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Separable potentials for the nucleon-nucleon $^3S_1$-$^3D_1$ channel*

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Separable potentials of rank one to five for the nucleon-nucleon $^3S_1$-$^3D_1$ channel are constructed using a method introduced by Ernst, Shakin, and Thaler. The potentials all have the same deuteron wave function as the Reid soft-core potential. In addition the phase shifts of the rank-four and -five potentials are quite close to the experimental phase shifts. A survey of previously published separable potentials of rank greater than one is made and it is concluded that most of these potentials have unacceptably poor deuteron properties. A number of errors that were made in the fitting of these potentials to the data are pointed out. A simplification of the Reid soft-core potential for the $^3S_1$-$^3D_1$ channel is also presented. The simplified potential consists entirely of Yukawa terms but has experimental properties that generally differ from the original Reid soft-core properties by less than the present experimental error.

NUCLEAR REACTIONS Separable potentials fitted to Reid $^3S_1$-$^3D_1$ phase shifts, deuteron wave function. Previous $^3S_1$-$^3D_1$ separable potentials surveyed, errors pointed out. Compared to UPE constructions. Simplified (local) Reid tensor force presented.

I. INTRODUCTION

Calculations of polarization phenomena in elastic nucleon-deuteron scattering have recently been quite successful in predicting much of the wealth of experimental data that is available in this area.[1-5] There are, however, certain areas (such as the nucleon polarization for energies above approximately 10 MeV) in which considerable improvement in the calculations is necessary. All of the calculations to date have used rank-one separable potentials and an obvious next step would be the use of more realistic potentials. In particular, the rank-one potentials of the Yamaguchi type are poor representations of the $^3S_1$-$^3D_1$ channel since they result in positive D-wave phase shifts[6] and excessively large values of the mixing parameter ϵ_1.[7] [One can show for an arbitrary rank-one potential that $\delta_2 > 0$ when $0 < \delta_0 < \frac{1}{2}\pi$ ($\delta_0 = \frac{1}{2}\pi$ for $E \sim 15$ MeV).]

This paper presents several separable potentials of rank one to five for the $^3S_1$-$^3D_1$ channel. They have been constructed using a technique described by Ernst, Shakin, and Thaler[8] (EST) that results in a separable potential that reproduces the half-off-shell T matrices of a given potential at selected energies. The EST method requires the use of a potential that adequately describes the $^3S_1$-$^3D_1$ channel and the Reid soft-core (SC) potential[9] was chosen for this purpose. However, it was found desirable for ease in the evaluation of certain integrals to simplify the structure of the subtracted one-pion-exchange potential that Reid used. The simplified potential, which consists entirely of Yukawa terms, is described in Sec. II. It is found that none of its predictions for experimentally measured properties (bound-state energy, quadrupole moment, phase shifts, etc.) differ significantly from those of the Reid SC potential.

Section III describes the potentials constructed using the EST method. Four potentials of rank one, two, four, and five are presented. All of the potentials have the same deuteron bound-state wave function as the Reid potential and the phase shifts for the rank-four and -five potentials are quite close to the experimental phase shifts. In making these separable potentials, the Reid potential was regarded simply as a necessary intermediary that defined the experimental data and the primary goal was to find separable potentials that had both acceptable deuteron properties and realistic phase shifts. For this reason, half-off-shell T matrices at negative energies different from the deuteron energy were not used as input to the potential even though this would have been possible.[10] Nevertheless, as will be seen in Sec. III, the negative energy properties of the potentials are reasonably close to those of the Reid potential.

A number of separable potentials of rank greater than one have already been published for this channel.[11-23] The properties of these potentials have been recalculated and the results are presented in Sec. IV. It was found that several errors were commonly made in previous fits and as a result the potentials are either very poor representations of the ${}^3S_1$-${}^3D_1$ channel or else require that the sign of their S-D coupling term be changed to get the quoted properties. In addition, in many of the fits the deuteron was largely ignored and as a result the potentials give quite bad values of the deuteron quadrupole moment (Q) and percent of D state (P_D). It is argued in Sec. IV that several multiparticle calculations show that the deuteron properties of a given potential strongly influence the results of multiparticle calculations made with that potential. Hence potentials whose deuteron properties differ significantly from experiment may give erroneous multiparticle results.

Section IV also contains a brief discussion of the Harms[24] and Afnan and Read[25,26] formulations of the unitary pole expansion (UPE). Harms's UPE appears to be very poor for this channel. Phase shifts for several of the Afnan and Read UPE's are presented and it is concluded that the separable potentials introduced here result in significantly better agreement with the two-body data.

II. SIMPLIFICATION OF REID SUBTRACTED ONE-PION EXCHANGE-POTENTIAL TERM

The Reid soft-core potentials consist[9] of a term that has the long-range behavior of the one-pion-exchange potential (OPEP) and phenomenologically fitted Yukawa potentials of shorter range than the OPEP. In the case of the tensor potential for the ${}^3S_1$-${}^3D_1$ channel, Reid uses

$$V_T(x) = V_{T,\mathrm{SOPEP}}(x) + V_{T,P}(x)\,, \tag{2.1}$$

where $x = \mu r$ with $\mu = 0.7\ \mathrm{fm}^{-1}$. The first term is the subtracted OPEP (SOPEP) and has the form

$$V_{T,\mathrm{SOPEP}}(x) = -h\left[\left(\frac{1}{x}+\frac{3}{x^2}+\frac{3}{x^3}\right)e^{-x} - \left(\frac{12}{x^2}+\frac{3}{x^3}\right)e^{-4x}\right], \tag{2.2}$$

where $h = 10.463$ MeV is determined by the pion-nucleon coupling constant. The second term is the phenomenological term ($V_{T,\mathrm{SOPEP}}$ is the same for both of Reid's soft-core potentials) and has the form

$$V_{T,P}(x) = a_2\frac{e^{-2x}}{x} + a_4\frac{e^{-4x}}{x} + a_6\frac{e^{-6x}}{x}, \tag{2.3}$$

where the a_i (some of which may be zero) were used in the fit to the ${}^3S_1$-${}^3D_1$ phase shifts. The large- and small-x behavior of the SOPEP term is

$$V_{T,\mathrm{SOPEP}}(x) \underset{x\to 0}{\sim} -\tfrac{47}{2}h/x\,, \tag{2.4}$$

$$V_{T,\mathrm{SOPEP}}(x) \underset{x\to \infty}{\sim} -h\left(\frac{1}{x}+\frac{3}{x^2}+\frac{3}{x^3}\right)e^{-x}\,. \tag{2.5}$$

Although the x^{-2} and x^{-3} terms in $V_{T,\mathrm{SOPEP}}$ cancel for small x, they make the evaluation of integrals involving V_T (such as the Fourier transform and expectation values of the potential between trial functions used in variational calculations) quite complicated. For this reason it would be desirable to be able to replace the SOPEP form given in Eq. (2.2) with a series such as that given in Eq. (2.3) (including a term a_1e^{-x}/x). The large-x behavior of such a series will differ appreciably from that given in Eq. (2.5) but, as will be seen, there is no significant degradation of the ability of the potential to predict the observed two-nucleon properties.

To find a function of the form given in Eq. (2.3) that will closely reproduce the properties of $V_{T,\mathrm{SOPEP}}$, we define

$$f(x) = \left(\frac{1}{x}+\frac{3}{x^2}+\frac{3}{x^3}\right)e^{-x} - \left(\frac{12}{x^2}+\frac{3}{x^3}\right)e^{-4x} \tag{2.6}$$

and

$$g(x) = (b_1e^{-x} + b_2e^{-2x} + b_4e^{-4x} + b_6e^{-6x})/x\,, \tag{2.7}$$

and we then choose the b_i so that $f(x) \approx g(x)$. In the present work it was required that $xg(x) \underset{x\to 0}{\sim} xf(x)$ so that one of the b_i was determined by setting

$$\sum_i b_i = \tfrac{47}{2}\,.$$

The other three parameters were found by requiring three integrals of the form

$$I_n(\alpha) \equiv \int_0^\infty dx\, x^n e^{-\alpha x}[f(x) - g(x)] \tag{2.8}$$

to be zero. This condition was used since integrals of this form will arise in variational calculations (such as the Raleigh-Ritz principle for the deuter-

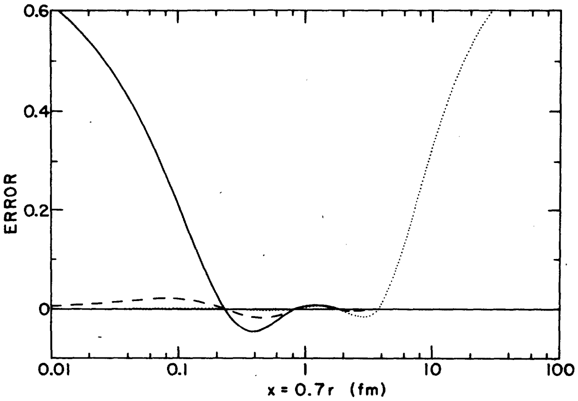

FIG. 1. Error in the approximate form of the SOPEP for V_T. Shown are $g-f$ (solid line), $(g-f)/f$ (dotted line), and $x(g-f)$ (dashed line) as a function of x. Equations (2.6) and (2.7) define f and g.

on binding energy).

A number of choices of α and the three values of n were made in an attempt to minimize $|f(x) - g(x)|$ in the region corresponding to $r < 5$ fm. Using $\alpha = 1$ and $n = 1, 3, 5$, the following values[27] of the b_i were found:

$$\begin{aligned} b_1 &= 1.7728\,, \\ b_2 &= 10.447\,, \\ b_4 &= 13.493\,, \\ b_6 &= -2.2124\,. \end{aligned} \tag{2.9}$$

Figure 1 shows the error in approximating $f(x)$ by the resulting $g(x)$. As can be seen, the relative error exceeds 1% only for $r > 3$ fm ($r = x/0.7$). Since the function values are quite small for $r > 3$ fm, the actual error in the function is still small and should not appreciably affect any physical observable. Similarly the rather big error for small x corresponds to a very small relative error and also will be given negligible weight in the integrals (involving r^2dr) that typically are used in two- and multi-particle calculations.

One can now construct a simplified Reid soft-core (SRSC) potential using

$$V_T(x) = (c_1e^{-x} + c_2e^{-2x} + c_4e^{-4x} + c_6e^{-6x})/x\,, \tag{2.10}$$

where

$$c_i = a_i - hb_i \tag{2.11}$$

and the a_i are the phenomenological coefficients found by Reid ($a_1 \equiv 0$) and the b_i are given in Eqs. (2.9). The resulting potential will then consist en-

TABLE I. Comparison of predictions of the SRSC potential (see text) with the values for the Reid SC potential. The SRSC values are given to the approximate accuracy with which they were found, while the SC values are given to as many places as were quoted by Reid. The SRSC values were computed using Reid's value of the nucleon mass (M=1/41.47 MeV^{-1} $\times$ fm^{-2}). A_D/A_S is the asymptotic ratio of the D- and S-wave components of the deuteron wave function. The phase shifts are Stapp nuclear-bar phase shifts, and here and elsewhere in this article the effective range parameters are defined by $k\cot\delta_0 = a^{-1} + \frac{1}{2}r_e k^2 + (Pr_e^{\,3})k^4 + \cdots$. The quoted experimental values for the phase shifts are the extremes (not including the assigned errors) of the values given in the Yale (Y-IV)$_{pp+np}$ fit (Ref. 26), both the constrained and unconstrained Livermore X fits (Ref. 27), and the 700-MeV Livermore IX fit (Ref. 28).

	SRSC value	Reid SC [a] value	Experimental value
Deuteron:			
E_B (MeV)	2.2298	2.224 60	2.224 61 ± 0.000 07 [b]
Q (fm^2)	0.281	0.2796	0.2875 ± 0.002 [c]
P_D (%)	6.49	6.470	2 → 8 [d]
A_D/A_S	0.0264	0.026 22	···
Effective range parameters:			
a (fm)	−5.39	−5.390	−5.423 ± 0.005 [e] −5.413 ± 0.005 [f]
r_e (fm)	1.73	1.720	1.765 ± 0.006 [e] 1.748 ± 0.006 [f] 1.735 [f]
Phase shifts (deg):			
24 MeV:			
δ_0	81.7	81.70	80 → 82
δ_2	−2.86	−2.86	−2.8 → −2.1
ϵ_1	1.83	1.83	−1.5 → +1.8
144 MeV:			
δ_0	29.8	29.85	29.0 → 29.5
δ_2	−15.9	−16.10	−14.8 → −15.4
ϵ_1	4.7	4.37	2.8 → 4.3
352 MeV:			
δ_0	−2.6	−2.41	−11 → +3
δ_2	−24.3	−24.69	−31.5 → −21
ϵ_1	9.2	8.61	3 → 14

[a] Reference 9.

[b] Reference 31.

[c] Reference 32.

[d] Reference 33.

[e] Reference 34.

[f] Reference 35.

TABLE II. The parameters used for the potentials in Eqs. (3.11)–(3.15). The E_i, l_i are the energies and angular momenta at which the potentials reproduce the half-off-shell K matrix of the SRSC potential. They do not have any particular relationship to the columns directly beneath them.

Designation	m	$3SD1A$	$3SD1B$		$3SD1C$	
E_i		−2.227	−2.227	200	−2.227	125
l_i		⋯	⋯	2	⋯	0
n		1	1	2	1	2
λ_n		−1.0	−1.0	1.0	−1.0	−1.0
$b_{n,0;m}$	1	−4.3208	−4.3734	2.1231×10^{1}	4.4258×10^{1}	−3.5537
	2	-7.3664×10^{2}	-7.5754×10^{2}	8.4050×10^{3}	9.9762×10^{2}	-8.0231×10^{2}
	3	-5.1796×10^{3}	-4.6083×10^{3}	-2.2853×10^{5}	-2.3439×10^{4}	-3.7581×10^{3}
	4	-3.4184×10^{4}	-3.9058×10^{4}	1.9536×10^{6}	-2.2279×10^{4}	-4.9248×10^{4}
	5	6.1120×10^{5}	6.3011×10^{5}	-7.6013×10^{6}	8.6645×10^{5}	6.8141×10^{5}
	6	-1.8643×10^{6}	-1.9015×10^{6}	1.4968×10^{7}	-2.6443×10^{6}	-2.0156×10^{6}
	7	2.1126×10^{6}	2.1484×10^{6}	-1.4443×10^{7}	2.9290×10^{6}	2.2625×10^{6}
	8	-8.2028×10^{5}	-8.3365×10^{5}	5.3947×10^{6}	-1.0978×10^{6}	-8.7560×10^{5}
$b_{n,2;m}$	1	1.2840×10^{1}	1.2862×10^{1}	−9.4241	-3.9543×10^{1}	1.1270×10^{1}
	2	1.0783×10^{3}	1.0631×10^{3}	6.0468×10^{3}	-3.2403×10^{3}	8.9827×10^{2}
	3	-3.7239×10^{3}	-3.4225×10^{3}	-1.2055×10^{5}	7.4255×10^{4}	-4.6341×10^{1}
	4	9.6414×10^{4}	9.3715×10^{4}	1.0762×10^{6}	-5.4675×10^{5}	6.7653×10^{4}
	5	-4.6721×10^{5}	-4.5651×10^{5}	-4.2624×10^{6}	2.2249×10^{6}	-3.5518×10^{5}
	6	8.7268×10^{5}	8.5198×10^{5}	8.2522×10^{6}	-4.8436×10^{6}	6.4575×10^{5}
	7	-7.4752×10^{5}	-7.2825×10^{5}	-7.6791×10^{6}	5.0781×10^{6}	-5.2143×10^{5}
	8	2.5535×10^{5}	2.4850×10^{5}	2.7315×10^{6}	-1.9936×10^{6}	1.6839×10^{5}

tirely of Yukawa terms. Table I compares some of the properties of this potential with the values found by Reid[9] for the SC potential and the experimental[28-35] values. It shows that the SRSC potential reproduces the Reid SC values with good accuracy.

III. FIT OF SEPARABLE POTENTIALS TO REID POTENTIAL

A. Description of method

Recently Ernst, Shakin, and Thaler[8] have described a method for constructing separable potentials that exactly reproduce the half-off-shell T matrix of an arbitrary given potential at selected energies. In a study of this construction for the case of an attractive S-wave square-well potential it was found that a potential of rank two would accurately reproduce the phase shifts and off-shell T matrices of the square-well potential over a large energy region.[36] In this section the generalization of the EST method for the case of a tensor potential will be written down and several separable potentials thusly derived from the Reid potential will be given.

Assume we have a potential V with a tensor component. Then the Lipmann-Schwinger equation for the scattering wave function with standing-wave boundary conditions is

$$\psi_{l,E}=\varphi_{l,k}+G_0(E)V\psi_{l,E}\,. \tag{3.1}$$

Here

$$\langle q'|G_0(E)|q\rangle=\delta(q'-q)q^{-2}P[E-q^2/(2\mu)]^{-1}, \tag{3.2}$$

where P denotes the principal value and μ is the reduced mass of the system. The asymptotic state is

$$\langle l',q'|\varphi_{l,k}\rangle=\delta_{l'l}\delta(q'-k)/k^2\,, \tag{3.3}$$

where $k=(2\mu E)^{1/2}$ is the scattering momentum. The half-off-shell K matrix is thus

$$\langle l',q'|K(E)|l,k\rangle=\langle\varphi_{l',q'}|\psi_{l,E}\rangle\,. \tag{3.4}$$

If we solve Eq. (3.1) for a given set of n energies E_i and corresponding angular momentum l_i, we can construct a rank-n potential as

$$\mathcal{V}=\sum_{i,j}V|\psi_{l_i,E_i}\rangle M_{ij}\langle\psi_{l_j,E_j}|V\,, \tag{3.5}$$

where

$$\sum_j M_{ij}\langle\psi_{l_j,E_j}|V|\psi_{l_k,E_k}\rangle=\delta_{ik}\,. \tag{3.6}$$

It is clear that since each ψ_{l_i,E_i} is an eigenstate of H_0+V (H_0 is the kinetic-energy operator), it is also an eigenstate of $H_0+\mathcal{V}$. Thus the half-off-shell K matrices $\langle l',q'|K(E_i)|l_i,(2\mu E_i)^{1/2}\rangle$ are the same for V and $\mathcal{V}$.

The construction in Eqs. (3.5) and (3.6) is not

TABLE II *(Continued)*

3*SD*1*C*		3*SD*1*D*					
125	400	−2.227	50	50	300	400	E_i
2	2	···	0	2	2	0	l_i
3	4	1	2	3	4	5	n
1.0	1.0	−1.0	−1.0	−1.0	1.0	1.0	λ_n
1.0736	-1.1198×10^{2}	-8.1050×10^{1}	−1.1188	−4.5656	−9.7073	-8.9669×10^{1}	$b_{n,0;m}$
-4.8865×10^{3}	8.3278×10^{3}	8.5116×10^{3}	-6.6479×10^{3}	-4.5477×10^{2}	-2.5115×10^{3}	1.0102×10^{4}	
9.4047×10^{4}	-7.0915×10^{4}	-1.3499×10^{5}	1.6261×10^{5}	-1.4047×10^{4}	3.9751×10^{4}	-2.3154×10^{5}	
-6.0674×10^{5}	-1.1703×10^{5}	7.3464×10^{5}	-1.2198×10^{6}	4.7029×10^{4}	-1.8771×10^{5}	2.0195×10^{6}	
1.8082×10^{6}	2.3552×10^{6}	-1.8861×10^{6}	4.1709×10^{6}	2.8798×10^{5}	3.4551×10^{5}	-8.1337×10^{6}	
-2.8654×10^{6}	-6.8638×10^{6}	2.8224×10^{6}	-7.3040×10^{6}	-1.2445×10^{6}	-3.7219×10^{5}	1.6196×10^{7}	
2.4172×10^{6}	7.5982×10^{6}	-2.6146×10^{6}	6.2714×10^{6}	1.5618×10^{6}	4.8497×10^{5}	-1.5419×10^{7}	
-8.6357×10^{5}	-2.8870×10^{6}	1.1328×10^{6}	-2.0489×10^{6}	-6.4285×10^{5}	-3.6268×10^{5}	5.5761×10^{6}	
-1.7832×10^{1}	-6.7286×10^{1}	1.2321×10^{2}	5.9229×10^{1}	2.8661	-2.9024×10^{1}	-1.7559×10^{2}	$b_{n,2;m}$
-4.0113×10^{3}	-2.7096×10^{3}	-3.8813×10^{3}	1.0492×10^{2}	1.1757×10^{3}	-3.1782×10^{3}	6.6298×10^{3}	
6.7715×10^{4}	1.0159×10^{5}	9.6396×10^{3}	3.2374×10^{4}	-6.0661×10^{3}	3.2000×10^{4}	-7.7784×10^{4}	
-5.4478×10^{5}	-9.5617×10^{5}	1.9965×10^{5}	-5.3017×10^{5}	1.2521×10^{5}	-1.3030×10^{5}	4.7809×10^{5}	
1.9390×10^{6}	4.4018×10^{6}	-1.2593×10^{6}	2.4685×10^{6}	-5.9725×10^{5}	9.0334×10^{4}	-1.4809×10^{6}	
-3.4505×10^{6}	-9.9337×10^{6}	3.0881×10^{6}	-4.9743×10^{6}	1.1272×10^{6}	3.8100×10^{5}	2.2224×10^{6}	
3.1186×10^{6}	1.0347×10^{7}	-3.5814×10^{6}	4.5091×10^{6}	-9.5692×10^{5}	-5.6172×10^{5}	-1.3802×10^{6}	
-1.1631×10^{6}	-3.9356×10^{6}	1.6212×10^{6}	-1.4619×10^{6}	3.0694×10^{5}	1.3777×10^{5}	1.7069×10^{5}	

possible if the matrix

$$\overline{V}\equiv\langle\psi_{l_j,E_j}|V|\psi_{l_i,E_i}\rangle$$

is singular. There is no guarantee that $\overline{V}$ will be nonsingular[37] and one may construct local potentials V (with no pathologies) such that for a certain energy E $\langle\psi_E|V|\psi_E\rangle=0$. This means that one cannot use the above prescription to construct a rank-one potential that reproduces ψ_E. In general one must make sure that $\overline{V}$ is not singular and avoid sets of energies $\{E_i\}$ in the vicinity of singularities of $\overline{V}$. In the present work with the ${}^3S_1$-${}^3D_1$ Reid potential, the determinant of $\overline{V}$ for a given rank fit and a given set of l_i was never found to change sign as the E_i were varied in the range 1 to 1000 MeV.

In the above discussion no use was made of the fact that l labels the angular momentum of the initial state and Eq. (3.5) may be regarded as the rather obvious extension of the EST construction to the case of multichannel scattering. Note that if there are N channels then to get all the $K_{l'l}(E_i)$ correct at n energies requires a rank nN separable potential.

For the ${}^3S_1$-${}^3D_1$ channel, l does label the orbital angular momentum and can have the two values 0 or 2. Thus to reproduce the three half-off-shell K (or T) matrices K_{00}, $K_{02}=K_{20}$, and K_{22} at a given energy would require a rank-two potential. In practice this may not be necessary and in some of the fits the energies and angular momenta were staggered so that fits were made at $(E_1, l_1=0)$, $(E_2, l_2=2)$, $(E_3, l_3=0)$, ... It is of course possible to include the deuteron bound-state energy as one of the energies E_i. In this case there is no inhomogeneous term in Eq. (3.1) and the label l_i does not apply. Including the deuteron in the set of energies to be fitted ensures that the deuteron wave function (both the S- and D-wave components) and binding energy will be reproduced. If one uses just the deuteron wave function as the input then the resulting rank-one potential is the same as Harms's UPA[24,38] approximation to the given potential.

The significant quantity in Eqs. (3.5) and (3.6) is $V|\psi\rangle$ which suggests that the Schwartz formulation[39] of the Schwinger variational principle[40] is a natural method for finding the required input from V. In this method we define

$$|U_{l,E}\rangle=V|\psi_{l,E}\rangle \tag{3.7}$$

and observe that

$$\begin{aligned}K_{l'l}(E)\equiv{}&\langle\varphi_{l',k}|U_{l,E}\rangle+\langle U_{l'E}|\varphi_{l,k}\rangle\\&+\langle U_{l',E}|G_0(E)-V^{-1}|U_{l,E}\rangle\end{aligned} \tag{3.8}$$

is a stationary (in both $\langle U_{l',E}|$ and $|U_{l,E}\rangle$) expression for the on-shell K matrix. If we expand U in a series of trial functions

$$|U_{l,E}\rangle=\sum_n a_{l,E;n}|W_n\rangle \tag{3.9}$$

and require that Eq. (3.8) be stationary under individual variations of the a's, then we get for each

(l, E) a set of linear equations which determine the $a_{l,E;n}$. Once these have been found it is straightforward to construct the separable potential given in Eq. (3.5).

Equation (3.7) and the occurrence of the term $\langle U | V^{-1} | U \rangle$ in Eq. (3.8) place limitations on the types of trial functions W_n that may be used in Eq. (3.9) since one must be able to compute $\langle U | V^{-1} | U \rangle$ even if V^{-1} is singular. For this reason it is useful to write

$$|W_n\rangle = V|W_n'\rangle , \tag{3.10}$$

where the W_n' are chosen to be square-integrable functions. Note that this should not be viewed as an expansion of $|\psi\rangle$ as $\sum a_n |W_n'\rangle$ since the W_n' do not properly represent the asymptotic form of ψ.

B. Potentials

The method described in Sec. III A was used to construct rank-one through rank-five potentials for the ${}^3S_1$-${}^3D_1$ channel. The SRSC potential of Sec. II was used as the input potential (V). In all cases the deuteron bound-state energy was used as one of the E_i in Eq. (3.5) (attempts in which the bound state was not used did not reproduce the deuteron with sufficient accuracy to be acceptable). The other E_i's (and l_i's) were chosen and varied by hand in an attempt to get a good fit to the phase shifts predicted by the SRSC. With the exception of the deuteron properties the off-shell predictions of the separable potentials were not considered in the choice of the E_i and l_i. Considerable difficulty was encountered in choosing the E_i and l_i for the higher rank potentials since the phase-shift curves tended to develop wiggles between the fitting energies. This is a difficulty of the EST construction that does not seem to be present in systematic constructions such as the UPE's[24-26] but the good fits that are possible make it worth contending with.

The trial functions of Eq. (3.10) are natural for the process of constructing the separable potentials and searching for the best E_i. However, they are rather unwieldy for subsequent uses of the separable potentials, particularly if the separable potential is to be evaluated in momentum space. For this reason, the separable potentials have been refitted in terms of form factors that have simple representations in both coordinate and momentum space. In addition the coupling constant matrix [M in Eq. (3.5)] has been diagonalized. The resulting final form of the separable potentials is

$$\mathcal{V} = \sum_n |v_n\rangle \lambda_n \langle v_n| \tag{3.11}$$

$$\langle l, q | v_n\rangle = \sum_m b_{n,l;m} w_{l,m}(q) \tag{3.12}$$

and

$$\begin{aligned} w_{0,m}(q) &= (q^2 + \alpha_m{}^2)^{-2} \\ w_{2,m}(q) &= q^2 (q^2 + \alpha_m{}^2)^{-3} . \end{aligned} \tag{3.13}$$

The Fourier transforms of Eq. (3.13) are (the norm-preserving Fourier transform is used)

$$\begin{aligned} w_{0,m}(r) &= -(\tfrac{1}{8}\pi)^{1/2} \frac{e^{-\alpha_m r}}{\alpha_m} , \\ w_{2,m}(r) &= -\tfrac{1}{8}(\tfrac{1}{2}\pi)^{1/2} r e^{-\alpha_m r} . \end{aligned} \tag{3.14}$$

In Eqs. (3.11)–(3.14) there is no reference to the SRSC potential (V) that was used to construct $\mathcal{V}$. Thus the user of these potentials can regard them simply as separable potentials that in some way were fitted to the ${}^3S_1$-${}^3D_1$ data.

It was found that eight terms in the sum (3.12) reproduced the form factors with sufficient accuracy. The α's used in Eqs. (3.13) and (3.14) were chosen to be

$$\alpha_m = \tfrac{3}{2} m \ (\mathrm{fm}^{-1}) , \tag{3.15}$$

where the units are indicated to be inverse femtometers. Table II contains the resulting coupling constants (λ_i) and expansion coefficients ($b_{n,l;m}$). Note that the b's have been normalized such that

$$|\lambda_i| = 1 \ \mathrm{MeV\,fm^{-5}} \tag{3.16}$$

and the b's are dimensionless. Table II does not contain a rank-three potential since one could not be found that significantly improved upon the results of the rank-two potential.

The bound-state properties and effective range parameters for these potentials are given in the first four lines of Table III. These values and all other values computed with these potentials were obtained with a nucleon mass of

$$M = 0.024\,112\,4 \ \mathrm{MeV^{-1} fm^{-2}} \tag{3.17}$$

which approximately corresponds to using the average of the neutron and proton mass (938.903 MeV/c^2) and to $\hbar c = 197.329$ MeV fm. The small deviations in the deuteron properties from the values obtained with the SRSC potential in Sec. II are principally due to the reexpression of the form factors in Eq. (3.12). Figures 2–4 compare the phase shifts of these potentials with the SRSC phases and Figs. 5–7 compare the off-shell scattering amplitudes $\langle p | F(E) | p\rangle$ for negative E of these potentials with the SRSC values. The off-shell scattering amplitude is defined as

$$\begin{aligned} \langle p | F(E) | p\rangle &= -\pi\mu \langle p | T(E) | p\rangle \\ &= -\tfrac{1}{2} i [\langle p | S(E) | p\rangle - 1] , \end{aligned} \tag{3.18}$$

TABLE III. Comparison of separable potentials for the $^3S_1$-$^3D_1$ channel. The column labeled "Sign V_{02}" shows when it was necessary to change the sign of the off-diagonal terms of previously published potentials to get the quoted results and is further explained in the text. The last line gives the SRSC values for comparison.

					Deuteron				Effective range			100 MeV		
Author	Ref. and Designation		Rank	Sign V_{02} [a]	E_B (MeV)	Q (fm^2)	P_D (%)	A_D/A_S	a (fm)	r_e (fm)	P	δ_0	δ_2	ϵ_1
		3*SD*1*A*	1		2.2235	0.2808	6.49	0.0264	−5.387	1.724	0.0118	39.1	+10.4	21.2
	This article	3*SD*1*B*	2		2.2269	0.2802	6.47	0.0264	−5.394	1.752	0.0376	34.1	−9.8	10.2
		3*SD*1*C*	4		2.2234	0.2804	6.49	0.0263	−5.389	1.731	0.0225	41.5	−11.6	5.5
		3*SD*1*D*	5		2.2241	0.2818	6.50	0.0266	−5.394	1.742	0.0319	41.6	−11.5	4.9
Afnan	11	*PD* 3.0	2	OK	2.224	0.279	3.00	0.0312	−5.399	1.732	−0.0043	46.2	−13.2	14.4
Afnan	11	*PD* 7.0	2	OK	2.223	0.280	6.98	0.0252	−5.400	1.729	−0.0071	49.0	−14.3	15.3
Faustmann	12		3	OK	2.224	0.338 [b]	3.81	0.0450	−5.261	1.547	−0.0285 [c]	45.7 [c]	−10.9 [c]	17.1 [c]
Hamman	13		4	OK	6.014 [c]	0.057 [c]	0.29 [c]	0.0190 [c]	−3.886 [b]	1.764	0.0392 [c]	41.4	−12.1 [b]	4.1
Kahana	14	a	4	Ch	2.208 [c]	0.092 [c]	0.32	0.0092 [c]	−5.461	1.809	0.0323 [c]	40.1	−11.7	5.2
Kahana	14	b	3	Ch	2.017 [c]	0.308 [c]	2.81	0.0327 [c]	−5.597 [b]	1.726	−0.0063 [c]	44.1 [c]	−15.4 [c]	14.6 [c]
Mitra	15		2	OK	2.233	0.274	3.08	0.0302 [c]	−5.382	1.719	−0.0108 [c]	43.3 [c]	+29.0 [c]	17.2 [c]
Mehrotra	16	$C+H+T$	2	OK	2.224	0.232	2.50	0.0237 [c]	−5.400 [b]	1.747 [b]	0.0236 [c]	37.8 [c]	+4.2 [c]	8.3 [c]
Mongan	17	I	2	N Ch	2.225	0.278	0.71	0.0692 [c]	−5.654	2.057	0.0009 [c]	36.0	−13.1	0.5 [b]
Mongan	17	II	2	N Ch	2.225	0.276	1.12	0.0481 [c]	−5.383	1.723	0.0066 [c]	39.3	−11.9	3.6 [b]
Mongan	17	IV	2	N Ch	2.228	0.279	1.40	0.0406 [c]	−5.376	1.717	0.0108 [c]	39.0	−12.1	5.4 [b]
Naqvi	18	$\beta=\gamma$	3	OK	2.225	0.275	3.22	0.0304 [c]	−5.388	1.717	0.0083 [c]	45.8 [c]	+7.2 [c]	17.2 [c]
Pradham	19	V_2	2	OK	2.228	0.283	3.77	0.0289 [c]	−5.421	1.778	0.0176 [c]	35.3 [c]	+6.9 [c]	18.0 [c]
Schaposnik	20		2	Ch	2.218 [c]	0.023	4.77	0.0033 [c]	−5.403 [c]	1.747 [c]	0.0346 [c]	40.6	−11.8	2.5
Sirohi	21		2	OK	2.379 [b]	0.107 [b]	0.61 [c]	0.0084 [c]	−5.378 [c]	1.861 [c]	−0.0102 [c]	47.7	−14.7	7.1
Strobel	22		2	Ch	2.213 [c]	0.038 [b]	0.09 [c]	0.0031 [c]	−5.369	1.699	0.0410 [c]	41.5	−7.5	3.5
Tabakin	23		2	OK	1.183 [b]	0.128 [b]	0.74 [b]	0.0040 [c]	−7.083	1.950	0.0215 [c]	39.8	−18.9	6.8
	This article	3*SD*1*E*	2		2.223	0.288	7.00	0.0437	−5.319	1.891	0.238	35.0	−16.2	4.3
	9				2.225	0.280	6.47	0.0262	−5.390	1.720	0.027	41.6	−12.6	3.5

[a] Ch indicates "changed," N Ch indicates "not changed."

[b] Differs from author.

[c] Not given by author.

where the S matrix is unitary on-shell.

As can be seen in Figs. 2–4, the phase shifts of the potentials become significantly better with each increase in rank. The rank-five 3*SD*1*D* potential has phase shifts that differ by less than 3° from those of the Reid potential for $E_{\rm lab} < 500$ MeV and produces a δ_0 phase shift that follows the Reid δ_0 significantly past 500 MeV. However, the 3*SD*1*D* phase shifts show a number of wiggles in the region 250 MeV $< E_{\rm lab} <$ 350 MeV. In addition, Figs. 5–7 show that the negative-energy T matrices for the 3*SD*1*C* potential are much closer to the Reid T matrices than are the 3*SD*1*D* values. For these reasons the rank-four 3*SD*1*C* potential may give more reliable results than the 3*SD*1*D* potential.

IV. COMPARISON WITH OTHER POTENTIALS

This section will compare the properties of the potentials described in Sec. III with those of previously published separable potentials of rank greater than one for the $^3S_1$-$^3D_1$ channel. In Sec. IV A we will consider separable potentials that were phenomenologically fitted to the $^3S_1$-$^3D_1$ data, while in Sec. IV B comparison will be made with the UPE's.

A. Survey of separable potentials for the ^{3}S$_1$-^{3}D$_1$ channel

In this part we survey a number of separable potentials of rank two or more that have been fitted to the $^3S_1$-$^3D_1$ channel.[11-23] Table III summarizes the present author's calculations of the properties of these potentials and may be regarded as an extension of a similar table published by Clement, Serduke, and Afnan[41] (CSA). The present results are in agreement with those of CSA for those potentials that appear in both tables.[42]

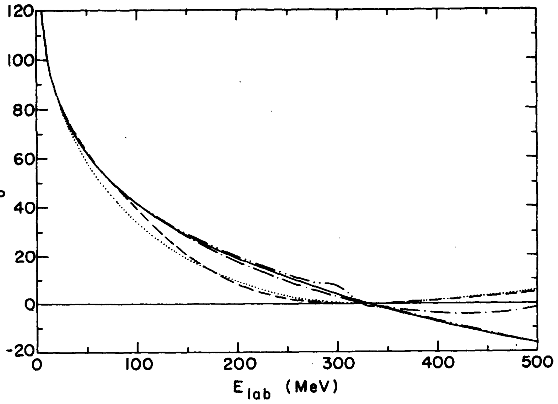

FIG. 2. Comparison of the S-wave phase shifts for the potentials in Table II with the SRSC phase shifts. The solid line is the prediction of the SRSC potential while the dashed line is 3*SD*1*A*, the dotted line is 3*SD*1*B*, the dash-dot line is 3*SD*1*C* and the dash-dot-dot line is 3*SD*1*D*.

In Table III values have been marked as different from the published values only for cases where the difference appears to be too large to be ascribable to the fact that the original authors may not have quoted their parameters to sufficient accuracy. Some general remarks on Table III seem in order.

A number of authors[21-23] have used the Biedenharn-Blatt approximation[43] for the deuteron quadrupole moment. As has been pointed out by Signell,[44] this is not a good approximation (it is typically in error by 50%) and should be avoided. One of the principle advantages of separable potentials is that they greatly simplify calculations; in particular it is very easy to get an analytic form for the two-body bound-state wave function in momentum space and to then use the Yamaguchi[45] formula for the quadrupole moment to compute Q to arbitrary accuracy.

The standard procedure[45,46] for fitting rank-one potentials to the $^3S_1$-$^3D_1$ channel is to fit exactly the deuteron properties E_B, Q, P_D, and the scattering length a. This results in a potential that fairly well reproduces the experimentally known deuteron properties but which gives completely wrong δ_2 and ϵ_1 phase shifts.[6] Exactly the opposite policy seems to have been followed for the rank-two potentials.[47] In one case[13] the authors did not report any bound-state properties of their potential while in most cases no attempt was made to constrain the potential to have reasonable values for Q and P_D. The result is that most of the previously published potentials listed in Table III have very poor deuteron properties.

This neglect of the deuteron properties of the separable potentials seems ill advised. Separable potentials were originally advocated for three-body

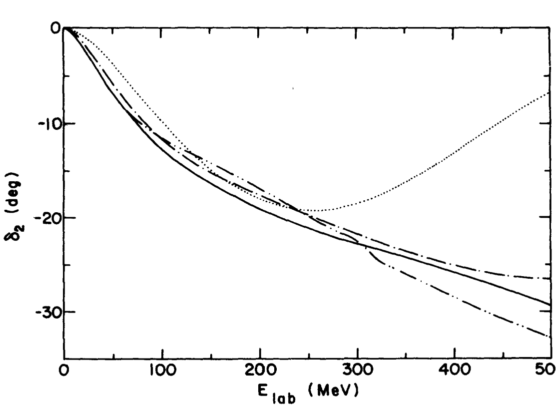

FIG. 3. D-wave phase shifts for the potentials in Table II. The lines are the same as for Fig. 2 except 3*SD*1*A* is not shown since it gives $\delta_2 > 0$.

scattering calculations on the basis of pole-dominance arguments[48] or the quasiparticle method.[49] Both of these approaches result in replacing the two-nucleon T matrix with its factorized form valid in the vicinity of a pole of the T matrix. In the case of the channel containing the deuteron, this factorized form is, of course, defined by the deuteron wave function and if there is any merit to these arguments, the deuteron properties of a given potential will strongly influence three-body calculations made with that potential. This appears to be substantiated by the success[1,3-5] of N-d calculations using rank-one $^3S_1$-$^3D_1$ interactions in predicting the deuteron tensor polarizations which are strongly dependent upon the presence of a tensor force in the N-N potential. (Aarons and Sloan[1] present a reasonable argument as to why these particular polarizations should depend on the tensor force.) Since both the δ_2 and ϵ_1 phase shifts of the rank-one potentials used are very poor,[6] it must be concluded that it is the D-wave part of the deuteron wave function that is producing the good tensor polarizations.

Afnan, Clement, and Serduke[11] have made nuclear matter and ^{4}He Brueckner-Goldstone calculations using several rank-two $^3S_1$-$^3D_1$ potentials in which the D-state probability of the deuteron was varied while the remaining deuteron and the scattering properties of the potentials remained relatively constant (the fifth and sixth entries in Table III are examples of the six potentials they considered). They found that the binding energy per particle in both cases was quite dependent on the D-state probability. The importance of the deuteron properties in nuclear matter calculations is further emphasized by their observation that the saturation curves for several local potentials (in particular the Reid soft- and hard-core potentials[9] and the Bryan-Scott[50] potential) are essentially the same as the curve for the separable potential of the corresponding D-state probability.

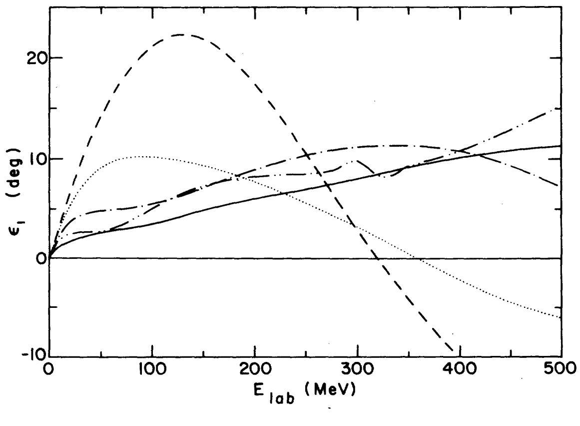

FIG. 4. The ϵ_1 phase shifts for the potentials in Table II. The lines are as described for Fig. 2.

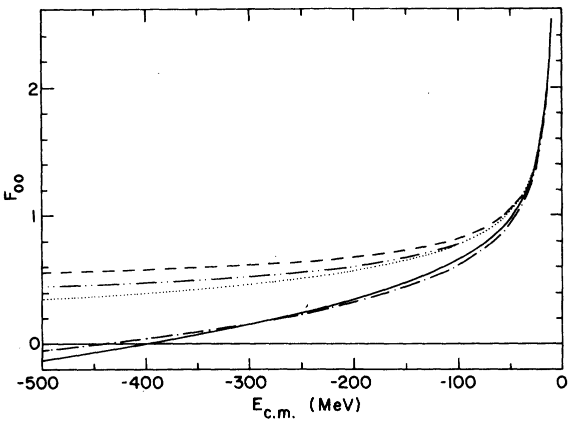

FIG. 5. The off-shell S-wave scattering amplitude at negative energies for the potentials in Table II. Shown are $\langle p|F(E)|p\rangle$ for $p=1$ fm^{-1} as a function of E. See Eq. (3.18) for the normalization of $\boldsymbol{F}$. The lines are the same as in Fig. 2.

These examples seem to cast considerable doubt on the significance of multiparticle calculations made with tensor potentials that contain a poor representation of the deuteron and indicate that the experimentally known properties of the deuteron should be included with rather high weight in any future fits of separable interactions to the $^3S_1$-$^3D_1$ channel.

Besides the potentials described in Sec. III, the author has attempted to fit the $^3S_1$-$^3D_1$ channel using rank-two potentials with form factors

$$\langle q|V\rangle = q^l/(q^2+\gamma^2)^m\,, \tag{4.1}$$

where various integers were used for m. In these fits three of the potential parameters were constrained to give good values of E_B, P_D, and Q

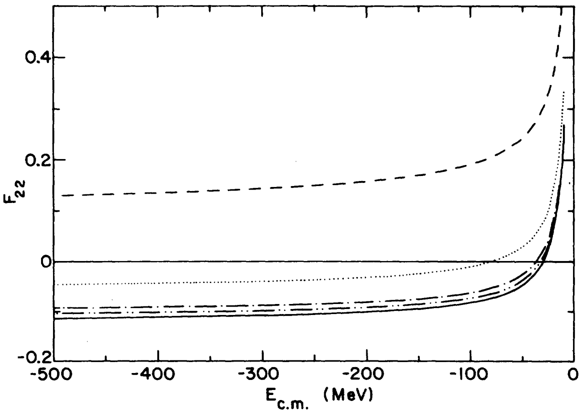

FIG. 6. The off-shell D-wave scattering amplitude at negative energies for the potentials in Table II. The lines are described in Fig. 2.

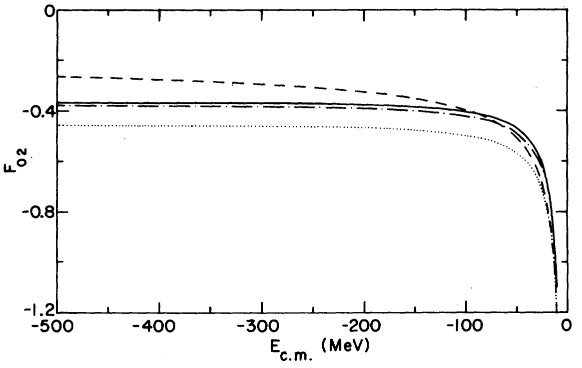

FIG. 7. The off-shell S- to D-wave scattering amplitude at negative energies for the potentials in Table II. The lines are as described for Fig. 2. The curve for $3SD1D$ is indistinguishable from that of the local SRSC potential.

while extensive searches were made on the remaining parameters in an attempt to fit the phase shifts. Potential $3SD1E$ in Table III is a typical result of such a search. The potential is

$$V=\sum_{m,i} |v_i^m\rangle \lambda_i \langle v_i^m| , \tag{4.2}$$

with

$$|v_i^m\rangle = |u_{0i}\mathcal{Y}_{101}^m\rangle + r_i |u_{2i}\mathcal{Y}_{121}^m\rangle \tag{4.3}$$

and

$$u_{li} = q^l/(q^2+\gamma_{li}^2)^{(l+2)/2} . \tag{4.4}$$

The parameters are

$$\begin{aligned} &\lambda_1=-5.30429, \quad \lambda_2=24.3648, \\ &r_1=-335.469, \quad r_2=2.55471, \\ &\gamma_{01}=0.836705, \quad \gamma_{02}=0.562653, \\ &\gamma_{21}=11.52618, \quad \gamma_{22}=1.02659. \end{aligned} \tag{4.5}$$

Figures 8–10 compare the phase shifts of this and several of the better potentials in Table III with the SRSC phase shifts. As can be seen, despite the fact that this potential was constrained to give reasonable deuteron properties, its phase shifts are comparable to those of the other potentials.

To compute the phase shifts for a tensor potential it is customary to start with the Lippmann-Schwinger equation for the half-off-shell T matrix:

$$T(E)|l,k\rangle = V|l,k\rangle + VG_0^+(E)T(E)|l,k\rangle , \tag{4.6}$$

where $|l,k\rangle$ is the initial state and again $k=(2\mu E)^{1/2}$ is the scattering momentum. Note that a $+i\epsilon$ prescription is now being used in G_0. Equation (4.6) may be solved to find the on-shell T matrix and then the scattering matrix is found as

$$S_{l'l}(E) = 1 - i2\pi\mu k T_{l'l}(E) , \tag{4.7}$$

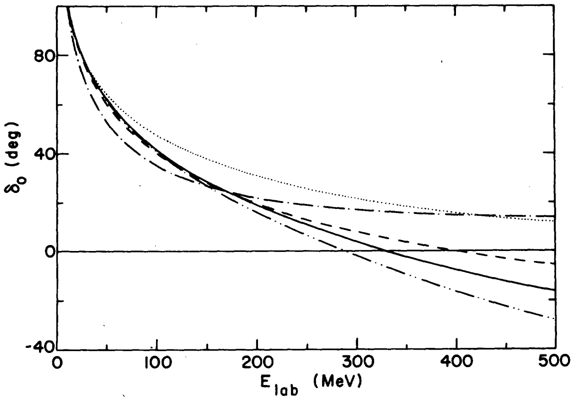

FIG. 8. Comparison of the S-wave phase shifts of some of the potentials in Table III with the Reid phase shift. The potentials shown are identified in Table III as: Ref. 21 (dotted line); Ref. 14 (dashed line); Ref. 20 (dash-dot-dot line); and the present article, $3SD1E$ (dash-dot). The solid line is the prediction of the SRSC potential.

where

$$T_{l'l}(E) = \langle l',k|T(E)|l,k\rangle . \tag{4.8}$$

[The particular factor that relates T to S in Eq. (4.7) will depend on the normalization of the initial states $|l,k\rangle$ and is not germane to the following discussion. Here we are only concerned with the absence of a factor $i^{l'-l}$ in Eq. (4.7).]

Using the Lippmann-Schwinger equation for the scattering wave function one can write the wave function in terms of the half-off-shell T matrix as

$$|\psi_{l,E}\rangle = |l,k\rangle + G_0^+(E)|l',k'\rangle\langle l',k'|T(E)|l,k\rangle , \tag{4.9}$$

where a complete sum over the intermediate states is understood. If the asymptotic form of the free Green's function is used in Eq. (4.9) it is straightforward to find the asymptotic form of the scatter-

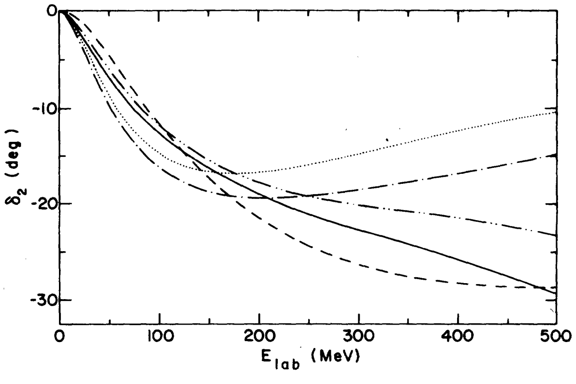

FIG. 9. D-wave phase shifts for some potentials in Table III. The lines are as described for Fig. 8.

ing wave function

$$\langle l, r | \psi_{l,E}\rangle \underset{r\to\infty}{\sim} i(2\pi)^{-1/2}[\delta_{l'l}(-)^l e^{-ikr} - S_{l'l}(E)e^{ikr}]/kr\,, \tag{4.10}$$

where Eqs. (4.7) and (4.8) have been used.

The phase shifts of both the Livermore[29] and Yale groups[28] are also defined in terms of an S matrix. However these groups use the S matrix introduced by Blatt and Biedenharn[51] which is defined by the relation

$$\langle l', r | \psi_{l,E}\rangle \underset{r\to\infty}{\propto} \delta_{l'l}(-)^l e^{-ikr} - i^{l'-l}\bar{S}_{l'l}e^{ikr}\,. \tag{4.11}$$

[This is evident from Eqs. (3.2)–(3.4) and (3.10) of Ref. 7 and from Eqs. (II-53) and (II-60) of Ref. 52.] Comparing Eqs. (4.10) and (4.11) we see that the phase shifts have been defined in terms of an S matrix ($\bar{S}$) whose off-diagonal elements are minus the off-diagonal elements of the S matrix that is customarily computed.

A number of authors[14,17,20,22] have overlooked this fact and as a result have fitted their potentials to the wrong sign of the mixing parameter ϵ_1. In these cases the correct value of ϵ_1 may be obtained by changing the sign of the off-diagonal potential (V_{02}) and column 3 of Table III shows when it was necessary to do this. Changing the sign of V_{02} will not change the values of δ_0, δ_2, E_B, or P_D but it will change both the sign and magnitude of the quadrupole moment Q. If Q had been found from the Biedenharn-Blatt approximation, this change will also be for the better but if Q was directly computed from the deuteron wave function, then the resulting Q will be quite bad (typically -0.3 fm^2). This is the case for the Mongan[17] potentials and for this reason column 3 indicates that the sign of V_{02} was not changed even though Mongan used the wrong sign of ϵ_1. (As has been emphasized by Breit and co-workers,[52,53] the Biedenharn-Blatt approximation does generally give the correct sign for Q. Hence one should be suspicious of a calculation that purports to give $Q > 0$ and $\epsilon_1 < 0$ for low energies.)

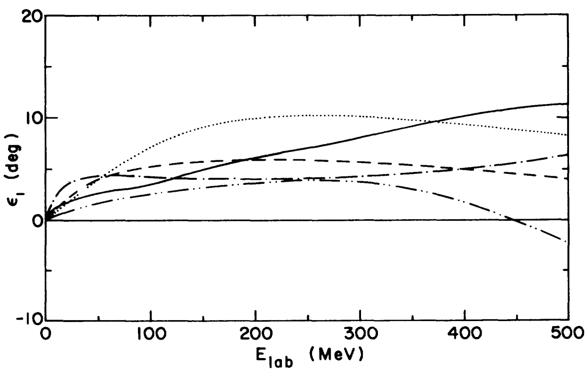

FIG. 10. The ϵ_1 phase shift for some of the potentials in Table III. The lines are as described for Fig. 8.

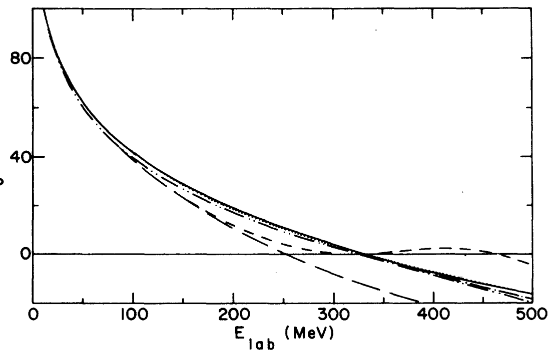

FIG. 11. Comparison of the S-wave phase shifts from the Afnan-Read form of the UPE with the SRSC phase shifts (solid line). The UPE potentials shown are: 1A (dashed line); 1A1R (long dashed line); 2A1R (dotted line); 2A2R (dash-dot line); and 3A2R (dash-dot-dot line).

Comparison of Figs. 2–4 with Figs. 8–10 shows that the potentials of Refs. 11–23 are generally comparable in terms of their phase shifts to the rank-two 3SD1B fit to the Reid potential but that the rank-four fit (3SD1C) seems to be clearly superior to all of them. If in addition one takes into account the deuteron properties listed in Table III then most of the previously published potentials listed in Table III are clearly inferior to the 3SD1B potential. The 3SD1E potential does, however, have good deuteron properties and has a significantly better ϵ_1 phase shift than the rank-two 3SD1B potential. However, one might prefer the latter potential since its off-shell behavior and deuteron wave function will be closer to that of the Reid potential.

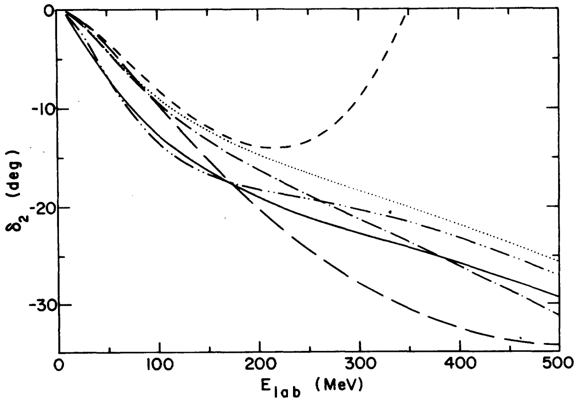

FIG. 12. D-wave phase shifts for the Afnan-Read UPE. The lines are as described for Fig. 11.

B. Comparison with the UPE potentials

A few years ago Harms[24] provided a systematic construction of separable potentials of increasing rank whose properties converged upon those of a given (local) potential. This unitary pole expansion (UPE) has met with considerable success in the construction of separable potentials to reproduce the off-shell effects of the Reid potential in the $^1S_0$ channel.[24] The first term of Harms's UPE is known as the unitary pole approximation (UPA) and for the $^3S_1$-$^3D_1$ channel with the Reid potential is the same as our 3*SD*1*A*. That is, it is a rank-one potential whose form factor is the deuteron wave function. It thus has the same poor phase shifts as any other rank-one potential in this channel (positive δ_2 and very large ϵ_1). The disappointing feature of Harms's UPE is that it requires a potential of quite high rank to correct the bad phase shifts associated with the first term in the expansion. Some calculations made by the author show that the rank-10 Harms's UPE consisting of three attractive terms and seven repulsive terms (which is designated as 3*A*7*R*) has $\delta_2 > 0$ and has ϵ_1 values that are typically a factor of 4 too large. Afnan and Read[26] have reported that a 12-term Harms's UPE potential still gives $\delta_2 > 0$ and point out[25] that such positive *D*-wave phase shifts will result in considerable over attraction in nuclear matter calculations. For these reasons it appears that the UPE as originally proposed by Harms does not seem suitable for the $^3S_1$-$^3D_1$ channel.

Afnan and Read[25,26] have proposed a second type of UPE for coupled channels. In their construction the eigenfunctions used by Harms are projected into *S*- and *D*-wave components and these partial wave components are then individually coupled to give the separable potential. (As opposed to the Harms's UPE, in the Afnan and Read UPE the coupling constant matrix is a function of the angular momenta appearing on the left and right.) The result is a potential with considerably better phase shifts than the Harms's UPE but which has twice the rank of the corresponding Harms potential. [Contrary to a statement in Ref. 25, the *nA mR* potential in the Afnan and Read construction is of rank $2(n+m)$.] The Afnan-Read construction also produces a deuteron wave function that is the same as that obtained with the Reid potential.

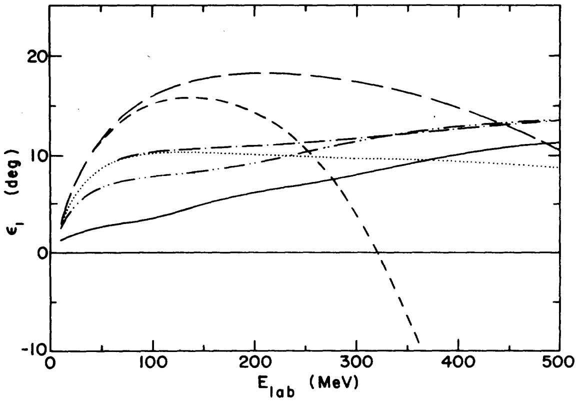

FIG. 13. The ϵ_1 mixing parameter for the Afnan-Read UPE. The lines are as described for Fig. 11.

Figures 11–13 show the phase shifts for Afnan-Read UPE's of rank 2–10. Comparing these with Figs. 2–4 demonstrates that the potentials introduced here give considerably better phase shifts for a given rank. This is not unexpected since the potentials in Table II have been directly fitted to the positive-energy features of the Reid potential. What is surprising is that the rank-four 3*SD*1*C* potential has negative-energy properties (Figs. 5–7) that are somewhat closer to those of the Reid potential than the Afnan-Read 1*A*1*R* and 2*A*1*R* potentials even though the latter is a rank-six potential. On the other hand, the rank-five 3*SD*1*D* potential has negative-energy properties that are considerably different from the Reid potential. In the Introduction (Sec. I) it was stressed that the Reid potential has been used principally as a method of getting at the experimentally known features of the $^3S_1$-$^3D_1$ channel. Nevertheless, using the methods of Ref. 10 it should be possible to "improve" the negative-energy properties of 3*SD*1*D* by adding one or more ranks to it that are determined by the negative-energy properties of the Reid potential.

V. CONCLUSIONS

The separable potentials 3*SD*1*B* through 3*SD*1*E* presented in Secs. III and IV reproduce with reasonable accuracy the experimental data for the $^3S_1$-$^3D_1$ channel and should prove useful for multiparticle calculations. These potentials appear to be superior to the previously published separable fits in this channel. The construction of Ernst, Shakin, and Thaler that was used in Sec. III is seen to be capable of yielding potentials that result in a better reproduction of the Reid phase shifts than UPE potentials of comparable rank.

The survey of previous separable potentials for this channel presented in Sec. IV shows that a number of errors have been made in fitting $^3S_1$-$^3D_1$ separable potentials. Despite the apparent importance of the deuteron properties to multiparticle calculations, many of these fits did not attempt to reproduce the experimentally known deuteron properties. As a result it is concluded that these potentials are not sufficiently realistic to be reliably used in multiparticle calculations.

Note added in proof: While this article was in press, a paper[54] by Doleschall was received in which two rank-two fits to the $^3S_1$-$^3D_1$ channel are presented. The properties given by Doleschall for these potentials have been reproduced by the present author and do not change the conclusions given in Sec. IV about the potentials presented in this article.

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