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Statistical analysis of intermediate structure

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We study several statistical tests which can be used to determine whether experimental data, typically a sequence of partial widths, are compatible with the statistical model of nuclear reactions or, on the contrary, imply the existence of intermediate structure. These tests are applicable to cases where the data can be ascribed to a definite partial wave. We give a brief, but critical, discussion of the few known methods, and develop several new tests. These new tests involve the study of the following quantities: (a) the length of the longest unbroken sequence (run) of partial widths lying either above or below their median, (b) the length of the longest run about a value that we call the value of optimal run length, (c) the distribution of runs up or down, i.e., of unbroken sequences composed of increasing or decreasing values, (d) the number of large adjacent partial widths, and (e) the ratio of the mean-square successive difference to the variance. We apply these new tests and the previously known ones, and discuss their merits and drawbacks, in the examples ^{40}Ca(p,p'), ^{56}Fe(n,n), ^{244}Cm + n, ^{187}Re(n,γ), ^{115}In(n,γ), ^{90}Zr + γ, Sn + γ, ^{70}Ge(p,p), ^{239}Pu(n,f), and ^{206}Pb(n,n). We find reliable evidence for intermediate structure in all these cases, except in the reactions ^{187}Re(n,γ) and ^{115}In(n,γ).

[NUCLEAR REACTIONS ^{40}Ca(p,p'), ^{56}Fe(n,n), ^{244}Cm + n, ^{187}Re(n,γ), ^{115}In(n,γ), ^{90}Zr + γ, Sn + γ, ^{70}Ge(p,p), ^{239}Pu(n,f), ^{206}Pb(n,n); calculated significance level of intermediate structure, using new statistical tests.]

I. INTRODUCTION

The partial widths of the compound-nuclear resonances in a given channel may appear to be enhanced in a certain energy domain. It has been proposed[1,2] that this feature may reflect the existence, at high excitation energy, of simple modes of motion (doorway states[3,4]) of the compound nucleus. This dynamical interpretation of the data is sometimes questioned, since the occurrence of these experimental features can be purely accidental, i.e., be compatible with the statistical model of nuclear reactions. *Intermediate structure* (IS) is a statistically significant deviation from the statistical model, which in addition takes place in a limited energy interval. Therefore, a dynamical interpretation of the data in terms of IS and doorway states should usually be attempted only when the data imply a *significant* deviation from the statistical model. Sometimes, a detailed statistical analysis is necessary in order to find the significance level of a tentative IS. A statistical test is said to be conducted at the α-significance level when there exists a risk α of rejecting the tested assumption (i.e., the validity of the statistical model) when it actually holds true. It is partly a matter of convention to set up a limit beyond which this risk α is considered to be significant. Here, we take this limit to be $\alpha = 0.05$, which is the usual choice in statistical analysis.

The main purpose of the present paper is threefold. First, we give a very brief critical review of the statistical tests which have been used in the past to identify IS. We correct or complete some of them. Secondly, we propose several new statistical methods. Finally, we apply these methods to a number of experimental data. In Sec. II, we briefly recall the main assumptions of the statistical model, in a form which is suited to the present context. We also give a more complete definition of IS. Section III is devoted to the methods which can be used when only energy-averaged quantities, for instance average total cross sections, are available. In Sec. IV, we discuss a number of statistical tests for randomness which can detect deviations from the statistical model. These tests can be applied to average quantities or to resonance parameters (e.g., partial widths). We briefly discuss two tests (Secs. IV C and IV H) which have been previously used, and one test (Sec. IV B) which was known but was, however, never applied to IS. We propose five new methods (Secs. IV D–IV G, and IV I). One of these (Sec. IV F) has been partly described in a recent letter.[5] Section V contains a statistical analysis of several specific reactions, namely ^{40}Ca(p,p'), ^{56}Fe(n,n), ^{244}Cm$+n$, ^{187}Re(n,γ), ^{115}In(n,γ), ^{90}Zr$+\gamma$, Sn$+\gamma$, ^{70}Ge(p,p), ^{239}Pu(n,f), and ^{206}Pb(n,n).

II. STATISTICAL MODEL

We write the collision matrix in the form

$$S_{cc'}=\exp(i\xi_c+i\xi_{c'})\left(Q_{c'c}-i\sum_{\lambda=1}^{N}\frac{\Gamma_{\lambda c}{}^{1/2}\Gamma_{\lambda c'}{}^{1/2}}{E-E_\lambda+\frac{1}{2}i\Gamma_\lambda}\right), \tag{II.1}$$

and assume that all the parameters are independent of energy. In a more accurate discussion, we could take into account a smooth energy dependence due to penetration effects (Sec. V B). Let us call E_1, E_2, ..., E_N the resonance energies ordered by increasing values and define the energy differences

$$d_\lambda=E_\lambda-E_{\lambda-1}\quad(\lambda=2,\ldots,N). \tag{II.2}$$

We call

$$F(\Gamma_{\lambda c})=(\Gamma_{1c},\Gamma_{2c},\ldots,\Gamma_{Nc})$$

and (II.3)

$$F(d_\lambda)=(d_1,d_2,\ldots,d_{N-1})$$

the sequences of partial widths and of level spacings ordered according to increasing resonance energies. Henceforth, we denote by $\tilde{x}$ a random variable, by x or x_λ one of its observed values, and by $\bar{x}$ the sample mean. The basic assumptions of the statistical model are the following:

(i) $F(\Gamma_{\lambda c})$ is a random series or a random sequence. This means that the observed values Γ_{1c}, Γ_{2c}, ..., Γ_{Nc} are compatible with the assumption that they are *independently* drawn from the same distribution law or, equivalently, that $F(\Gamma_{\lambda c})$ is a random permutation of the N numbers Γ_{1c}, ..., Γ_{Nc}, with each permutation having the same probability.

(ii) $F(d_\lambda)$ is also a random series.

(iii) The partial widths $\Gamma_{\lambda c}$ and $\Gamma_{\lambda c'}$ in different channels are not correlated.

The specification in (i) above that the $\Gamma_{\lambda c}$ are obtained from the *same* distribution implies in particular that they refer to resonances with the same J^π. This remark will be of importance in Secs. V D and V E. In some of the tests described below, it is further assumed that the variables $\tilde{\Gamma}_{\lambda c}$ and $\tilde{d}_\lambda$ follow Porter-Thomas and Wigner distributions, respectively. Most of the tests discussed in Sec. IV can be applied to assumptions (i) and (ii). However, we shall mainly discuss violations of assumption (i), since most dynamical models of IS predict a violation of only that assumption.

Let us first consider a situation where only one channel is open. Using the unitarity property of the scattering matrix, we can write Eq. (II.1) in the two equivalent forms

$$S=\exp(2i\xi)\frac{\text{c.c.}}{1+\frac{1}{2}i\sum_{\lambda=1}^{N}[\Gamma_\lambda/(E-E_\lambda)]}, \tag{II.4}$$

$$S=\exp(2i\xi)\frac{\text{c.c.}}{E-\epsilon_0+\frac{1}{2}i\Gamma\uparrow-\sum_{j=1}^{N-1}[v_j^2/(E-e_j)]}, \tag{II.5}$$

where c.c. denotes the complex conjugate of the denominator. In the dynamical interpretation of Eq. (II.5) in terms of the one-doorway-state model, v_j is the real matrix element

$$v_j=\langle\phi_0|H|\phi_j\rangle, \tag{II.6}$$

which couples the doorway configuration ϕ_0 to the complicated modes of motion ϕ_j.[2,4] If it is assumed that $F(v_j^2)$ is a random series, it can be shown that assumption (i) is violated.[4] This is the usual dynamical interpretation of IS.

We call I an averaging interval, centered on E, and introduce the strength function $s_c(E)$ in channel c

$$s_c(E)=I^{-1}\sum_{\lambda\in I}\Gamma_{\lambda c}=\frac{\overline{\Gamma}_c}{\overline{d}}, \tag{II.7}$$

where the bar denotes a local ensemble average. For $s_c\ll1$ and $I\gg\overline{d}$, the average total cross section in channel c is given by

$$\langle\sigma_{\text{tot},c}(E)\rangle_I=\frac{2\pi^2}{k_c^2}g_J s_c(E), \tag{II.8}$$

where g_J is the familiar spin statistical factor.

III. ENERGY-AVERAGED CROSS SECTIONS

In the present section, we discuss three methods which can be applied to the analysis of data corresponding to averages over several resonances. For definiteness, we take the example of an average total cross section.

A. Monte Carlo calculations

One can generate average cross sections from random choices of the parameters appearing in Eq. (II.1). For this purpose, one must assume a given probability distribution for the quantities $\Gamma_{\lambda c}{}^{1/2}$ and d_λ. Such calculations have been performed by Singh, Hoffman-Pinther, and Lang[6] in the case $\overline{\Gamma}/\overline{d}\gg1$, by Baglan, Bowman, and Berman[7] in a one-channel case with $\overline{\Gamma}/\overline{d} \ll 1$, and by Schrack, Schwartz, and Heaton[8] in various cases. Their results indicate that the frequency of occurrence of bumps of statistical origin in average cross sections is fairly large. Baglan, Bowman, and Berman[7] take $\overline{\Gamma}/\overline{d} \simeq 0.1$, plot $\langle\sigma_{tot}(E)\rangle_I$ versus E, and count the mean number Λ of bumps per 1000 resonances. Their definition of a "bump" implies two somewhat arbitrary criteria[7]:

(a) The size of the averaging interval I: They choose $I = 8\overline{d}$.

(b) The ratio H of the minimum height of a bump to the mean cross section $\langle\sigma_{tot}\rangle_{I\to\infty}$: They take $H = 1.5$.

The numbers quoted by Baglan, Bowman, and Berman depend rather sensitively upon these conventional criteria. One result of general validity emerges, however, from their calculations: The mean number (Λ), per 1000 resonances, of statistical bumps of width larger than or equal to $M\overline{d}$ decreases fairly rapidly when M increases: $\Lambda = 15$ for $M = 10$, $\Lambda = 10$ for $M = 14$, and $\Lambda = 2$ for $M = 20$. We emphasize that the results of these Monte Carlo calculations cannot be directly compared with the frequency of occurrence of bunching of large partial widths. Indeed, Eq. (II.8) shows that $\langle\sigma_{tot}\rangle_I$ is influenced by variations of $\overline{d}$ as well as of $\overline{\Gamma}$.

In practice, the problem appears in the following form. Suppose that one bump is observed in $\langle\sigma\rangle_I$, in an energy interval of size ΔE. Is the probability negligible (i.e., according to our convention, less than 0.05) that at least one bump of similar or of larger width and height is generated in ΔE from a sample of level spacings and widths drawn from the distribution laws of the statistical model? In order to answer this question on the basis of Monte Carlo calculations, the latter must be performed with the experimental values of $\overline{\Gamma}$, $\overline{d}$, H, and I. The first three quantities are rarely known with good accuracy. Moreover, one would need to calculate the probability distribution of the number $\tilde{n}$ of statistical structures in ΔE. This would require lengthy Monte Carlo calculations. One could, as a first approximation, assume that the bumps occur randomly, with an average of $\overline{\Lambda}$ bumps in an energy interval D. Then, the probability distribution of $\tilde{n}$, in an energy interval ΔE, is given by Poisson's law

$$\Pr[\tilde{n} = n] = \exp(-\alpha)\frac{\alpha^n}{n!}\ . \qquad \text{(III.1)}$$

Here, α is the mean of $\tilde{n}$ and is related to the variance σ^2 of $\tilde{n}$ and to $\overline{\Lambda}$ by

$$\alpha = \sigma^2 = \overline{\Lambda}\Delta E/D. \qquad \text{(III.2)}$$

Let us take as an example the reaction $^{206}Pb(n,n)$.[7] The observed enhancement contains about $M = 14$ resonances. If one assumes $H \simeq 1.5$, Baglan, Bowman, and Berman[7] give $\overline{\Lambda} = 10$, for $D = 1000\ \overline{d}$. Then the probability of observing at least a statistical bump in an energy interval containing K resonances is obtained from Eqs. (III.1) and (III.2)

$$\Pr[\tilde{n} \geq 1] = 1 - \Pr[\tilde{n} = 0] = 1 - \exp(-K\overline{\Lambda} \times 10^{-3})\ . \qquad \text{(III.3)}$$

For $K = 14$, one has

$$\Pr[\tilde{n} \geq 1] = 0.13, \qquad \text{(III.4)}$$

and for $K = 30$, one finds

$$\Pr[\tilde{n} \geq 1] = 0.26\ . \qquad \text{(III.5)}$$

We conclude from Eq. (III.4) that the observed enhancement does not imply a significant deviation from the statistical assumptions, since this would require $\Pr[\tilde{n} \geq 1] \leq 0.05$.

In summary, the Monte Carlo calculations are quite instructive but lengthy. They require the knowledge of $\overline{\Gamma}$, $\overline{d}$, and H, which are usually only poorly known. If the fine structure is resolved, the methods described in Sec. IV are usually preferable.

B. Autocorrelation function

Pappalardo[9] proposed that, in the case $\overline{\Gamma}/\overline{d} \gg 1$, the autocorrelation function should display two steps if two basic coherence widths exist. This method was improved and used by Carlson and Barschall[10] and by Schrack, Schwartz, and Heaton.[8] It presents the drawback of not giving the significance level of a tentative deviation from the statistical model. Carlson and Barschall[10] have also calculated the variance of the average cross section, due to the fluctuations in the number and width of resonance levels.

C. Correlation between $\langle\sigma(E_k)\rangle_I$ and $\langle\sigma(E_{k-1})\rangle_I$

The average total cross section is given by Eq. (II.8). If the statistical model holds, the quantities

$$\sigma_k = \langle\sigma(E_k)\rangle_I\ , \qquad \text{(III.6)}$$

and σ_{k-1} are independent for $I \gg \overline{\Gamma}$, $I \gg \overline{d}$, and for

$$E_k - E_{k-1} \gtrsim I\ . \qquad \text{(III.7)}$$

If $I \gg \overline{\Gamma}$, while $I \simeq \overline{d}$, condition (III.7) should be replaced by

$$E_k - E_{k-1} \gg I\ . \qquad \text{(III.8)}$$

We return to the relations (III.7) and (III.8) in Sec. V, where we discuss several examples. We shall see that condition (III.8) is in practice hard to fulfill, since the difference $E_k - E_{k-1}$ must in any case be kept smaller than the width of the tentative IS. The statistical tests described in Sec. IV can be used to investigate whether the sequence

$$F(\sigma)=(\sigma_1, \sigma_2, \ldots, \sigma_N) \tag{III.9}$$

is random, as it should be if the statistical model holds true.

IV. STATISTICAL TESTS FOR RANDOMNESS

A. Introduction

The purpose of the present section is to develop several statistical tests which can be used to investigate whether a sequence of numbers, for instance of partial widths, is random. The relative merits and drawbacks of these tests must be evaluated in each specific case. It is sufficient that only one test give a significant deviation from the statistical assumptions to conclude the existence of a significant deviation from the statistical model.

We call $x_1, x_2, \ldots, x_N$ the observed values of a random variable $\tilde{x}$, which has a continuous probability distribution. Let $F(x)$ denote the sequence

$$F(x)=(x_1, \ldots, x_N)\ , \tag{IV.1}$$

where the observables are ordered in a prescribed way, for instance with increasing values of the associated energy. Usually, we illustrate the discussion by the example $x_\lambda = \Gamma_{\lambda c}$, since assumption (i) is the one which is expected to be violated in the vicinity of an IS. In most of the following tests, it is checked whether $F(x)$ is a random sequence. One could also study whether the partial widths and the level distances follow Porter-Thomas and Wigner distributions, respectively. This type of test, however, appears to be both less practicable and less powerful, for two main reasons. First, the available samples are usually too small. Secondly, Monte Carlo calculations indicate that IS introduces only a small deviation from the Porter-Thomas distribution of partial widths.[11, 12]

B. Comparison between two samples

Wald and Wolfowitz[13] proposed a method to determine whether two samples of data are drawn from the same population. This test may, for instance, be used to compare two sets of partial widths, with one group $\Gamma^{(1)}_{\lambda c}$ corresponding to the resonances lying in the region of the tentative IS, while the other one $\Gamma^{(2)}_{\lambda c}$ contains the other resonances. The values of the two sets are arranged in a single array, by order of increasing magnitude. A "run" is defined as a series of consecutive values belonging to the same set, either $\Gamma^{(1)}$ or $\Gamma^{(2)}$. The probability distribution of the total number $\tilde{U}$ of runs has been calculated by Wald and Wolfowitz.[13] To our knowledge, this test has never been applied to IS. It requires two samples, i.e., a rather large number of partial widths. Moreover, and mainly, we saw in Sec. IV A that IS gives rise to only a small deviation from the probability distribution which prevails in the background.

C. Number of runs about a reference value

James[14] proposed to use the following method to identify IS. Let us choose some reference value R, and call "run above" an unbroken series of observed quantities lying above R. A "run below" is defined in a similar way. The probability distribution of the total number U of runs above and below is the same as that calculated by Wald and Wolfowitz.[13] Here $\tilde{U}$ is the total number of runs above and below, in a sample drawn from a binomial population whose probabilities p and $q(=1-p)$ are unknown. The present method gives the significance level of the assumption that the probability p remains constant throughout the sample. If the numbers m and n of values lying above and below R, respectively, are both larger than about 10, one can assume that the quantity

$$\tilde{X}=\frac{|\tilde{U}-E(\tilde{U})|-\frac{1}{2}}{\sigma(\tilde{U})} \tag{IV.2}$$

is a Gaussian variable, whose expected value $E(\tilde{U})$ and variance $\sigma^2(\tilde{U})$ are given by[13]

$$E(\tilde{U})=\frac{2mn}{m+n}+1\ , \tag{IV.3a}$$

$$\sigma^2(\tilde{U})=\frac{2mn(2mn-m-n)}{(m+n)^2(m+n-1)}\ . \tag{IV.3b}$$

For $m=n=N/2$, the reference value is identical to the median of the sample. This is the choice which has been made in the literature,[14-16] where the test was applied to neutron-induced fission reactions. However, we shall see in Sec. VI that a choice of R different from the median is sometimes preferable. In the presence of IS, we expect, as the alternative to randomness, too large a proportion of long runs above and below the median and, hence, too few runs. When m or n is smaller than about 10, the assumption that $\tilde{X}$ is a Gaussian variable is no longer justified. We found, however, that Swed and Eisenhart[17] have tabulated significance levels for $m, n \leq 20$. We give in Table I the values of U such that $\Pr[\tilde{U} \leq U]$ is equal to 0.05 and 0.01, respectively, for $m \leq n \leq 20$, $m=2, 3, \ldots, 10$. We can assume without loss of generality that $m \leq n$. We apply the present test in Secs. V A, V B, V F, V G, and VI for cases where m and n can be smaller than 10.

D. Longest run above the median

The length of a run is equal to the number of values x_λ that it contains. Moore[15] argues that the existence of a run above the median with a length larger than or equal to 5 implies that $F(x)$ [Eq. (IV.1)] is not a random series, on a 0.03 significance level. We first describe why this estimate is incorrect, and then show how to obtain a correct evaluation of the significance level of the randomness hypothesis of $F(x)$ from the length of the longest run relative to the median.

Moore identifies the quantity $P(k)$ in Eq. (17) of Ref. 15 with the probability of finding one run of length k in a sample of size N. This quantity $P(k)$ can be seen to give instead the ratio of the average number of runs of length k to the average number of runs with arbitrary length, when one considers the totality of the runs among all the possible sequences $F(x)$ obtained by reordering the quantities x_j. The quantity which gives the level of significance of the randomness hypothesis of $F(x)$ is quite different: It can be identified with the probability of finding, in a sample of size N, at least one run of length larger than or equal to k, on one (or on either) side of the median. Mosteller[18] and Olmstead[19] have calculated the minimum value that the length of a run (on one side of the median, and on either side of the median, respectively) must take, in order to reject the randomness hypothesis for a sample of size N, on a significance level α. We give in Table II these minimum lengths for $\alpha = 0.05$ and 0.01 and for various values of N. It is also useful to have the maximum value that the sample size N must take, in order to conclude, on a significance level α, that the existence of one run of length $\geqslant d$ implies nonrandomness. These values are given in Table III, for $\alpha = 0.01$ and 0.10, for runs on only one side of the median, and on either side of the median, respectively. This test is applied in Secs. VA–VH.

E. Longest run relative to the value of optimal run length

We define the value of optimal run length (VORL) in the following way. Let a_m be the length of the longest run above some given value m, and b_m denote the length of the longest run below m. This value m is identical to the VORL, k, when the quantity $f_m = \min(a_m, b_m)$ is maximum. Table IV gives the value of the maximum size N of a sample such that the observation of one run of length at least f_k, relative to the VORL, implies a deviation from randomness, on a significance level α.[19]

TABLE I. Values of U such that $\alpha = \Pr[\tilde{U} \leq U]$ is equal to 0.05 and 0.01, respectively, for $m = 2$–10 and $20 \geq n \geq m$ (Sec. IV C).

n	$\alpha = 0.05$	$\alpha = 0.01$
	$m=2$	
2	...	...
7	...	...
8	2	...
18	2	...
19	2	2
20	2	2
	$m=3$	
3	...	...
4	...	...
5	2	...
8	2	...
9	2	2
10	3	2
20	3	2
	$m=4$	
4	2	...
5	2	...
6	3	2
11	3	2
12	4	3
20	4	3
	$m=5$	
5	3	2
8	3	2
9	4	3
13	4	3
14	5	3
15	5	4
20	5	4
	$m=6$	
6	3	2
7	4	3
9	4	3
10	5	3
11	5	4
14	5	4
15	6	4
16	6	4
17	6	5
20	6	5
	$m=7$	
7	4	3
8	4	3
9	5	4
11	5	4
12	6	4
13	6	5
16	6	5
17	7	5
18	7	5
19	7	6
20	7	6
	$m=8$	
8	5	4
9	5	4
10	6	4
11	6	5
13	6	5
14	7	5
15	7	5
16	7	6
17	7	6
18	8	6
20	8	6
	$m=9$	
9	6	4
10	6	5
11	6	5
12	7	5
13	7	6
14	7	6
15	8	6
16	8	6
17	8	7
19	8	7
20	9	7
	$m=10$	
10	6	5
11	7	5
12	7	6
13	8	6
14	8	6
15	8	7
16	8	7
17	9	7
18	9	7
19	9	8
20	9	8

TABLE II. Minimum length of a run relative to the median which implies nonrandomness, on a significance level α, for samples of size N (Sec. IV D).

N	One side of median $\alpha=0.05$	One side of median $\alpha=0.01$	Either side of median $\alpha=0.05$	Either side of median $\alpha=0.01$
10	5	...	5	...
20	7	8	7	8
30	8	9	8	9
40	8	9	9	10
50	8	10	10	11
60	9	10	10	11
100	10	12	11	13
200	11	13	12	14

Table V gives the minimum value of f_k which implies nonrandomness, on a significance level α, for a sample of given size N.[19] This test will be used in Secs. V A, V B, V C, V F, and V G. It has not been applied before to IS.

F. Runs up and down

A "run up" of length d is a group of $d+1$ consecutive values such that

$$x_{i+j} < x_{i+j+1} \quad (j=0,\ 1,\ \ldots,\ d-1)\ . \tag{IV.4}$$

A "run down" is defined in the same way. The probability distribution of runs up and down has been studied by Levene and Wolfowitz,[20] Wolfowitz,[21] and by Olmstead.[22] In Ref. 5, we showed how this method can be used to identify IS. In the vicinity of an IS, one expects a smaller number of short runs up and down for the partial widths, and a larger number of long ones, than in the case when $F(\Gamma)$ is a random series. This was applied to the cases ^{206}Pb(n,n) and ^{115}In(n,γ) in Ref. 5.[23] There we show how one can calculate, from the observed number $K(d,N)$ of runs up and down of length d in a sample of size N, the level of significance of the hypothesis that the sequence $F(x)$ is a random sequence. We also give in Ref. 5 a table showing the minimum value that the length of a run up or down must have in order that a sample of size N displays a nonstatistical behavior, with significance levels $\alpha=0.05$ and 0.10, respectively.

It can also be useful to study the number $K'(d,N)$ of runs up and down of length larger than or equal to d. The expected value I' and the variance σ^2 of $\tilde{K}'$ are given by[20]

$$I'(d,N)=a'N-b'\ , \tag{IV.5a}$$

$$\sigma_{I'}{}^2(d,N)=c'I'+e'\ . \tag{IV.5b}$$

The coefficients a', b', c', and e' are listed[24] in Table VI for $d\leq 5$. Their values for $d\leq 11$ can be found in Ref. 24. For N large, the ratio

$$\tilde{R}'(d,N)=\frac{\tilde{K}'(d,N)-I'(d,N)}{\sigma_{I'}(d,N)} \tag{IV.6}$$

is close to a Gaussian variable. Moreover, the fact that $c'\simeq 1$ and $e'\simeq 0$ for $d\geq 4$ indicates that one can then approximate the probability distribution of $K'(d,N)$ by a Poisson law of mean I', since one has then $\sigma^2\simeq I'$.

These tests involving runs up and down will be used in Secs. V B–V H.

TABLE III. Maximum size of a sample that contains one run of length at least d relative to the median, which implies nonrandomness, on a significance level α (Sec. IV D).

d	One side of median $\alpha=0.01$	One side of median $\alpha=0.10$	Either side of median $\alpha=0.01$	Either side of median $\alpha=0.10$
3	6	6	6	6
4	8	10	8	8
5	10	16	10	14
6	14	26	12	20
7	18	44	16	32
8	26	78	22	52
9	38	142	32	86
10	56	256	42	150

TABLE IV. Maximum size of a sample that contains one run of length at least f_k, on each side of a cut k, which implies nonrandomness, on a significance level α, (Sec. IV E).

f_k	$\alpha=0.01$	$\alpha=0.10$
3	6	8
4	8	12
5	12	18
6	16	34
7	24	58
8	38	108
9	66	204
10	118	400

TABLE V. Minimum length of a run, on each side of a cut k, which implies nonrandomness, on a significance level α, for samples of size N (Sec. IV E).

N	$\alpha=0.05$	$\alpha=0.01$
10	5	5
20	6	7
40	7	8
100	9	10

G. Mean-square successive difference

The mean-square successive difference δ^2 of the array $x_1, \ldots, x_N$ is defined by

$$\delta^2 = (N-1)^{-1} \sum_{j=1}^{N-1} (x_{j+1} - x_j)^2 \ . \qquad \text{(IV.7)}$$

This quantity is less sensitive to slow variations of the mean than the sample variance s^2:

$$s^2 = (N-1)^{-1} \sum_{j=1}^{N} (x_j - \bar{x})^2 \ . \qquad \text{(IV.8)}$$

On the contrary, δ^2 is more sensitive than s^2 to rapid variations of the mean. Hence, the value of the ratio

$$\eta = \frac{\delta^2}{s^2} \qquad \text{(IV.9)}$$

can indicate the existence of a nonrandom variation of the mean of a *normal* population. If the quantities x_j are normally distributed, the expected value of δ^2 is $2\sigma^2$. In this case, confidence intervals corresponding to given significance levels α have been computed for η.[24,25] In Refs. 24 and 25, confidence intervals are listed, which are such that if the observed value of η is smaller than or equal to the lower limit of the confidence interval, the mean has a slow and nonrandom variation, on the significance level α. If η is larger than or equal to the upper limit of the interval, the variation of the mean is cyclic and rapid, on the significance level α. To the best of our knowledge, this test has not been applied before to IS. We emphasize that this test can only be applied to normal variables. Hence, it is usually not applicable to partial widths in a given channel c, which have a Porter-Thomas distribution. However, it can be applied, for instance, to the sums of many partial widths. We shall use it in Secs. V D and V E for the reactions ^{187}Re(n,γ) and ^{115}In(n,γ), respectively.

TABLE VI. Expected value $I'(d,N)$ and variance $\sigma_{I'}^2(d,N)$ of the number of runs up and down of length at least d, in a random sample of size N (Sec. IV F).

	$I'(d,N) = a'N - b'$		$\sigma_{I'}^2(d,N) = c'I' + e'$	
d	a'	b'	c'	e'
1	6.6×10^{-1}	3.3×10^{-1}	2.6×10^{-1}	-2.3×10^{-1}
2	2.5×10^{-1}	4.16×10^{-1}	3.16×10^{-1}	7.2×10^{-2}
3	6.6×10^{-2}	1.83×10^{-1}	7.11×10^{-1}	1.7×10^{-2}
4	1.38×10^{-2}	5.27×10^{-2}	9.18×10^{-1}	···
5	2.38×10^{-3}	1.15×10^{-2}	9.82×10^{-1}	···

H. Serial correlation coefficient

The serial correlation coefficient of lag h is given by

$$R_h = \frac{\sum_{j=1}^{N} x_j x_{j+h} - [(\sum_{j=1}^{N} x_j)^2/N]}{\sum_{j=1}^{N} x_j^2 - [(\sum_{j=1}^{N} x_j)^2/N]} \ , \qquad \text{(IV.10)}$$

where x_{j+h} should be replaced by x_{j+h-N} for $j+h>N$. The probability distribution of $\tilde{R}_h$ has been calculated by Anderson,[26] when $\tilde{x}$ is a normal variable. The method based on the calculation of R_h has been applied in Refs. 14, 27–29. We found out that Wald and Wolfowitz[30] have calculated the distribution of R_h when $\tilde{x}$ has a continuous, but otherwise arbitrary, probability distribution. We show in Sec. V D that the values of R_1 and of η are very sensitive to the possibility that only one resonance (even among many) has been missed. This limits, in some cases, the reliability of the tests based on the calculation of η and R_1 and is related to the fact that the tests described in Secs. IV G and IV H are very powerful. The test of Sec. IV H is applied in Secs. V D and V E.

I. Large adjacent values

It happens that one observes in a sample of widths, for instance, an unbroken sequence of large widths. It may be that the length d of this run above the median is not sufficiently large to imply a nonstatistical behavior (Sec. IV D), and that the occurrence of d nonadjacent large widths in the sample would also not be significant (Sec. V A). However, the fact that there exists d large *and* adjacent widths may imply a nonstatistical behavior. The purpose of the present section is to develop a method to deal with such a case.

Let us assume that μ, the actual average value of $\tilde{x}$, can be calculated with a sufficiently narrow confidence interval. From the distribution of $\tilde{x}$, for instance from the Porter-Thomas distribution in the case of partial widths, one can calculate the probability p of the event

$$\tilde{x}_\lambda > t \ , \qquad \text{(IV.11)}$$

where t is chosen in such a way that Eq. (IV.11) is fulfilled for all the observed values, x_λ, in the "run." The values of x_λ can then be divided into two categories, $x_\lambda > t$ (with probability p) and $x_\lambda < t$ (with probability $q = 1-p$). This yields a sample from a binomial population whose probabilities p and q are known, if the probability distribution of $\tilde{x}$ is known. Von Bortkiewicz[31] and Mood[32] have calculated the probability distribution of runs of length larger than or equal to a given value, for random events arising from a binomial population of known probabilities p and q. Let $\tilde{S}_k$ be the number of runs of length larger than or equal to k in a sample of size N. The expectation value $E(\tilde{S}_k)$ and the variance $\sigma^2(\tilde{S}_k)$ of $\tilde{S}_k$ are given by[32]

$$E(\tilde{S}_k)=p^k[(N-k)q+1], \tag{IV.12}$$

and

$$\begin{aligned}\sigma^2(\tilde{S}_k)=p^{2k}\{&(N-2k+1)(N-2k)\\&-2(N-2k)(N-2k-1)p\\&+(N-2k)(N-2k-1)p^2-[(N-k)q+1]^2\}\\&+p^k[(N-k)q+1]\,.\end{aligned} \tag{IV.13}$$

In the limit $N\to\infty$, the variable

$$\tilde{x}_k=N^{-1/2}(\tilde{S}_k-Np^kq) \tag{IV.14}$$

is normally distributed with zero mean and with variance[32]

$$\sigma^2(\tilde{x}_k)=p^kq-(2k+1)p^{2k}q^2\,. \tag{IV.15}$$

These expressions can be used to test the randomness of the x_λ's if the probability distribution of $\tilde{x}_\lambda$ is known. This method will be applied in Secs. V A and V H to the reactions ^{40}Ca(p,p') and ^{239}Pu-(n,f). We also show in Sec. V A that the occurrence of d nonadjacent large values may be not significant, while a *run* of d large values implies a nonstatistical behavior.

V. ANALYSIS OF EXPERIMENTAL DATA

A. ^{40}Ca(p,p')

The cross section ^{40}Ca(p,p') shows twelve $\frac{5}{2}^+$ resonances between 6 and 8 MeV, among which four levels, grouped between 7.1 and 7.3 MeV, have particularly large widths.[33,34] These states decay predominantly to the 3^- level at 3.73-MeV excitation energy in ^{40}Ca. Table VII shows the partial widths of the resonances in this inelastic channel. These resonance states have been interpreted in Ref. 34 as resulting from the spreading of a doorway state, whose configuration corresponds to the coupling of a $2p_{1/2}$ single-particle state to the 3^- collective excited state of ^{40}Ca. Here, we show that the enhancement of the widths cannot be ascribed to statistical fluctuations. Our analysis hinges upon the assumption, whose validity is discussed in Ref. 34, that a very broad $\frac{5}{2}^+$ resonance at 8.135 MeV ($\Gamma_{p'}=22$ keV) should not be included in the analysis.

TABLE VII. Resonance parameters for ^{40}Ca(p,p'), from Ref. 34.

E_λ (MeV)	$\Gamma_{\lambda p'}$ (keV)	E_λ (MeV)	$\Gamma_{\lambda p'}$ (keV)
6.146	(0.1)	7.105	7.6
6.395	(0.1)	7.140	8
6.530	(0.1)	7.198	13
6.818	(0.1)	7.276	6
6.969	1.2	7.344	(1.)
7.032	(0.5)	7.647	(1.4)

We first apply the test of Sec. IV D based on the longest run above the median. Table VII shows that the median of the sample is located between 1 and 1.2 keV. The longest run above has a length equal to 4 which does not imply nonrandomness (Table II). The test involving runs up and down (Sec. IV F) cannot be applied, since several $\Gamma_{\lambda p'}$ take equal values. The test based on the length f_k, of the longest run relative to the VORL (Sec. IV E) leads to a deviation from randomness of $F(\Gamma_{\lambda p'})$, on the level of significance $\alpha=0.10$ only. This is too large to imply the existence of IS, according to our conventional limit $\alpha=0.05$. The tests based on the serial correlation coefficient and on the mean-square successive difference are not reliable, because of the experimental errors and also because the $\tilde{\Gamma}_{\lambda p'}$'s are not normally distributed.

Since the peculiar feature of the data consists in the occurrence of four large and adjacent widths, it is natural to apply the test described in Sec. IV I. The observed mean ($m_{\rm obs}$) of the sample of 12 widths is 3.26 keV. We neglect the experimental errors. It is important to estimate a confidence interval for the actual mean μ of the distribution of $\tilde{\Gamma}_{\lambda p'}$. We assume that $\tilde{\Gamma}_{\lambda p'}/\mu$ is a χ^2 variable with one degree of freedom. The sample mean

$$\tilde{m}=\frac{\mu}{12}\sum(\tilde{\Gamma}_{\lambda p'}/\mu) \tag{V.1}$$

is the product of a χ^2 variable with 12 degrees of freedom by the constant $\mu/12$. Thus, we have

$$\mathrm{E}(\tilde{m})=\mu\,, \tag{V.2}$$

$$\sigma(\tilde{m})=6^{-1/2}\mu\,. \tag{V.3}$$

Making use of the approximation $\mu=m_{\rm obs}$ in (V.3), we find

$$\sigma(\tilde{m})\simeq 1.33\text{ keV}\,.$$

Since the sample mean $\tilde{m}$ is close to a normal variable, we can calculate the following approximate confidence interval for μ:

$$\Pr[3.26\text{ keV}-\sigma<\mu<3.26\text{ keV}+\sigma]=\Pr[1.93\text{ keV}<\mu<4.59\text{ keV}]=0.68. \tag{V.4}$$

Table VII and Eq. (V.4) show that the four values in the run are larger than 1.3 times the upper value of the confidence interval (V.4) for μ. We have:

$$p = \Pr[\tilde{\Gamma}_{\lambda p'} > 1.3\mu] = \Pr[\chi^2_1 > 1.3] = 0.25 . \quad (V.5)$$

Using Eqs. (IV.12) and (IV.13), where $N=12$, $k=4$, $p=0.25=1-q$, we find

$$E(\tilde{S}_4) = 0.027, \quad \sigma^2(\tilde{S}_4) = 0.027. \quad (V.6)$$

According to the Tchebycheff inequality, which is valid for any distribution, we have

$$\Pr[\,|\tilde{S}_4 - E(\tilde{S}_4)| > 4.5\sigma(\tilde{S}_4)] \leq 0.05 , \quad (V.7)$$

from which we conclude, in the present case, that

$$\Pr[\tilde{S}_4 > 0.77] < 0.05 . \quad (V.8)$$

Comparing this result with the observed value of S_4 (i.e., with unity) we conclude that the existence of a set of four large consecutive values of $\Gamma_{\lambda p'}$ implies a deviation from the statistical assumption that $F(\tilde{\Gamma}_{\lambda p'}/\mu)$ is a random variable drawn from a χ_1^2 population. We recall that this conclusion is based on the assumptions that all resonances listed in Table VII are $\frac{5}{2}^+$ states and that the broad resonance at 8.135 MeV should be treated on a separate footing.[34] These assumptions appear quite well justified. We also emphasize that it is essential to use the fact that the four large values are adjacent. Indeed, the probability of finding at least four such large values in an arbitrary order, in a sample of size 12, is 0.35, which is quite large. Finally, we note that we used the approximation $\mu = m_{obs}$ in Eq. (V.3) and that we did not require a very stringent confidence interval for μ. Despite these approximations, we believe that this statistical analysis is valuable in the present case. This method should moreover be very useful in other cases.

We also apply to the $\Gamma_{\lambda p'}$ the test of Wald and Wolfowitz described in Sec. IV C. We choose a reference value R between 1.5 and 6 keV. Table VII shows that the total number of runs about this value is equal to 3 and the numbers of $\Gamma_{\lambda p'}$ lying above and below this reference value, respectively, are $m=4$ and $n=8$. We conclude from Table I that $F(\Gamma_{\lambda p'})$ is not a random series, on a significance level $\alpha < 0.05$.

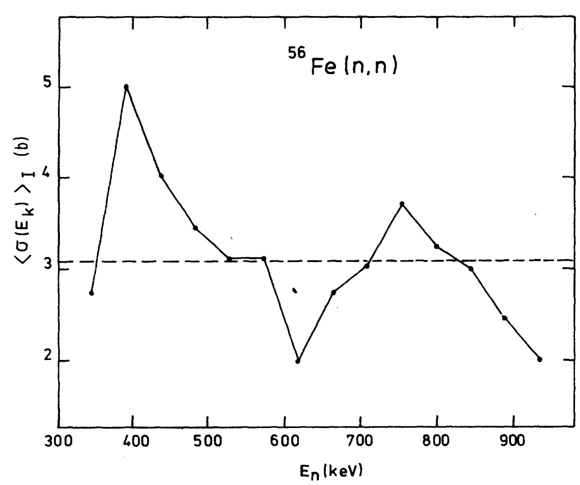

FIG. 1. Values of the total cross section averaged over 20 keV, for $^{56}Fe(n,n)$, for energies separated by 45 keV. The dashed line is the sample median.

B. $^{56}Fe(n,n)$

Elwyn and Monahan[35,36] gave some evidence for the existence of an IS in the reaction $^{56}Fe(n,n)$, for $E_n \simeq 360$ keV. They did not, however, find significant deviations from the Porter-Thomas and Wigner distributions of the partial widths and level spacings, respectively. They did not perform other statistical analysis of the data and neglected to take into account the finite range of data effects.[37] Schrack, Schwartz, and Heaton[8] analyzed the $^{56}Fe(n,n)$ cross section for $E_n > 500$ keV, with the method described in Sec. III B and report the existence of an IS at $E_n \simeq 770$ keV. In the present section, we give a statistical analysis of the experimental data based on the methods described in Sec. IV. We first discuss the average cross section, then the fine structure parameters.

The energies E_k and E_{k-1} which are such that $\langle\sigma(E_k)\rangle_I$ and $\langle\sigma(E_{k-1})\rangle_I$ are not correlated, in the frame of the statistical model, should fulfill the relation (III.8). In the present case, $d \simeq I \simeq 20$ keV. From the investigation of a simple model with equidistant resonances, we concluded that one must take $E_k - E_{k-1} \gtrsim 45$ keV. Figure 1 shows the values of $\langle\sigma(E_k)\rangle_I$ for $E_k - E_{k-1} = 45$ keV. We note the presence of a run down of length 5, for $400 < E_n < 620$ keV, provided that the point at 530 keV lies higher than that at 575 keV. Since the total number of points is $N=14$, Table 1 of Ref. 5 shows that the sequence $F(\langle\sigma(E_k)\rangle_I)$ is not random, on a significance level 0.05. Actually, the values $\langle\sigma(530 \text{ keV})\rangle$ and $\langle\sigma(575 \text{ keV})\rangle$ are practically equal, and it appears dangerous to draw any definite conclusion from this test alone. We therefore apply the method involving the number of runs up and down of given length (Sec. IV F). We now assume the existence of a run *up* of length 1 between 530 and 575 keV, since we just saw that the other alternative implies a deviation from the statistical model. The results of this analysis, based on Table 2 of Ref. 5 and on Table VI of the present paper, are shown in Table VIII. The quantity P,

TABLE VIII. Runs up and down for $\langle\sigma(E_k)\rangle_I$, for $^{56}Fe(n,n)$.

d	$K(d,14)$	$I(d,14)$	σ_I	P
1	3	5.91	2.37	0.11
2	0	2.33	1.19	0.03
≥3	3	0.74	0.74	0.001

TABLE X. Runs up and down for Γ_λ^0, for $^{56}Fe(n,n)$.

d	$K(d,13)$	$I(d,13)$	σ_I	P
1	1	5.49	2.28	0.02
2	2	2.15	1.14	0.45
≥3	2	0.675	0.705	0.03

defined in Ref. 5, gives the approximate level of significance for nonrandomness for the analyzed quantities. Thus, we conclude to the existence of nonrandomness, on a 0.001 level of significance.

We apply to the data of Fig. 1 the test of Wald and Wolfowitz (Sec. IV C). We choose the reference value $R=3.1$ b. Then, we have $U=5$, $m=n=7$. Table I shows that the randomness hypothesis cannot be rejected with this test. The same conclusion is reached with the tests of Secs. IV D and IV E.

We now turn to the analysis of the fine structure parameters, which were measured by Bowman, Bilpuch, and Newson[38] for $360 < E_n < 650$ keV. The values of Γ_λ and E_λ for the s-wave resonances located between 186 and 614 keV are given in Table IX. It is appropriate to extract the penetration factor for s wave, thereby defining the quantities

$$\Gamma_\lambda^0 = [1 \text{ eV}/E_\lambda(\text{eV})]^{1/2}\Gamma_\lambda. \qquad (V.9)$$

We first apply the test based on the runs up and down (Sec. IV F) to the quantities Γ_λ^0, for a sample of size 13. The results are shown in Table X. They imply a deviation from randomness of the type expected for IS, i.e., a too small number of runs of length 1, and a too large number of long runs. We note that we cannot apply the test to the partial widths Γ_λ proper, because several take equal values. In Ref. 36, the quantities v_j^2 defined by Eq. (II.6) are extracted from the identity between Eqs. (II.4) and (II.5), disregarding the energy dependence of the Γ_λ. With this proviso in mind, we give in Table XI, the results of the test involving runs up and down applied to the quantities v_j^2 given in Ref. 36. They show that the number of runs of length 1 is too small so that $F(v_j^2)$ is not a random sequence. This may indicate that the doorway interpretation outlined in Sec. II is not valid. However, we note that several values of v_q^2 are practically equal[36] and the penetration effects might thus considerably modify the results of this test. A fit of the experimental data, using Eq. (II.5), with an energy dependent $\Gamma\uparrow$ [see Eq. (V.9)], would be of interest.

TABLE IX. Fine structure parameters for $^{56}Fe(n,n)$.

E_λ (keV)	Γ_λ (keV)	Γ_λ^0 (eV)	v_q^2 (keV^2)
186.5	3.5	8.1	1465.02
220.0	1.3	2.8	1143.02
243.5	0.3	0.6	213.43
273.0	3.5	6.7	1003.98
314.5	5.5	9.8	1027.68
360.5	9.3	15.5	274.11
382.0	10.0	16.3	185.43
406.0	2.5	3.9	362.12
438.0	1.5	2.3	654.32
469.5	1.5	2.2	969.92
499.5	2.5	3.5	1652.40
559.5	2.5	3.3	1383.67
614.0	2.0	2.6	

We also apply the test involving the length of the longest run above the median (Sec. IV D) to the quantities $\Gamma_\lambda, \Gamma_\lambda^0$, and v_q^2. The value of the sample median is such that half of the observed values of $\Gamma_\lambda, \Gamma_\lambda^0$, or v_q^2 are larger than this value. We have

$$\text{med}\Gamma_\lambda = 2.5 \text{ keV},\quad \text{med}\Gamma_\lambda^0 = 3.5 \text{ keV}, \qquad (V.10)$$

and

$$\text{med}v_q^2 \simeq 1000 \text{ keV}^2. \qquad (V.11)$$

Table IX shows that there is an unbroken sequence of five Γ_λ^0 above the median. Table II shows that the randomness hypothesis cannot be rejected for these parameters ($N=13$). The lengths of the longest run above the median are 4 and 2, for the Γ_λ and v_q^2, respectively; they are compatible with randomness of these parameters.

We also apply to the quantities Γ_λ^0 and v_q^2 the test based on the VORL (Sec. IV E). For the Γ_λ^0's, the VORL (k) is located between 3.5 and 3.9 eV

TABLE XI. Runs up and down for v_q^2, for $^{56}Fe(n,n)$.

d	$K(d,12)$	$I(d,12)$	σ_I	P
1	1	5.08	2.18	0.03
2	3	1.97	1.09	0.17
≥3	1	0.609	0.670	0.28

(Table IX) and the length f_k is equal to 5. Table IV shows that this corresponds to a deviation from randomness for the sequence $F(\Gamma_\lambda^0)$, on a significance level $\alpha \simeq 0.01$. The VORL (k) for the $v_q{}^2$ is located between 362 and 969 keV^2 with $f_k = 3$. This does not imply a nonstatistical behavior.

Finally, we apply to the quantities Γ_λ^0 and $v_q{}^2$ the test of Wald and Wolfowitz (Sec. IV C). For the Γ_λ^0 and a reference value $R = 3.6$ eV, we obtain $U=4$, $m=6$, and $n=7$. For the $v_q{}^2$ and a reference value $R = 1000$ keV^2, $U=5$, $m=n=6$. Table I shows that the randomness hypothesis is rejected for the Γ_λ^0 but not for the $v_q{}^2$. This confirms the possible validity of the doorway-state interpretation.

C. ^{244}Cm + n

The ratio of average fission to capture in ^{244}Cm + n for $E_n \lesssim 5$ keV shows[12] a few structures which were identified by Moore[15] as IS, on the basis of the test of Wald and Wolfowitz described in Sec. IV C, on a 0.006 significance level. Here, we apply to the data plotted in Fig. 4 of Ref. 12 (ratio of average fission to capture cross sections) the test involving runs up and down described in Sec. IV F. The averaging interval ($I = 100$ eV) contains about eight resonances[15] and the conditions of Sec. III C are fulfilled. The sample size is equal to 49.

Table XII shows the results. The (too small) number of short runs and the existence of one run down of length 5 both indicate significant deviations from the statistical model. We also apply the tests of Secs. IV D and IV E to the data of Fig. 4 of Ref. 12. The length of the longest run above the median is 6 which does not imply nonrandomness (Table II). The VORL, k, has the value 0.7×37 and f_k is equal to 7. This implies nonrandomness on a significance level $\alpha \simeq 0.05$ (Table V).

D. ^{187}Re(n, γ)

Stolovy, Namenson, and Godlove[28] measured, for the s-wave resonances in the reaction ^{187}Re(n, γ), the ratio B_λ of the yield of photons with energy higher than 4 MeV to that of photons with energy higher than 1 MeV. A bump appears, in the vicinity of $E_n = 120$ eV, in the plot of B_λ versus E_λ, with a width of only 30 eV. We discuss whether this bump implies the existence of an IS.

Because of the large number of final states involved, the distribution of B_λ is close to a normal one. Only the first 46 measured values of B_λ, corresponding to $0 < E_\lambda < 200$ eV, are suitable for the statistical analysis because, above 200 eV, some resonances are being missed.[29] The serial correlation coefficient R_1 [Eq. (IV.10)] is equal to 0.34 and corresponds to a nonstatistical behavior, on a 0.005 significance level.[29]

Here, we discuss the reliability of this analysis and apply three other statistical tests. First, we emphasize that, even if one assumes that only s-wave capture resonances are included in the analysis, they correspond to two possible spins for the compound nucleus, namely 2 and 3. Hence the data, in the frame of the statistical model, are actually drawn from two different populations (approximately normal populations but with different means and standard deviations) and the validity of *any* of the statistical tests described in Sec. IV is highly questionable. This remark also applies to the reaction ^{115}In(n, γ) discussed in the following section.

We notice from Table II of Ref. 28 that one resonance at $E_\lambda = 108$ eV (in the region of the tentative IS) has been left out of the analysis, probably because the corresponding value of B_λ is somewhat too small to be accurately determined. We calculate the variation of the serial correlation coefficient R_1 when a value $B_\lambda = 0.09$ is assumed for this resonance (the average value of B_λ for the sample of 46 resonances is 0.10 and the standard deviation is 0.01). We then find that R_1 becomes equal to 0.20, which no longer implies a nonstatistical behavior for $F(B_\lambda)$. We return below to the sensitivity of this test to the omission of only one resonance even in a large sample. We now apply the test involving the mean-square successive difference (Sec. IV G). We find, for the first 46 values of B_λ, $\eta = 1.29$, which implies a nonstatistical behavior, on a level of significance $\alpha < 0.01$.[24,25] However, if a value 0.09 is assumed for the B_λ at 108 eV, η becomes equal to 1.56 which no longer corresponds to a nonrandom series for the B_λ.

We conclude that the tests described in Secs. IV G and IV H appear to show the same essential features. They apply to normal populations and are very powerful. This last characteristic is not always a quality in practice. As emphasized in Ref. 24 (page 159), a less powerful test gives often more reliable conclusions than a powerful

TABLE XII. Runs up and down for the ratio of average fission to capture, for ^{244}Cm + n.

d	$K(d, 49)$	$I(d, 49)$	σ_I	P
1	9	20.47	4.52	0.006
2	14	8.74	2.31	0.01
3	2	2.45	1.40	0.37
4	0	0.52	0.70	0.23
5	1	0.09	0.30	0.001

one. Indeed, the results of a very powerful test have the counterpart of being very dependent upon experimental errors, as shown by the present example.

We also apply the test of runs up and down (Sec. IV F) to the two sets of the 46 first values of B_λ, with and without the resonance at 108 eV. We do not find any significant deviation from randomness in these two cases. The results are shown in Tables XIII and XIV. This reflects the fact that this test may not be very powerful to detect nonrandomness, in some cases. Again, we emphasize that this may in practice be an advantage, in the sense that the conclusions can be more reliable. We expect, on general grounds, the test of Sec. IV F will lead to negative conclusions concerning the presence of an IS where the dispersion of the measured quantities $[B_\lambda, \Gamma_\lambda, \text{ or } \sigma(E_\lambda)]$ around their local mean is of about the same magnitude as the variation of this local mean due to the IS.

We also apply the test of Sec. IV D. For $90 < E_n < 130$ eV, there exists a run above the median of length 9, which implies nonrandomness, on a significance level $\alpha < 0.05$ (Table II). However, if the resonance at 108 eV has a B_λ value lying below the median, the length of the run above decreases to 5 which no longer implies nonrandomness.

After completion of this study, we learned that Stolovy, Namenson, and Godlove[29] refined and extended their measurements of neutron resonances up to about 3 keV in the target nucleus ^{187}Re. They found an order of magnitude more resonances than previously reported[28] and no evidence for IS. This confirms the conclusion drawn above concerning the fact that the tests of Secs. IV G and IV H may be too powerful and thereby too sensitive to experimental errors. For this reason, we believe that our analysis is of interest, even if it was applied to unreliable data, and decided to report it for the purpose of illustration.

E. ^{115}In(n, γ)

Coceva *et al.*[27] measured the ratio B_λ of the yield of photons with energy larger than 4 MeV to that of photons with energy higher than 1.6 MeV, for the reaction ^{115}In(n,γ), in the neutron energy range 40 eV–1 keV. They found a nonstatistical behavior for the B_λ associated to s-wave resonances, between 40 and 500 eV, on a 0.01 significance level. This conclusion is based on the tests of Secs. IV C and IV H. The serial correlation coefficient R_1 is equal to 0.41, for the first 34 values of B_λ ($E_\lambda < 500$ eV).[27] We include all the 56 levels below 1000 eV, obtain $R_1 = 0.284$, and still find nonrandomness, on a 0.01 significance level. We also compute the mean-square successive difference and find $\eta = 1.29$, for $N = 56$, which implies a nonrandom energy dependence of the mean of the B_λ, on a 0.01 significance level.[24,25] If only the first 34 values of B_λ are taken into account, we find $\eta = 1.11$, which implies nonrandomness on a 0.01 significance level. This confirms our remark in Sec. V D, that the tests of Secs. IV G and IV H usually lead to similar conclusions. We mentioned in Ref. 5 that the test involving runs up and down (Sec. IV F) does not lead to the rejection of the randomness hypothesis for the first 34 values of B_λ. When all the 56 values of B_λ are included in the analysis of runs up and down, the same conclusion is reached, as shown by Table XV. This again shows that the test of Sec. IV F is less powerful than the tests of Secs. IV G and IV H. We have discussed in Sec. V D the reliability of the conclusions drawn from the different tests, when experimental errors are taken into account. We also apply the test of Sec. IV D. For $170 < E_n < 330$ eV, a run above the median of length 10 is present, in the sample of 56 values of B_λ. This implies nonrandomness, on a significance level $\alpha = 0.01$ (Table III). As in the case of ^{187}Re(n,γ), we emphasize that all the results of the statistical analysis are very questionable, because the values of B_λ are drawn from at least two different popu-

TABLE XIII. Runs up and down for B_λ, for ^{187}Re(n,γ). The resonance at 108 eV is not included.

d	$K(d,46)$	$I(d,46)$	σ_I	P
1	14	19.22	4.37	0.12
2	7	8.19	2.24	0.30
3	4	2.29	1.35	0.10
≥4	1	0.58	0.73	0.28

TABLE XIV. Runs up and down for B_λ, for ^{187}Re(n,γ). A value $B_\lambda = 0.09$ is associated with the resonance at 108 eV.

d	$K(d,46)$	$I(d,46)$	σ_I	P
1	16	19.22	4.37	0.23
2	8	8.19	2.24	0.47
3	3	2.29	1.35	0.30
≥4	1	0.58	0.73	0.28

TABLE XV. Runs up and down for B_λ, for ^{115}In(n,γ), when all the measured values of B_λ are included.

d	$K(d,56)$	$I(d,56)$	σ_I	P
1	28	23.38	4.83	0.17
2	10	10.02	2.48	0.50
≥3	2	3.51	1.59	0.17

lations corresponding to levels of different angular momenta. Hence, we believe that a measurement of the spins of the resonances is necessary before concluding to the existence of IS.

F. $^{90}Zr+\gamma$ and $Sn+\gamma$

Axel, Min, and Sutton[39] have studied the dipole photointeraction cross section, $\sigma_{\gamma t}$, in $^{90}Zr+\gamma$, for $8.5 < E_\gamma < 12.5$ MeV, with an energy resolution of 70 keV. They found local enhancements near 9 and 11.5 MeV, and tentatively associated them with IS. We subtract the tail of the giant dipole resonance ($\sigma_{g.d.}$) from the data given in Fig. 4 of Ref. 39. The resulting values are plotted in Fig. 2, for energies separated by 70 keV. Since the conditions listed in Sec. III C are fulfilled, we can apply to the data in Fig. 2 the test of Sec. IV F based on runs up and down. We note that the numbers of runs up and down of a given length are fairly independent of the precise way in which the tail of the giant dipole resonance is drawn. Table XVI shows the results of the test. We conclude to nonrandomness for the sequence $F(\sigma_{\gamma t}-\sigma_{g.d.})$. The long runs lie between 8.5 and 10 MeV, where IS can thus be located. The test of Sec. IV F gives no significant deviation from randomness between 11.4 and 11.8 MeV, where Axel, Min, and Sutton[39] have assumed IS by visual inspection of the data. This disagreement is due to the fact that some points plotted in Fig. 4 of Ref. 39, in this energy domain, are correlated (Sec. III C) and have to be left out of the statistical analysis (for these points, $E_k - E_{k-1} < 70$ keV and $I \simeq 70$ keV).

We also apply the test of Sec. IV C to the data of Fig. 2. The median of the sample has a value close

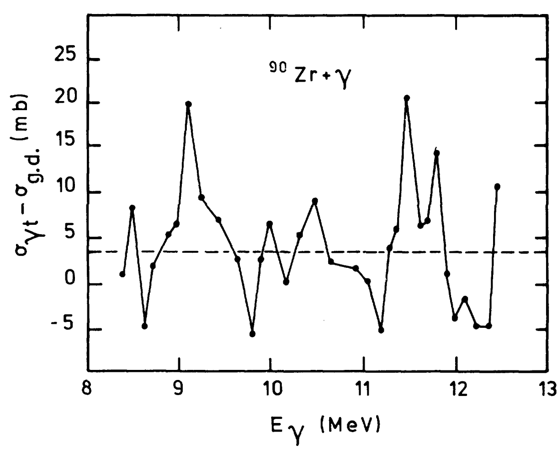

FIG. 2. Values of the difference between the photointeraction cross section and the tail of the giant dipole resonance, for $^{90}Zr+\gamma$, for energies separated by 70 keV. The dashed line is the sample median.

TABLE XVI. Runs up and down for $^{90}Zr+\gamma$, for the difference between the measured cross section and the tail of the giant dipole resonance (Fig. 2).

d	$K(d,32)$	$I(d,32)$	σ_I	P
1	6	13.40	3.63	0.02
2	5	5.63	1.85	0.38
3	1	1.56	1.11	0.31
4	3	0.33	0.55	$<10^{-4}$

to 3.4 mb. For this reference value, we have (Sec. IV C and Ref. 17) $U=12$, $m=n=16$. This result implies nonrandomness, on a 0.05 significance level (Table I of Ref. 17). The tests of Secs. IV D and IV E do not give significant deviations from randomness.

We apply similar analyses to the reaction $Sn+\gamma$, for which $\sigma_{\gamma t}-\sigma_{g.d.}$ can be obtained from Fig. 3 of Ref. 39, for $6 < E_\gamma < 9.5$ MeV. The values of $\sigma_{\gamma t}-\sigma_{g.d.}$ separated by 70 keV are plotted in Fig. 3. The results of the test of Sec. IV F are given in Table XVII. We conclude to the existence of an IS between 7.5 and 8.8 MeV. The results of the test of Sec. IV C applied to the data of Fig. 3 are the following, for a reference value equal to 7 mb (Sec. IV C and Ref. 17): $U=10$, $m=n=17$, and $\Pr[\tilde{U}\le 10] = 0.004$. They indicate nonrandomness, on a 0.004 significance level. There exists a run of length 11 above the median, for $7.5 < E_\gamma < 8.8$ MeV. Table I shows that this run above implies IS, on a significance level $\alpha < 0.01$. However, the present analysis is less reliable than the preceding one ($^{90}Zr+\gamma$), because natural Sn is used as a target and several isotopes contribute to the cross section. Thus, it is not certain that we are dealing with a sample of values drawn from a single population.

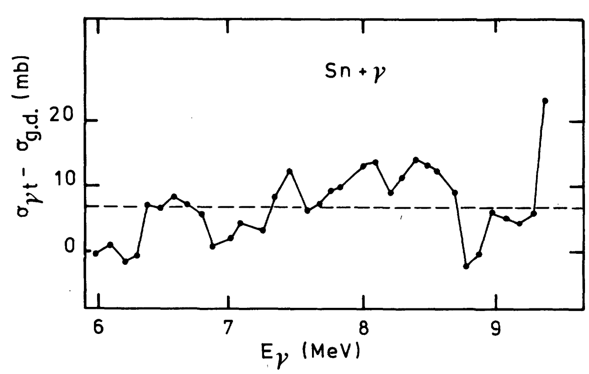

FIG. 3. Values of the difference between the photointeraction cross section and the tail of the giant dipole resonance, for $Sn+\gamma$, for energies separated by 70 keV. The dashed line is the sample median.

G. $^{70}Ge(p,p)$

Temmer *et al.*[40] found five substructures within one $\frac{1}{2}^+$ isobaric analog resonance at 5.05 MeV in $^{70}Ge(p,p)$; the energy resolution was about 3 keV. The widths of these substructures ($\simeq$ 20 keV) are much larger than the individual widths ($\simeq$ 1 keV) of the compound nuclear states. The average spacing of the compound nuclear resonances is also about 1 keV. Refined measurements of the cross section of $^{70}Ge(p,p)$, with an energy resolution of 400 eV, were recently performed by Meyer[41] *et al.* We subtract from the data of Meyer[41] *et al.*, averaged over 3 keV, a smooth cross section, corresponding to an average over an energy interval $I > 20$ keV, for a scattering angle $\theta = 90°$. The points which fulfill the conditions given in Sec. III C are separated by about 4.4 keV. These values, $\Delta(d\sigma/d\Omega)_{c.m.}$, are shown in Fig. 4, for $4.94 < E_p < 5.10$ MeV. The substructure at 5.13 MeV[40] is excluded from the analysis, since it does not correspond to a $\frac{1}{2}^+$ contribution.[41] We apply to the data of Fig. 4 the test of runs up and down (Sec. IV F). It has the advantage of being fairly independent of the precise value and shape of the smooth cross section obtained by averaging over $I > 20$ keV. The results are shown in Table XVIII. We see that the number of short runs is too small, and the number of long runs too large, thus implying nonrandomness. The most striking structure is located at about 5.04 MeV (Fig. 4). Even if the values corresponding to this structure are excluded from the statistical analysis, four runs of length 3 remain, in a sample of size 29. This still implies nonrandomness, on a 0.02 significance level, which shows that significant substructures, other than the one at 5.04 MeV exist, presumably at about 4.97 and 5.06 MeV. The test of Sec. IV C applied to the data of Fig. 4 gives the following results[17] for the reference value at 5 mb/sr: $U = 17$, $m = 19$, $n = 18$, $\Pr[\tilde{U} \leq 17] = 0.25$. The randomness hypothesis cannot be rejected with the test of Wald and Wolfowitz. The same conclusion is reached if we apply the tests of Secs. IV D and IV E. Finally, we note that some of the substructures appear to be excited in inelastic scattering,[40] in a way which depends upon the nature of the outgoing channel. It is only in the elastic channel, however, that at least three significant structures exist.

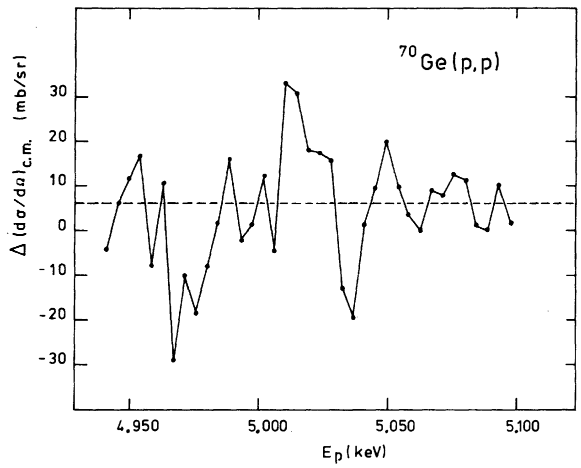

FIG. 4. Values of the difference between the differential cross section ($\theta = 90°$) averaged over 3 keV and over $I > 20$ keV, respectively, for $^{70}Ge(p,p)$, and for energies separated by 4.4 eV. The dashed line is the sample median.

H. $^{239}Pu(n,f)$

The total, elastic, and fission cross sections for $^{239}Pu + n$ were extensively studied by the Saclay group[42,43,16] for $0 < E_n < 660$ eV. The spins and parities of the resonances were identified. Two statistical tests indicate a nonstatistical behavior for the 1^+ resonances.[16] The first one involves the fission widths, averaged over 110 eV ($\langle\langle\Gamma_f\rangle\rangle$). By a Monte Carlo method, the very small value of $\langle\Gamma_f\rangle$ obtained between 550 and 660 eV was found incompatible with pure statistical fluctuations of the Γ_f.[16] The second test was that of Wald and Wolfowitz (Sec. IV C). It was applied to the individual fission widths for the 1^+ resonances and it was found that $F(\Gamma_f)$ is not a random series.[16] With this test it is, however, not possible to find the position of IS. We therefore apply the test described in Sec. IV D, which involves the length of the longest run above the median. Figure 5

TABLE XVII. Runs up and down for $Sn + \gamma$, for the difference between the measured cross section and the tail of the giant dipole resonance (Fig. 3).

d	$K(d,34)$	$I(d,34)$	σ_I	P
1	7	14.23	3.75	0.03
2	7	5.99	1.91	0.30
3	1	1.66	1.15	0.28
≥4	2	0.42	0.62	0.005

TABLE XVIII. Runs up and down for $^{70}Ge(p,p)$, for the difference between the cross section averaged over 3 keV and over $I > 20$ keV, respectively.

d	$K(d,37)$	$I(d,37)$	σ_I	P
1	13	15.48	3.91	0.26
2	1	6.54	2.00	0.003
≥3	6	2.26	1.27	0.002

shows all the fission widths calculated for the identified 1^+ resonances, between 0 and 660 eV.[43,44] The dashed line is the median of the sample of the 145 fission widths. We see that a run, above the median, of length 12 is present between 355 and 430 eV. Table II shows that this run implies nonrandomness for the $(\Gamma_f)_{1^+}$, on a significance level $\alpha \simeq 0.01$. We also apply the test based on runs up and down (Sec. IV F) to the $(\Gamma_f)_{1^+}$. We do not find significant deviations from randomness with this test. We also consider the mean of the fission widths, $\langle\Gamma_f\rangle$, on each 110-eV energy interval. We estimate the standard deviation of this mean, in each 110-eV interval, by the method described in Sec. V A, which assumes a Porter-Thomas distribution for the Γ_f. We have:

$$\sigma_{\langle\Gamma_f\rangle} \simeq \langle\Gamma_f\rangle_{\text{obs}}\, 2^{1/2} n^{-1/2}, \qquad \text{(V.12)}$$

where n is the number of resonances in the corresponding 110-eV interval. Assuming then a normal distribution for $\langle\Gamma_f\rangle$, we calculate confidence intervals, on a 0.05 significance level, for the corresponding actual mean, on each interval. These values are shown in Fig. 6 (full lines). We also compute the mean m of the full sample of $(\Gamma_f)_{1^+}$ ($N = 145$) and a confidence interval for the corresponding actual mean, on a 0.05 significance level. They are represented by the dashed lines in Fig. 6. We see that the region between 550 and 660 eV implies a nonstatistical behavior. This result is in agreement with the Monte Carlo calculations of Ref. 16. We return to this point below, where we show that the apparent anomaly around 600 eV is probably due to the existence of IS around 400 eV, in the sense that the anomaly at 600 eV disappears if the points around 400 eV are omitted from the analysis. We note that some fission widths for the 1^+ resonances could not be calculated and that some values of $(\Gamma_f)_{1^+}$ used in our analysis are maximum values.[44] It is clear that if many 1^+ fission widths have been omitted in the analysis, all the results of the statistical tests are very questionable.

We also performed an analysis of the 103 values of $(\Gamma_f)_{1^+}$ published in Refs. 42 and 43, which do not include 42 values of Γ_f which are not accurately known and may be maximum values. We first applied to this sample of 103 resonances the test of Sec. IV E, which involves the VORL, k. In the present case, $k = 15$ meV, $f_k = 9$. Table V shows that the sample is not random, on a 0.05 significance level. The longest run above k (length 9) lies around 400 eV, while the longest run below k (length 12) is located around 600 eV. It is the latter set of points between 550 and 660 eV which was found responsible for a significant deviation from the statistical be-

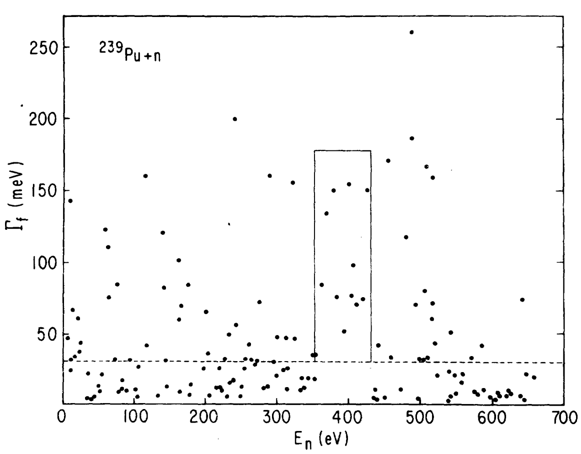

FIG. 5. Fission widths for the 1^+ resonances between 0 and 660 eV, for ^{239}Pu + n. The dashed line is the sample median.

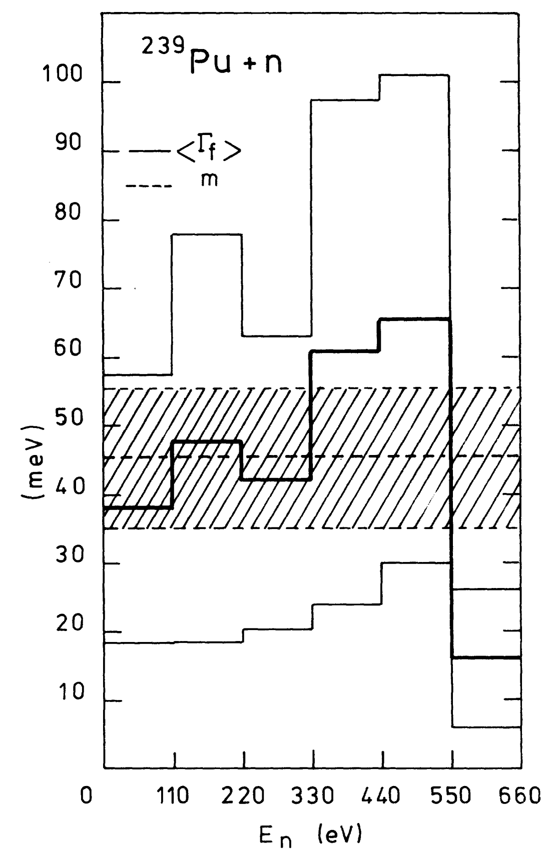

FIG. 6. In full lines: $\langle\Gamma_f\rangle_j = \frac{1}{n}\sum_{\lambda\in\Delta E_j}(\Gamma_{\lambda f})_{1^+}$ and confidence interval for the corresponding actual mean ($\alpha = 0.05$); n is the number of 1^+ resonances in each interval $\Delta E_j = 110$ eV. In dashed lines: $m = \frac{1}{145}\sum_\lambda (\Gamma_{\lambda f})_{1^+}$ and confidence interval for the corresponding actual mean ($\alpha = 0.05$).

havior in Ref. 16 (Monte Carlo method). Such a set of abnormally *small* widths appears difficult to reconcile with the doorway state model of IS, which would lead to an *enhancement* of the widths. One may therefore conjecture that the anomaly is due to the long run *above* the VORL. We investigated this possibility in applying the test which is based on the existence of large adjacent values (Sec. IV I) to the run above k.

We first calculate, using the same method as in Sec. V A, a confidence interval ($\alpha = 0.05$) for the mean μ of the sample of 103 values of $(\Gamma_f)_{1^+}$, and obtain

$$\Pr[23.1 \text{ meV} < \mu < 40.5 \text{ meV}] = 0.95\,. \qquad (\text{V}.13)$$

Between 380 and 430 eV, we find an unbroken sequence of six values of Γ_f larger than 1.3 times the upper limit of the confidence interval (V.13). Assuming a Porter-Thomas distribution for the Γ_f, we have (Secs. IV I and V A)

$$p = 0.25, \quad q = 0.75\,. \qquad (\text{V}.14)$$

The expectation value and the variance of the number of runs of length equal to or larger than 6 are [Eqs. (IV.12) and (IV.13)]

$$E(\tilde{S}_6) = 0.018, \;\; \sigma^2(\tilde{S}_6) = 0.018. \qquad (\text{V}.15)$$

The Tchebycheff inequality then gives

$$\Pr[\tilde{S}_6 > E(\tilde{S}_6) + 4.5\sigma(\tilde{S}_6)] = \Pr[\tilde{S}_6 > 0.621] \le 0.05. \qquad (\text{V}.16)$$

Hence, the observed value $S_6 = 1$ shows that the six adjacent values of Γ_f between 380 and 440 eV imply a significant deviation from randomness for $F(\Gamma_f)$. The same method also leads to the conclusion that the existence of 12 small adjacent values between 575 and 645 eV implies a significant deviation from randomness. The latter conclusion is also reached when the six large widths between 380 and 440 eV are omitted in the analysis. However, this long run below k disappears if we consider the sample[44] of 145 values of $(\Gamma_f)_{1^+}$ which was studied above. This is gratifying, since a long run *below* has no known theoretical interpretation.

I. ^{206}Pb(n, n)

In the elastic scattering of neutrons by ^{206}Pb, 11 $\frac{1}{2}^+$ resonances have been observed between 200 and 650 keV, and 3 further ones between 650 and 750 keV.[45] We leave out the latter resonances from the present discussion, since their parameters are somewhat uncertain because of a sizable p-wave background. We have previously applied[5] to the 11 values of Γ_λ obtained from Ref. 46 and to the 10 values of v_j^2 taken from Ref. 37, the test based on runs up and down (Sec. IV F) and found a significant deviation from randomness for the Γ_λ ($\alpha = 0.04$) but not for the v_j^2. We now apply to these two sets of quantities the test based on the number of runs about a reference value (Sec. IV C). We obtain (Fig. 1 of Ref. 5 and Table I) for the Γ_λ and a reference value lying at 2.2 eV: $m = 8$, $n = 3$, $U = 2$, and $\Pr[\tilde{U} \le 2] < 0.05$; for the v_j^2 and a reference value at 1.3 MeV^2: $m = 3$, $n = 7$, $U = 2$, and $\Pr[\tilde{U} \le 2] < 0.05$. These results imply nonrandomness for the Γ_λ and for the v_j^2. We notice that the reference values are not the median of the samples. If the reference value is chosen equal to the median, the randomness hypothesis cannot be rejected. We also applied the tests of Secs. IV D and IV E and did not find significant deviations from randomness with these tests, for the Γ_λ and v_j^2.

VI. CONCLUSIONS

We developed in the present paper several statistical methods for testing whether a set of data, ordered in a prescribed way, is a random sequence. These tests can be applied to the determination of the significance level of an assumed IS, whenever the data can be associated with a given angular momentum and parity. This restriction is essential, because the observed quantities should be drawn *from the same population*. In practice, these methods are therefore applicable at low energy, or when the background due to other angular momentum components can be reliably evaluated. These tests can be applied to resonance parameters, or to a sequence of *averaged* quantities. In the latter case, however, one must choose the sequence of points in such a way that they should form a random series in the frame of the statistical model (Sec. III C).

Among the methods discussed here, two have been applied previously: the test of Wald and Wolfowitz (Sec. IV C) and the test based on the serial correlation coefficient (Sec. IV H). The test of Wald and Wolfowitz is valid for an arbitrary distribution. The test based on the serial correlation coefficient has been developed for normal populations. It is a very powerful one; this unfortunately also implies that the conclusions drawn from this test are very sensitive to experimental errors. We described and applied several new tests. The first two are based on the longest run above the median (Sec. IV D) and about the VORL (Sec. IV E), and are valid for any distribution. In Sec. IV G, we proposed the use of a test based on the mean-square successive difference; it shares the same qualities and drawbacks as the method based on the serial correlation coefficient (Sec. IV H). These two tests are well adapted to exhibit a smooth nonrandom energy dependence of the mean of a normal population, but are very powerful, and therefore very sensitive to experimental errors. For instance, we showed in Sec. V D that the significance level obtained from these tests can be drastically modified by the omission of only one resonance in the region of the assumed IS, even in a large sample. In Sec. IV I, a new method based on the existence of several large and consecutive observed quantities is proposed. It appears to be very useful in some cases, but requires the knowledge of the probability distribution of the tested quantities. Another test (Sec. IV F) involves runs up and down. This test is valid for arbitrary distributions; it is not very powerful, and is therefore less sensitive than others to the presence of experimental errors, in the case of large samples. In general, the test of Wald and Wolfowitz (Sec. IV C) cannot determine the position of the IS. On the contrary, the tests involving the length of the runs above the median (Sec. IV D), or around the VORL (Sec. IV E), and the method based on the existence of adjacent large values (Sec. IV I) can locate the IS. The test involving runs up and down (Sec. IV F) can also localize the IS phenomenon when runs up and down of abnormally large length are present. Finally, we recall that it is sufficient that only *one* statistical test yields *significant* deviations from the statistical model to establish the existence of IS.

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