Pion-nucleus coordinate-space potential

Leonard S. Kisslinger*
Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213
and
Frank Tabakin*
University of Pittsburgh, Pittsburgh, Pennsylvania 15260
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A first-order pion-nucleus optical potential which is convenient for the analysis of pion-nucleus data is presented. This configuration-space potential incorporates a number of important theoretical features. A Lorentz transformation of the pion-nucleon interaction from the pion-nucleon to the pion-nucleus center-of-mass system is shown to yield a first-order potential which is almost local in coordinate space, and which differs from the originally proposed gradient potential, by containing an additional term proportional to the Laplacian of the density. We then show how this form is modified by off-shell and high-energy considerations. It is emphasized that proper off-shell and threshold behavior can be included and that energy and momentum variables must be treated carefully in deriving the optical potential. Typical numerical results are given and the new potential is shown to yield significant build-up of large-angle cross sections.

[NUCLEAR REACTIONS $^{12}C(\pi, \pi)$, $E = 120-280$ MeV; calculated $\sigma(\theta)$: Optical potential deduced from multiple-scattering theory.]

I. INTRODUCTION

It has long been recognized that the scattering of pions by nuclei can very likely be understood in terms of the basic pion-nucleon interaction and multiple-scattering theory. Numerous fits to experimental data in the past few years are most encouraging. In relating pion-nucleus elastic scattering to pion-nucleon dynamics it is useful to obtain an optical potential; and for the treatment of inelastic scattering and reactions, an accurate optical potential is essential. In view of the forthcoming precision experiments to be performed at meson factories, it is timely to improve the optical-model description.

The basic theoretical framework for the derivation of an optical potential as a multiple-scattering expansion is well established,[1] and various series expansions are possible.[2] However, in carrying out this expansion there are a number of important theoretical questions. Most of the basic theoretical problems involve the first-order potential. Our objective in the present work is to obtain a first-order potential which is convenient to use for the analysis of data, and is accurate to the same order as the contributions of the higher-order terms.

The fundamental theoretical problem is that the expansion occurs in terms of a "bound t matrix,"[1] the scattering of a pion from a nucleon bound in the nucleus. There are several important questions involved in attempting to relate this to the scattering of pions by free nucleons. These include the relation of the basic interaction in the nucleus to the free two-body interaction, the questions of the state dependence of the t matrix, many-body operators, and so forth. We are concerned with intermediate (or high) energies, where the impulse approximation is adequate to give at least a starting point. The optical potential is thereby represented in terms of a two-body scattering matrix. The many-body interactions will mainly contribute to the absorptive part of the potential, and must be treated separately.

Within the framework of the impulse approximation, the fundamental problem reduces to determining the effective pion-nucleon scattering matrix as a function of the variables and quantum numbers. Essentially, this means determining the off-shell t matrix in the pion-nucleus center-of-mass system. Recently there has been a considerable interest and progress in this program.[3-5] In the present work, we make use of the recent work of Landau, Phatak and Tabakin, where a solution to the inverse problem for π-nucleon scattering was carried out with a separable form for the interaction. This gives a scattering matrix as a function of the initial and final relative momenta and scattering energy which should be adequate for

the nuclear problem, where many-body dynamics inhibit far off-shell scatterings.

We are led to consider first the effect of the Lorentz transformation from the pion-nucleon to the pion-nucleus center-of-mass system (an effect which has been considered by a number of workers[3-5,6]), keeping the proper threshold behavior in each partial wave, and then the proper form of the potential which results from the consideration of the off-shell and high-energy behavior. Our first conclusion is that from the Lorentz transformation we obtain for the first-order potential one which is almost local in coordinate space, and which differs from the gradient potential which was originally proposed[7] by containing an additional term proportional to the Laplacian of the density. We then see how this form is modified by off-shell and high-energy considerations. Typical numerical results are given and discussed. A series of appendices give full details to assist in the application of our results for the analysis of experimental data.

II. DERIVATION OF THE OPTICAL POTENTIAL

A. General considerations

We shall proceed in coordinate space. Although in general the pion-nucleus potential is nonlocal, as will be discussed in some detail below, it turns out that our new form is "almost local," and has a convenient coordinate-space representation. This not only allows physical insight, but it is quite practical in allowing Coulomb effects to be accurately and conveniently included. Making use of the impulse approximation and assuming state independence of the scattering (i.e., assuming that the scattering depends only on the relative momenta, as well as spin and isospin quantities), one can express the first-order optical potential as

$$\langle \vec{x}|V|\vec{x}'\rangle = [A/(2\pi)^3]\int d^3k\,d^3k'\,d^3z\langle \vec{k}'|t(E)|\vec{k}\rangle \times e^{i\vec{k}'\cdot(\vec{x}-\vec{z})}e^{-i\vec{k}\cdot(\vec{x}'-\vec{z})}\rho(z), \quad (1)$$

where $\rho(z)$ is the nuclear single-particle density normalized to one, $\int d^3z\rho(z)=1$, A is the number of nucleons, $\langle \vec{k}'|t(E)|\vec{k}\rangle$ is the collision matrix (averaged over nucleons in the nucleus) as a function of relative momenta, $\vec{k}$ and $\vec{k}'$ and energy E evaluated in the pion-nucleus center-of-mass system. Here E is the total pion-nucleon energy equal to $E_\pi + E_N = (p_{\rm lab}{}^2+\mu^2)^{1/2} + (p_N{}^2+m^2)^{1/2} \approx E_\pi + m$, where μ and m are the pion and nucleon masses, respectively.

The objective of the present work can now be clearly stated. It is to use the best present information on the collision matrix $\langle \vec{k}'|t(E)|\vec{k}\rangle$ to derive a convenient optical potential in coordinate space from the expression (1). There are some qualitative observations that one can make about the potential (1), for typical dependence of $\langle \vec{k}'|t(E)|\vec{k}\rangle$ on the variables.

(i) For s-wave two-body scattering the scattering matrix has the form

$$\langle \vec{k}'|t(E)|\vec{k}\rangle = a(E, k^2, k'^2).$$

If the scattering matrix depends only on the energy,

$$\langle \vec{k}'|t(E)|\vec{k}\rangle = a(E), \quad (2a)$$

then

$$\langle \vec{x}|V|\vec{x}'\rangle = \delta(\vec{x}-\vec{x}')V_0\rho(x), \quad (2b)$$

a local potential proportional to the density. This is the usual low-energy assumption.

(ii) If the scattering matrix depends only on the magnitude of the momentum transfer and the energy

$$\langle \vec{k}'|t(E)|\vec{k}\rangle = t(E, |\vec{k}'-\vec{k}|), \quad (2c)$$

then the optical potential is also local, i.e.,

$$\langle \vec{x}|V|\vec{x}'\rangle = \delta(\vec{x}'-\vec{x})V(x). \quad (2d)$$

(iii) For

$$\langle \vec{k}'|t(E)|\vec{k}\rangle = a(E) + b(E)\vec{k}'\cdot\vec{k}, \quad (2e)$$

then

$$\langle \vec{x}|V|\vec{x}'\rangle = \delta(\vec{x}-\vec{x}')[\alpha\rho(x) + \beta\vec{\nabla}\cdot\rho\nabla]. \quad (2f)$$

This form (2e), which was assumed in Ref. 7, gives an "almost local" potential of the gradient form (2f). The p-wave part, $b(E)\vec{k}'\cdot\vec{k}$, is approximately of correct form on shell, if $\vec{k}, \vec{k}'$, and E are the pion-nucleon c.m. variables, due to the angular dependence $\vec{k}\cdot\vec{k}'$ and the p-wave threshold behavior kk'.

(iv) Generally, if the amplitude depends upon k^2 and k'^2 separately, even for s-wave two-body scattering one can expect a nonlocal potential.

We now turn to the general treatment of the π-nucleon collision matrix needed in (1). It is important to realize that the pion-nucleon collision matrix in (1) is defined in the pion-nucleus center-of-mass system, which we approximate as the pion-nucleon laboratory frame. To relate $t(E)$ to the corresponding operator $t(\omega)$ defined in the pion-nucleon c.m. system, a transformation to the pion-nucleon c.m. system is required. That transformation is defined by

$$\langle \vec{k}'|t(E)|\vec{k}\rangle = \gamma\langle \vec{\kappa}'|t(\omega)|\vec{\kappa}\rangle, \quad (3)$$

where $\vec{\kappa}, \vec{\kappa}'$ denote momenta in the pion-nucleon c.m. system. The transformed collision energy is $\omega = (p^2+\mu^2)^{1/2} + (p^2+m^2)^{1/2} = \omega_\pi + \omega_N = \sqrt{s}$. The fac-

tor γ is determined using the Lorentz invariance fo probability to be

$$\gamma=\left[\frac{E_\pi(\kappa)E_\pi(\kappa')E_N(\kappa)E_N(\kappa')}{E_\pi(k)E_\pi(k')E_N(k_N)E_N(k_N')}\right]^{1/2}\cong\frac{\omega_\pi\omega_N}{E_\pi m}\ . \tag{4}$$

The approximation given in (4) in which γ is a function only of the scattering energy is most useful in what follows. It has been verified in the numerical work of Ref. 5. The transformation (3) is completely specified by the states $|\vec{k}\rangle$, $|\vec{k}'\rangle$, $|\vec{\kappa}\rangle$, and $|\vec{\kappa}'\rangle$. Consistent with the concept of potential scatterings, we define these as on-mass shell momentum states. This uniquely determines the transformation. The momentum (κ, κ') in the pion-nucleon c.m. are related to the pion-nucleon laboratory momenta (k, k') in the standard way, $\kappa = (m/\sqrt{s})k$; here we make use of the approximate equivalence of the pion-nucleus laboratory (and c.m.) system to the pion-nucleon laboratory system at low and medium energy.

We must now write the function $\langle\vec{\kappa}'|\,t(\omega)|\vec{\kappa}\rangle$ in terms of the variables k, k', and E, taking care to distinguish energy and momentum variables. In doing this we shall use a functional form approximating the results of Ref. 5, but the main conclusions of the present work will only make use of the qualitative features of this form, not on the details.

To carry out the transformation (3), it is important to recognize that in addition to changing the wave numbers, a change in angles results from transforming from the pion-nucleon laboratory to c.m. systems. The angles $\hat{k}\cdot\hat{k}'$ in the laboratory and $\hat{\kappa}\cdot\hat{\kappa}'$ (the caret is used to designate unit vectors) in the c.m. for the pion-nucleon system are related by

$$\hat{\kappa}\cdot\hat{\kappa}'=C_0+C_1\hat{k}\cdot\hat{k}'\equiv\frac{E_\pi(\kappa)E_\pi(\kappa')-E_\pi(k)E_\pi(k')}{\kappa\kappa'}+\frac{kk'}{\kappa\kappa'}\,\hat{k}\cdot\hat{k}'\ . \tag{5}$$

The above relation, although derived using the invariance of the four-vector product $t=(k'-k)^2$, is not a purely relativistic effect. It is well known that for any speed an isotropic term (C_0) occurs when transforming a c.m. p wave to the laboratory system. An important point of this paper is to correctly include the "isotropic p wave" term (C_0) along with proper "threshold" behavior of t (see later).

Our next step is to partial wave decompose t in the pion-nucleon c.m. system and to introduce simple off-shell models with correct low momenta behavior. We have for each isospin I

$$\langle\vec{\kappa}'|t_I(\omega)|\vec{\kappa}\rangle=\sum_{l,j=l\pm1/2}(j+\tfrac{1}{2})\langle\kappa'|\,t^I_{lj}(\omega)|\kappa\rangle P_l(\hat{\kappa}'\cdot\hat{\kappa}). \tag{6}$$

The collision matrix can now be extended off shell using the simple model

$$\langle\kappa'|t^I_{lj}(\omega)|\kappa\rangle=-\frac{1}{4\pi^2}\frac{\sqrt{s}}{\omega_\pi\omega_N}\frac{\alpha^I_{lj}(\omega)}{\kappa}\frac{g^I_{lj}(\kappa')g^I_{lj}(\kappa)}{[g^I_{lj}(\omega)]^2}\ , \tag{7}$$

where $\alpha^I_{lj}(\omega)=e^{i\delta^I_{lj}(\omega)}\sin\delta^I_{lj}(\omega)$ is the pion-nucleon partial wave amplitude. Equation (7) includes the relativistic density of states. The functions $g^I_{lj}(\kappa)$ can be obtained for each state from Ref. 5. The main qualitative feature is that these $g_{lj}(\kappa)$ serve as form factors which provide an off-shell extrapolation based on our knowledge of pion-nucleon dynamics and which cut off the interaction at high momentum in a theoretically sound manner. This is discussed in detail by Landau and Tabakin, Ref. 5. A study of the results of Ref. 5 led us to the form

$$g_{lj}(\kappa)=\kappa^l e^{-a_{lj}{}^2\kappa^2}\ . \tag{8}$$

The κ^l give the threshold behavior and the a_{lj} can be roughly determined numerically from Ref. 5. Note that in addition to defining the off-shell behavior, the high-energy behavior is also strongly modified by the a_{lj}. Let us refer to the parameters a_{lj} as the form-factor parameters. Substituting (3)–(8) into (1) results in a nonlocal potential of the form

$$\langle\vec{x}\,|V|\vec{x}'\rangle\sum_i c_i\,\rho_i(\vec{x},\vec{x}')+\sum_j d_j\vec{\nabla}_x\cdot\rho_j(\vec{x},\vec{x}')\nabla_{x'}\ . \tag{9}$$

Important details of the derivation of this potential and an explicit form are given in Appendix B. This potential is being studied and will be reported on later.

B. Limit of vanishing form-factor parameters

In this paper we wish to make use of the observation that the form-factor parameters a_{lj} are quite small, a fraction of a Fermi for the s and p waves. We therefore wish to look in the limit as $a_{lj}\to 0$. In Appendix C the limit for small a_{lj} is derived, from which the results of this section can be obtained. Restricting ourselves to s and p waves and using Eqs. (6), (7), and (5), one observes (See Appendix A) that

$$\langle\vec{k}'|t(\omega)|\vec{k}\rangle=a(\omega)+b(\omega)\vec{k}\cdot\vec{k}'+C_0b(\omega), \tag{10}$$

where

$$a(\omega)=-b_0(\omega)p_{\rm lab}{}^2/[2E_\pi(2\pi)^3],$$

$$b(\omega)=-b_1(\omega)p^2/[2E_\pi(2\pi)^3p^2],$$

with $b_0(\omega)$ and $b_1(\omega)$ being the parameters of Auerbach and Sternheim.[6] Substituting (10) into (1)

gives

$$\langle \vec{x}|V|\vec{x}'\rangle = V_0 + V_1 + V_{10}$$
$$\cong -\frac{A}{2E_\pi}\left\{b_0(\omega)p_{\rm lab}{}^2\rho(x) - b_1(\omega)\vec{\nabla}\cdot\rho\vec{\nabla} + \frac{1}{2}\left(\frac{s}{m^2}-1\right)b_1(\omega)[\nabla^2\rho(x)]\right\}\delta(\vec{x}-\vec{x}'). \quad (12)$$

Equation (12) is the potential which arises from our observation that the nonlocality of the pion-nucleus potential is small. Thus it should contain much of the physics of Ref. 5, even though it is of incredibly simple form. One should note that although the term V_{10} arises from the isotropic p-wave potential, $V_{p(s)}$ of Appendix B, it is only part of that term. The remainder of that term in the $a \to 0$ limit removes a factor of s/m^2 from the gradient term. See Eqs. (C3), (C5), and (C7) for greater details concerning the origin of the form (12).

The potential given in Eq. (12) is the major result of this paper. The new feature of this portion of our work is the existence of the term V_{10} in Eq. (12). These results show that the main effect of transforming the collision matrix from the laboratory to the center-of-mass system, while keeping track of the threshold factor κ^l, is to add a term $\frac{1}{2}(1 - s/m^2)b_1(\nabla^2\rho)$ to the usual Kisslinger potential. Other authors have considered the transformation effect, but the threshold effect was not previously treated correctly.

The new term has a relatively small coefficient and because of its surface-peaked nature, we anticipate that the new term will mainly affect the large-angle scattering. It is interesting to note that with both the off-shell extrapolations

$$\langle \vec{k}'|t(\omega)|\vec{k}\rangle = r(\omega) + s(\omega)\vec{k}'\cdot\vec{k},$$

leading to the gradient potential, and

$$\langle \kappa'|t(\omega)|\kappa\rangle = u(\omega) + v(\omega)|\vec{k}-\vec{k}'|^2,$$

leading to a potential of the form $\alpha\rho + \beta\nabla^2\rho$, one obtains the identical extra isotropic p-wave term from the Lorentz transformation. This also suggests that the result is not critically dependent upon the off-shell extrapolation.

Numerical results with this potential will be given in Sec. III.

C. Lorentz transformation dropping mass-shell conditions

Let us return to the expression for $\langle \vec{k}'|t(\omega)|\vec{k}\rangle$. Using Eqs. (3)–(8) one can write [see Eqs. (3)–(8) and the definitions in Appendix B]

$$\langle k'|t(E)|k\rangle = a(\omega)e^{2a_s^2p^2}e^{-2a_s^2(\kappa'^2+\kappa^2)} + b(\omega)e^{2a_p^2p^2}e^{-a_p^2(\kappa'^2+\kappa^2)(\kappa'^2+\kappa^2)}[E_\pi(\kappa)E_\pi(\kappa') - E_\pi(k)E_\pi(k) + \vec{k}\cdot\vec{k}']. \quad (13)$$

Recall that the quantities $E_\pi(\kappa)$, $E_\pi(\kappa')$, $E_\pi(k)$, and $E_\pi(k')$ are the energies associated with the three-vectors κ, κ', k, and k', respectively. Since the pion is assumed to be on the mass shell throughout the scattering process, the relationship $E_\pi{}^2(\kappa) = \kappa^2 + \mu^2$ etc., must be retained. This we have done in the preceding discussion. Let us, however, now arbitrarily take a different (and incorrect) ansatz. Let us assume that $E_\pi(\kappa) = \omega = E_\pi(\kappa')$ and $E_\pi(k) = E_\pi = E_\pi(k')$; i.e., that these quantities are the center-of-mass and, laboratory scattering energies, respectively.

Then, using the methods of Appendix C and the previous section, it is simple to show that in the limit of $a_{jl} \to 0$

$$V(x) = -\frac{A}{2E_\pi}\left[\left\{b_0(\omega)p_{\rm lab}{}^2 - \left(\frac{s}{m^2}-1\right)b_1(\omega)\right\}\rho - \frac{s}{m^2}b_1(\omega)\vec{\nabla}\cdot\rho\vec{\nabla}\right]. \quad (14)$$

This is the form assumed by Fäldt, and used to compare two different off-shell assumptions. However, as will be evident in Sec. III, this form is not at all equivalent to the one obtained keeping the mass-shell condition. We would like to emphasize the derivation of Eq. (14), for it shows how carefully one must keep track of the variables to avoid error in relating the many-body problem to the two-body problem.

D. Form-factor correction

Returning to Eqs. (1), (B2), and (B3) we observe that the integrations over $\vec{K}$ and $\vec{q}$ cover quite different ranges. Because nuclei are weakly bound systems, the range of $\vec{q}$ is restricted by $\rho(q)$ to quite small values. Thus the magnitudes of $a_s q$ and $a_p q$ are small. We thus can safely drop the factors $e^{-a_s{}^2q^2}$ and $e^{-a_p{}^2q^2}$. However, the values $a_s{}^2K^2$ and $a_p{}^2K^2$ are not small for important parts of the integrals involved in obtaining the potentials. Thus we are led to consider the potential derived in Appendix C in the limit of vanishing nonlocality and nonvanishing product $a_p K$. It was shown there [Eq. (C13)] that one can expect the form

$$\langle \vec{x}|V|\vec{x}'\rangle = -\frac{A}{2E_\pi}\{a_0(\omega)p_{\rm lab}{}^2\rho(x) - \tfrac{1}{2}a_2(\omega)[\nabla^2\rho(x)] - [a_1(\omega)+a_2(\omega)]\vec{\nabla}\cdot\rho\vec{\nabla}\}\delta(\vec{x}-\vec{x}'), \quad (15)$$

with $a_0(\omega) = b_0(\omega) - \frac{3}{4}(p_{\rm lab}\, a_p)^{-2}\ (s/m^2 - 1)\ b_1(\omega)$, $a_2(\omega) = \frac{1}{2}(1 - s/m^2)\ b_1(\omega)$, and $a_1(\omega) + a_2(\omega) = -(s/m^2) b_1(\omega)$. It should be emphasized that the derivation of Eq. (15) is not entirely satisfactory. However, we suggest that it could be considered as a phenomenological form, with guidance for the theoretical parameters. When compared to Eq. (12), it suggests that the major improvement to account for the high-energy behavior of the pion-nucleon amplitude might be to modify the term proportional to the density as indicated by (C13), i.e., an excellent phenomenological form might be obtained by adding a potential

$$V^H(x) = \alpha(E)(s/m^2 - 1) b_1(\omega)\rho(x) \qquad (16)$$

to the potential given in Eq. (12). We do not consider the form (15) in the discussion of the numerical results which follow.

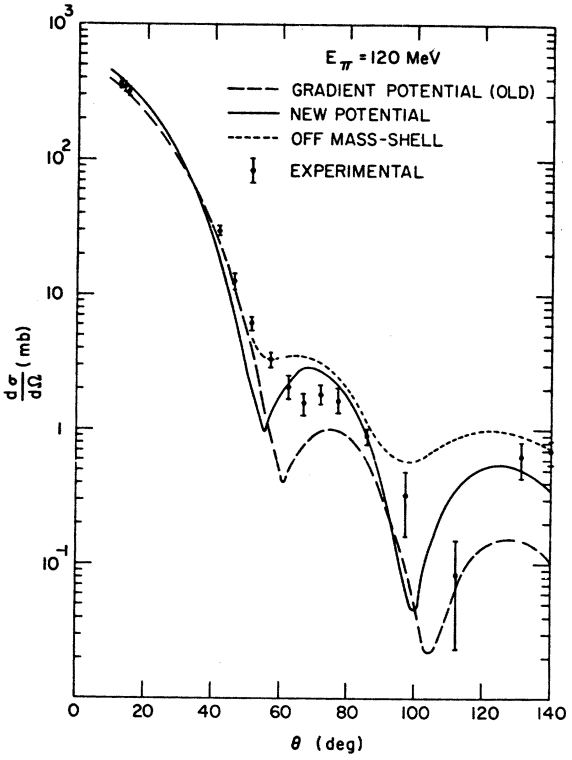

FIG. 1. The differential cross sections for pion-^{12}C elastic scattering at 120-MeV laboratory kinetic energy. The experimental points are the Binon *et al.* (Ref. 8) data. Results are shown for the old (Ref. 7) gradient potential and for the new potential [Eq. (12)] using the Fermi-averaged Sternheim-Auerbach parameters (Ref. 6). Calculations were performed on the corrected version of ABACUS-M. The new potential tends to build up backward-angle cross sections and to move the minima to smaller angles, which agrees with the corresponding momentum space calculation (Ref. 5). Also shown are the results obtained using the (incorrect) off-mass-shell [$E \neq (k^2 + \mu^2)^{1/2}$] assumptions.

III. RESULTS AND DISCUSSION

It has been shown that the first-order pion-nucleus optical potential, including off-shell and high-energy behavior derived from a solution to the inverse pion-nucleon scattering problem, is of the nonlocal form (9), given in detail by expression (B11). The explicit form (B11) is expected to be approximately equivalent to the potential in momentum space given in Ref. 5. However, the nonlocality is of short range, and nonlocal corrections also arise from corrections to the impulse approximation and from higher-order terms which seem to be of the same order as these corrections to a local potential. Thus we expect that one can use an almost local potential in the first order and be consistent with multiple-scattering expansion. Our main result [Eq. (12)] is that this is possible and that it is realized by adding a potential proportional to the Laplacian of the density to the gradient potential of Ref. 7.

Numerical results for pion-^{12}C scattering are presented in Figs. 1–7 along with the experimental data of Binon *et al.*[8] Our object is not, in fact, detailed comparison with experiment, but is to study the accomplishment of the theoretical modifications

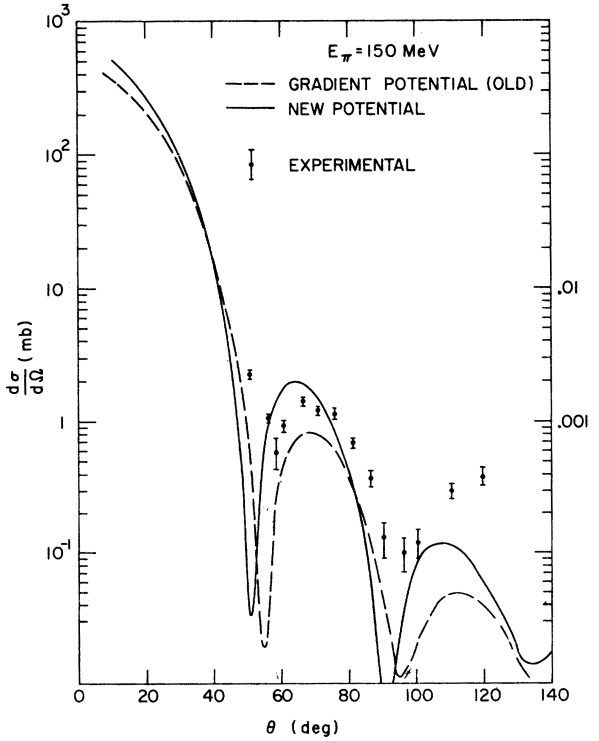

FIG. 2. The differential cross sections for pion-^{12}C elastic scattering at 150-MeV laboratory kinetic energy. See Fig. 1 for other details.

being considered. In the figures are shown the results with the form (12) and also with the usual gradient form (A3). The Fermi-averaged theoretical parameters of Sternheim and Auerbach[6] are used.

Let us first observe that the main feature for which we have worked has been accomplished, that with the new potential the large-angle scattering has been increased to be of the order of the experimental data where measured. In comparison with the old gradient potential, the new potential gives a large enhancement for the scattering in the backwards hemisphere by 1 or 2 orders of magnitude or more for the higher energies considered.

Moreover, a comparison with the results of Landau, Phatak, and Tabakin shows that the general shape of the cross section is in good agreement and for the most part the magnitude. At all energies, all of the qualitative features of the differential cross section of Ref. 5 (see Fig. 6 in that reference) are reproduced. The major discrepancies are that the new potential has deeper diffraction minima and that the large-angle scattering somewhat less than in Ref. 5 (the largest difference in the large-angle scattering being at 150 MeV). The Coulomb interference, which was neglected in Ref. 5, could account for this. Note also that the potential (14) derived with an off-mass-shell assumption gives quite different results (see Figs. 1, 4, and 6 for sample comparisons with the correct form).

There are many corrections which must be considered, many of which have been discussed in the literature. One point of theory which can be made concerns one general discrepancy between the theoretical results obtained with the potential (12) derived here and the experimental cross sections, the fact that the theoretical shape tends to be peaked toward smaller angles. This probably can be accounted for by many-body corrections. This is perhaps most easily seen in the doorway isobar model which has been recently proposed.[9] In that work it was observed that a difference between the binding energy of the $\Delta(1236)$ isobar and a nucleon in the nucleus corresponding to a weaker interaction for the isobar tends to produce just such a change in the theoretical cross section.

Other results obtained in the paper are relevant to the phenomenological use of our "almost local" potential. It is suggested that the addition of a term proportional to the density, given in Eq. (16) with parameters guided by considerations in Appendix C, might account for important features of

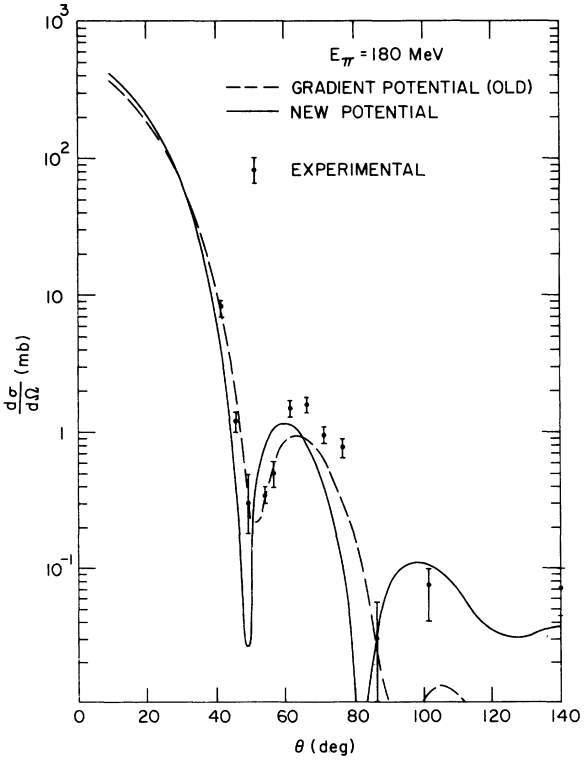

FIG. 3. The differential cross sections for pion-^{12}C elastic scattering at 180-MeV laboratory kinetic energy. See Fig. 1 for other details.

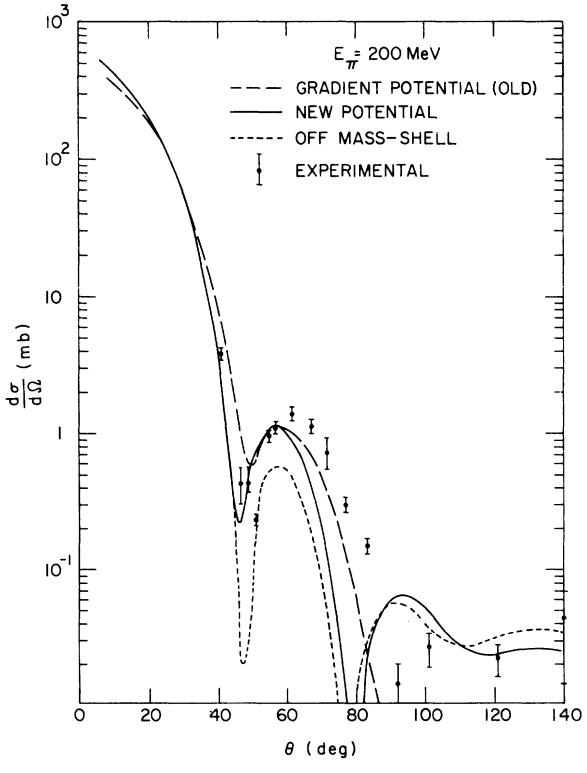

FIG. 4. The differential cross sections for pion-^{12}C elastic scattering at 200-MeV laboratory kinetic energy. See Fig. 1 for other details.

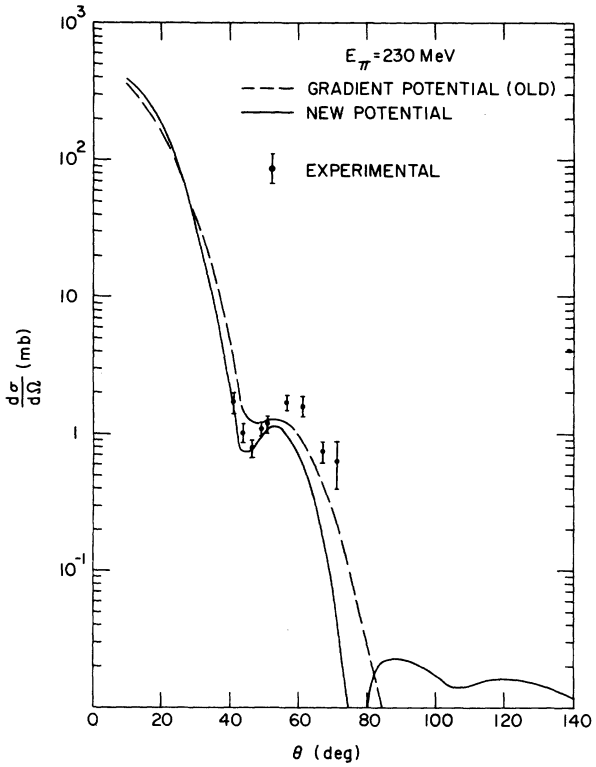

FIG. 5. The differential cross sections for pion-^{12}C elastic scattering at 230-MeV laboratory kinetic energy. See Fig. 1 for other details.

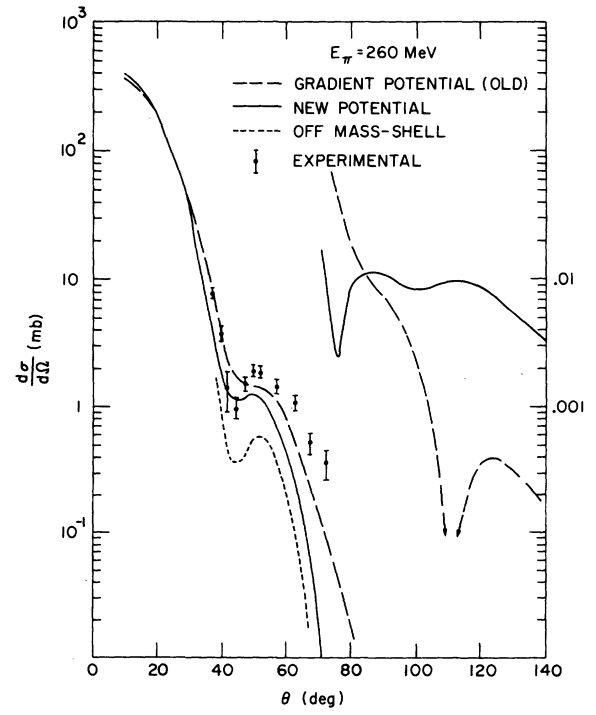

FIG. 6. The differential cross sections for pion-^{12}C elastic scattering at 260-MeV laboratory kinetic energy. See Fig. 1 for other details.

the pion-nucleon form factor not included in Eq. (12). Another approach is to use the improved potential defined by (C3), (C5), and (C7), which is derived by an expansion in the form-factor parameters. The local momentum approximation could be used for the higher derivatives, giving a practical potential.

A theoretical point of great importance should be stressed here. Our work has been guided by the solution for the off-shell t matrix by Landau and Tabakin.[5] Yet our "almost local" potential, which represents all of the qualitative and most of the quantitative features of the momentum-space calculation corresponding to the nonlocal potential (B13), makes use only of the general structure of that two-body amplitude. This is most encouraging. It leads us first to have strong hope that the nuclear potential is weakly enough dependent upon the off-shell behavior so that it might be possible to explore nuclear correlations with a careful theoretical and experimental study of elastic scattering. Secondly, it now seems more likely that with a convenient, "almost local" coordinate-space potential one can calculate accurate pion wave functions to use in inelastic scattering and reaction studies. One must remember, however, that some of the basic theoretical problems associated with the treatment of the scattering of pions in a nuclear

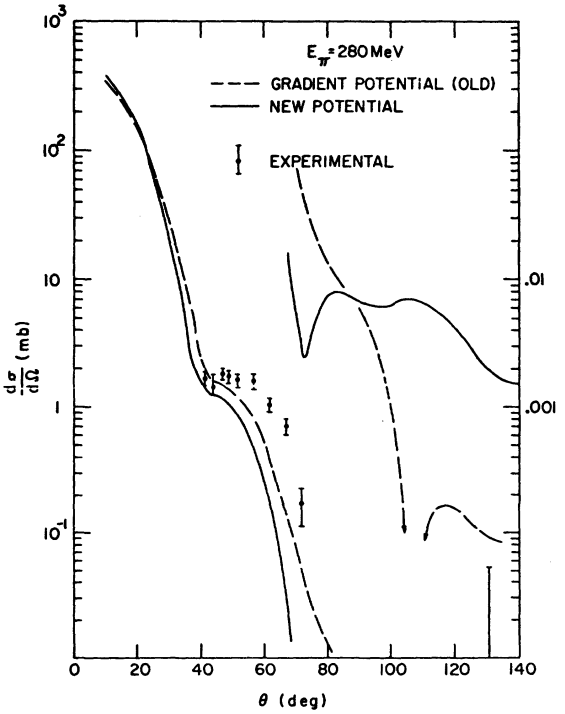

FIG. 7. The differential cross sections for pion-^{12}C elastic scattering at 280-MeV laboratory kinetic energy. See Fig. 1 for other details.

medium by a bound nucleon might give important modifications. Thus, even if the suggested potential is correct in form, the magnitude and energy dependence of the parameters can be expected to differ from the theoretical values. We have not attempted here to review the work which has been done in estimating such corrections.

In summary, the potential of Eq. (12) is presented as being a first-order potential which incorporates a number of important theoretical features. Its form is that of a correct Lorentz transformation on the gradient potential of Ref. 7. However, it takes into consideration not only the correct Lorentz transformation, but also the proper threshold behavior of the individual partial waves and the general features of the solution of the off-shell form of the pion-nucleon scattering matrix derived from a separable potential. It is suggested that our new form is a satisfactory first-order potential, and that for theoretical improvements a great number of many-body effects should be incorporated. (However, improved phenomenological forms are also suggested.) This is an important program in which we, as well as a number of others, are involved.

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APPENDIX A. KINEMATICS AND DEFINITIONS

In the body of the paper two amplitudes are defined, $\langle\vec{k}'|\,t(E)\,|\vec{k}\rangle$ and $\langle\vec{\kappa}'|\,t(\omega)\,|\vec{\kappa}\rangle$, the off-shell scattering matrices in the pion-nucleus and pion-nucleon center-of-mass systems, respectively. Associated with these there are 12 momentum four-vectors, which are now defined. For the pion and nucleon in the states $|\vec{\kappa}\rangle$, $|\vec{\kappa}'\rangle$, $|\vec{k}\rangle$, and $|\vec{k}'\rangle$ the pion has three-momenta $\vec{\kappa}$, $\vec{\kappa}'$, $\vec{k}$, and $\vec{k}'$, respectively; while in these states the nucleon has momenta $-\vec{\kappa}$, $-\vec{\kappa}'$, $\vec{k}_N$, and $\vec{k}'_N$, respectively. Since we deal here entirely with mass-shell quantities, we can define the eight momentum four-vectors

$$\begin{aligned}&(E_\pi(\kappa),\vec{\kappa}), \quad (E_N(\kappa),-\vec{\kappa}),\\ &(E_\pi(\kappa'),\vec{\kappa}'), \quad (E_N(\kappa'),-\vec{\kappa}'),\\ &(E_\pi(k),\vec{k}), \quad (E_N(k_N),\vec{k}_N),\\ &(E_\pi(k),\vec{k}), \quad (E_N(k'_N),\vec{k}'_N),\end{aligned} \tag{A1}$$

where $E_\pi(\kappa)=(\kappa^2+\mu^2)^{1/2}$ etc., and $E_N(\kappa)=(\kappa^2+m^2)^{1/2}$ etc. The units $\hbar=c=1$ are used throughout. In the pion-nucleon center-of-mass system the pion and nucleon have three-momenta $\vec{p}$, $-\vec{p}$, respectively, while in the pion-nucleus center-of-mass system they have momenta $\vec{p}_L$ and $\vec{p}_N$, respectively. We thereby define the momentum four-vectors for the pion and the nucleon in the center-of-mass and laboratory systems as

$$\begin{aligned}&(\omega_\pi,\vec{p}), \quad (\omega_N,-\vec{p}),\\ &(E_\pi,\vec{p}_L), \quad (E_N,\vec{p}_N).\end{aligned} \tag{A2}$$

The invariant variable $s=(\omega_\pi+\omega_N)^2=(\omega)^2$.

Working at low and intermediate energies one can take advantage of the large mass of the nucleus and the nucleon and approximately equate the pion-nucleon laboratory and the pion-nucleus center-of-mass systems. This is done in the numerical calculations presented here.

A small correction could be made to account for the transformation from the pion-nucleus laboratory to the pion-nucleus center-of-mass systems. This effect can be included approximately in Eq. (12) by introducing a factor $\xi(\omega)\equiv p_{\rm lab}/p_{\rm c.m.}$, which is the ratio of the pion momentum in the pion-nucleus laboratory system ($p_{\rm lab}$) to the pion momentum in the pion-nucleus center-of-mass system ($p_{\rm c.m.}$). The optical potential including this transformation is given by

$$\langle\vec{x}|\,V\,|\vec{x}'\rangle=-\frac{A}{2E_\pi(p_{\rm c.m.})}\,\xi(\omega)\Big\{b_0(\omega)p_{\rm c.m.}{}^2\rho(x)-b_1\vec{\nabla}\cdot\rho(x)\vec{\nabla}+\frac{1}{2}\Big[\frac{s}{m^2}\,\xi^{-1}(\omega)-1\Big]b_1(\omega)(\nabla^2\rho(x))\Big\}\delta(\vec{x}-\vec{x}').$$

Since $\xi\simeq 1$ at the energies considered in this paper, we do not include this correction. Note also that in the present work we use $E_N\approx m$; however, this does not restrict the nuclear Fourier components, which appear in nuclear form factors.

Here we consider scattering only from spin-zero systems, although the work can easily be extended to the general case. This results in the optical potential depending only upon the two parameters introduced in Ref. 7. All of the results of the present work are expressed in terms of the familiar parametrization of Auerbach and Sternheim, in such a manner that the widely distributed computer code ABACUS-M[10] can readily be used. The Auerbach-Sternheim parameters are defined by the gradient potential

$$2E_\pi V(x)=-Ab_0(\omega)p_{\rm lab}{}^2\rho(x)+Ab_1(\omega)\vec{\nabla}\cdot\rho(x)\vec{\nabla}, \tag{A3}$$

where for a spinless nucleus with Z protons and $N=A-Z$ neutrons

$$A b_0(\omega) p_{\rm lab}{}^2 = (4\pi/3 p_{\rm lab})(s/m^2)[Z(2\alpha_1+\alpha_3)+3N\alpha_3],$$
$$A b_1(\omega) = (4\pi/3 p_{\rm lab}{}^3)(s/m^2)[Z(2\alpha_{33}+\alpha_{31}+4\alpha_{13}+2\alpha_{11}) + 3N(2\alpha_{33}+\alpha_{31})]. \quad \text{(A4)}$$

The conventional notation is used for the s- and p-wave phase shifts with δ_1 and δ_3 the $I=\frac{1}{2}$ and $\frac{3}{2}$ $l=0$ phase shifts, and for $l=1$, the phase shifts are labeled by $\delta_{(2I)(2J)}$.

As a consequence of the theory presented here, it is shown that the "almost local" form of the first-order optical potential should be of the form [Eq. (12)]

$$2E_\pi V(x) = -A[a_0 p_{\rm lab}{}^2 \rho(x) - \tfrac{1}{2}a_2 \nabla^2 \rho(x) - (a_1+a_2)\vec{\nabla}\cdot\rho(x)\vec{\nabla}]. \quad \text{(A5)}$$

The form of "Klass 3" in the ABACUS-M code has been used with a_0, a_1, a_2 being used instead of b_0, b_1, b_2 in Ref. 10 to try to minimize confusion.[11]

APPENDIX B. GENERAL FORM FOR THE FIRST-ORDER OPTICAL POTENTIAL

In this Appendix we derive the first-order optical potential corresponding to the π-nucleon off-shell t matrix given by Eqs. (3)–(8). Only s and p waves are considered for the two-body scattering. Further, we make use of the numerical results of Ref. 5 to recognize that the two s-state form-factor parameters are almost equal, and that the four p-state form factors are also approximately the same. Thus Eq. (8) can safely be replaced by

$$g_{lj}(\kappa) = \begin{cases} e^{-a_s{}^2\kappa^2} & l=0 \\ \kappa e^{-a_p{}^2\kappa^2} & l=1. \end{cases} \quad \text{(B1)}$$

Note also that the form factor parameters which appear in (B1) are $a_{1j}{}^2 m^2/s$, which are approximately constant.

Inserting Eqs. (3)–(7) and (B1) into (1), one obtains

$$\langle \vec{x}|V|\vec{x}'\rangle = \langle \vec{x}|V_s|\vec{x}'\rangle + \langle \vec{x}|V_p|\vec{x}'\rangle + \langle \vec{x}|V_{p(s)}|\vec{x}'\rangle, \quad \text{(B2)}$$

with

$$\langle \vec{x}|V_s|\vec{x}'\rangle = A e^{+2a_s{}^2 p_{\rm lab}{}^2} a(\omega)(2\pi)^{-3/2} \int d^3k d^3k' e^{i(\vec{k}'\cdot\vec{x}-\vec{k}\cdot\vec{x}')} \rho(|\vec{k}-\vec{k}'|) e^{-a_s{}^2(k^2+k'^2)},$$
$$\langle \vec{x}|V_p|\vec{x}'\rangle = A e^{+2a_p{}^2 p_{\rm lab}{}^2} b(\omega)(2\pi)^{-3/2} \int d^3k d^3k' \vec{k}'\cdot\vec{k} e^{i(\vec{k}'\cdot\vec{x}-\vec{k}\cdot\vec{x}')} \rho(|\vec{k}-\vec{k}'|) e^{-a_p{}^2(k^2+k'^2)}, \quad \text{(B3)}$$
$$\langle \vec{x}|V_{p(s)}|\vec{x}'\rangle = A e^{+2a_p{}^2 p_{\rm lab}{}^2} b(\omega)(2\pi)^{-3/2} \int d^3k d^3k' [E_\pi(\kappa)E_\pi(\kappa') - E_\pi(k)E_\pi(k')] e^{i(\vec{k}'\cdot\vec{x}-\vec{k}\cdot\vec{x}')} \rho(|\vec{k}-\vec{k}'|) e^{-a_p{}^2(k^2+k'^2)},$$

where $\rho(\vec{q}) = (2\pi)^{-3/2}\int d^3z e^{-i\vec{q}\cdot\vec{z}}\rho(z)$ and using the notation of Appendix A

$$a(\omega) = -b_0(\omega) p_{\rm lab}{}^2/[2E_\pi(2\pi)^3],$$
$$b(\omega) = -b_1(\omega)\left(\frac{s}{m^2}\right)/[2E_\pi(2\pi)^3]. \quad \text{(B4)}$$

In order to give a convenient explicit form for the potential, we express the density as a sum of modified Gaussians

$$\rho(z) = \sum_{I,J} A(I,J) z^{2J} e^{-C_I z^2}, \quad \text{(B5a)}$$

which has the Fourier transform

$$\rho(q) = \sum_{I,J} A(I,J)(-1)^J \frac{\partial^J}{\partial C_I{}^J} \frac{1}{2\sqrt{2}\, C_I{}^{3/2}} e^{-q^2/4C_I}. \quad \text{(B5b)}$$

Introducing center-of-mass and relative coordinates

$$\vec{K} = (\vec{k}+\vec{k}')/2, \quad \vec{q} = \vec{k}-\vec{k}'$$
$$\vec{R} = (\vec{x}+\vec{x}')/2, \quad \vec{r} = \vec{x}-\vec{x}',$$

one can immediately evaluate the first two terms in (B2). For the s wave (B3) becomes

$$\langle \vec{x}|V_s|\vec{x}'\rangle = \frac{Aa(\omega)}{e^{-2a_s^2 p_{\rm lab}^2}}(2\pi)^{-3/2}\int d^3K d^3q e^{i(\vec{K}\cdot\vec{r}+\vec{q}\cdot\vec{R})}\rho(q)e^{-a_s^2(2K^2+q^2/2)}, \tag{B6}$$

which upon introducing the form (B5b) can easily be shown to be

$$\langle \vec{x}|V_s|\vec{x}'\rangle = \frac{-Ab_0(\omega)p_{\rm lab}^2(\pi/2)^{3/2}}{2E_\pi e^{-2a_s^2 p_{\rm lab}^2}(2\pi)^3}\sum_{IJ}A(I,J)(-1)^J\frac{\partial^J}{\partial C_I^J}\frac{1}{[1+2C_I a_s^2]^{3/2}a_s^3}e^{-r^2/8a_s^2}e^{-C_I R^2/(1+2C_I a_s^2)}. \tag{B7}$$

The second term can also be done trivially, resulting in

$$\langle \vec{x}|V_p|\vec{x}'\rangle = \frac{Ab_1(\omega)(s/m^2)(\pi/2)^{3/2}}{2E_\pi e^{-2a_p^2 p_{\rm lab}^2}(2\pi)^3}\sum_{I,J}A(I,J)(-1)^J\frac{\partial^J}{\partial C_I^J}\frac{1}{[1+2C_I a_p^2]^{3/2}a_s^3}\vec{\nabla}_{x'}e^{-r^2/8a_p^2}e^{-C_I R^2/(1+2C_I a_p^2)}\vec{\nabla}_{x'}. \tag{B8}$$

The third term in (B3), $V_{p(s)}$, which we refer to as the isotropic p-wave term, is more difficult to evaluate. We derive a convenient form for this term making use of an approximation for the factor $[E_\pi(\kappa)E_\pi(\kappa') - E_\pi(k)E_\pi(k')]$ which results from two observations. First, since the nucleus is a loosely bound system, $\rho(q)$ is a rapidly decreasing function of q, and large values of $|\vec{k}-\vec{k}'|$ do not contribute much to the momentum integrals defining the optical potential. It is this observation which forms much of the basis for the present work, and gives us confidence that a useful optical potential can be derived by incorporating the main features of the off-shell and high-energy π-nucleon behavior into the many-body formalism without completely reproducing the details of the off-shell two-body collisions. As a result, for large magnitude of k and k' in the integral (B3) (i.e., k and $k' \gg \mu$) one can use

$$E_\pi(\kappa)E_\pi(\kappa') - E_\pi(k)E_\pi(k') \approx \tfrac{1}{2}\left(\frac{m^2}{s}-1\right)(k^2+k'^2). \tag{B9}$$

Observing, secondly, that the identical form (B9) is correct for the low k and k' parts of the integral, we make use of the approximation (B9) in evaluating the isotropic p-wave term in (B3). The result is

$$\langle \vec{x}|V_{p(s)}|x'\rangle \approx -\frac{Ab_1(\omega)(1-s/m^2)(\pi/2)^{3/2}}{2E_\pi e^{-2a_p^2 p_{\rm lab}^2}(2\pi)^3}\sum_{I,J}A(I,J)(-1)^J\frac{\partial^J}{\partial C_I^J}(\nabla_r^2+\tfrac{1}{4}\nabla_R^2)e^{-r^2/8a_p^2}e^{-C_I R^2/(1+2C_I a_p^2)}. \tag{B10}$$

Summarizing our results, the optical potential is

$$\begin{aligned}\langle \vec{x}|V|\vec{x}'\rangle = &-[(A/2E_\pi)8(2\pi)^{3/2}]\sum_{I,J}A(I,J)(-1)^J\frac{\partial^J}{\partial C_I^J}\frac{1}{[1+2C_I a_s^2]^{3/2}a_s^3}\\ &\times[b_0(\omega)p_{\rm lab}^2 e^{-r^2/8a_s^2}e^{-C_I R^2/(1+2C_I a_s^2)}\\ &+b_1(\omega)(1-s/m^2)(\nabla_r^2+\tfrac{1}{4}\nabla_R^2)e^{-r^2/8a_s^2}e^{-C_I R^2/(1+2C_I a_s^2)}\\ &+b_1(\omega)\frac{s}{m^2}\vec{\nabla}_x\cdot e^{-r^2/8a_p^2}e^{-C_I R^2/(1+2C_I a_p^2)}\vec{\nabla}_{x'}].\end{aligned} \tag{B11}$$

This potential includes the effect of the Lorentz transformation and the off-shell and high momentum behavior given in Eqs. (6)–(8), and thus should be quite similar to the momentum space formulation of Ref. 5. There is no difficulty including the Coulomb interaction, which is essential especially for higher-mass nuclei.

Although these nonlocal potentials involve a lengthier calculation than the "almost local" potentials now being used, there is no real difficulty in principle or practice in using the potential (B11). Moreover, the higher-order terms in the multiple-scattering formulation of the optical potential involve nonlocal potentials. Thus, if one is to study

particle correlations, which are introduced by the higher-order terms, it is essential to use a proper nonlocal potential for the first-order potential. In particular, it is the large-angle scattering from which one hopes to learn about particle correlations. Since the nonlocality plays a specially important role for large-angle scattering, it is essential to include these corrections to the first-order potential in any program which studies correlations. We suggest that the potential (B11) should be an adequate starting point.

APPENDIX C. LIMITS FOR SMALL FORM-FACTOR PARAMETERS

In this Appendix we explore some of the consequences of the observations that the form-factor parameters a_{ij} are small compared to the nuclear radius, and that the nonlocality found in the previous Appendix is small compared to the nuclear dimensions. Referring to Eqs. (B2), (B3), and (B6), one sees that the form-factor parameters appear in the integrals as a factor

$$e^{-a^2(2K^2+q^2/2)}.$$

As has been discussed above, the nuclear form factor $\rho(q)$ limits the q integral so that it is safe to drop the factor $e^{-a^2q^2/2}$. However, one cannot use this argument to drop the factor $e^{-2(aK)^2}$. We look at the effect of this factor in two ways.

(1) Limit of a_{ij}, very small. Assuming that the $a_{ij} \to 0$ we use the expansion $e^{-a^2(2K^2+q^2/2)} \approx 1 - 2a^2K^2$. Let us look first at the potential V_s form (B3).

$$\langle \vec{x}|V_s|\vec{x}'\rangle_{a_s\to 0} \cong Aa(\omega)(2\pi)^{-3/2}\int d^3k d^3q(1+a_s{}^2\nabla_r{}^2) \times e^{i(\vec{k}\cdot\vec{r}+\vec{q}\cdot\vec{R})}\rho(q). \tag{C1}$$

In carrying out the integrals, which involve derivatives of δ functions, one must remember that this is an operator to be used in a Schrödinger-like equation. One finds that (C1) becomes

$$\langle \vec{x}|V_s|\vec{x}'\rangle \cong -\frac{A}{2E_\pi}[b_0(\omega)p_{\rm lab}{}^2\rho(x)+2a_s{}^2(\vec{\nabla}\cdot\rho_{(x)}\vec{\nabla}+\tfrac{1}{4}\nabla^2\rho(x)]\delta(x-x'). \tag{C2}$$

The meaning of this operator as used in a Schrödinger-like equation is

$$\langle \vec{x}|V_s|\phi\rangle = -\frac{A}{2E_\pi}b_0(\omega)p_{\rm lab}{}^2\Big(\rho(x)\phi(x)+2a_s{}^2\{\vec{\nabla}\cdot\rho(x)\vec{\nabla}\phi(x)+[\tfrac{1}{4}\nabla^2\rho(x)]\phi(x)\}\Big). \tag{C3}$$

In a similar manner V_p has the approximate form

$$\langle \vec{x}|V_p|\vec{x}'\rangle \cong -Ab(\omega)(2\pi)^{-3}\int d^3k d^3k' d^3z(1-2k^2a_p{}^2)\ \vec{\nabla}_x e^{i\vec{k}'\cdot(\vec{x}-\vec{z})}\cdot\vec{\nabla}_{x'}e^{i\vec{k}\cdot(\vec{x}'-\vec{z})}\rho(z). \tag{C4}$$

With the same considerations used to obtain (C3), one finds

$$\langle \vec{x}|V_p|\ \phi\rangle \cong \frac{A}{2E_\pi}b_1(\omega)\frac{s}{m^2}\left[\vec{\nabla}\cdot\rho\vec{\nabla}\phi+2a_s{}^2\left\{\vec{\nabla}\rho\cdot\vec{\nabla}\nabla^2\phi+\tfrac{1}{4}\vec{\nabla}(\nabla^2\rho)\cdot\nabla\phi+\Big(\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\rho(x)\Big)\Big(\sum_{k,l=1}^{3}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}\phi(x)\Big)\right\}\right], \tag{C5}$$

and finally, for the isotropic p wave

$$\langle \vec{x}|V_{p(s)}|\vec{x}'\rangle \cong \frac{A}{2E_\pi}\Big(1-\frac{s}{m^2}\Big)b_1(\omega)\frac{1}{(2\pi)^3}\int(\nabla_r{}^2+\tfrac{1}{4}\nabla_R{}^2)\rho(R)(1+2a_p{}^2\nabla_r{}^2)\delta(r). \tag{C6}$$

Proceeding as above, one finds

$$\langle x|V_{x(s)}|\phi\rangle = \frac{A}{2E_\pi}\Big(1-\frac{s}{m^2}\Big)b_1(\omega)[\vec{\nabla}\rho\cdot\vec{\nabla}\phi+\tfrac{1}{2}\nabla^2\rho)\phi+2a_p{}^2\{\tfrac{1}{2}\vec{\nabla}(\nabla^2\rho)\cdot\vec{\nabla}\phi+\tfrac{1}{2}(\nabla^2(\nabla^2\rho))\phi+\dots\}]. \tag{C7}$$

From (C3), (C5), and (C7), one has the small a_{ij} limit of the potential [see Eq. (12)].

(2) Small a_{ij} with high-energy behavior. It is also interesting to look at the form one obtains from directly using the approximation

$$e^{-a^2(2K^2+q^2/2)} \approx e^{-2a^2K^2} \tag{C8}$$

discussed above, without letting $a \to 0$. This incorporates the high momentum cutoff of the pion-nucleon form factor into the optical potential. Using (C8) in (B2), (B3), one finds (let us use $a_s = a_p = a$ for simplicity)

$$\langle \vec{x}|V|\vec{x}'\rangle = -\frac{A}{2E_\pi}\left\{b_0(\omega)p_{\rm lab}{}^2\rho - b_1(\omega)\frac{s}{m}\vec{\nabla}_x\,\rho(R)\vec{\nabla}_{x'} - b_1(\omega)\left(1-\frac{s}{m^2}\right)(\nabla_r{}^2+\tfrac{1}{4}\nabla_R{}^2)\rho(R)\right\}\frac{1}{(2\pi)^{3/2}8a^3}e^{-r^2/8a^2}. \tag{C9}$$

Looking at the last term we see that the isotropic p wave becomes in this approximation

$$\langle \vec{x}|V_{p(s)}|\vec{x}'\rangle = \frac{Ab_1(\omega)}{2E_\pi}\left(1-\frac{s}{m^2}\right)\left\{\tfrac{1}{4}\nabla_R{}^2\rho(R)+\frac{1}{4a^2}\left[-3+\frac{r^2}{4a^2}\right]\rho(R)\right\}\frac{e^{-r^2/8a^2}}{8(2\pi)^{3/2}a^3}. \tag{C10}$$

Noticing that

$$\frac{1}{8(2\pi)^{3/2}a^3}e^{-r^2/8a^2}\xrightarrow[a\to 0]{}\delta(r), \tag{C11}$$

one observes that the first two terms in (C9) go to the $a \to 0$ terms in (C3) and (C5) for the s and p wave, respectively. Of course, in this limit the form (C10) reduces to the $a \to 0$ limit of $V_{p(s)}$ in Eq. (C7). This can be seen by evaluating the derivatives of the $\delta(r)$, as discussed above.

It is interesting to consider the form (C10) without taking the limit $a \to 0$. The first term is approximately

$$V_{p(s)}^{(1)} \approx \tfrac{1}{4}\left(1-\frac{s}{m^2}\right)\frac{Ab_1(\omega)}{2E_\pi}(\nabla^2\rho)\delta(\vec{x}-\vec{x}'), \tag{C12}$$

or one half of the $a \to 0$ limit of $V_{p(s)}$. Let us study the remainder of (C10) without taking the limit as $a \to 0$. The last term in (C10) is small for small a, but in general this term is not small in the $a \to 0$ limit. However, in the limit of fixed a, dropping nonlocality, we would drop this term. In this approximation

$$\langle \vec{x}|V_{p(s)}|\vec{x}'\rangle \approx \frac{A}{2E_\pi}\left(1-\frac{s}{m^2}\right)\left[\tfrac{1}{4}\nabla^2\rho(x)-\frac{3}{4a^2}\rho(x)\right]\delta(\vec{x}-\vec{x}'). \tag{C13}$$

Combining results, one has the form

$$\langle \vec{x}|V|\vec{x}'\rangle \approx -\frac{A}{2E_\pi}\left\{\left[b_0(\omega)p_{\rm lab}{}^2-\frac{3}{4a^2}\left(\frac{s}{m^2}-1\right)b_1(\omega)\right]\rho(x) + \tfrac{1}{4}\left(\frac{s}{m^2}-1\right)b_1(\omega)\nabla^2\rho(x)-\frac{s}{m^2}b_1(\omega)\vec{\nabla}\cdot\rho\vec{\nabla}\right\} \times\delta(\vec{x}-\vec{x}'). \tag{C14}$$

Clearly, the potential (C13) does not represent a well-defined limit, but suggests a form for a local potential.

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[11]We are grateful to Dr. R. A. Eisenstein for pointing out an error in Ref. 10. In the description of "Klass 3" there is an error in using $+b_2$ instead of $-b_2/2$. This has been verified by Dr. E. Auerbach (private communication). Since the potential present in this paper is of this type, it is important that users of ABACUS-M correct this error.