Coulomb corrections to nuclear beta decay through induced terms

A. Bottino* and G. Ciocchetti*
Istituto di Fisica dell' Università, Torino, Italy
and Istituto di Fisica Nucleare, Sezione di Torino, Italy

C. W. Kim†
Department of Physics, The Johns Hopkins University, Baltimore, Maryland 21218

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We present an analytical expression for the shape factor for the β decay of ^{12}B with the Coulomb corrections to order αZRW_0, including the correction through the induced weak magnetism term and examine effects of this induced Coulomb correction.

[RADIOACTIVITY ^{12}B; calculated $\sigma(E_e)$, Coulomb corrections.]

Analysis of β-decay electron spectra for the purpose of testing the validity of general properties of the weak interactions [such as the hypothesis of the conserved vector current (CVC) and the existence of second-class currents] generally requires a precise knowledge of the Coulomb corrections to the spectra. For example, in the experimental determination of the weak magnetism term F_M (for the test of CVC) from the measured electron shape factor for the β decay $^{12}B \to {}^{12}C + e^- + \bar{\nu}_e$, the Coulomb correction could change the value of F_M by as much as 15%.

In the past, several authors[1-8] have calculated the electron shape factors including the Coulomb corrections of order $(\alpha Z)(RW_0)$ where R is the nuclear radius and W_0 the maximum energy available in β decay. However, they have neglected the contribution of the part of the Coulomb corrections due to the presence of various induced terms, i.e., the Coulomb correction through the induced terms such as the weak magnetism term in the above example. In this article we present an analytical expression for the shape factor for the β decay of ^{12}B with the Coulomb corrections to order αZRW_0 *including* the correction through the induced term F_M and examine effects of this induced Coulomb correction.

In order to explain precisely what we mean by the Coulomb correction through the induced terms, let us consider nuclear β decay in the elementary-particle treatment of nuclei.

In the absence of Coulomb corrections, the transition matrix element for the decay $i \to f + e^- + \bar{\nu}_e$ is given by

$$\begin{aligned}\mathfrak{M} &= \langle f(\vec{p}_f), e^-(\vec{p}_e), \bar{\nu}(\vec{p}_\nu)| \int d\vec{x}\, H_W(\vec{x}, 0)| i(\vec{p}_i)\rangle \\ &= \frac{G\cos\theta_c}{\sqrt{2}} (2\pi)^3\delta^{(3)}(\vec{p}_i - \vec{p}_f - \vec{p}_e - \vec{p}_\nu)\langle f(\vec{p}_f), e^-(\vec{p}_e), \bar{\nu}(\vec{p}_\nu)| J_\alpha^{(+)}(0) l_\alpha(0)| i(\vec{p}_i)\rangle \\ &= \frac{G\cos\theta_c}{\sqrt{2}} (2\pi)^3\delta^{(3)}(\vec{p}_i - \vec{p}_f - \vec{p}_e - \vec{p}_\nu)\langle f(\vec{p}_f)| J_\alpha^{(+)}(0)| i(\vec{p}_i)\rangle \bar{u}_e(\vec{p}_e)\gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu),\\ H_W(x) &= \frac{G\cos\theta_c}{\sqrt{2}} J_\alpha^{(+)}(x) l_\alpha(x),\end{aligned} \tag{1}$$

where $G = 1.02\times 10^{-5}/m_p^2$, $\cos\theta_c = 0.98$, and $J_\alpha^{(+)}(x)$ and $l_\alpha(x)$ are, respectively, the hadron and lepton weak currents. The rest of the notation is self-evident. In the presence of the final-state Coulomb interaction, Eq. (1) is modified to[7]

$$\begin{aligned}\mathfrak{M} \cong \frac{G\cos\theta_c}{\sqrt{2}} (2\pi)^3\delta^{(3)}(\vec{p}_i - \vec{p}_f - \vec{p}_e - \vec{p}_\nu) \bigg\{ \int d\vec{p}\bigg[\int d\vec{r}\, \frac{\bar{\psi}_e(\vec{r}, \vec{p}_e)}{(2\pi)^3} e^{i\vec{r}\cdot\vec{p}}\bigg] \\ \times \langle f(\vec{p}_f + \vec{p}_e - \vec{p})| J_\alpha^{(+)}(0)| i(\vec{p}_i)\rangle\bigg\} \gamma_\alpha(1+\gamma_5)v_\nu(\vec{p}_\nu),\end{aligned} \tag{2}$$

where $\psi_e(\vec{r}, \vec{p}_e)$ is the Coulomb-distorted electron wave function. As can be seen from Eq. (2), the hadron part is modified in three ways:
(1) Nuclear form factors which characterize the hadron matrix element are now functions of $q' = [(p_f + p_e - p) - p_i]$ instead of $q = (p_f - p_i)$ (later averaged over $\vec{p}$).
(2) There appear additional kinematic terms due to the replacement of q by q' in the matrix element, in particular for every induced term.
(3) Final nuclear spinors (generalized) which describe nuclei with nonzero spins are modified.
The last modification can easily be neglected since the nuclei involved are extremely nonrelativistic. In all the previous calculations of the Coulomb correction the modification (2) has also been neglected based either on the argument that this contribution is small since this is a correction through already small induced terms or on the mathematical simplicity. Therefore, the correction due to the modification (1) has been considered dominant.

As mentioned already, the modification (2) amounts to introducing new terms associated with the induced terms in the matrix element. For example, the weak magnetism term $[F_M(q^2)/2m_p] \times\sigma_{\alpha\beta}q_\beta$ will be replaced, symbolically, by

$$\frac{1}{(2\pi)^3}\int d\vec{p}\left[\int d\vec{r}\left(\frac{\overline{\psi}_e(\vec{r},\vec{p}_e)}{\overline{u}_e(\vec{p}_e)}\right)e^{i\vec{r}\cdot\vec{p}}\right]\frac{F_M(q'^2)}{2m_p} \times\sigma_{\alpha\beta}[q_\beta+(p_e-p)_\beta] \tag{3}$$

thus introducing a new term

$$\frac{1}{(2\pi)^3}\int d\vec{p}\left[\int d\vec{r}\left(\frac{\overline{\psi}_e(\vec{r},\vec{p}_e)}{\overline{u}_e(\vec{p}_e)}\right)e^{i\vec{r}\cdot\vec{p}}\right]\frac{F_M(q'^2)}{2m_p} \times\sigma_{\alpha\beta}(p_e-p)_\beta \tag{4}$$

which clearly reduces to zero when the Coulomb correction is absent [i.e., when $\overline{\psi}_e(\vec{r},\vec{p}_e)=\overline{u}_e(\vec{p}_e)\times e^{-i\vec{r}\cdot\vec{p}_e}$].

We discuss the contribution of the term given in Eq. (4) using the β decay ${}^{12}\mathrm{B}(1^+)\to{}^{12}\mathrm{C}(0^+)+e^-+\overline{\nu}_e$ as an example. The hadron matrix element for this transition is given by

$$\langle{}^{12}\mathrm{C}|J_\alpha^{(+)}|{}^{12}\mathrm{B}\rangle=\langle{}^{12}\mathrm{C}|V_\alpha^{(+)}(0)|{}^{12}\mathrm{B}\rangle+\langle{}^{12}\mathrm{C}|A_\alpha^{(+)}(0)|{}^{12}\mathrm{B}\rangle,$$

$$\langle{}^{12}\mathrm{C}|V_\alpha^{(+)}(0)|{}^{12}\mathrm{B}\rangle=\sqrt{2}\,m\epsilon_{\alpha\beta\gamma\delta}q_\beta\xi_\gamma\frac{Q_\delta}{2m}\frac{F_M(q^2)}{2m_p},$$

$$\langle{}^{12}\mathrm{C}|A_\alpha^{(+)}(0)|{}^{12}\mathrm{B}\rangle=\sqrt{2}\,m\left[F_A(q^2)\xi_\alpha+\frac{q_\alpha}{m_\pi^2}\xi qF_P(q^2)\right],$$

$$Q_\alpha=(p_i+p_f)_\alpha, \tag{5}$$

where $F_M(q^2)$, $F_A(q^2)$, and $F_P(q^2)$ are, respectively, the weak magnetism, axial-vector, and induced pseudoscalar nuclear form factors and ξ_α is the polarization four-vector of the initial nucleus of spin one. Also, m is the nuclear mass ($m\cong m_i\cong m_f$). One can calculate the spectrum factor from Eq. (2) together with Eq. (5) but this procedure is very tedious.[7] On the other hand, as demonstrated in Ref. (8), using a perturbative expansion of the distorted electron wave function and the low-energy approximation of nuclear form factors of the single-pole form, one can calculate the effect of the Coulomb correction to order αZ in the finite size contribution. The final result can be given, in this approach, in an analytic form. The result is

$$d\Gamma=\frac{G^2\cos^2\theta_c}{4\pi^3}|F_A(0)|^2S'(E_e,Z)p_eE_e(W_0-E_e)^2dE_e,$$

$$\begin{aligned}S'(E_e,Z)=1&+\tfrac{8}{3}\left(\frac{1}{2m_p}\right)\left(\frac{F_M(0)}{F_A(0)}\right)\left(E_e-\tfrac{1}{2}W_0-\frac{m_e^2}{2E_e}\right)+\frac{\alpha Z}{\pi^2}\frac{1}{3E_eE_\nu}\frac{1}{2}\int d(\cos\theta)\\&\times\mathrm{Re}\int d\vec{p}\Bigg\{[3E_\nu(m_e^2+E_e^2)+3E_\nu\vec{p}_e\cdot\vec{p}-E_e\vec{p}_\nu\cdot\vec{p}-E_e\vec{p}_e\cdot\vec{p}_\nu]+\frac{1}{m_p}\left(\frac{F_M(0)}{F_A(0)}\right)\\&\times[(E_e^2+m_e^2)\vec{p}_\nu-E_eE_\nu\vec{p}_e-E_eE_\nu\vec{p}+(\vec{p}_e\cdot\vec{p})\vec{p}_\nu]\;(-\vec{p}_e-2\vec{p}_\nu-\vec{p})\\&\times\frac{F_c[(p-p_e)^2]}{(\vec{p}-\vec{p}_e)^2}\frac{F_A[(p_\nu+p)^2]}{F_A(q^2)}\frac{1}{\vec{p}^2-\vec{p}_e^2-i\epsilon}\Bigg\},\end{aligned} \tag{6}$$

where $F_C(q^2)$ is the charge form factor (normalized to one) of the final nucleus. In Eq. (6) the F_P term has been neglected (which is always justified in β decay) and we have also used the relation[9]

$$\frac{F_M(q^2)}{F_A(q^2)}\cong\frac{F_M(0)}{F_A(0)}. \tag{7}$$

Following the method described in Ref. 8, we represent $F_C(q^2)$ and $F_M(q^2)$ by

$$F_C(q^2)=\frac{1}{1+a^2q^2}, \quad a=\frac{1}{\sqrt{6}}\langle r^2\rangle^{1/2}=0.976 \text{ fm},$$

$$F_M(q^2)=\frac{F_M(0)}{1+b^2q^2}, \quad b=1.16 \text{ fm}, \tag{8}$$

where the last equation is obtained from inelastic electron scattering experiment $e^-+{}^{12}C\to{}^{12}C^*(15.1$ MeV$)+e^-$. Keeping terms up to order of $p_e a$ and $p_e b$ in Eq. (6), we finally obtain

$$d\Gamma\cong\frac{G^2\cos^2\theta_c}{4\pi^3}|F_A(0)|^2F_0(Z,E_e)$$
$$\times S(E_e,Z)p_eE_e(W_0-E_e)^2dE_e,$$

$$S(E_e,Z)=1+\frac{8}{3}\left(\frac{1}{2m_p}\right)\left(\frac{F_M(0)}{F_A(0)}\right)\left(E_e-\tfrac{1}{2}W_0-\frac{m_e^2}{2E_e}\right)$$
$$-\alpha Z\left(CW_0+C'E_e+C''\frac{m_e^2}{E_e}\right)$$
$$+\left\{\frac{8}{3}\left(\frac{1}{2m_p}\right)\left(\frac{F_M(0)}{F_A(0)}\right)\frac{\alpha Z}{2(a+b)}\right\}, \tag{9}$$

where

$$C=-\frac{2}{9}\frac{(2ab^2+b^3)}{(a+b)^2},$$

$$C'=\frac{8}{3(a+b)^2}(a^3+2a^2b+\tfrac{8}{3}ab^2+\tfrac{4}{3}b^3),$$

$$C''=\frac{2}{3}\frac{1}{(a+b)^2}(2a^3+4a^2b+2ab^2+b^3). \tag{10}$$

The last term in Eq. (9) (in braces) is the Coulomb correction through the induced weak magnetism term. Note that this is of the form

$$\frac{\alpha Z}{m_pR}=\left(\frac{W_0}{m_p}\right)\left(\frac{\alpha Z}{W_0R}\right), \tag{11}$$

where the first factor (W_0/m_p) represents the magnitude of the contribution of the induced term, implying that for the induced terms the Coulomb correction factor is $(\alpha Z/W_0R)$ instead of $(\alpha Z)(W_0R)$ (remember $W_0R\ll 1$ for β decay). Since, for practical purpose, $a=b$, we set in Eq. (10) for simplicity

$$a=b=\frac{1}{\sqrt{6}}\left(\frac{3}{5}\right)^{1/2}R=\frac{R}{\sqrt{10}}. \tag{12}$$

Equation (10) becomes then

$$C=-\frac{1}{6\sqrt{10}}R,$$

$$C'=\frac{14}{3\sqrt{10}}R,$$

$$C''=\frac{3}{2\sqrt{10}}R. \tag{13}$$

We make the following remarks concerning the effect of the Coulomb correction through the induced term, $F_M(0)\alpha Z$, in Eq. (9):

(1) The $F_M(0)\alpha Z$ term is of the same order of magnitude as the remaining Coulomb correction terms, since we have

$$RW_0\sim(m_pR)^{-1}.$$

This term, therefore, introduces an additional correction of about one percent in the ft value (the same direction for all decays).

(2) The $F_M(0)\alpha Z$ term remains with *the same sign* for β^+ and β^- decays while the $F_M(0)$ and αZ terms in Eq. (9) change their signs for β^+ decay. Hence, this term does not contribute to the ratio $(ft)_+/(ft)_-$ which has been used for the test of the second-class currents.

(3) The $F_M(0)\alpha Z$ term has no energy dependence so that it does not affect the determination of the numerical value of the weak magnetism term from the observed spectrum shape.

(4) The numerical value of the $F_M(0)\alpha Z$ term is more sensitive to the type of the form factors used than the usual finite-size correction term [third term in Eq. (9)]. This term changes, for example, by about 15% if dipole form factors instead of single-pole form factors are used, while the usual finite-size correction term changes by only a few percent.

(5) For the $0^+\to0^+$ transition such as ${}^{14}O\to{}^{14}N^*+e^+ +\nu_e$, the Coulomb correction through induced terms is absent since there is no induced term due to CVC. For forbidden transitions, in particular, for the $0^-\to0^+$ transition, the correction becomes relatively more important. Further discussions on these transitions will be given in a forthcoming paper.

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[1]M. E. Rose, *Relativistic Electron Theory* (Wiley, New York, 1961).

[2]E. J. Konopinski, *The Theory of Beta Radioactivity* (Clarendon, Oxford, England, 1966).

[3]J. N. Huffaker and C. E. Laird, Nucl. Phys. A92, 584 (1967).
[4]H. Behrens and W. Bühring, Nucl. Phys. A106, 433 (1968).
[5]D. A. Dicus and R. E. Norton, Phys. Rev. D 1, 1360 (1970).
[6]D. H. Wilkinson, Nucl. Phys. A158, 476 (1970).
[7]L. Armstrong, Jr., and C. W. Kim, Phys. Rev. C 5, 672 (1972).
[8]A. Bottino and G. Ciocchetti, Phys. Lett. 43B, 170 (1973).
[9]C. W. Kim and H. Primakoff, Phys. Rev. 140, B566 (1965).