PHYSICAL REVIEW C VOLUME 9, NUMBER 2 FEBRUARY 1974

Nucleon-nucleon scattering analyses. I. Proton-proton scattering, from 1 to 500 MeV*

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(Received 22 June 1973)

Energy-independent and energy-dependent analyses are given for the proton-proton scattering data from 1 to 500 MeV. These analyses are similar to those of the Livermore series, but a number of modifications have been made, particularly in the low-energy representation. First, the range of the analyses has been extended by 50 MeV (i.e., from 450 to 500 MeV) and provisions have been made to include total cross sections in the analyses. This and new data additions have resulted in a 20% expansion of the data base. The other modifications concern the low-energy data and the representation bases. To help alleviate normalization difficulties with the low-energy data, the normalization errors of all experiments in the 0- to 10-MeV energy range were floated. The S-wave scattering-length and effective-range parameters were adjusted to give the best fit to the data, and the energy-dependent representation was modified to account for low-energy Coulomb effects. A 28-parameter energy-dependent solution is obtained which is quite good, with $\chi^2 = 1052$ for 1233 data. This is a definite improvement over the Livermore result, but the phase shifts in the 50- to 400-MeV range are virtually unchanged (with the exception of the $^3F_4$ and $^1G_4$ phases) despite the modifications to the bases functions, a strong indication that the data place strong restrictions on the phases.

[NUCLEAR STRUCTURE energy-dependent, energy-independent phase-shift analyses of 1–500-MeV p-p scattering data.]

I. INTRODUCTION

A number of additions to the proton-proton (p-p) scattering data between 1 and 500 MeV have occurred since the last Livermore p-p analyses.[1] In addition, it has been shown[2] that the Livermore energy-dependent representation is inadequate for the description of low-energy ($E < 10$ MeV) cross sections, which we believe is due to Coulomb barrier effects which have not been accounted for.

In the present paper we present an analysis of the expanded p-p data set, including total cross sections, from 1 to 500 MeV. The energy-dependent representation has been altered to include low-energy charge effects and the low-energy S-wave parameters (effective range and scattering length) have been adjusted in order to obtain a reasonable fit to the low-energy cross sections.

For completeness, we list the results of single-energy analyses at 25, 50, 95, 142, 210, 330, and 425 MeV. Most of these results are similar to the last Livermore analysis [hereafter referred to as L(X)], although there are some differences, due to data additions and modifications.

Finally, we list completely, with relevant comments, the data as used in the indicated analyses. The reason for such throughness is that we wish to provide a self-contained and complete basis for future partial-wave analyses at higher energies in anticipation of the new measurements to be done on high-intensity proton accelerators, such as those at Los Alamos (LAMPF) and Vancouver (TRIUMPH).

II. MODIFICATIONS TO THE LOW-ENERGY REPRESENTATION

In L(X), the phase shifts were parametrized with the form

$$\delta_l = \delta_l^{\text{OPEC}} + \sum_i \alpha_{l,i} f_{li}, \qquad (1)$$

where l is an angular momentum index and the energy arguments have been suppressed. The expansion bases f_{li} were defined in paper VII of the Livermore series[3] and δ_l^{OPEC} represents the one-pion-exchange contribution calculated with the Born approximation (for $l > 0$). In the $^1S_0$ state δ_l^{OPEC} was given by an effective-range expansion which included both Coulomb and vacuum polarization effects. The low-energy parameters $a = -7.815$ fm and $r = 2.795$ fm (the scattering length and effective range, respectively) were taken from Noyes's[4] fit to the low-energy scattering data.

It has subsequently been determined that the low-energy cross sections (below 10 MeV) are internally inconsistent[3] and that the L(X) energy-dependent representation does not give a "good" fit to the data. [Actually the L(X) solution produced χ^2/datum

~1, but as Sher, Signell, and Heller[3] (which we shall refer to as SSH) point out, a better fit can be obtained.] To remedy this situation, we shall introduce three modifications to the L(X) phenomenological representations.

Consider Eq. (1) for $l>0$. The threshold energy dependence of this expression is k^{2l+1} (k the barycentric nucleon momentum). This is proper for an uncharged reaction, but it is well known[5] that the Coulomb effects in p-p scattering introduce an additional factor of $C_l(k^2)$, where

$$C_0(k^2)=\frac{2\pi\eta}{e^{2\pi\eta}-1},$$

$$\eta=\frac{e^2}{\hbar c}\left(\frac{c}{\nu_r}\right),$$

ν_r is the relative velocity of the protons, and

$$C_l=\left(1+\eta^2/l^2\right)C_{l-1}\,. \qquad (2)$$

Clearly, to optimize our low-energy fits, our phenomenological representation must have the proper threshold behavior. To achieve this, we introduce new basis functions

$$f_{il}\rightarrow C_l f_{il} \qquad (3)$$

and modify the OPEC

$$\delta_l^{\mathrm{OPEC}}=C_l B_l,$$

where B_l is the Born approximation to the one-pion-exchange contribution. The entire effect of these modifications is contained in the low-energy behavior of the low-angular-momentum ($l\leq 1$) states. The higher-angular-momentum states are so small at low energies that the additional Coulomb suppression of such phases is unimportant in fitting the data.

The second step taken to improve the low-energy results involved varying the S-wave threshold parameters to fit the data. We shall continue to call these low-energy parameters a (scattering length) and r (effective range), since they are the parameters which are used in an effective-range expansion. It must be remembered, however, that they have been adjusted for a best fit to all of the data; therefore their values depend upon both the form of our phenomenological representation and upon the complete data base used.

The procedure for determining a and r is as follows: We construct a "data" set consisting of the scattering data up to some maximum energy E_{max} and the S- and P-wave phase shifts as given by L(X) single-energy analyses above E_{max}. We then fit these "data" sets by varying only the S- and P-wave parameters, all other partial waves being given by their energy-dependent L(X) values. The results were remarkably stable as E_{max} varied from 10 to 150 MeV, and all values were consistent with $a=-7.761\pm0.0098$ fm and $r=2.687\pm$

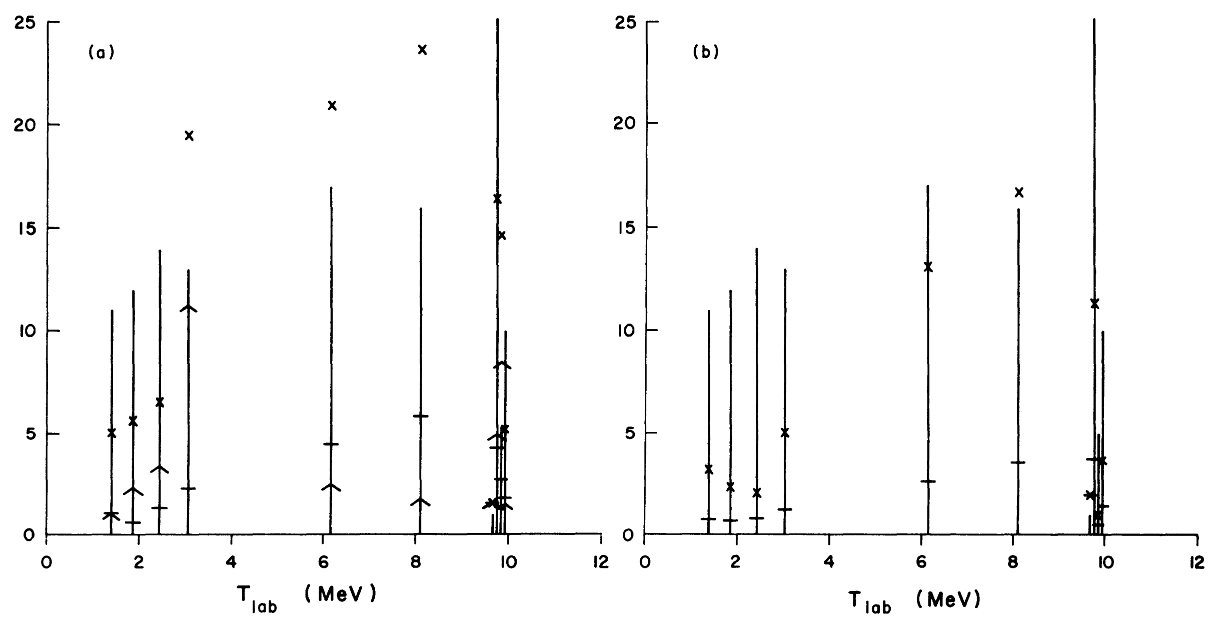

FIG. 1. The low-energy χ^2 distribution (a) before floating the low-energy data and (b) after floating. In the above figure, × represents the total χ^2 contribution from each data set, — represents the largest single contribution to χ^2 from each data set, and ∧ represents the contribution to χ^2 from the over-all normalization. The total number of data points in each experiment is given by the vertical line.

±0.0146 fm. These values, which were obtained for $E_{\max} = 27.6$ MeV, were subsequently used for the energy-dependent analysis.

Our final observation is that much of the internal inconsistency between the low-energy (1 to 10 MeV) cross sections in our representation can be relieved by freely renormalizing (floating) this data; in fact the entire effect of such a renormalization is to reduce the χ^2 contribution from the 125 data points by about 60 (a factor of 2). The values of a and r which were reported above are for floated data. A determination of a and r from the unfloated

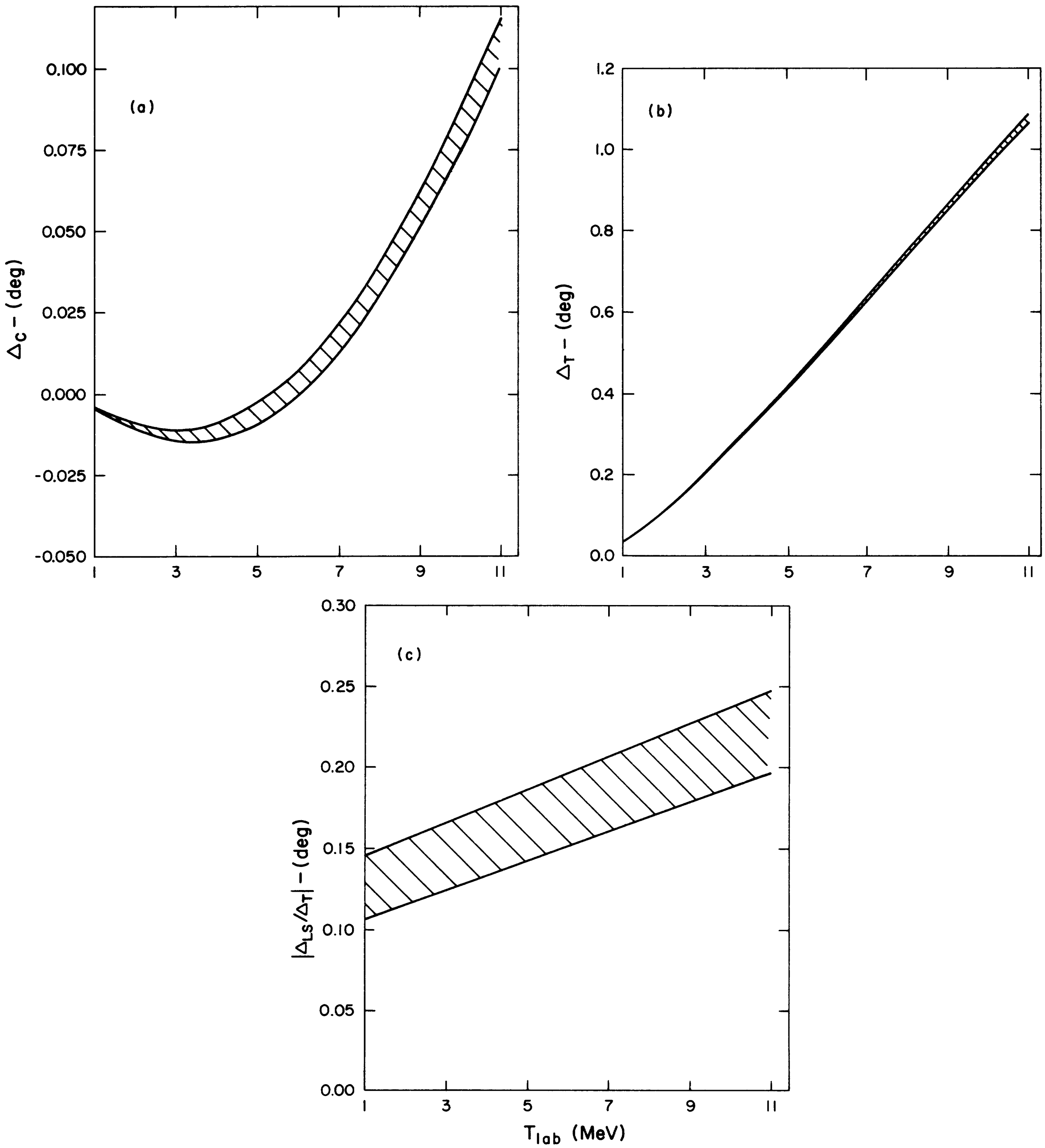

FIG. 2. The low-energy scattering parameters (a) Δ_C, (b) Δ_T, and (c) $-\Delta_{LS}/\Delta_T$ as determined from the 0 – 500-MeV solution.

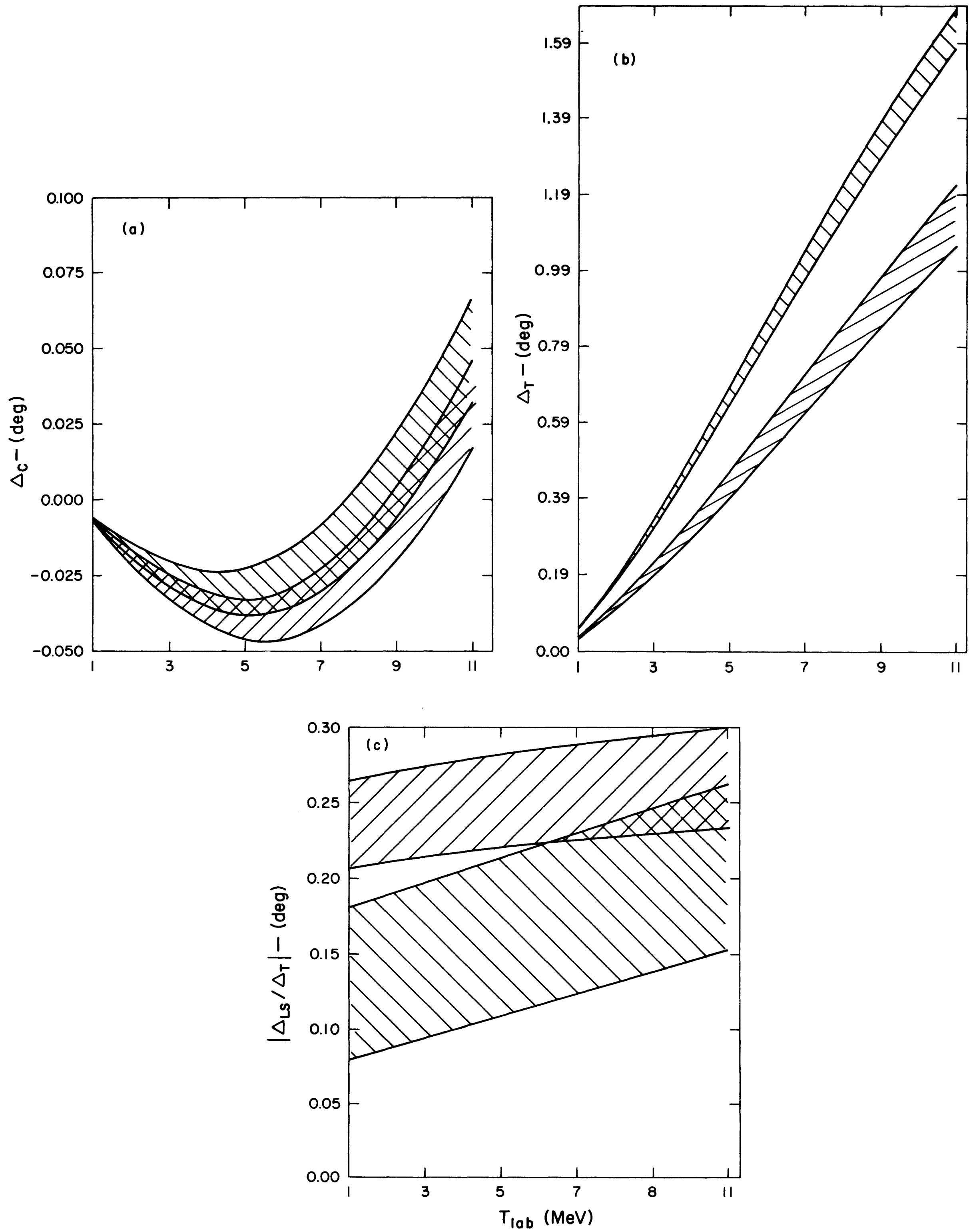

FIG. 3. A comparison of the scattering parameters (a) Δ_C, (b) Δ_T, and (c) $-\Delta_{LS}/\Delta_T$, for floated and unfloated low-energy data (see text). In all instances, the uppermost solutions correspond to the floated data.

TABLE I. Table I lists the experiments, the experiment type (polarization, P; differential cross section, σ; total cross sections, σ_T; etc.), their normalization errors, the predicted normalization, the χ^2 per datum (M), the references, and relevant comments.

Energy	No. and type of data	Angular range (c.m.) (deg)	% data std error	% norm. std. error	M value	Predicted norm.	References	Comments
1.397	11 σ	12–70	0.2	100.1	0.25	0.998	7	a, b
1.855	12 σ	12–80	0.2	100.1	0.15	1.001	7	c, a, b
2.425	14 σ	12–100	0.2	100.1	0.16	1.001	7	a, b
3.037	14 σ	12–90	0.2	100.1	0.24	1.003	7	a, b
6.141	17 σ	12–100	1.6	100.1	1.14	0.992	8	d, e, a
8.097	16 σ	12–90	1.1	100.1	1.41	0.990	8	d, e, a
9.680	1 σ	90	1.2	0.0	2.12	1.000	9	f
9.690	26 σ	10–89	0.9	100.7	0.65	0.978	10	f, a
9.690	5 σ	26–60	0.7	100.3	0.52	0.975	11	g, a, b
9.918	10 σ	25–100	0.6	100.3	0.78	0.989	11	g, h, a, b
11.400	1 C_{NN}	90	1.6	0.0	0.03	1.000	12	f
11.400	1 A_{XX}	90	10.0	0.0	0.01	1.000	12	f
13.600	11 σ	20–110	0.8	0.3	0.83	0.995	11	g, b
14.160	17 σ	18–114	5.4	10.0	0.08	0.996	13	
16.200	1 P	50	0.7	0.0	0.76	1.000	14	f
18.200	8 σ	30–90	1.1	1.5	0.74	0.874	15	f
19.200	1 C_{NN}	90	1.8	0.0	0.06	1.000	12	f
19.200	1 A_{XX}	90	1.9	0.0	0.17	1.000	12	f
20.200	8 P	34–90	0.2	11.8	0.16	1.013	13	d
23.500	1 C_{NN}	90	2.0	0.0	0.37	1.000	12	f
23.500	1 A_{XX}	90	2.4	0.0	0.27	1.000	12	f
25.630	23 σ	10–89	1.0	100.9	0.36	0.973	16	f
26.500	1 C_{NN}	90	1.1	0.0	0.76	1.000	12	f
26.500	1 A_{XX}	90	1.3	0.0	4.01	1.000	12	f
27.000	1 C_{NN}	90	7.0	0.0	0.26	1.000	17	f
27.600	2 R	23–54	4.4	3.0	0.57	1.010	18	
27.600	3 A	23–54	2.6	3.0	1.75	1.006	18	
28.160	1 σ	90	1.9	0.0	0.03	1.000	9	f
30.000	1 P	45	0.3	0.0	4.30	1.000	19	
31.150	1 σ	90	1.5	0.0	0.52	1.000	9	f
34.200	1 σ	90	1.5	0.0	2.64	1.000	9	f
36.900	1 σ	90	1.5	0.0	1.17	1.000	9	f
37.230	1 C_{NN}	90	1.3	0.0	0.00	1.000	20	g
37.230	1 A_{XX}	90	1.6	0.0	3.47	1.000	20	g
39.400	27 σ	8–89	1.4	0.9	1.17	1.000	21	f
39.600	1 σ	90	1.5	0.0	0.98	1.000	9	f
41.000	1 σ	90	7.6	0.0	1.01	1.000	22	
44.660	1 σ	90	1.7	0.0	0.54	1.000	9	f
46.000	1 P	45	1.2	0.0	1.52	1.000	23	
47.500	4 A	39–87	2.7	5.0	0.96	1.039	24	
47.500	1 C_{NN}	90	3.9	10.0	0.02	0.984	25	g
47.500	1 A_{XX}	90	3.2	10.0	3.60	1.217	25	g
47.800	5 R	23–87	3.8	5.0	1.34	1.006	18	
47.800	5 A	23–87	3.5	5.0	0.16	0.994	18	
49.400	28 σ	13–90	0.6	0.3	1.14	0.999	26	
49.900	1 P	34	0.2	0.0	0.01	1.000	19	
50.000	1 D	70	7.5	0.0	0.68	1.000	27	f
50.170	1 σ	90	1.7	0.0	0.27	1.000	9	f
51.500	1 σ	90	7.5	0.0	0.92	1.000	22	
51.500	9 σ	16–35	3.8	4.5	0.89	1.063	28	

TABLE I *(Continued)*

Energy	No. and type of data	Angular range (c.m.) (deg)	% data std. error	% norm. std. error	M value	Predicted norm.	References	Comments
51.700	1 P	60	0.9	0.0	0.49	1.000	29	f
51.800	9 σ	35–90	1.9	2.5	1.67	1.054	28	
52.000	1 C_{NN}	90	9.5	0.0	2.32	1.000	30	
52.000	1 C_{KP}	90	11.0	0.0	2.50	1.000	30	
52.340	25 σ	14–90	0.5	0.5	0.72	1.002	31	
53.200	1 P	75	0.8	0.0	1.76	1.000	29	f
56.000	1 P	45	0.6	0.0	0.03	1.000	23	f
56.150	1 σ	90	1.6	0.0	0.30	1.000	9	f
58.500	1 P	45	1.0	0.0	0.73	1.000	29	f
61.920	1 σ	90	1.6	0.0	0.12	1.000	9	f
66.000	10 σ	25–71	1.6	Float	0.13	0.970	23	f
66.000	11 P	20–71	0.8	2.8	1.00	0.970	23	f
68.300	25 σ	10–88	0.9	1.1	1.08	0.999	31	f
68.420	1 σ	90	1.6	0.0	0.12	1.000	9	f
69.500	1 σ	90	5.9	0.0	0.10	1.000	22	
70.000	1 P	45	0.6	0.0	2.73	1.000	29	f
71.000	1 P	45	0.8	0.0	0.60	1.000	23	f
73.500	1 C_{NN}	90	6.1	0.0	0.10	1.000	33	f
78.000	1 P	45	0.7	0.0	0.61	1.000	23	f
78.500	1 σ	90	5.9	0.0	0.02	1.000	22	
86.000	1 P	45	0.7	0.0	0.02	1.000	23	f
95.000	6 σ	40–90	3.5	Float	0.13	1.017	22	
95.000	6 σ	25–90	3.0	Float	0.36	1.000	22	f
95.000	1 σ	90	5.3	0.0	0.01	1.000	22	
95.000	13 σ	25–86	1.7	Float	0.19	1.037	23	f
95.000	14 P	20–86	0.6	2.8	1.33	0.998	23	f
97.000	1 P	45	0.5	0.0	2.53	1.000	29	f
97.700	13 P	16–88	0.3	0.8	0.70	1.002	34	d
98.000	14 P	10–81	2.1	2.0	1.07	1.019	25	f
98.000	5 D	20–61	10.3	0.0	0.88	1.000	36	
98.000	5 R	31–72	13.2	0.0	1.38	1.000	37	
98.000	1 C_{NN}	90	5.3	0.0	0.51	1.000	33	f
98.000	4 R'	31–62	22.8	0.0	0.20	1.000	37	
98.800	19 σ	22–88	1.2	1.0	0.82	1.006	34	d
102.000	3 σ	30–66	1.7	Float	2.25	1.022	23	f
102.000	3 P	30–66	0.6	2.8	1.07	0.971	23	f
107.000	3 σ	30–66	1.8	Float	0.16	1.083	23	f
118.000	14 σ	25–88	1.4	Float	1.08	1.053	23	f
127.000	3 σ	31–66	1.4	Float	0.28	1.052	23	f
130.000	4 P	20–81	2.0	3.3	0.54	1.015	38	f
137.000	3 σ	31–66	1.5	Float	0.02	1.022	23	f
137.000	3 P	31–66	0.6	2.8	0.79	1.014	23	f
137.500	5 R'	43–82	8.0	0.0	0.07	1.000	39	
138.000	4 D	31–82	12.4	0.0	1.37	1.000	40	
139.000	6 A	31–82	4.9	4.0	0.43	1.021	41	f
140.000	6 R	31–82	4.8	0.0	1.19	1.000	42	
140.400	5 R'	31–82	9.7	0.0	0.17	1.000	43	
140.700	19 P	16–87	0.5	0.8	0.96	0.986	44	d
142.000	26 P	5–82	1.8	102.0	0.81	0.932	35	f
142.000	7 D	12–71	4.5	0.0	1.23	1.000	45	

TABLE I *(Continued)*

Energy	No. and type of data	Angular range (c.m.) (deg)	% data std. error	% norm. std. error	M value	Predicted norm.	References	Comments
142.000	8 R	24–90	7.8	0.0	0.94	1.000	46,47	
143.000	7 D	31–92	10.0	0.0	0.37	1.000	48	
143.000	6 A	32–84	8.2	0.0	1.04	1.000	49	
143.000	2 C_{NN}	60–90	5.8	0.0	0.19	1.000	33	f
144.000	25 σ	4–41	1.4	0.6	1.07	1.002	50	g
144.100	6 σ	16–36	0.7	0.9	0.51	1.004	44	d
144.100	14 σ	46–87	0.4	0.6	1.05	0.992	44	d
147.000	28 P	6–87	0.9	2.8	1.09	0.981	23	f
155.000	21 σ	10–89	2.2	4.0	1.00	1.018	51	
170.000	7 P	31–82	1.8	3.3	0.41	1.015	38	f
174.000	5 P	20–72	3.3	6.6	0.80	1.062	52	
179.000	1 σ_T		0.5	0.8	0.22	0.996	53	g
210.000	7 σ	30–90	1.1	Float	0.90	1.025	54	
210.000	6 P	30–80	0.9	3.6	0.33	1.017	54	f
214.000	12 σ	9–38	1.6	1.3	1.09	1.005	54	
213.000	13 P	8–38	1.7	3.1	1.35	1.011	53	f
213.000	7 D	30–90	7.5	0.0	0.58	1.000	54	
213.000	7 R	30–90	6.0	0.0	0.15	1.000	54	f
213.000	2 A	80–90	11.7	0.0	0.89	1.000	54	f
213.000	5 R'	30–90	5.9	0.0	1.49	1.000	54	f
213.000	5 E	30–70	4.2	0.0	0.84	1.000	54	f
267.500	1 σ_T		0.4	0.8	0.00	1.000	53	g
276.000	6 P	19–76	2.2	7.5	1.68	1.135	55	f
305.010	14 C_{NN}	59–103	9.0	8.0	1.18	1.135	86	d
307.000	6 P	33–79	2.6	100.0	0.10	1.030	57	i
310.000	6 σ	7–21	6.6	Float	1.18	0.970	85	
310.000	7 P	6–21	16.0	4.0	0.60	1.012	85	f
310.000	6	23–80	6.5	0.0	1.29	1.000	55	
310.000	6 R	22–80	8.8	0.0	1.65	1.000	55	
315.000	7 σ	21–89	1.7	Float	0.93	1.037	55	
315.000	6 P	21–76	2.2	4.0	1.06	1.077	55	f
315.000	1 C_{NN}	90	15.0	0.0	0.44	1.000	56	
315.000	1 C_{NN}	45	51.0	0.0	0.04	1.000	57	
315.000	1 C_{KP}	45	51.0	0.0	0.00	1.000	57	
315.000	1 σ_T		4.1	0.0	0.00	1.000	58	
316.000	3 A	25–76	5.4	0.0	0.65	1.000	59	
320.000	1 C_{NN}	90	11.0	0.0	1.20	1.000	18	
328.000	5 σ	40–83	5.3	2.9	1.68	0.965	60	g
328.000	13 P	49–88	2.7	6.2	0.52	1.031	61	
330.000	15 σ	6–29	7.9	900.0	1.14	1.133	85	f
330.000	13 C_{NN}	59–99	12.6	8.0	0.35	1.070	62,68	d
330.000	1 σ_R		20.0	0.0	0.22	1.000	68	d
334.500	10 σ	43–89	10.0	5.0	0.53	1.036	63	g
334.500	11 P	43–89	5.1	6.0	1.08	1.070	63	g
342.500	1 σ_T		0.4	0.8	1.68	1.010	53	g
345.000	9 σ	15–53	3.2	Float	0.75	1.132	64	
345.000	15 σ	35–89	4.1	5.0	1.11	0.986	64	
370.000	1 σ_R		20.0	0.0	0.86	1.000	68	d
380.000	10 σ	14–30	1.6	101.3	0.63	0.976	65	j
380.000	6 σ	30–90	1.0	1.5	0.73	0.997	66	j

TABLE I *(Continued)*

Energy	No. and type of data	Angular range (c.m.) (deg)	% data std. error	% norm. std. error	M value	Predicted norm.	References	Comments
382.000	1 C_{NN}	90	8.4	0.0	2.29	1.000	18	j
382.000	1 C_{KP}	90	10.0	0.0	3.90	1.000	18	j
386.000	14 C_{NN}	58–100	6.3	8.0	1.70	1.219	62, 86	d
388.000	1 σ_T		0.4	0.8	0.04	1.001	53	g
394.000	7 P	33–82	3.1	100.0	0.10	0.965	57	i
394.000	7 P	33–83	6.7	100.0	0.09	0.981	57	i
399.000	2 C_{NN}	60–90	33.8	0.0	0.41	1.000	67	
399.000	2 C_{KP}	60–90	33.1	0.0	0.42	1.00	67	
400.000	1 σ_R		20.0	0.0	1.37	1.000	68	d
406.500	1 σ_T		0.4	0.8	0.27	0.996	53	g
408.000	1 σ_T		3.8	0.0	4.94	1.000	69	g
410.000	1 σ_T		5.3	0.0	0.02	1.000	70	g
410.000	1 σ_T		2.7	0.0	0.79	1.000	71	g
415.000	14 P	51–96	2.2	4.7	0.79	1.003	62	d
415.000	7 P	15–75	2.5	5.7	0.51	1.027	72	
415.000	1 D	90	9.0	0.0	5.07	1.000	73	
415.000	14 C_{NN}	51–96	4.9	108.0	0.92	1.277	62	d
418.000	1 σ_T		3.9	0.0	0.72	1.000	52	g
418.000	1 σ_R		24.4	0.0	3.82	1.000	52	g
418.600	10 σ	40–87	9.3	5.0	0.35	1.021	63	
418.500	9 P	40–87	4.4	6.0	1.21	0.994	63	
419.000	7 σ	28–90	4.9	10.0	1.80	1.134	74	
425.000	1 P	65	1.9	10.0	0.20	0.963	75	g
425.000	2 D	65–90	5.1	0.0	1.04	1.000	74	g
425.000	1 R	65	3.9	0.0	0.01	1.000	75	g
425.000	1 A	65	3.9	0.0	3.55	1.000	75	g
428.000	6 P	30–120	6.4	10.0	1.24	1.029	76	
429.000	7 D	30–120	17.3	0.0	0.36	1.000	76	
429.000	7 R	30–120	10.9	0.0	0.43	1.000	76	
429.000	6 A	45–120	13.4	0.0	1.45	1.000	76	
429.000	2 A'	65–115	2.5	0.0	0.88	1.000	77	
429.000	6 A'	30–120	11.3	0.0	2.09	1.000	76	g
430.000	1 P	65	1.3	0.0	2.05	1.000	77	d
430.000	2 D	65–115	3.2	0.0	0.01	1.000	77	d
430.000	2 R	65–115	3.0	0.0	2.43	1.000	77	d
430.000	2 A	65–115	2.5	0.0	1.65	1.000	77	d
430.000	2 R'	65–115	4.1	0.0	0.55	1.000	77	d
431.000	1 σ_R		22.2	0.0	0.19	1.000	78	e
437.000	8 σ	17–90	4.7	5.0	0.24	1.038	79	
439.500	1 σ_T		0.5	0.8	4.26	0.986	53	g
449.000	1 C_{NN}	90	15.0	0.0	1.93	1.000	67	
449.000	1 C_{KP}	90	14.0	0.0	0.09	1.000	67	
450.000	1 σ_R		15.0	0.0	3.18	1.000	80	j
460.000	2 σ	30–90	9.9	0.0	0.54	1.000	81	
460.000	9 σ	20–90	6.0	10.0	1.07	1.034	82	
460.000	1 σ_T		1.4	0.0	0.91	1.000	71	g
460.000	1 σ_R		70.0	0.0	0.01	1.000	83	
469.900	8 σ	33–87	5.6	3.4	1.33	0.997	60	
498.000	7 P	33–81	4.0	100.0	0.15	0.795	57	i
498.000	7 P	33–82	6.5	100.0	0.17	0.845	57	i

TABLE I *(Continued)*

Energy	No. and type of data	Angular range (c.m.) (deg)	% data std. error	% norm. std. error	M value	Predicted norm.	References	Comments
500.000	10 σ	42–89	9.3	5.0	0.47	1.010	63	g
500.000	8 P	36–84	5.5	6.0	0.92	0.940	63	
500.000	23 P	34–87	2.6	10.0	0.94	0.881	84	
500.000	1 σ_T		1.3	0.0	0.10	1.000	71	g

[a] All data below 10 MeV were floated for the purpose of the analysis (with the exception of the single cross section at 9.68 MeV).

[b] These data were not used as originally published: we floated them, meaning we added to each set a notmalization· "datum" with a norm standard error of 100%. The experimentally determined norm errors quoted in Ref. 2 were not removed from the (total) individual angle errors before floating. Thus the quoted value of χ^2 should be slightly corrected. If we denote by χ^2 the χ^2 value obtained in the analysis, then the corrected value of χ^2 may be estimated by

$$\chi^2 \simeq \overline{\chi}^2\left[1+\left(\frac{E_N}{E_i}\right)^2\right]=\overline{\chi}^2+\Delta\overline{\chi}^2,$$

where E_N and E_i are the normalization and individual angle errors, respectively. Using the approximate values in Ref. 2, we find

E_i (MeV)	$\Delta\overline{\chi}^2$
1.397	0.769
1.855	0.220
2.425	0.143
3.037	0.332

[c] The 90° datum was removed as suggested by M. S. Sher, P. Signell, and L. Heller (Ref. 2).

[d] Comments pertinent to this data set were given by M. H. MacGregor, R. A. Arndt, and R. M. Wright (Ref. 1).

[e] Data set BGS was selected in preference to Data set D since it was more compatible with neighboring measurements. The 9.918-MeV cross sections were excluded as suggested by M. S. Sher, P. Signell, and L. Heller (Ref. 2).

[f] Comments pertinent to this data set were given by M. H. MacGregor, R. A. Arndt, and R. M. Wright (Ref. 3).

[g] New data which were not included in the analyses referenced in comments d, f, and j.

[h] The 20.05° datum was removed as suggested by the authors.

[i] Angular errors for these measurements are important and in some instances swamp the statistical errors. We estimate $\Delta P/\Delta\theta$ by fitting

$$P(\theta)=\sin\theta\sum_{n,l}\alpha_{n,l}E^nP_l(\cos\theta)$$

to the entire set of polarization measurements between 300 and 500 MeV and differentiating. These errors were folded in with the statistical errors.

[j] Comments pertinent to this data set were given by M. H. MacGregor, R. A. Arndt, and R. M. Wright (Ref. 68).

data produces $a=-7.745\pm0.007$ and $r=2.669\pm0.009$ fm, which are within overlapping errors of the above values. Figure 1 is a graphic comparison of the fits obtained with and without the floating procedure.

In Fig. 2 we present the usual low-energy parameters (Δ_c, $-\Delta_T$, $-\Delta_{LS}/\Delta_T$) as calculated from the energy-dependent representation. These results (particularly Δ_c) are in essential agreement with the L(X) values but in apparent disagreement with some of the SSH values over the 1–10-MeV energy range. Holdeman, Signell, and Sher[6] have indicated that the disagreement is a consequence of the parameter boundedness of the L(X) solution. In order to clarify this situation, we analyzed the data from 1–27.6 MeV (207 data points) using two parameters in each P state and one variable in the $^1S_0$ state, for a total of seven phenomenological parameters. We feel that this eliminates any question of extreme form limitation. Two versions of the data were considered; one with the low-energy cross sections freely normalized (floated), and one with these data with fixed normalization. The low-energy parameters from these analyses are presented in Fig. 3. It would appear from our results that the low-energy P-wave parameters, particularly Δ_T and $-\Delta_{LS}/\Delta_T$, depend essentially upon our decision to float the low-energy data. We feel that the quality of the fit of our energy-dependent representation ($\chi^2=107$ for the 207 data points in the 1 – 27.6-MeV energy range) and any reasonable interpretation of the errors (defined in terms of χ^2 changes of 1), depicted in Figs. 2 and 3, are convincing evidence that the values obtained are reflections of our choice of data (floated low-energy cross sections) and do not result from an overly restrictive parametrization.

III. DATA BASE

The data base used for the present analyses and their references[7–89] is given in Table I. Most of these data were included in previous analyses[68] and the interested reader is referred to those analyses for appropriate comments. There are, however, a number of new additions, exclusions, and alterations of the old data.

The more notable of the additions are the differential cross sections of Jarmie *et al.*[11] at 9.69, 9.918, and 13.6 MeV; of Jarvis, Whitehead, and Shah[50] at 144 MeV; and of Ryan *et al.*[60] at 328 and 469.9 MeV; the differential cross sections and polarization data of Albrew *et al.*[63] at 334.5, 418.6, and 500 MeV; and the total cross sections of Schwaller *et al.*[53] at 179, 267.5, 342.5, 388, 406.5, and 439.5 MeV. In addition, there are triple-scattering data at a variety of energies. These additions result in a total of 243 new datum not used in the Livermore analyses.[90,91]

We have excluded the differential cross sections of Slobodrian *et al.*[8] at 9.918 MeV as suggested by SSH.[2] We find that even with the low-energy floating procedure described in the previous section, these data are not consistent with our solution or the surrounding data. This exclusion does not appear to be particularly significant, in view of the new data[11] available at this energy.

During the analysis, we found that the two polarization measurements of Cheng *et al.*[57] at 400 MeV were at variance both with the predicted values and with each other. A realistic appraisal of the error indicated that the rather large angular uncertainties (as much as 6.9°) were sufficient to swamp the statistical errors given. To estimate this source of error, we parametrize all existing p-p polarization data between 300 and 700 MeV with the form

$$P(\theta, E) = \sin\theta \sum_{n,l} \alpha_{n,l} E^n P_l(\cos\theta),$$

where E is the lab kinetic energy, θ the c.m. scattering angle, and P_l the Legendre polynomials. Our fits to the data are quite reasonable (average χ^2/datum ~ 1), but the parameters we obtain are definitely in disagreement with those listed in Ref. 57. We then fold this error, due to angular and energy uncertainties, in quadrature with the statistical error, i.e.,

$$(\Delta P)^2 = \left(\frac{dP}{dE}\Delta E\right)^2 + \left(\frac{dP}{d\theta}\Delta\theta\right)^2 + (\Delta P_s)^2 ,$$

where ΔP_s is the statistical error. These data and their revised errors are given in Table II.

In our final data selection, we arbitrarily exclude any datum whose χ^2 contribution is greater than 5, and we float any data set whose normalization χ^2 is greater than 5. This eliminates 24 of 1257 total data points and floats 4 of 206 total experiments (see Table I). This procedure has no effect on our solutions, but merely results in a reduction of the total χ^2 to 1052 (a drop of approximately 200).

TABLE II. The polarization data of Cheng and the revised errors as used in the analyses (see text).

Beam energy (MeV)	θ (deg)	Polarization
	(a) Liquid hydrogen	
310	33.6	0.402 ± 0.027
	47.3	0.374 ± 0.017
	50.1	0.362 ± 0.019
	62.4	0.276 ± 0.024
	66.3	0.217 ± 0.029
	79.4	0.117 ± 0.034
394	33.8	0.442 ± 0.017
	47.8	0.419 ± 0.023
	48.0	0.419 ± 0.028
	63.5	0.275 ± 0.035
	65.2	0.272 ± 0.040
	80.6	0.105 ± 0.034
	82.5	0.084 ± 0.035
498	33.7	0.490 ± 0.014
	36.8	0.510 ± 0.059
	46.9	0.461 ± 0.030
	48.7	0.452 ± 0.028
	64.6	0.313 ± 0.048
	81.3	0.107 ± 0.042
	81.8	0.113 ± 0.041
	(b) Liquid deuterium	
394	33.1	0.442 ± 0.080
	48.3	0.387 ± 0.054
	48.9	0.419 ± 0.052
	63.5	0.246 ± 0.070
	66.6	0.270 ± 0.078
	80.3	0.076 ± 0.065
	83.1	0.063 ± 0.066
498	33.4	0.507 ± 0.070
	48.5	0.426 ± 0.057
	49.1	0.412 ± 0.049
	64.7	0.264 ± 0.076
	65.0	0.275 ± 0.075
	81.2	0.116 ± 0.067
	82.1	0.110 ± 0.059

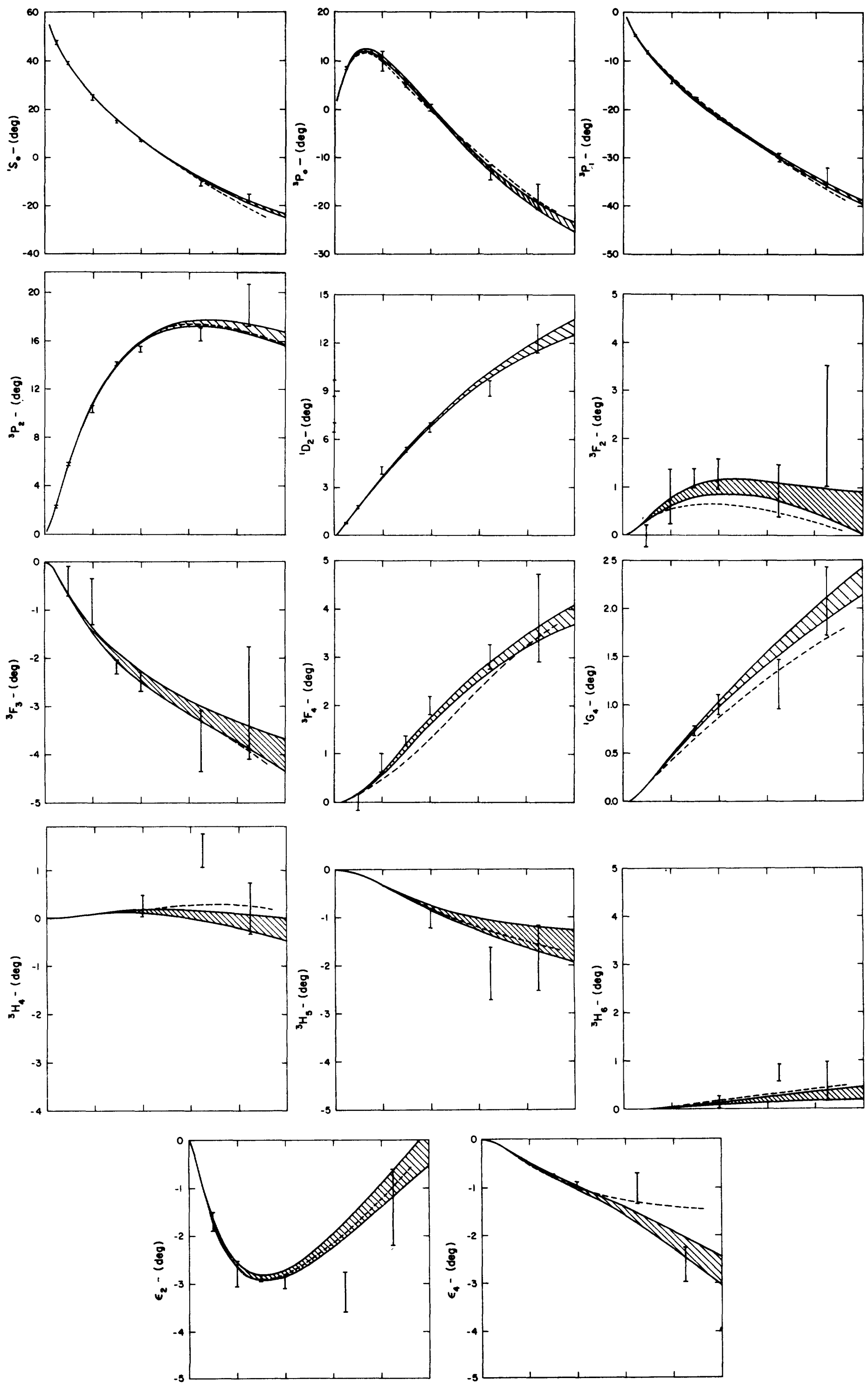

FIG. 4. Phase shifts from the energy-dependent and energy-independent solutions. The dashed lines refer to the L(X) energy-dependent solutions. The range of the horizontal axis is 1–500 MeV. The error bands on the energy-dependent solution, which are an admittedly rather nebulous measure of uncertainty, are defined in terms of a χ^2 change of 1.

TABLE III. Energy-independent solutions.

Energy (MeV)	25	50	100	150	200	325	425
No. of data	43	94	111	205	64	159	145
χ^2	17.6	83.3	80.6	173.8	47.4	118.7	119.1
Energy band (MeV)	20.2–30	47.5–56.15	95–107	130–155	210–213	305–345	394–439.5
$^1S_0$	48.03±0.37	39.20±0.37	24.68±0.97	14.72±0.56	7.01±0.53	−10.42±1.42	−17.15±1.76
$^1D_2$	0.79±0.03	1.79±0.10	4.09±0.23	5.34±0.17	6.74±0.30	9.18±0.48	12.27±0.88
$^1G_4$				0.73±0.05	1.00±0.10	1.22±0.26	2.07±0.35
$^3P_0$	8.55±0.35	11.46±0.61	9.93±2.05	5.17±0.50	0.17±0.55	−13.03±1.54	−18.10±2.68
$^3P_1$	−4.71±0.18	−8.17±0.30	−14.19±0.50	−17.64±0.15	−21.49±0.33	−29.87±0.89	−34.06±2.10
$^3P_2$	2.32±0.12	5.81±0.14	10.38±0.30	14.12±0.10	15.34±0.24	16.56±0.55	18.98±1.70
ϵ_2		−1.70±0.19	−2.79±0.26	−2.89±0.07	−2.93±0.16	−3.17±0.41	−1.40±0.79
$^3F_2$		−0.01±0.23	0.81±0.57	1.18±0.20	1.27±0.32	0.93±0.54	2.28±1.25
$^3F_3$		−0.41±0.31	−0.82±0.47	−2.18±0.15	−2.48±0.20	−3.71±0.62	−2.93±1.17
$^3F_4$		0.00±0.18	0.82±0.20	1.28±0.09	2.01±0.19	3.02±0.25	3.83±0.91
ϵ_4				−0.75±0.03	−0.97±0.90	−1.02±0.32	−2.61±0.37
$^3H_4$					0.26±0.22	1.40±0.35	0.21±0.54
$^3H_5$					−1.02±0.21	−2.17±0.55	−1.86±0.68
$^3H_6$					0.13±0.14	0.76±0.18	0.57±0.41

IV. PHASE-SHIFT RESULTS

A. Energy-dependent analysis

In the present energy-dependent 1- to 500-MeV analysis, we considered a total of 1233 data points. Our final solution contains 28 adjustable parameters (exclusive of a and r), 27 for the description of the elastic phases and 1 for the inelastic $^1D_2$ phase. This results in a total χ^2 contribution of 1042 for the data set, or an average χ^2 per datum (M) of 0.85. This represents a definite improvement over the L(X) solution, primarily in the fit to the low-energy data. Presumably, this is due to our modifications of the low-energy representation and to the floating procedure for the low-energy data.

During the course of the analysis, we examined, in some detail, the problem of the number of free parameters required to fit the data. Although an objective quantitative criterion is, of course, impossible, in our opinion the 28-parameter solution we present here represents the optimum for our present phenomenology.

The data used in the present analysis, the average χ^2 per datum, the predicted normalization and over relevant information are given in Table I. Figure 4 presents a graphical representation of our energy-dependent solutions and their error corridors.

B. Energy-independent analyses

In the interests of completeness, we have obtained new energy-independent solutions in narrow bands about 25, 50, 100, 150, 200, 325, and 425 MeV. To accommodate the finite width of each band, energy slopes are assigned to the phase shifts from the energy-dependent solution.

Of the 1233 total data points, 832 were incorporated into these analyses. The results are summarized in Table III and compared with the energy-dependent results in Fig. 4.

*Research sponsored by the National Science Foundation.

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[90]We have not included the low-energy data of J. E. Broiley, Jr., J. D. Seagrave, and J. G. Beery [Phys. Rev. 135, B1119 (1964)] in our analysis due to the approximate nature of our vacuum polarization corrections and the fact that our analysis code does not contain provisions for the simultaneous normalization of related experiments at different energies.
[91]It has been suggested that we include the 134- and 147-MeV integrated cross sections of R. Goloskie and J. N. Palmieri [Nucl. Phys. 55, 463 (1964)] in our analysis. Unfortunately our analysis code does not have provisions for such measurements. To compare the consistency of our predictions with these data we give the following table:

Average Energy (MeV)	(mb)	Prediction (mb)
147	23.69 ± 0.15	23.73 ± 0.29
134	24.89 ± 0.22	24.44 ± 0.28