Evaluation of the curvature-correction term from the equation of state of nuclear matter

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Based on the nuclear equation of state, the curvature correction term to the surface tension coefficient is calculated. Tolman's δ correction is shown to be sensitive to the Skyrme force parametrization. The temperature dependence of the Tolman length, important in heavy ion collision experiments, is derived. In the present approach the curvature term is related to the bulk properties of the nuclear matter through the equation of state. The results are compared with the existing theoretical calculations based on the Gibbs-Tolman formalism and with the theoretical predictions concerning its dependence on the interparticle distance.

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During recent decades, rapid progress has been achieved in macroscopic description of the nuclear matter [1,2]. A number of papers devoted to the thermodynamics of small systems or hydrodynamics of nuclear matter appeared [3–5]. Among the macroscopic models used in nuclear physics, theories based on the droplet model of nuclei [6] play a special role. They make possible the description of the average properties of a saturated system, such as a nucleus consisting of two components (neutrons and protons), with account for the boundary effects and the presence of the diffuse layer. For nuclei with mass number A, in studies of their surface properties, the account for curvature effects is important, which means the inclusion of additional terms proportional to $A^{\frac{1}{3}}$ in an expansion (e.g., Eq. (2.13) in Ref. [6]) concerning the nuclear properties in terms of the fundamental dimensionless ratio, i.e., the ratio of the interparticle spacing r_0 to the nuclear radius R, $\frac{r_0}{R} = A^{-\frac{1}{3}}$.

Corrections due to curvature may play an important role when studying light nuclei or processes where surface terms are important. Particularly important are those corrections in the interpretation of multifragmentation experiments [5,7], in which light nuclei necessarily appear. The exponential dependence of the yield of fragments on the surface tension makes this process sensitive to the curvature corrections [8]. Other important phenomena that may be affected by the changes in the surface tension due to curvature corrections are the following:

(1) the appearance of the neck region in the fission processes and

(2) the hydrodynamic instability of the structures formed in heavy ion experiments, governed by surface effects [9].

Quite a number of papers dedicated to studies of the surface energy and the properties of the surfaces in nuclear matter [9–11] appeared. Furthermore, the dependence of the surface tension (and surface energy) on the surface curvature and studies of its influence on different physical properties was also studied by various groups of authors [12,13]. Still, for decades it remains a controversial issue in mesoscopic thermodynamics [14–16].

Let us first recall that the thermodynamic description of the curvature correction, originating from the difference between the equimolar surface and the surface of tension [17,18], dates back to the 1940s. The Tolman length δ was originally introduced in Ref. [19] to describe the curvature dependence of the surface tension of a small liquid droplet. It was defined as a correction term in the surface tension σ of the liquid-vapor droplet in the isothermal case:

$$\sigma(R) = \sigma_\infty \left(1 - \frac{2\delta}{R} + \cdots\right), \quad (1)$$

where R is the droplet radius, equal to the radius of the surface tension [17,18], and σ_∞ is the surface tension of the planar interface. Equation (1) originates from the Gibbs-Tolman-Kenig-Buff's thermodynamic equation and from the assumption that δ is independent of R for $\delta \ll R$ [20]. This physics should work not only for liquid droplets but also for any system with curved interface of a non-negligible boundary layer [14]. This situation corresponds to nuclei and nuclear systems with a finite diffuse layer [1]. The value of the Tolman length has the same order of magnitude as the average interparticle distance r_0 [21,22], typical of a nuclear systems $r_0 \sim 0.7$ fm at normal density $\rho \sim 0.17$ fm^{-3}. Hence, mathematically the term $\frac{2\delta}{R}$ in Eq. (1) becomes important for the systems with $R < 14$ fm, i.e., important for nuclear systems with $R < 8$ fm. The above estimations show that this approximation works well in a wide range of radii and can play an important role in all known nuclei and structures formed in heavy ion collision experiments.

Within the above approximation, the dependence of the surface tension on the curvature of the interface is defined only by the Tolman length δ. Therefore, the knowledge (evaluation) of δ is quite important. However, the sign of the known

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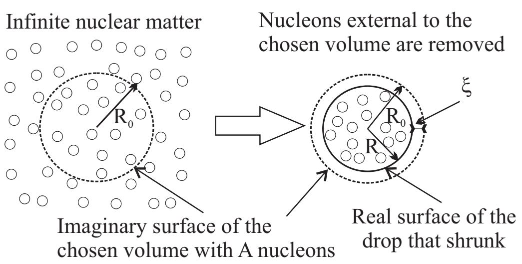

FIG. 1. Schematic picture of the gedanken experiment.

(calculated) values of the Tolman length are not unique: Both negative and positive values can be found in the literature [15,17,20]. At the same time, there are no reliable experimental methods to evaluate it. This paper aims to introduce a method allowing the evaluation of δ from the experimental data [23].

In studying the curvature-correction term for the nuclear matter one should keep in mind the connection between the surface and bulk properties of the matter [6,12]. As shown in the droplet model, the coefficients in the term proportional to $A^{\frac{1}{3}}$ in the expansion of nuclear properties in terms of the fundamental dimensionless ratio $A^{-\frac{1}{3}}$ are connected to the bulk properties of the nuclear matter, described by terms proportional to A and $A^{\frac{2}{3}}$. This justifies our approach to the evaluation of the curvature correction (Tolman's length δ) from the equation of state (EOS) of nuclear matter.

Let us consider infinite nuclear matter ($P_0, T = \text{const.}$) with the chosen spherical volume $V_0 = \frac{4}{3}\pi {R_0}^3$ in it, consisting of A nucleons. Next, one may perform the following gedanken experiment. When all nucleons external to the chosen volume are removed, one gets a "nuclear droplet" that, due to surface tension, shrinks to the volume $V = \frac{4}{3}\pi R^3$, where R is the final radius of the chosen volume (Fig. 1).

This droplet remains in equilibrium in one of the following regimes:

(i) The time scale of the particles evaporation is big enough and the evaporated particles are removed from the surface ($P(V,T) = 0$).

(ii) The "nuclear liquid" is surrounded by the saturated "nuclear vapor" with $P^{\text{liq}} = P^{\text{vap}}$ and chemical potential $\mu^{\text{liq}} = \mu^{\text{vap}}$. The superscripts liq and vap denote liquid and vapor phases, respectively.

Another gedanken experiment is to consider a big nucleus ${V_0}^{\text{big}} = 2V_0 = 2(\frac{4}{3}\pi {R_0}^3)$ in equilibrium, consisting of $2A$ nucleons, subsequently divided in two equal parts V each containing A nucleons. Due to surface effects the smaller nuclei will shrink, so that $2V = 2(\frac{4}{3}\pi R^3) < {V_0}^{\text{big}}$.

Next we introduce the parameter $\xi = R_0 - R > 0$ (Fig. 1), assumed to be independent of R. Let us consider the EOS the same as the form in Ref. [23]:

$$\Delta P = P - P_0 = f\left(\frac{V_0 - V}{V_0}\right), \tag{2}$$

where P_0, V_0 and P, V are the initial and final points of the system evolution along the coexistence curve. Within our gedanken experiment, a small parameter $\frac{\xi}{R}$ can be introduced and, therefore, the function f in Eq. (2) with the argument $\frac{V_0 - V}{V_0} = 1 - [1 + \frac{\xi}{R}]^{-3}$ can be expanded in the series:

$$\begin{aligned}\Delta P &= f(0) + \frac{\partial f(0)}{\partial\left(\frac{1}{R}\right)}\frac{1}{R} + \frac{1}{2}\frac{\partial^2 f(0)}{\partial\left(\frac{1}{R}\right)^2}\left(\frac{1}{R}\right)^2 + \cdots \\ &= f(0) + 3\dot{f}(0)\left(\frac{\xi}{R}\right) + \frac{1}{2}(9\ddot{f}(0) - 12\dot{f}(0))\left(\frac{\xi}{R}\right)^2 + \cdots .\end{aligned} \tag{3}$$

On the other hand, the excess pressure due to the surface tension in the left-hand side of Eq. (3) can be found from the Laplace equation [24]:

$$\Delta P = \frac{2\sigma(R)}{R}, \tag{4}$$

By substituting $\sigma(R)$ from Eq. (1) to Eq. (4), restricting ourself by the second order in the expansion (3), and equating the coefficients of the same order of $\frac{1}{R}$, one gets from Eqs. (3) and (4)

$$\delta = \left[\frac{4\dot{f}(0) - 3\ddot{f}(0)}{6(\dot{f}(0))^2}\right]\sigma_\infty. \tag{5}$$

To get numerical results for the Tolman length from Eq. (5), it is necessary to chose the appropriate EOS of nuclear matter. In the present work we adopt the EOS of nuclear matter of low-temperature and high-density limit where $\lambda^3\rho \gg 1$, that is, when the average de Broglie thermal wavelength λ is larger than the average interparticle separation $\rho^{-\frac{1}{3}}$ in a form [25]

$$\begin{aligned}&P(\rho_q, T) \\ &= \sum_q \left[\frac{5}{3}{\varepsilon^*}_{kq}(\rho_q, T) - \varepsilon_{kq}(\rho_q, T)\right] \\ &\quad + \frac{t_0}{2}\left(1 + \frac{x_0}{2}\right)\rho^2 + \frac{t_3}{12}\left(1 + \frac{x_3}{2}\right)(\alpha + 1)\rho^{\alpha+2} \\ &\quad - \frac{t_0}{2}\left(x_0 + \frac{1}{2}\right)\sum_q {\rho_q}^2 - \frac{t_3}{12}\left(\frac{1}{2} + x_3\right)(\alpha + 1)\rho^\alpha \sum_q {\rho_q}^2 \\ &\quad + C(\beta + 1)\rho^\beta {\rho_p}^2 + C_s(\eta - 1)\rho^\eta,\end{aligned} \tag{6}$$

with

$$\begin{aligned}\varepsilon_{kq} &= \frac{{m^*}_q}{m}\frac{1}{\beta}\frac{2g}{\sqrt{\pi}}{\lambda_q}^{-3}F_{\frac{3}{2}}(\eta_q), \\ {\varepsilon^*}_{kq} &= \frac{1}{\beta}\frac{2g}{\sqrt{\pi}}{\lambda_q}^{-3}F_{\frac{3}{2}}(\eta_q), \\ \eta_q(\rho_q, T) &= {F_{\frac{1}{2}}}^{-1}\left(\frac{\sqrt{\pi}}{2g}{\lambda_q}^3\rho_q\right), \\ C\rho^\beta &= \frac{4\pi}{5}e^2R^2, \\ C_s\rho^\eta &= \frac{4\pi {r_0}^2\sigma}{V^{\frac{1}{3}}}\rho^{\frac{2}{3}},\end{aligned} \tag{7}$$

where m and m^* are the mass and effective mass respectively, T and ρ are the temperature and density, q is the proton or neutron, F is the Fermi integral, $\lambda = \sqrt{\frac{2\pi\hbar^2}{m^* T}}$ is the average de Broglie thermal wavelength, $g = 2$ is the spin degeneracy factor, t_0, t_3, x_0, x_3 and α are the Skyrme force parameters, $\beta = \frac{1}{T}$, and $C\rho^\beta$ and $C_s\rho^\eta$ the approximate Coulomb and surface effects for a finite uniform sphere of radius $R = r_0 A^{\frac{1}{3}}$ with total charge Z ($U_c = \frac{3}{5}\frac{e^2Z^2}{RV}$). From Eqs. (5) and (6) for the symmetric nuclear matter with the isospin-independent effective mass in the case $T = 0$ at normal density ρ_0, one gets for δ:

$$\delta = \frac{2}{3}\frac{1}{{\rho_0}^2}\frac{-33t_0 - 160W{\rho_0}^{-1/3} + t_3(1+\alpha){\rho_0}^{\alpha}\frac{1}{12}(7(3\alpha+6) - 3(3\alpha+6)^2)}{\left(15t_0 + \frac{1}{12}t_3(1+\alpha)((3\alpha+6) - (3\alpha+6)^2)\right)^2}\sigma_\infty, \tag{8}$$

where

$$W = \frac{h^2}{10m}\left(\frac{3}{8\pi g}\right)^{\frac{2}{3}}\left(\frac{5 - 3\frac{m*}{m}}{\frac{m*}{m}}\right). \tag{9}$$

As a check of our approach, we have calculated the values of the Tolman length δ for different effective interactions, SLy6, SkM*, and SV-min (Table I) with the surface tension of the planar interface at $T = 0$ set to $\sigma_\infty = 1.1$ Mev/fm^2 [26]. The obtained values appear to be negative for all chosen parametrizations (Table II), which means that the surface of tension is located closer to the liquid phase with respect to the equimolar surface. Those results are consistent with those obtained from the Gibbs-Tolman approach [16,29]. All values derived in the present approach agree in sign with those calculated in Refs. [16,29] from the Gibbs-Tolman formalism applied to the charged Fermi-liquid droplets. As to the absolute value, they are slightly higher but consistent in the order of magnitude. Even though the authors of Refs. [16,29] performed a detailed analysis of the positions of the surface of tension and equimolar surface in the nuclei in order to calculate the Tolman length; the result $\delta = -0.3703$ fm seem to underestimate the Tolman length, as it is smaller than the internucleon distance $r_0 \sim 0.7$ fm.

As seen from Table II, the obtained value of δ in the case of SkM* parametrization, introduced to account for the surface properties of nuclear matter, is close to the distance between the nucleons,

One can see from Table II that the suggested approach is very sensitive to parametrization of the Skyrme force, and the results may differ by more than a factor of two. In our opinion, this discrepancy

(i) may originate from the different sets of phenomenological inputs used when calibrating the nuclear effective energy functionals or

(ii) may be related to the same constant surface tension coefficient σ_∞ used in calculations with all the parametrizations when its value depends on the energy functional used.

Although this point requires further study, a brief analysis of the observed difference is still possible. For example, the chosen parametrizations represent three typical sets:

(i) SkM$*$ is a representative of the group of Skyrme forces developed to study the surface energy and fission barriers [30]. SkM$*$ was constructed by using the fission barriers in ^{208}Pb and surface coefficients $a_{\rm surf}$ as phenomenological inputs. It improves the description of surface effects and the high-precision description of nuclear ground states considerably. Therefore, it looks quite reasonable that the results for the Tolman length obtained for that parametrization are very close to the internuclear distance $r_0 \sim 0.7$ fm, in accordance with the theoretical predictions of Refs. [21,22].

(ii) SLy6 is a parametrization particularly adapted to neutron-rich matter and neutron stars [27,31] and it is not surprising that the Tolman length δ evaluated in the present paper for the symmetric matter is less consistent with the theoretical predictions of its value as compared to the SkM$*$ force. The reason why this force produces an overestimated value may come from the bigger difference between the neutron and proton distributions in the neutron-rich matter as compared to symmetric nuclear matter and, therefore, from the possible increase of the distance between the equimolar surface and surface of tension.

(iii) As to the third, SV-min force, it is the most recent parametrization that was constructed using a large set of spherical nuclei as well as some detailed observables such as neutron skin, isotope shifts, and superheavy elements [28]. The r.m.s. errors in the charge distribution form factor, radius, and surface thickness for that force are very close to those of the SkM$*$ force [32]. That may suggest an explanation for the similar deviation of the Tolman length from the theoretical predictions for this two forces.

TABLE I. Sets of Skyrme parameters and corresponding nuclear properties used in the present paper [27,28]

Skyrme forces	SkM* [27]	SLy6 [27]	SV-min [28]
K (MeV)	216.6	229.8	222.0
$m*/m$	0.79	0.69	0.95
t_0 (MeV fm^3)	−2645.0	−2479.50	−2112.248
x_0	0.09	0.825	0.243 886
t_3 (MeV fm$^{3(1+\alpha)}$)	15 595.0	13 673.0	13 988.567
x_3	0.0	1.355	0.258 070
α	1/6	1/6	0.255 368

TABLE II. Tolman's length δ for different Skyrme force parametrizations and from Ref. [29] for the SkM* force

	SkM*	Sly6	SV-min	Ref. [29]
Tolman length δ (fm)	−0.8869	−1.5600	−0.5512	−0.3703

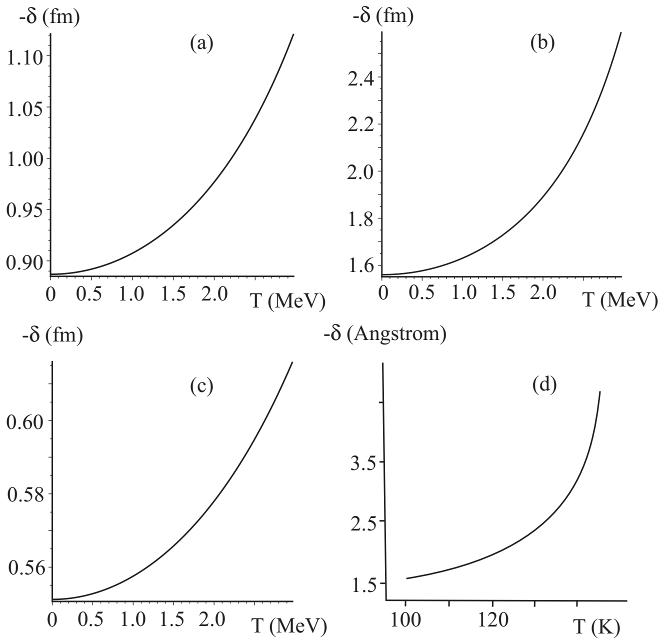

FIG. 2. Temperature dependence of the Tolman length δ. Panels (a), (b), and (c) show the nuclear matter with the parametrizations SkM*, SLy6, and SV-min respectively. The initial values ρ_0 correspond to the equilibrium condition $P(\rho_0,T)=0$. (d) The same for the ordinary liquid Ar [23].

The suggested approach may provide realistic values for the Tolman's δ correction that can be used as a test for the validity of each parametrization in treating the surface properties of the nuclei.

The temperature dependence $\delta(T)$ is shown in Fig. 2. One can see that the correction term in nuclear matter [Figs. 2(a), 2(b), and 2(c)] increases with temperature for all studied parametrizations of nuclear forces. All the curves far away from the critical point can be approximated by the equation

$$\delta(T)=\delta(0)(1+aT^b), \tag{10}$$

where a and b are free parameters that slightly vary for all forces. Any simple analytical approximation explaining the physics of such a behavior is highly desirable. At the same time, our semiqualitative picture corresponds to the data for ordinary liquids [Fig. 2(d)] known in the literature [23] and makes it important for heavy ion collision experiments, where the yield of fragments is exponentially dependent on the surface tension, and the nuclear matter is hot.

In this paper we have calculated the curvature correction term in the surface tension from the nuclear equation of state for the SLy6, SkM∗, and SV-min parametrizations. The obtained results show the importance of that correction for light nuclei.

To summarize, our study shows that the present approach makes possible calculations of the Tolman length δ from a simple thermodynamic equations. The obtained values are consistent with the existing data and with the theoretical predictions. We remind that the agreement improves if EOS accounting for the surface effects (SkM* and SV-min) instead of those adapted for neutron matter (e.g., SLy6) are used. The temperature dependence $\delta(T)$ for the nuclear matter shows the same behavior as that of ordinary liquids. The possibility of evaluating the temperature dependence of the curvature correction term makes the suggested approach useful in analyzing the results of heavy ion collision experiments and in calculating yields of light fragments. The present approach, based on a minimal set of assumptions, provides a simple and reliable way to calculate the curvature correction term, and it may be used to study the properties of light nuclei as well as of complicated nuclear processes, sensitive to the surface tension.

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