Symplectic group structure of the ^{48}Cr, ^{88}Ru, and ^{92}Pd ground states

K. Neergård

Fjordtoften 17, 4700 Næstved, Denmark

The ground states of ^{48}Cr, ^{88}Ru, and ^{92}Pd are studied in the $1f_{7/2}$ or $1g_{9/2}$ shell model with effective interactions from the literature. They are found to be composed, quite independently of the shell and the interaction, roughly of 75% of $(s,t) = (0,0)$ and 25% of $(s,t) = (4,0)$, where s is the seniority and t the reduced isospin. Other irreps of the symplectic group $Sp(2j + 1)$, where j is the single-nucleon angular momentum, make only very small contributions. The state χ obtained by antisymmetrization and normalization of the ground state in the stretch scheme of Danos and Gillet [M. Danos and V. Gillet, Phys. Rev. **161**, 1034 (1967)] has a very different structure where the $Sp(2j + 1)$ irreps other than $(s,t) = (0,0)$ and $(4,0)$ contribute 20% and 41% for $j = 7/2$ and 9/2, respectively. The contributions of χ and the $s = 0$ state to the calculated states are about equal for ^{48}Cr. For ^{88}Ru and ^{92}Pd the $s = 0$ state is unambigously a better approximation to the calculated states than χ. A state χ' obtained by antisymmetrization and normalization of the product of two stretch-scheme ground states of the system with two valence nucleons or nucleon holes of each type has much larger overlaps with the calculated ground states than χ but a deviating $Sp(2j + 1)$ decomposition.

I. INTRODUCTION

In an impressive experiment, Cederwall *et al.* [1] measured the γ decay of three excited states of ^{92}Pd, which has four neutrons and four protons less than the doubly magic ^{100}Sn. They interpreted the spectrum in terms of the "stretch scheme" proposed in the 1960s by Danos and Gillet [2] to describe deformed nuclei in the shell model. Qi *et al.* [3] made shell model calculations in support of this interpretation employing a valence space composed of the shells $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$, and $1g_{7/2}$ as well as smaller spaces. The stretch scheme applies to nuclei with equal even numbers of valence neutrons and protons. All valence neutron orbits are supposed to belong to the same j shell and so also for the protons. The valence nucleons are divided into two "chains," each of them formed by half of the valence neutrons and half of the valence protons. Within a chain the nucleonic angular momenta are coupled to the maximal total angular momentum through pairwise coupling of a neutron and a proton to their maximal combined angular momentum. In the ground state the chain angular momenta are opposite, and "rotational" excitations are formed by a bending of them towards each other to form a nonzero total angular momentum. Nucleon holes in a j shell may replace valence nucleons without changing the scheme essentially. This is its version relevant to ^{92}Pd, which may be seen as a system of four neutron holes and four proton holes in the $1g_{9/2}$ shell. In the following I use the term *quasinucleon* to denote either a valence nucleon or a nucleon hole, and I call two quasinucleons *equivalent* if either both of them are valence nucleons or both of them are nucleon holes.

Generalizing the adaption to nuclei independently by Bohr, Mottelson, and Pines [4] and Bogolyubov and Solov'yov [5] of the theory of superconductivity of Bardeen, Cooper, and Schrieffer [6], Goswami and Kisslinger [7] introduced in the 1960s a concept of "isoscalar pairing" different from the "isovector pairing" described by the Bardeen-Cooper-Schrieffer theory. This concept is much discussed in the subsequent literature; see the review by Frauendorf and Macchiavelli [8]. Referring

to predictions of isoscalar pairing in nuclei with equal numbers N and Z of neutrons end protons, Cederwall *et al.* [1] state that their results "reveal evidence for a spin-aligned, isoscalar neutronproton coupling scheme" and "suggest that this coupling scheme replaces normal superfluidity (characterized by seniority coupling) in the ground and lowest excited states of the heaviest $N = Z$ nuclei."

In support of this suggestion, Qi *et al.* [3] point out that in single-j-shell calculations for the system of two quasineutrons and two equivalent quasiprotons in the $1f_{7/2}$, $1g_{9/2}$, or $1h_{11/2}$ shell with empirical effective interaction, the product of the state of two quasineutrons with combined angular momentum zero and the similar state of the two quasiprotons makes up only a little more than half of the calculated ground states. Consideration of this product is motivated by its resemblance to the product of neutron and proton Bardeen-Cooper-Schrieffer states conventionally employed to model nuclear superfluidity. It is, however, not an eigenstate of isospin. In a meaningful adaption of the concept of nuclear superfluidity to the single-j-shell model of $N = Z$ nuclei one should rather see the unique state with isospin and seniority zero as the manifestation of isovector pairing. I show in a previous article [9] that this state makes up about 80% of the calculated ground states of two quasineutrons and two equivalent quasiprotons in the $1f_{7/2}$ or $1g_{9/2}$ shell.

The stretch-scheme ground state is not antisymmetric in the quasineutrons or the quasiprotons. Qi *et al.* [3] consider the antisymmetrized and normalized state and get overlaps of 92–95% with the calculated ground states of two quasineutrons and two equivalent quasiprotons. This finding is essentially confirmed by my calculations in Ref. [9]. I find, moreover, that the overlaps are larger in the $1f_{7/2}$ shell than in the $1g_{9/2}$ shell. The overlap of the seniority zero state with the antisymmetrized and normalized stretch-scheme state is 62% and 52%, respectively, so to this extent seniority zero and the stretch scheme are different visualizations of the same physics, in this case of two quasineutrons and two equivalent quasiprotons.

TABLE I. Expectation values and overlaps in percentages. Due to rounding off the sum of these percentages may differ slightly from 100 and a zero only means that the percentage is less than 0.5. The operator $P_{s,t}$ is the projection onto the subspace of the $I = T = 0$ space with the given (s,t) and $\langle P_{s,t}\rangle_a$ is its expectation value in state a. State ψ is the calculated ground state and states χ and χ' are defined by Eqs. (22) and (23). In the lasts four columns pairs of percentages are shown. The first percentage is for $a = \chi$ and the second one is for $a = \chi'$. The rows "$\psi = \chi$" and "$\psi = \chi'$" show the $\mathrm{Sp}(2j + 1)$ decompositions of these states and the row "Dimension" shows the subspace dimension.

	$\langle P_{0,0}\rangle_\psi$	$\langle P_{4,0}\rangle_\psi$	$\langle P_{4,2}\rangle_\psi$	$\langle P_{6,1}\rangle_\psi$	$\langle P_{8,0}\rangle_\psi$	$|\langle a|\psi\rangle|^2$		$\dfrac{|\langle a|P_{4,0}|\psi\rangle|^2}{\langle P_{4,0}\rangle_a\langle P_{4,0}\rangle_\psi}$		$\dfrac{|\langle a|P_{6,1}|\psi\rangle|^2}{\langle P_{6,1}\rangle_a\langle P_{6,1}\rangle_\psi}$		$\dfrac{|\langle a|P_{8,0}|\psi\rangle|^2}{\langle P_{8,0}\rangle_a\langle P_{8,0}\rangle_\psi}$	
^{48}Cr													
Dimension	1	2	0	2	1								
SchTr, emp.	75	24		0	1	78	93	97	98				
SchTr, fit	79	20		0	0	77	93	97	98				
ZR I	80	20		0	0	77	92	98	98				
ZR II	73	26		0	1	80	92	100	94				
$\psi = \chi$	49	30		19	2								
$\psi = \chi'$	67	26		6	1								
$(^{88}$Ru,$)$ ^{92}Pd													
Dimension	1	2	1	5	7								
SchTr, emp.	70	27	1	0	2	65	86	97	99	77	36	86	98
SchTr, fit	70	27	0	0	2	63	87	95	100	76	36	85	98
QLW	70	27	1	0	2	66	84	99	96	71	39	89	96
ZE I	83	16	0	0	1	50	80	93	100	1	93	82	99
ZE II	72	25	0	0	2	61	86	97	99	61	46	86	98
ZE III	85	14	1	0	1	44	76	100	93	36	7	89	95
ZE IV	76	22	0	0	1	61	82	99	97	60	46	88	97
CCGI	76	22	0	0	2	56	84	92	100	49	63	82	98
SLGT0	73	25	0	0	2	63	85	96	99	75	40	86	98
GF	68	28	1	0	3	67	86	97	99	83	31	87	98
Nb90	70	27	1	0	2	64	86	96	99	77	36	86	98
$\psi = \chi$	25	34	10	20	10								
$\psi = \chi'$	47	36	0	6	10								

In their analysis of calculated states of ^{92}Pd, Qi *et al.* [3] counted the numbers of pairs of $1g_{9/2}$ holes with definite combined angular momentum. I show in Ref. [9] that for two quasineutrons and two equivalent quasiprotons in the $1g_{9/2}$ shell, the seniority zero state has a large content of quasinucleon pairs with high angular momenta, and the antisymmetrized and normalized stretch-scheme state has a large content of pairs with low angular momenta. Inferring a pairing mode from such counts is thus not straightforward.

The present text presents an analysis similar to the one in Ref. [9] addressing the case of ^{92}Pd. I thus consider the system of four quasineutrons and four equivalent quasiprotons in a single j shell. Besides ^{92}Pd this is the single-j-shell configuration of its $1g_{9/2}$ cross conjugate, ^{88}Ru, and of ^{48}Cr in the $1f_{7/2}$ shell. The particle-hole symmetry of the single-j-shell model is, in the $1f_{7/2}$ shell, only approximately obeyed by the data. Van Isacker [10], in his $1f_{7/2}$ shell model calculations, accordingly makes an interpolation between the empirical two-valence-nucleon and two-nucleon-hole interactions. As seen from Table I, these interactions, denoted there by ZR I and II, give qualitatively similar results in the present type of analysis. Effects breaking the particle-hole symmetry are thus apparently minorly important in this context. The situation is somewhat different in the $1g_{9/2}$ shell because the observed ^{80}Zr spectrum is clearly rotational and thus not that of a closed-shell nucleus. The nucleus ^{88}Ru seems to the sit on the edge of

an onset of deformation with $N = Z$ decreasing from 50, so modeling it by the $1g_{9/2}$ shell model may be questionable. The focus of my study as concerns the $1g_{9/2}$ shell is on ^{92}Pd.

In the next Sec. II I describe the method used to construct the interaction matrix in the space of isospin and angular momentum zero and decompose the calculated ground state into irreps of the symplectic group $\mathrm{Sp}(2j + 1)$, where j is the single-nucleon angular momentum. The results of this decomposition are shown and discussed in Sec. III. Section IV discusses the stretch-scheme ground state. It is found that only a small part of it belongs to the space of states antisymmetric in the quasineutrons and in the quasiprotons. Following Qi *et al.* [3] I antisymmetrize and normalize this part and then discuss the decomposition of the antisymmetrized and normalized state into irreps of $\mathrm{Sp}(2j + 1)$ and its overlaps with the calculated states. In Sec. V a similar analysis is applied to the state obtained by antisymmetrization and normalization of the product of two stretch-scheme ground states of the system of two quasineutrons and two equivalent quasiprotons. The article is summarized in Sec. VI.

II. METHOD

The eight quasinucleons are labeled with numbers 1–8 so that quasinucleons 1–4 are quasineutrons and quasinucleons 5–8 are quasiprotons. The angular momentum of the ith

quasinucleon is denoted by j_i, and all these angular momenta are equal to $j = 7/2$ in the $1f_{7/2}$ shell and $j = 9/2$ in the $1g_{9/2}$ shell. States of the system with total angular momentum $I = 0$ may be expanded in a basis of states:

$$|\alpha\rangle = |[(j_e\beta)_n(j_e\gamma)_p]0\rangle, \tag{1}$$

where $|j_e\beta\rangle_n$ is a totally antisymmetric state with angular momentum j_e of the quasineutrons. The index β labels a complete, orthonormal set of such states. The definition of $|j_e\gamma\rangle_p$ is analogous for quasiprotons. The outmost (square) brackets in Eq. (1) followed by the value of I indicate vector coupling with the total magnetic quantum number suppressed. A similar notation is employed throughout this article.

The states $|j_e\beta\rangle_n$ may be expanded on a basis of states:

$$|j_{12}j_{34}\rangle_{j_e} = |[(j_1j_2)j_{12}(j_3j_4)j_{34}]j_e\rangle, \tag{2}$$

with even j_{12} and j_{34}. To determine the subspaces with a given symmetry of the span of this basis one can use that the sum of transpositions

$$K_4 = \sum_{1\leqslant i<k\leqslant 4}(ik) \tag{3}$$

is in the symmetric group S(4), a class sum, and therefore within each irrep a constant dependent only on the irrep. Because the states (2) carry the irrep $[1^2] \times [1^2]$ of the product of the S(2) of quasinucleons 1 and 2 and quasinucleons 3 and 4, the Young frames of the irreps of S(4) present in their span have at most two columns. Let such a frame have column lengths λ and μ. One can calculate its K_4 by evaluating in Yamanouchi's [11] realization of the irrep the diagonal matrix element of the sum (3) in the tableau where the indices 1–4 appear successively from top to bottom in the columns from left to right. The result, which I denote by just K because it is not limited to the case $n = \lambda + \mu = 4$, is

$$K = n - n^2/4 - d(d + 1), \tag{4}$$

with $d = (\lambda - \mu)/2$. Because K as given by this expression decreases with d, different (λ,μ) with the same n have different K. Indicating by a prime the restriction of operators to the span of the states (2) with a given j_e, we have

$$K_4' = -2 + 4(23)'. \tag{5}$$

The subspaces of definite symmetry are thus the eigenspaces of $(23)'$. The totally antisymmetric states $|j_e\beta\rangle_n$, which have $(\lambda,\mu) = (4,0)$, in particular form a basis for the eigenspace with eigenvalue -1. They are therefore obtained by diagonalization of $(23)'$ in the basis (2). The matrix elements are

$$\langle j_{12}j_{34}|(23)'|j_{12}'j_{34}'\rangle_{j_e} = \langle j_{12}j_{34}|j_{12}'j_{34}'\rangle_{jjjjj_e} \tag{6}$$

in terms of what Zamick and Escuderos [12] call the unitary nine-j symbol:

$$\langle ef|gh\rangle_{abcdi} = \langle[(ab)e(cd)f]i|[(ac)g(bd)h]i\rangle. \tag{7}$$

A charge-independent interaction of two quasinucleons in the same j shell can be written

$$V = \sum_J c_J P_J, \tag{8}$$

with

$$P_J = \sum_{1\leqslant i<k\leqslant 8} P_{j_{ik}=J}, \tag{9}$$

where $P_{j_{ik}=J}$ denotes the projection onto the eigenspace with eigenvalue J of the combined angular momentum j_{ik} of the ith and kth quasinucleons. Indicating by a double prime the restriction of operators to the span of the states (1), we have

$$P_J'' = 12P_{j_{12}=J}'' + 16P_{j_{15}=J}''. \tag{10}$$

In the basis of states

$$|j_{12}j_{34}j_{56}j_{78}j_e\rangle = |[(j_{12}j_{34})j_e(j_{56}j_{78})j_e]0\rangle \tag{11}$$

the projection $P_{j_{12}=J}$ is diagonal with matrix elements $\delta_{j_{12}J}$. The projection $P_{j_{15}=J}$ has the matrix elements

$$\langle j_{12}j_{34}j_{56}j_{78}j_e|P_{j_{15}=J}|j_{12}'j_{34}'j_{56}'j_{78}'j_e'\rangle$$
$$= \delta_{j_{34}j_{34}'}\delta_{j_{78}j_{78}'}\sum_{j_{26},j_d}\langle j_{12}j_{56}j_e|Jj_{26}j_d\rangle_{j_{34}j_{78}}$$
$$\times\langle j_{12}'j_{56}'j_e'|Jj_{26}j_d\rangle_{j_{34}j_{78}}, \tag{12}$$

with

$$\langle j_{12}j_{56}j_e|Jj_{26}j_d\rangle_{j_{34}j_{78}}$$
$$= \langle j_ej_e|j_dj_d\rangle_{j_{12}j_{34}j_{56}j_{78}0}\langle j_{12}j_{56}|Jj_{26}\rangle_{jjjjj_d}. \tag{13}$$

Because the states (1) carry the irrep $[1^4] \times [1^4]$ of the product of the quasineutron and the quasiproton S(4), the Young frames of the irreps of S(8) present in their span have at most two columns. In Eq. (4) we now have $n = 8$, and d is the isospin T. By the relation

$$K_8'' = -12 + 16(15)'', \tag{14}$$

where

$$K_8 = \sum_{1\leqslant i<k\leqslant 8}(ik), \tag{15}$$

the subspaces with definite T are thus obtained by diagonalization of (15)''. In particular the $T = 0$ space has (15)'' = 1/4. The matrix elements of (15)'' are obtained from

$$(15)'' = -\sum_J(-)^J P_{j_{15}=J}'' \tag{16}$$

and the restriction of Eq. (12) to the span of the states (1).

Each eigenspace of T is the intersection with the $I = 0$ space of an irrep of the unitary group U($2j + 1$) characterized by n and T, and these irreps split into irreps of Sp($2j + 1$) characterized by a seniority s and a reduced isospin t [13]. Racah's seniority operator [14], generalized to jj coupling and nuclei by Edmonds and Flowers [15],

$$Q = (2j + 1)P_0 \tag{17}$$

is within each such irrep a constant dependent for a given j only on the U($2j + 1$) and Sp($2j + 1$) irreps. Edmonds and Flowers [15] derive a closed expression which can be written

$$Q = f(j,n,T) - f(j,s,t), \tag{18}$$

with

$$f(j,x,y) = (j + 2)x - x^2/4 - y(y + 1). \tag{19}$$

Using Flowers's method [13] one finds that $(n,T) = (8,0)$ is composed for $j \geqslant 7/2$ of $(s,t) = (0,0)$, $(2,1)$, $(4,0)$, $(4,2)$, $(6,1)$, and $(8,0)$. These are seen from Eqs. (18) and (19) to have distinct Q. The corresponding subspaces of the $I = T = 0$ space are therefore obtained by diagonalization of the restriction of the operator (17). Because $(s,t) = (2,1)$ is composed of $I = 2,4,\ldots,2j-1$ [13], its intersection with the $I = T = 0$ space is the null space.

III. Sp($2j+1$) DECOMPOSITIONS OF CALCULATED GROUND STATES

Calculations were made with the same effective $1f_{7/2}$ and $1g_{9/2}$ interactions as in Ref. [9]. Thence I repeat a brief description of each of them. The interactions SchTr are from the appendix of the classic study by Schiffer and True [16] with "emp." referring to the empirical matrix elements and "fit" to those derived from a universal interaction fitted to the data. ZR I and II are Models I and II of Zamick and Robinson [17]. They were derived from the spectra of ^{42}Sc and ^{54}Co, respectively. QLW is $0g_{9/2}$ of Qi, Liotta, and Wyss [18]. It was extracted from an interaction for the $2p_{1/2} + 1g_{9/2}$ configuration space provided by Johnstone and Skouras [19]. ZE I–IV are from Zamick and Escuderos [20]. Specifically, ZE I and II are their INTc and INTd. The former consists of a $T = 1$ part from the spectrum of ^{98}Cd and a $T = 0$ part from a δ interaction. The latter has a lower c_9. ZE III and IV are from the spectrum of ^{90}Nb with different choices of the 1^+ level. CCGI is adapted from the $V_{\text{low-}k}$ of Coraggio, Covello, Gargano, and Itaco [21]. This is not charge independent. To conserve isospin, I use their neutron-proton matrix elements in all channels. SLGT0, GF, and Nb90 (named Nb90ZI in Ref. [9]) are from Zerguine and Van Isacker [22]. Specifically, SLGT0 and GF were constructed by renormalization to the $1g_{9/2}$ subspace of interactions for the $2p_{1/2} + 1g_{9/2}$ configuration space provided, respectively, by Serduke, Lawson, and Gloeckner [23] and Gross and Frenkel [24], and Nb90 is from the spectrum of ^{90}Nb with yet another choice of 1^+ level.

Table I shows for each interaction the decomposition of the ground state into Sp($2j+1$) irreps. It is seen that quite independently of the shell and the interaction the ground state is composed roughly of 75% of $(s,t) = (0,0)$ and 25% of $(s,t) = (4,0)$. Other irreps make only small contributions, which tend, however, to be somewhat larger in the $1g_{9/2}$ shell than in the $1f_{7/2}$ shell. The typical contribution of about 75% of the $s = 0$ state found here for $n = 8$ is slightly less than the typical 80% found in Ref. [9] for $n = 4$. Yet this state, which, as explained in the Introduction, may be conceived of as the manifestation of perfect isovector pairing in the single-j-shell model of $N = Z$ nuclei, remains a fairly good first approximation also for $n = 8$.

Comparison with the case $n = 4$ studied in Ref. [9] reveals a striking similarity: In that case only $(s,t) = (0,0)$ and $(s,t) = (4,0)$ occur; for $n = 8$ they dominate the calculated states in about the same ratio. A hint to an understanding of this similarity may conceivably be found in Qi's calculations [25] with the interaction QWL of multihole states in the $1g_{9/2}$ shell, which show that the antisymmetrized and normalized product

of a pair of ^{96}Gd ground states makes up in this model 96% of the ^{92}Pd ground state. (Clearly from comparison with Ref. [3] the quantity denoted by x^2 in Table I of Ref. [25] is just x.)

Because a two-quasinucleon interaction can break at most two $J = 0$ pairs, its matrix elements between Sp($2j + 1$) irreps differing by more than four in s vanish. In an expansion where the terms in the interaction (8) other than the pairing force, $J = 0$, are treated as perturbations, the ground state components with $s > 4$ are therefore of second order. This explains their small size. That the $(s,t) = (4,2)$ components in the $1g_{9/2}$ shell are much smaller than the $(s,t) = (4,0)$ components is due to smaller matrix elements from $s = 0$. For $j = 7/2$ all matrix elements involving $(s,t) = (6,1)$, and therefore this component, vanish within the numeric accuracy for all the interactions. Some fundamental selection rule thus seems to be active in this case. The same is not true for $j = 9/2$ and I have no explanation for this apparent partial conservation of seniority, which bears a resemblance to the much discussed case of $j = 9/2$, $n = 2T = s = 2t = 4$, and $I = 4$ and 6; see Van Isacker and Heinze [26] and references therein.

IV. STRETCH-SCHEME GROUND STATE

The stretch-scheme ground state is

$$|\sigma\rangle = |\{[(j_1 j_5)\bar{j}_d (j_2 j_6)\bar{j}_d]j_d[(j_3 j_7)\bar{j}_d(j_4 j_8)\bar{j}_d]j_d\}0\rangle, \quad (20)$$

with $\bar{j}_d = 2j$ and $j_d = 4j - 2$. Its image by the projection P onto the span of the states (1) has the components

$$\langle\alpha|\sigma\rangle = \langle\beta|\bar{j}_e\bar{j}_e\rangle_{j_e}\langle\gamma|\bar{j}_e\bar{j}_e\rangle_{j_e}\langle j_e j_e|j_d j_d\rangle_{\bar{j}_e\bar{j}_e\bar{j}_e\bar{j}_e 0}$$
$$\times\langle\bar{j}_e\bar{j}_e|\bar{j}_d\bar{j}_d\rangle^2_{jjjjj_d}, \quad (21)$$

with $\bar{j}_e = 2j - 1$. The squared norm $\||P|\sigma\rangle\|^2$ is 1.5% for both j. This is much less than for $n = 4$ [9], where the corresponding squared norm is about 50%. It means that 98.5% of $|\sigma\rangle$ carries irreps of the quasineutron $\times$ quasiproton S(4) $\times$ S(4) other than $[1^4] \times [1^4]$. Several factors reduce $\||P|\sigma\rangle\|^2$. First the unitary nine-j symbol $\langle\bar{j}_e\bar{j}_e|\bar{j}_d\bar{j}_d\rangle_{jjjjj_d}$ in Eq. (21) is about 0.7 for both j's and enters the squared norm to the power of 4. Second $\langle\beta|\bar{j}_e\bar{j}_e\rangle_{j_e}$ vanishes for odd j_e because a totally antisymmetric $|j_e\beta\rangle_n$ is symmetric in j_{12} and j_{34}. This gives another factor of about $(1/2)^2$. Third the totally antisymmetric part of $|\bar{j}_e\bar{j}_e\rangle_{j_e}$ for even j_e is typically about 1/3. With the product $\langle\beta|\bar{j}_e\bar{j}_e\rangle_{j_e}\langle\gamma|\bar{j}_e\bar{j}_e\rangle_{j_e}$ in Eq. (21) this factor enters $\||P|\sigma\rangle\|^2$ to the power of 2.

Following Qi *et al.* [3] I consider the state χ obtained by normalization of $P|\sigma\rangle$, that is,

$$|\chi\rangle = \frac{P|\sigma\rangle}{\||P|\sigma\rangle\|}. \quad (22)$$

Quite generally antisymmetrization in the quasiprotons and in the quasineutrons of a product of $T = 0$ states gives a $T = 0$ state because each factor in the product has a symmetry $[2^q]$ and the only S(n) irrep with a Young frame with at most two columns containing a product of such S($2q$) irreps is $[2^{n/2}]$. In particular because each pair of quasinucleons with indices i and $i + 4$ in the state (20) has $T = 0$ (symmetry $[2]$) the state χ has $T = 0$.

The Sp($2j + 1$) decomposition of χ is shown in Table I. It is seen to be for both j's markedly different from those of the calculated states ψ. In particular the irreps other than $(s,t) = (0,0)$ and $(4,0)$, which are almost absent from ψ, contribute 20% and 41%, respectively, of χ, and $(s,t) = (6,1)$, which makes only very small contributions to ψ—for $j = 7/2$ vanishing within the numeric accuracy—gives for both j the largest of these contributions to χ, about 20% of the total. While the calculated states are distributed in an approximate ratio 3 : 1 on $(s,t) = (0,0)$ and $(4,0)$, these irreps have more equal weights in χ with the contribution of $(s,t) = (4,0)$ being for $j = 9/2$ even the larger of the two. With 49% and 25%, respectively, for the two j, the overlap of χ with the $s = 0$ state is considerably less for $n = 8$ than for $n = 4$, where it amounts to 62% and 52%, respectively, as mentioned in the Introduction.

The overlaps of ψ with χ are in the $1f_{7/2}$ shell about the same as their overlaps with the $s = 0$ state, 78% and 77% on average over the interactions. In the $1g_{9/2}$ shell they are 60% on average over the interactions and the $s = 0$ state is unambiguously a better approximation to ψ than χ. The result $|\langle \chi | \psi \rangle|^2 = 66\%$ for the interaction QLW agrees with Ref. [3].

When the subspace of the $I = T = 0$ space belonging to a given Sp($2j + 1$) irrep has a dimension larger than one, one may ask whether the images of χ and ψ by projection onto this subspace have the same directions. This question is addressed in the last three columns in Table I. Due to the numeric vanishing, mentioned in the last paragraph of Sec. III, of the $(s,t) = (6,1)$ components of ψ for $j = 7/2$, this case is omitted. It is seen that the directions are very much the same for $(s,t) = (4,0)$ and almost as much so for $(s,t) = (8,0)$ in the $1g_{9/2}$ shell, while the situation is more ambiguous for $(s,t) = (6,1)$ in the $1g_{9/2}$ shell with almost exactly orthogonal projected states for the interaction ZE I. Once more a similarity with the case $n = 4$ studied in Ref. [9] is revealed: There, as well, the $(s,t) = (4,0)$ components of ψ and χ have almost exactly the same directions.

V. PRODUCT OF STRETCH-SCHEME GROUND STATES

In a $1f_{7/2}$ shell-model calculation for ^{48}Cr with an interaction interpolated from ZR I and II, Van Isacker [10] finds that the state

$$|\chi'\rangle = \frac{P|\sigma'\rangle}{\|P|\sigma'\rangle\|}, \tag{23}$$

with

$$|\sigma'\rangle = |[(j_1 j_5)\bar{J}_d (j_2 j_6)\bar{J}_d]0\rangle \times |[(j_3 j_7)\bar{J}_d (j_4 j_8)\bar{J}_d]0\rangle, \tag{24}$$

makes up 92.7% of the calculated ground state. The factors in the product (24) are recognized as stretch-scheme ground states of the $n = 4$ system considered in Ref. [9]. Like χ the state χ' has $T = 0$. The components of $P|\sigma'\rangle$ in the basis (1) are

$$\langle \alpha | \sigma' \rangle = \sum_{j_a j_b} \langle \beta | j_a j_b \rangle_{j_e} \langle \gamma | j_a j_b \rangle_{j_e} \langle j_e j_e | 00 \rangle_{j_a j_b j_a j_b 0}$$
$$\times \langle j_a j_a | \bar{J}_d \bar{J}_d \rangle_{jjjj0} \langle j_b j_b | \bar{J}_d \bar{J}_d \rangle_{jjjj0}, \tag{25}$$

which gives $\|P|\sigma'\rangle\|^2 = 0.9\%$ and 1.0% for $j = 7/2$ and 9/2, respectively. Properties of χ' are displayed in Table I. The overlaps $|\langle \chi' | \psi \rangle|^2$ are seen to be considerably larger than $|\langle \chi | \psi \rangle|^2$, about 93% and 85% in the $1f_{7/2}$ and $1g_{9/2}$ shells, respectively. The overlaps calculated with the interactions ZR I and II are consistent with Van Isacker's [10] with the interpolated interaction. The overlap of χ' with the $s = 0$ state is also closer to the one found in Ref. [9] for the $n = 4$ antisymmetrized and normalized stretch-scheme ground state. Like χ the images of χ' by projection onto the $(s,t) = (4,0)$ and $(8,0)$ spaces have practically the same directions as those of ψ. Its Sp($2j + 1$) decomposition deviates, however, significantly from that of ψ, especially in the $1g_{9/2}$ shell, although not quite as much as that of χ. In particular χ' has like χ fairly large components of the irreps other than $(s,t) = (0,0)$ and $(4,0)$, which are almost absent in ψ.

The relative success of χ' in reproducing ψ might be understood from Qi's [25] observation that the ^{92}Pd ground state is well described in the $1g_{9/2}$ shell model as an antisymmetrized and normalized product of two ^{96}Gd ground states. Because the ^{96}Gd ground states have in this model a very large overlap with the corresponding antisymmetrized and normalized stretch-scheme ground state [3,9,10,22], one would then anticipate that an antisymmetrized and normalized product of copies of the latter has also a large overlap with the calculated ^{92}Pd ground states. That χ' is a better approximation to ψ in the $1f_{7/2}$ than in the $1g_{9/2}$ shell is in this understanding consistent with the finding in Ref. [9] that the same holds for $n = 4$ in the comparison of the calculated states and the antisymmetrized and normalized stretch-scheme ground state. The $n = 8$ overlaps are indeed fairly close to the squares of the $n = 4$ overlaps.

VI. SUMMARY

In the $1f_{7/2}$ or $1g_{9/2}$ shell model with effective interaction from the literature, I calculated the ground states of the system of four neutrons and four protons or four neutron holes and four proton holes, briefly four quasineutrons and four equivalent quasiprotons. This is the single-j-shell configuration of the nuclei ^{48}Cr, ^{88}Ru, and ^{92}Pd. The calculated states ψ were decomposed into the irreps of the symplectic group Sp($2j + 1$), which are characterized by the seniority s and the reduced isospin t. Here j is the single-nucleon angular momentum, equal in the shells considered to 7/2 and 9/2, respectively. The states ψ are found to be composed roughly of 75% of $(s,t) = (0,0)$ and 25% of $(s,t) = (4,0)$ independently of the shell and the interaction. This is similar to the case of two quasineutrons and two equivalent quasiprotons studied in Ref. [9], where the corresponding parts are about 80% and 20%. Other Sp($2j + 1$) irreps, which may occur for $n = 8$, make only very small contributions. This was understood from the exact vanishing of the matrix elements of any two-quasinucleon interaction between irreps with a difference in s larger than 4 and small matrix elements between $(s,t) = (0,0)$ and $(4,2)$ in the $1g_{9/2}$ shell. For $j = 7/2$ also all matrix elements involving $(s,t) = (6,1)$, and therefore these ground state components, vanish within the numeric accuracy.

The ground state in the stretch scheme of Danos and Gillet [2] was antisymmetrized in the quasineutrons and in the quasiprotons. The antisymmetrized state is found to make up 1.5% of the total for both j's. Following Qi *et al.* [3], I considered the state χ given by normalization of this antisymmetrized state. It is found to contain 20% and 41% of Sp($2j + 1$) irreps other than $(s,t) = (0,0)$ and $(4,0)$ for $j = 7/2$ and 9/2, respectively, much unlike ψ. For both j's the major part of this contribution, about 20% of the total in both cases, resides in $(s,t) = (6,1)$. Unlike ψ the irreps $(s,t) = (0,0)$ and $(s,t) = (4,0)$ contribute roughly equally to χ and $(s,t) = (4,0)$ makes for $j = 9/2$, the larger of these two contributions.

The overlaps of ψ with χ are found to be for ^{48}Cr similar to their overlaps with the $s = 0$ state. For ^{88}Ru and ^{92}Pd they are significantly less, so that the $s = 0$ state is there unambiguously a better approximation to ψ than χ. For ^{92}Pd and the interaction employed by Qi *et al.* in Ref. [3], their result for $|\langle\chi|\psi\rangle|^2$ is confirmed.

The Sp($2j + 1$) irreps $(s,t) = (4,0)$ and $(s,t) = (6,1)$ have for both j intersections of dimensions larger than one with the space with angular momentum and isospin zero. So does the irrep $(s,t) = (8,0)$ for $j = 9/2$. The images of ψ and χ by projection onto these multidimensional spaces are found to have in a good approximation the same directions for $(s,t) = (4,0)$ and $(8,0)$, whereas for $j = 9/2$ and $(s,t) = (6,1)$ the result in this respect varies with the interaction. As to $(s,t) = (4,0)$, this is similar to the case of two quasineutrons and two equivalent quasiprotons studied in Ref. [9]. Due to the aforesaid vanishing for $j = 7/2$ of the $(s,t) = (6,1)$ components of ψ, no comparison of directions is possible in this case.

The state χ' obtained by antisymmetrization and normalization of the product of two stretch-scheme ground states of the system of two quasineutrons and two equivalent quasiprotons was discussed briefly. It has much larger overlaps with ψ than χ but a deviating Sp($2j + 1$) decomposition. The large overlaps $|\langle\chi'|\psi\rangle|^2$ might be understood from Qi's observation [25] that the ^{92}Pd ground state is well described in the $1g_{9/2}$ shell model as an antisymmetrized and normalized product of two ^{96}Gd ground states.

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