Triton binding energies with two-body radial-distortion unitary transforms*

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We determine the triton binding energies for a class of potentials that arise from radial-distortion unitary transformations. These potentials are phase-shift equivalent to a two-term Yukawa potential, which represents an average of the $^3S_1$ and $^1S_0$ nucleon-nucleon potentials. We solve an angular-momentum-decomposed version of the Faddeev-Lovelace equations that we have developed, in order to obtain the three-body binding energy E_T. We observe that E_T varies slightly with these potentials and we find that they yield similar deuteron wave functions in accord with the results reported by Haftel. We discuss some evidence on the sensitivity of nuclear matter and three-body calculations to off-shell variations of the two-body t matrices.

[NUCLEAR STRUCTURE ^{3}H; calculated binding energy. Solved Faddeev-Lovelace equations, angular-momentum decomposed. Deduced deuteron form factors.]

I. INTRODUCTION

Recently there has been a great deal of interest in explorations of the arbitrariness of the off-energy-shell nucleon-nucleon scattering amplitude. These t matrices are at present only specified on the energy shell by the two-nucleon phase shifts. Many-body calculations require t matrices at various momenta and energies and offer, in principle, some hope of determining this off-shell nucleon-nucleon amplitude. Such a determination may come about when such calculations yield either a highly nonphysical result or a wide spread in the theoretical predictions for an experimentally determined quantity. Then the off-shell t matrices are examined and *if* the cause of the discrepancy can be isolated, some insight into the off-shell nucleon-nucleon interaction results. A difficulty with this approach is that many approximations are usually necessary before a many-body quantity is computed. Hence, any discrepancy may be due to an approximation and attempts to untangle such affairs are fraught with difficulty.

There is also the question of how to go from on-shell t matrices to off-shell t matrices. The simplest solution is to assume a potential and then solve the resulting Lippmann-Schwinger equation. Various alternative proposals have been advanced,[1-5] and while work continues on these approaches none of them is fully operational as yet. Hence we assume a potential that fits certain two-nucleon on-shell data. Off-shell variations are introduced by generating phase-shift equivalent potentials. The various methods for doing this are reviewed in Bahethi and Fuda,[6] which contains an excellent set of references to previous work.

We use unitary transforms which are sufficiently short ranged to preserve the two-body phase shifts. Such transforms are generally either of finite rank or of the radial distortion variety.[7] The finite-rank transforms are easier to apply and they appear in calculations for finite nuclei,[8] electron-deuteron scattering,[9] deuteron photodisintegration,[10] muon capture by deuterons,[11] pion production,[12] nuclear matter,[13,14] and the triton.[15-18] Several references to the applications of radial distortion transforms are given in Ref. 6. In addition, such transforms are found in calculations for the two-body problem,[19] deuteron photodisintegration,[20] and nuclear matter.[21]

We address ourselves here to a study of the three-body problem with potentials arising from the application of radial-distortion unitary transforms to a two-term Yukawa potential. We choose to investigate this problem since the Faddeev-Lovelace equations[22,23] provide a complete nonrelativistic theory for such systems. This is in contrast to the situation which exists for finite nuclei or for nuclear matter where higher-order terms are required and are only poorly known. We hope, therefore, that any possible conclusions on off-shell effects the three-body problem supplies will be free

of conflicting interpretations. For simplicity we shall deal with an average of the central $^1S_0$ and $^3S_1$ potentials. The results of Ref. 18 (see their Table II) indicate that our qualitative conclusions will not be affected by this approximation. We find that there are only slight variations in the three-body binding energy E_T in contrast with the differences of several MeV in the binding energy per particle that are found in current nuclear matter approaches with radial-distortion transforms.[7,21]

In Sec. II we discuss the form of the three-body equations that we solve, since we utilize a different form than that normally encountered. The details of the transformed potential and our numerical procedures are also given in this section, while Sec. III contains our results. In Sec. IV, we briefly discuss our findings and some related investigations in an effort to understand our results and those from nuclear matter.

II. EQUATIONS AND DEFINITIONS

A. Three-body equations

The Faddeev-Lovelace formulation of the three-body problem is based on two-body scattering amplitudes, and these are obtained from a partial-wave Lippmann-Schwinger equation

$$t_l(p, p', w) = v_l(p, p') + \int_0^\infty q^2 dq \frac{v_l(p,q) t_l(q, p', w)}{q_w^2 - q^2 + i\epsilon} . \tag{1}$$

The two-body energy $w = \hbar^2 q_w^2/(2m)$ MeV, while the t_l and the v_l have units of femtometers. The Faddeev-Lovelace equations are[23]

$$U_{\alpha\beta}(S) = \sum_{\delta \neq \beta} V_\delta(S) + \sum_{\gamma \neq \alpha} T_\gamma(S) G_0(S) U_{\gamma\beta}(S) . \tag{2}$$

V_δ and T_γ are the two-body potential operator and the two-body scattering operator in the three-body space, $G_0(S)$ is the free three-particle Green function, and S is the total three-particle energy. The subscripts in Eq. (2) run from 0 to 3 and represent the various three-body channels.

Let the mass and the momentum of each particle be m_α and $\vec{K}_\alpha$, respectively, and let $M = m_1 + m_2 + m_3$. We work in the total center-of-momentum frame

$$\sum_{\alpha=1}^{3} \vec{K}_\alpha = 0 , \tag{3}$$

and we define the spatial dependence of a three-body state by two momentum vectors, for example,

$$\vec{p}_1 = (m_3\vec{K}_2 - m_2\vec{K}_3)/[2m_2m_3(m_2+m_3)]^{1/2} ,$$
$$\vec{q}_1 = [m_1(\vec{K}_2 + \vec{K}_3) - (m_2+m_3)\vec{K}_1]/[2m_1(m_2+m_3)M]^{1/2} \tag{4}$$

$\vec{p}_1$ is the relative momentum of particles 2 and 3, while $\vec{q}_1$ is the relative momentum of particle 1 with respect to the 2-3 pair. Any two of the six vectors $\{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{q}_1, \vec{q}_2, \vec{q}_3\}$ may be used to define a basis and we denote the linear transformations between two such sets by

$$\vec{p}_\alpha = f_{\alpha\beta}\vec{p}_\beta + g_{\alpha\beta}\vec{q}_\beta ,$$
$$\vec{q}_\alpha = \tilde{f}_{\alpha\beta}\vec{p}_\beta + \tilde{g}_{\alpha\beta}\vec{q}_\beta . \tag{5}$$

The two chosen momentum vectors are combined with six spin coordinates and six isospin coordinates for a complete specification of a three-body state. A subscript on a bra or a ket labels the choice of basis.

We reduce Eq. (2) to manageable form through an angular-momentum decomposition which starts by sandwiching Eq. (2) between an $_\alpha\langle|$ state and a $|\rangle_\beta$ state. The decomposition of the integral term of Eq. (2) proceeds in a manner similar to the detailed decompositions of the analogous terms in Refs. 24 and 25. Our approach differs somewhat from these in that they work with the Faddeev equations and transform $_\alpha\langle|\, T_\alpha(S)\,|\rangle_\gamma$ into $_\alpha\langle|\, T_\alpha(S)\,|\rangle_\alpha$ while we transform $_\alpha\langle|\, T_\gamma(S)\,|\rangle_\gamma$ into $_\gamma\langle|\, T_\gamma(S)\,|\rangle_\gamma$. A complete discussion of the latter approach is contained in Ref. 26. Our immediate concern is with the case in which only two-body s-wave interactions and relative s states are allowed and we give the reduced form of Eq. (2) for this case when the three particles are identical spinless bosons without isospin. We let $U \equiv U_{\beta\beta} + 2U_{\alpha\beta}$ and

$$U(p_1, q_1, \tilde{p}_3, \tilde{q}_3, S) = \{\text{Inhomog. term}\} + 2\int_0^\infty dp_2' \int_L^U dq_2' f_2(p_1, q_1, p_2', q_2') t_2(\tilde{p}, p_2', S - (q_2')^2)[S - (p_2')^2 - (q_2')^2]^{-1} \times U(p_2', q_2', \tilde{p}_3, \tilde{q}_3, S) . \tag{6}$$

Here

$$f_2(p_1, q_1, p_2', q_2') \equiv (p_2')^2 q_2'/(2|\tilde{f}_{21}\tilde{g}_{21}|\, p_1 q_1), \quad \tilde{p}^2 = p_1^2 + q_1^2 - (q_2')^2, \quad \left.\begin{matrix} U \\ L \end{matrix}\right\} = (\tilde{f}_{21}^2 p_1^2 + \tilde{g}_{21}^2 q_1^2 \pm 2|\tilde{f}_{21}\tilde{g}_{21}|\, p_1 q_1)^{1/2} . \tag{7}$$

We do not specify the exact form of the inhomogeneous term {Inhomog. term} since the method we use to solve Eq. (6) does not require it. A distinguishing feature of Eq. (6) is that the limits of integration are independent of the integration variables. A formulation of the three-body breakup problem which takes advantage of this situation is presented in Ref. 26.

B. Method for locating bound states

We find the three-body bound state through Eq. (6) with a modification of the method used by Malfliet and Tjon.[27] They locate the bound-state pole with the iterated Faddeev equations by looking at the successive terms that contribute to a selected matrix element. The ground-state energy is the energy at which the ratio of successive terms tends to one. In this approach any initial term except one which is orthogonal to the eigenvector associated with the ground state may be used to start the iteration.

We found that the ratio method led to oscillations and that the successive ratios did not converge for the transformed potentials we consider here. We overcome this obstacle by calculating the three-body matrix element through diagonal Padé approximants,[28] which are based on the terms that give the ratios. A bound state is indicated by a pole in this matrix element, so we find where the matrix element changes sign and increases in magnitude. Convergence is generally achieved by the seventh approximant and all of the series terms except the first are used. Both the ratio and the Padé methods give the same three-body results when they are both applicable. Haftel[17] has also used this Padé approach to three-body bound states.

The double integral of Eq. (6) is replaced by a double sum through Gaussian integration rules. We let N_p and N_q be the number of sample points for the p_2' and the q_2' integrations, respectively. The needed parts of the integrand at the required momenta and energy values are obtained by interpolation on the set of terms from the previous iteration of the Faddeev-Lovelace equation. The necessary two-body scattering amplitudes are found by interpolation on a previously calculated grid, which was constructed by solving Eq. (1) at 30 energies. All the interpolations are done with quadratic polynomials.

C. Radial-distortion unitary transformations

We base our discussion on Coester, Cohen, Day, and Vincent[7] and we let R be a function of r such that

$$\frac{dR}{dr} \equiv [\mu(r)]^{-1/2} \geq 0 \tag{8}$$

for all r, and $R-r$ goes to zero for large r. The transformed two-body radial wave function is related to the untransformed by

$$\tilde{\phi}(r) = U_T \phi = [\mu(r)]^{-1/4} \phi(R(r)) . \tag{9}$$

The phase shifts are the same if $\tilde{\phi}$ and ϕ have the same asymptotic behavior, and this occurs if $R-r \to 0$ faster than r^{-1} for large r.

The transformed two-body Hamiltonian $\tilde{H}$ is defined by

$$U_T H U_T^{-1} U_T \phi = \tilde{H}\tilde{\phi} \equiv (T + \tilde{V})\tilde{\phi} , \tag{10}$$

and is obtained explicitly by changing the radial variable from R to r and by applying H to $[\mu(r)]^{1/4}\tilde{\phi}(r)$. In Eq. (10) T is the kinetic energy operator and we define $\tilde{V} \equiv \tilde{H} - T$. We use only s-wave potentials so

$$\tilde{H} = -\frac{1}{2}\left(\frac{d^2}{dr^2}\mu + \mu\frac{d^2}{dr^2}\right) + \frac{1}{4}\left[\left(\frac{d^2\mu}{dr^2}\right) + \frac{1}{4}\frac{1}{\mu}\left(\frac{d\mu}{dr}\right)^2\right] + V(R(r)) , \tag{11}$$

and we require the s-wave projections of the potential $\tilde{V}$ in momentum space for Eq. (1). These are

$$\begin{aligned} v_0(p,p') &= (2/\pi)\int_0^\infty dr\, r j_0(pr)(\tilde{V}(r))\, r j_0(p'r) \\ &= (1/\pi pp')\int_0^\infty dr \sin pr\{[\mu(r) - 1.0](p^2 + p'^2) \\ &\quad + \frac{1}{2}\frac{d^2\mu}{dr^2} + [1/8\mu(r)]\left(\frac{d\mu}{dr}\right)^2 + 2V(R(r))\}\sin p'r , \end{aligned} \tag{12}$$

where $j_0(x) = (\sin x)/x$. For the cases considered here $\mu(r) > 0$ for all r. At large r, $R(r)$ is required to go to r and this forces $\mu(r)$ to 1, $d\mu/dr$ to 0, and $d^2\mu/dr^2$ to 0 for large r. Hence the radial integration is finite and the integral may be cut off at a finite upper limit.

The integrand of Eq. (12) oscillates too much for a straightforward application of Simpson's

TABLE I. Three-nucleon binding energies for the transformed potentials based upon Eqs. (13) and (14). $\hat{S} = 0.0$ represents the untransformed potential. The binding energies are in MeV.

Transform parameters $(\hat{S}, \alpha, \beta)$	Number of sample points for each integration variable			
	10	12	14	16
(0.2, 0.5, 0.05)	−7.257	···	−7.34	···
(0.0, −, −)	−7.45	−7.57	···	−7.52
(−1.0, 0.6, 0.4)	−6.71	−8.58	−7.14	−7.23
(−0.4, 2.0, 1.4)	−7.49	−7.63	···	···

rule, so the sine factors are put into the weights. The curly-bracketed part of the integrand is fitted with a parabola for each set of three integration sample points. For each interval, the analytic expressions for the three integrals $\int dr \sin pr \times \sin p'r$, $\int r\,dr \sin pr \sin p'r$, and $\int r^2\,dr \sin pr \sin p'r$, are carefully evaluated in double precision. For those situations where a cancellation and a loss of numerical significance are possible, the analytic expressions are expanded in series. This exposes the pieces which cancel and allows us to circumvent them. Such an approach is used when $(p+p')r$ is less than 0.1, for example. The radial integration is generally cut off at 20 fm; and 1999 sample points are used for 0 to 1 fm, while 749 are used for 1 to 20 fm. In the case of the longest range transform a cut off of 25 fm is used with 949 sample points for 1 to 25 fm. This procedure gives sufficient accuracy for the present calculations.

III. RESULTS

We consider a superposition of Yukawa potentials

$$V(r) = -578.1[\exp(-1.552\,88r)/r] + 1458.05[\exp(-3.1057r)/r]. \quad (13)$$

This potential is quite close to Case V of Malfliet and Tjon[27] and it is an average potential in MeV

TABLE II. Two-body scattering amplitudes, $\langle p|t(w)|p'\rangle$, for the untransformed $(0.0,-,-)$ and three transformed potentials $(\hat{S},\alpha,\beta)$. The matrix elements are in fm. $p = 0.0132$ fm^{-1}.

p' (fm^{-1})	(0.2, 0.5, 0.05)	(0.0, −, −)	(−1.0, 0.6, 0.4)	(−0.4, 2.0, 1.4)
		$w = -2.59$ MeV		
0.0132	−5.87	−6.14	−6.11	−6.51
0.191	−5.74	−6.00	−5.97	−6.36
0.395	−5.31	−5.56	−5.53	−5.88
0.741	−4.12	−4.33	−4.26	−4.49
1.350	−1.76	−1.82	−1.46	−1.69
2.530	0.526	0.792	1.71	1.10
5.238	0.219	0.689	0.710	0.673
13.822	−0.055	0.041	−0.040	0.028
		$w = -9.71$ MeV		
0.0132	−3.07	−3.14	−3.13	−3.26
0.191	−2.99	−3.06	−3.06	−3.18
0.395	−2.76	−2.82	−2.82	−2.92
0.741	−2.12	−2.17	−2.14	−2.19
1.350	−0.888	−0.890	−0.706	−0.812
2.530	0.263	0.389	0.840	0.524
5.238	0.108	0.333	0.346	0.318
13.822	−0.027	0.020	−0.019	0.013
		$w = -45.97$ MeV		
0.0132	−2.02	−2.04	−2.03	−2.13
0.191	−1.96	−1.98	−1.98	−2.07
0.395	−1.80	−1.81	−1.81	−1.87
0.741	−1.36	−1.37	−1.34	−1.37
1.350	−0.543	−0.53	−0.397	−0.480
2.530	0.159	0.238	0.517	0.317
5.238	0.064	0.197	0.216	0.190
13.822	−0.016	0.012	−0.011	0.008
		$w = -177.09$ MeV		
0.0132	−1.72	−1.72	−1.68	−1.80
0.191	−1.67	−1.67	−1.63	−1.75
0.395	−1.52	−1.52	−1.47	−1.57
0.741	−1.13	−1.13	−1.06	−1.12
1.350	−0.436	−0.412	−0.258	−0.361
2.530	0.131	0.204	0.450	0.270
5.238	0.053	0.165	0.215	0.166
13.822	−0.013	0.011	−0.086	0.008

for the $^3S_1$ and $^1S_0$ nucleon-nucleon states. With these parameters the deuteron is bound at 0.356 MeV. The radial distortion we use is the same as one used by Coester, Cohen, Day, and Vincent[7] in nuclear matter:

$$R = r + \hat{S}[\exp(-r/\alpha) - \exp(-r/\beta)] . \tag{14}$$

$\hat{S}$ greater than zero and $\hat{S}$ less than zero are referred to as the positive and negative transformations, respectively. The positive transform is a radial expansion, while the negative transform is a radial compression. Our transform parameters are given in Table I. We choose $\hat{S} = -1.0$ fm, instead of $\hat{S} = -1.2$ fm,[7] so that $\mu(0) \neq 0$ but is greater than zero. The two-body t matrices associated with these potentials are shown in Table II at selected momenta for four energies. The table illustrates the type of variation these transforms induce in the negative-energy t-matrix elements.

Table I also gives the three-body binding energies Eq. (6) yields for these transformed potentials and for the untransformed potential. All the three-body ground states have quite similar energies, and the differences between them are within the numerical uncertainty of such a calculation. This table also shows that it is important to check whether large apparent variations are in fact due to nonconvergence. In all of our cases the convergence was adequate for our purposes. In an effort to explain this lack of sensitivity, we follow a suggestion of Haftel[17] and examine the deuteron wave functions and form factors for these potentials.

We first obtain the deuteron wave function in momentum space $\psi(p)$ from

$$\psi(p) = -(q_w^2 + p^2)^{-1} \int_0^\infty q^2 dq\, v(p, q)\psi(q) . \tag{15}$$

We put in our two-body binding energy of 0.356 MeV and we use 80 Gaussian sample points in the replacement of the integral. We then apply a Fourier transform to $\psi(p)$ to get the deuteron radial wave function $u(R) = R\psi(R)$ and

$$u(R) = (2/\pi)^{1/2} \int_0^\infty q\, dq \sin qR\psi(q) . \tag{16}$$

Once we have $u(R)$, Eq. (9) provides the transformed wave function $\tilde{u}$ and an inverse Fourier transform gives

$$\tilde{\psi}(p) = (2/\pi)^{1/2} p^{-1} \int_0^\infty dr \sin pr\tilde{u}(r) , \tag{17}$$

the transformed-momentum-space wave function.

The Fourier transforms are done by Simpson's rule and the sine factor is taken into the weights. The accuracy is helped considerably by taking advantage of the fact that both u and $\tilde{u}$ approach an exponential at large r. Generally 201 or 401 sample points are used for each interval of length 2π. When necessary we increased this number of sample points until the result was stable to at least three figures. We used a sixth-order Lagrange interpolation formula to do the required interpolations.

In Figs. 1 and 2 we present the deuteron radial wave functions and the deuteron form factors

$$g(p) = (p^2 + q_w^2)\psi(p) . \tag{18}$$

The figures show that the transforms defined by Eq. (14) do not introduce much variation into these deuteron functions. In particular the form factor

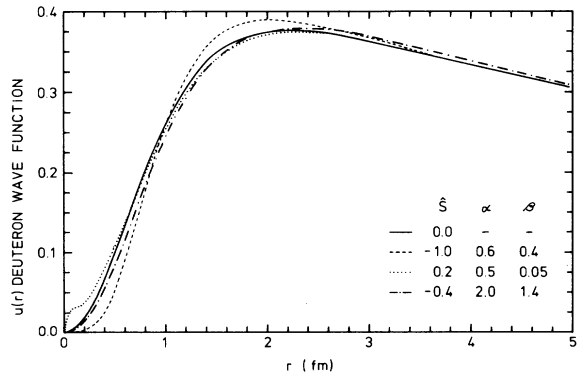

FIG. 1. The deuteron radial wave functions for the potential given in Eq. (13) and for three potentials which arise from it with the radial-distortion unitary transforms of Eq. (14). $\hat{S} = 0.0$ represents the untransformed potential. The deuteron is bound at 0.356 MeV with these potentials.

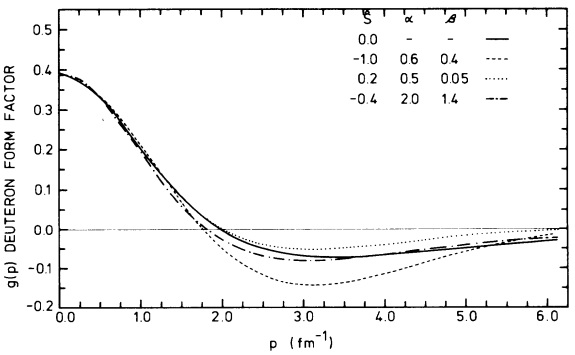

FIG. 2. The deuteron momentum space form factors associated with the deuteron radial wave functions of Fig. 1.

agrees very well out to 2 fm^{-1}, in contrast to those cases in Ref. 17 for which there are noticeable changes in E_T.

IV. DISCUSSION

Our results are consistent with those of several recent investigations. Haftel[17] has studied finite-rank unitary transforms of a potential with parameters slightly different from those of Eq. (13). He finds that for potentials with roughly similar deuteron form factors the triton binding energy only varies by 1 MeV. Fiedeldey and McGurk[29] and Afnan and Serduke[15] find slight variations in E_T for phase-shift equivalent rank-two separable potentials, when the deuteron $\psi(p)$ and E_{deut} are pinned down. In addition, there is evidence that when the deuteron wave function is severely distorted E_T,[15,17,29] pion production,[12] and electron-deuteron scattering results[9] vary widely. All of this points to the importance of computing the deuteron properties that accompany the generation of phase-shift equivalent potentials, both to see if the above picture endures and to determine the predictive power of deuteron variations.

We have presented the three-body binding energies that result from the application of the radial-distortion unitary transforms of Eq. (14) to the potential of Eq. (13). Table I shows that variation of the binding energy is slight and we have commented on a possible explanation of this. Transforms like Eq. (14) were used by Coester, Cohen, Day, and Vincent[7] in their investigation of nuclear matter with a potential[30] similar to that of Eq. (13). They found variations in the binding energy per particle of several MeV and changes of several tenths of an inverse femtometer in the Fermi momentum at saturation. Their results and those of the previous section indicate that nuclear matter is much more sensitive than E_T to off-shell variations of the t matrix. A similar pattern was observed for finite-rank transforms and the Reid soft-core potential[31] by Haftel and Tabakin for nuclear matter[14] and Harper, Kim, and Tubis for E_T.[18]

Haftel and Tabakin[14] show that large changes in the half-shell t matrices for momenta ≥ 6 fm^{-1} lead to large variations in the nuclear matter binding energy calculated in the ladder approximation. There are indications that the triton binding energy is most sensitive to negative-energy t matrices with momenta of about 2.0 fm^{-1} or less.[32,33] Some tentative support for this view occurs when the slight variations in E_T (see Table I) are coupled with our Table II, which shows that $t(p,p',w)$ varies more at $p' > 2.0$ fm^{-1} than for $p' < 2.0$ fm^{-1}. The effects on such negative-energy t-matrices of large changes in the half-shell t-matrix elements is not yet known, since the relation between them is quite involved (see, e.g., Ref. 5). In addition, nuclear matter calculations are performed without higher-order corrections, while three-body calculations such as those done here use a complete theory. A step to close this gap has been taken by Coester, Day, and Goodman,[34] who include some three-body graphs in a calculation of nuclear matter with finite-rank transforms. Their findings suggest there is little change in the results of Ref. 7 and this is consistent with some earlier work by Ristig[21] with radial-distortion transforms. An explanation of the increased sensitivity of the binding energy per particle in nuclear matter is very desirable and would probably require studies like those in Refs. 32 and 33 applied to nuclear matter.

In conclusion, we have shown that radial-distortion unitary transforms lead to off-shell variations in the two-body t matrices which in turn affect the three-body binding energy only slightly. We explain this by noting that our deuteron wave functions are quite similar to the untransformed $\psi(p)$ and that our t matrices are similar to each other at low momenta. Further work is necessary to clarify the connection between deuteron wave functions, half-shell t matrices, and those t matrices that enter many-body calculations.

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