m$
(6)	$(J+1-m)$	$-\frac{1}{2}\ln\sqrt{2\pi J}$	$-\ln(J)$	$m\ln J$	$-J\ln(J)$	$-1+m$
Total	$\frac{1}{2}$	$\ln\left(\frac{2}{\pi J}\right)^{1/4}$	$\ln\frac{1}{\sqrt{J}}$	$-m\ln 2$	0	$-\frac{1}{2}$

We use the Stirling approximation to calculate FAC. The detailed results are given in Table II.

We next combine Tables I and II. There are many cancellations when we add the totals of $\ln \mathrm{FAC}$ and $\ln 3j$ in Table I and Table II . The result is

$$\ln \mathrm{FAC} + \ln 3j = -(m-1)\ln 2 = -n\ln 2. \tag{14}$$

The antilog is

$$e^{-n\ln(2)} = \frac{1}{2^n}. \tag{15}$$

The j dependence comes from

$$\sqrt{\frac{(2J+1)(2J-3)}{2}} \tag{16}$$

and PROD

$$\sqrt{\mathrm{PROD}} \to (8j)^m, \tag{17}$$

putting everything together we obtain the result

$$U9j \to \frac{(-1)^n}{2\sqrt{216^n}}\frac{\sqrt{[(2n+2)!(2n)!]}}{(n!)j^n}. \tag{18}$$

In the different limit of fixed I and $j \gg I$, we get the behavior

$$U9j \to \sqrt{6\pi}\, j^{3/2} e^{-4\ln(2)j}. \tag{19}$$

The best way to demonstrate the power-law behavior of the $U9j$ symbol is to plot the logarithm of $U9j$ versus the logarithm of j. We plot this in Fig. 1. Note the independence of the slopes of the curves for different values of n.

TABLE II. ln (FAC).

	$-C_i$	$\ln\sqrt{2\pi C_i}$	$\alpha_i \ln C_i$	$\gamma_i J \ln()$	$\gamma_i J \ln()$
(1)	$-2J$	$2\ln(\sqrt{2\pi J})$	0	$2J\ln J$	0
(2)	$+2J$	$-\ln\sqrt{4\pi J}$	0	$-2J\ln(2J)$	0
(3)	$-2J-\frac{1}{2}$	$\ln\sqrt{8\pi J}$	$\frac{1}{2}\ln(4J)$	$2J\ln(4J)$	$\frac{1}{2}$
(4)	$J+\frac{1}{2}$	$-\frac{1}{2}\ln\sqrt{4\pi J}$	$-\frac{1}{2}\ln(2J)$	$-J\ln(2J)$	$\frac{1}{2}$
(5)	$J-\frac{1}{2}$	$-\frac{1}{2}\ln\sqrt{4\pi J}$	$\frac{1}{2}\ln(2J)$	$-J\ln(2J)$	$-\frac{1}{2}$
Total	$-\frac{1}{2}$	$\ln\left(\frac{\pi J}{2}\right)^{1/4}$	$\ln(2\sqrt{J})$	0	$\frac{1}{2}$

We present results of the percent deviation of our approximate values of $3j$ and $U9j$ from the exact values in Tables III and IV.

We note other work on asymptotics of CG coefficients by Reinsch and Morehead [3]. In their work they defined

$$\beta = [(j_1+j_2-j)(j+j_2-j_1) \times (j+j_1-j_2)(j_1+j_2+j)]^{1/2}. \tag{20}$$

They found an approximate expression for the CG coeffecients in their Eq. (B9).

$$\mathrm{CG} = \langle j_1 j_2 00 | j 0\rangle \approx 2(-1)^{\frac{j_1+j_2-j}{2}} \times \sqrt{\frac{2j+1}{2\pi\beta}}\sqrt{\frac{j+j_1+j_2}{j+j_1+j_2+1}}(1+\delta_4+\delta_6) \times\left[1+\frac{1}{24}\left(\frac{2}{j}+\frac{2}{j_1}+\frac{1}{j_2} - \frac{1}{j+j_1+j_2} - \frac{1}{-j+j_1+j_2} - \frac{1}{j-j_1+j_2} - \frac{1}{j+j_1-j_2}\right)\right]. \tag{21}$$

We quickly run into trouble in making a comparison to our results, especially for $n = 0$. In their Eq. (B12) they have in the leading term CG proportional to $\frac{1}{\sqrt{\beta}}$. However, for the case $j = j_1 + j_2$, that is to say $I = I_{\max}$, with our $n = 0$, we see that β vanishes and hence their expression for CG is disproved.

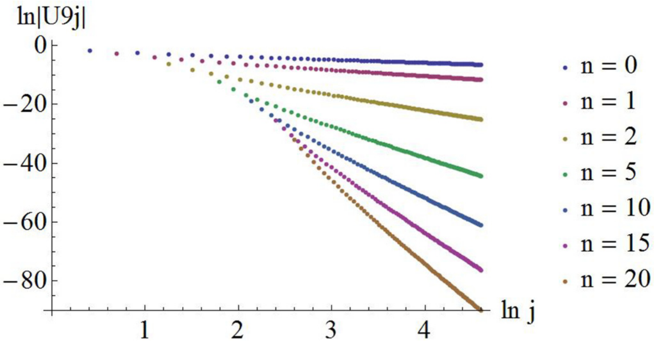

FIG. 1. (Color online) $\ln|U9j|$ vs. $\ln j$ for many values of n.

TABLE III. Comparison of the exact and asymptotic values of the $3j$ symbols.

	j	Accepted $3j$	Approximate $3j$	Percent error
$n = 0$				
	9/2	0.0917186951	0.0859524287	6.28690401
	99/2	0.0143074760	0.0142302863	0.539505856
	999/2	0.00251476295	0.00251342493	0.0532063347
	9999/2	0.000446679154	0.000446655420	0.00531331215
$n = 1$				
	9/2	−0.0703281160	−0.0607775452	13.5800180
	99/2	−0.0101817625	−0.0100623320	1.17298491
	999/2	−0.00177932008	−0.00177725981	0.115789693
	9999/2	−0.000315869604	−0.000315833077	0.0115641443
$n = 2$				
	9/2	0.0667864681	0.0526348981	21.1892774
	99/2	0.00887471327	0.00871423511	1.80826305
	999/2	0.00154190275	0.00153915215	0.178390316
	9999/2	0.000273568204	0.000273519468	0.0178151485
$n = 10$				
	99/2	0.00642003383	0.00597328117	6.95872744
	999/2	0.00106225244	0.00105503104	0.679819882
	9999/2	0.000187614589	0.000187487331	0.0678293749
$n = 100$				
	999/2	0.000637437519	0.000596632653	6.40139073
	9999/2	0.000106699870	0.000106026325	0.631251795

Evidently their formula is not valid in this region. On the other hand, our expression Eq. (13) from [1] works just fine.

In this work we have used an explicit expressions for the $9j$ symbol in question by Varshalovitch *et al.* [4]. We have given the details of how the asymptotic behaviors of selected $3j$ and $9j$ coefficients and their unitary counterparts are obtained. There are some subtleties, e.g., in the second column of Table I, although term-by-term we get nonzero results, the entire

TABLE IV. Comparison of the exact and asymptotic values of the $U9j$ symbols.

	j	Accepted $U9j$	Approximate $U9j$	Percent error
$n = 0$				
	9/2	0.492152957	0.500000000	1.59443179
	99/2	0.499361854	0.500000000	0.127792280
	999/2	0.499937371	0.500000000	0.0125274006
	9999/2	0.499993749	0.500000000	0.00125027349
$n = 1$				
	9/2	−0.0378955625	−0.0340206909	10.2251328
	99/2	−0.00312046463	−0.00309279008	0.886872805
	999/2	−0.000306761485	−0.000306492711	0.0876166329
	9999/2	−0.0000306243639	−0.0000306216840	0.00875116429
$n = 2$				
	9/2	0.00606563844	0.00448261961	26.0981402
	99/2	0.0000379552583	0.0000370464431	2.39443810
	999/2	3.64686293×10^{-7}	3.63819464×10^{-7}	0.237691695
	9999/2	3.63251097×10^{-9}	3.63164818×10^{-9}	0.0237519144
$n = 10$				
	99/2	7.33668833×10^{-17}	5.24669432×10^{-17}	28.4868855
	999/2	4.95097802×10^{-27}	4.79272848×10^{-27}	3.19632873
	9999/2	4.76517144×10^{-37}	4.74976392×10^{-37}	0.323335927
$n = 20$				
	99/2	1.75313503×10^{-27}	5.21781167×10^{-28}	70.2372517
	999/2	4.88682624×10^{-48}	$4.35394087*10^{-48}$	10.9045287
	9999/2	$4.32566*10^{-68}$	$4.27622870*10^{-68}$	1.143

sum is zero and so we must expand further as in the following column. There are similar points for Table II. We further note that one can take asymptotic limits in more than one way. Here the emphasis is on when the total angular momentum I is large ($I = I_{\max} - 2n, n \ll j$), and one obtains a power-law behavior $1/j^n$. This is most easily seen by plotting $\ln|U9j|$ versus $\ln j$. On the other hand, if one keeps I fixed and increases j one gets a dominantly exponential behavior, as shown in Eq. (19). This is most easily seen by plotting $U9j$ versus j. Last, we recall the physics motivation for this work: how maximum-j pairing manifests itself in nuclei [5].

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