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Particle-hole interactions and vibrational states*

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A method is presented for the treatment of particle-hole residual interactions that is both simple and extremely accurate. This method is compared with the random-phase approximation, Tamm-Dancoff approximation, and other approximations in the schematic model of Brown and the monopole model of Lipkin, Meshkov, and Glick.

I. INTRODUCTION

The Tamm-Dancoff approximation (TDA) and the random-phase approximation (RPA) have served as a basis for the calculation of the properties of vibrational states in spherical and deformed nuclides. In spite of the widespread use of these approximations, they suffer from serious inadequacies. The TDA does not take into account the ground-state correlations induced by residual interactions. For a sufficiently large interaction, the RPA gives unphysical imaginary eigenvalues. Further, half of the eigenvalues obtained with the RPA are spurious for all interaction strengths. In this work we develop an approach that has none of these shortcomings.

Our approach to the problem of particle-hole residual interactions is based on the treatment of particle-particle residual interactions that we have presented previously. This approach is more accurate than the RPA and other higher-order approximations that we have encountered. In the case of a separable residual interaction, our method gives dispersion relations of the TDA form; i.e., there are not any spurious solutions. Because of the simplicity and accuracy of our approach, we feel that it can provide a useful starting point for the treatment of more complicated particle-hole residual interactions than we consider here.

In Sec. II we develop the formalism for treating particle-hole residual interactions. In Sec. III we apply this treatment to schematic models and compare our results to other approximations.

II. FORMALISM

The treatment of particle-hole interactions consists of two parts: (1) the derivation of equations for the description of excited states and (2) the construction of an explicitly correlated ground state. We consider a Hamiltonian with a single multipole particle-hole residual interaction. All orbitals are assumed to have unique partners in the residual interaction.

The Hamiltonian of the system we study is

$$H=\sum_{\alpha}\epsilon_{\alpha}N_{\alpha}-\sum_{2b;\,3c}V_{2b;\,3c}T_{2b}T_{c3}\,, \tag{1}$$

where $T_{\alpha\beta}$ is defined as

$$T_{\alpha\beta}=a_{\alpha}^{\dagger}a_{\beta}+a_{-\beta}^{\dagger}a_{-\alpha}\,. \tag{2}$$

The letter N is used to denote a fermion occupation probability and $a^{\dagger}$ (a) are used to denote fermion creation (annihilation) operators. Numbers are used as indices for particle orbitals and Latin letters are used as indices for hole orbitals. Greek letters are used as indices for both particle and hole orbitals. The negative indices are used to indicate time reversal partners in the doubly degenerate Nilsson orbitals that we have in mind.

The starting point for our treatment of vibrational states is the well-known commutation relation[1] of a particle-hole pair with the Hamiltonian

$$[H,a_4^{\dagger}a_d]=(\epsilon_4-\epsilon_d)a_4^{\dagger}a_d-\sum_{2b}{}' V_{2b;\,4d}T_{2b}[N_d(1-N_4)-N_4(1-N_d)]-V_{4d;\,4d}(a_4^{\dagger}a_d+a_{-d}^{\dagger}a_{-4})[N_d(1-N_4)-N_4(1-N_d)]\,, \tag{3}$$

where the prime on the summation in Eq. (3) indicates that the summation does not include $4d$. We shall ignore the final term in Eq. (3) and the prime on the summation in our development in order to keep the equations uncluttered. It is a simple matter to take these features into account, and we do in the numerical results that we present. Denoting the excited state of interest as $|\phi\rangle$ and the correlated ground state as $|0\rangle$, we obtain from Eq. (3) the relations

$$[(\epsilon_4-\epsilon_d)-\omega]\langle\phi|a_4^{\dagger}a_d|0\rangle=\sum_{2b}V_{2b;\,4d}\langle\phi|T_{2b}[N_d(1-N_4)-N_4(1-N_d)]|0\rangle \tag{4}$$

and

$$[-(\epsilon_4-\epsilon_d)-\omega]\langle\phi|a^\dagger_{-d}a_{-4}|0\rangle =\sum_{2b}V_{2b;4d}\langle\phi|T_{2b}[N_{-4}(1-N_{-d})-N_{-d}(1-N_{-4})]|0\rangle, \quad (5)$$

where ω is the excitation energy of the state $|\phi\rangle$. It should be noted that Eq. (5) has meaning only when the particle-hole correlations induced by the residual interaction are included in the description of the ground state. In order to obtain a set of useful relations for the amplitudes $\langle\phi|a^\dagger_4 a_d|0\rangle$ and $\langle\phi|a^\dagger_{-d}a_{-4}|0\rangle$, we must decompose the expressions on the right-hand sides of Eqs. (4) and (5). Because each orbital has a unique partner in the residual interaction, we have the operator relations

$$N_{-4}(1-N_{-d})+N_{-d}(1-N_{-4})=1\,, \qquad N_d(1-N_4)+N_4(1-N_d)=1 \quad (6)$$

and we also note that

$$N_{-d}(1-N_{-4})=a^\dagger_{-d}a_{-4}a^\dagger_{-4}a_{-d} \quad (7)$$

independent of any features of the interaction. Making use of Eqs. (6) and (7), we rewrite the terms on the right-hand side of Eq. (5) as

$$\langle\phi|T_{2b}[N_{-4}(1-N_{-d})-N_{-d}(1-N_{-4})]|0\rangle =\langle\phi|T_{2b}|0\rangle-2\langle\phi|a^\dagger_{-d}a_{-4}T_{2b}a^\dagger_{-4}a_{-d}|0\rangle\,. \quad (8)$$

At this point, we introduce a complete set of intermediate states into the final term of Eq. (8) to obtain

$$\langle\phi|a^\dagger_{-d}a_{-4}T_{2b}a^\dagger_{-4}a_{-d}|0\rangle =\langle\phi|a^\dagger_{-d}a_{-4}|0\rangle\langle T_{2b}a^\dagger_{-4}a_{-d}\rangle +\sum_{|\gamma\rangle\neq|0\rangle}\langle\phi|a^\dagger_{-d}a_{-4}|\gamma\rangle\langle\gamma|T_{2b}a^\dagger_{-4}a_{-d}|0\rangle\,, \quad (9)$$

where we use angled brackets to denote expectation values in the correlated ground state. It should be noted that our treatment of the problem is exact to this point. However, Eq. (9) is not very useful as it stands.

The approximation that we introduce[2,3] at this point is to replace the summation over intermediate states in Eq. (9) with a single intermediate state $|\gamma_0\rangle$ (SIS approximation). For $|\gamma_0\rangle$, we choose the normalized state

$$|\gamma_0\rangle=\frac{N_{-4}(1-N_{-d})-\langle N_{-4}(1-N_{-d})\rangle}{\{\langle N_{-4}(1-N_{-d})\rangle[1-\langle N_{-4}(1-N_{-d})\rangle]\}^{1/2}}|0\rangle\,. \quad (10)$$

The state $|\gamma_0\rangle$ is orthogonal to the ground state $|0\rangle$. Our motivation for this choice of $|\gamma_0\rangle$ is that both $\langle\phi|a^\dagger_{-d}a_{-4}|\gamma_0\rangle$ and $\langle\gamma_0|T_{2b}a^\dagger_{-4}a_{-d}|0\rangle$ are large. An equally plausible choice for the intermediate state is

$$|\gamma_0'\rangle=\frac{N_{-d}(1-N_{-4})-\langle N_{-d}(1-N_{-4})\rangle}{\{\langle N_{-d}(1-N_{-4})\rangle[1-\langle N_{-d}(1-N_{-4})\rangle]\}^{1/2}}|0\rangle\,. \quad (10')$$

However, making use of Eq. (6), we see that these choices are equivalent.

Inserting the intermediate state $|\gamma_0\rangle$ into Eq. (9), we obtain the result of interest

$$\langle\phi|a^\dagger_{-d}a_{-4}T_{2b}a^\dagger_{-4}a_{-d}|0\rangle=\frac{\langle\phi|a^\dagger_{-d}a_{-4}|0\rangle\langle T_{2b}a^\dagger_{-4}a_{-d}\rangle}{\langle N_{-4}(1-N_{-d})\rangle}\,. \quad (11)$$

Making an analogous approximation, we find that

$$\langle\phi|a^\dagger_4 a_d T_{2b}a^\dagger_d a_4|0\rangle=\frac{\langle\phi|a^\dagger_4 a_d|0\rangle\langle T_{2b}a^\dagger_d a_4\rangle}{\langle N_d(1-N_4)\rangle}\,. \quad (12)$$

Substituting Eqs. (11) and (12) into Eqs. (4) and (5), we obtain the set of linear equations for the amplitudes $\langle\phi|a^\dagger_4 a_d|0\rangle$ and $\langle\phi|a^\dagger_{-d}a_{-4}|0\rangle$:

$$\left[(\epsilon_4-\epsilon_d)-\omega+2\sum_{2b}V_{2b;4d}\frac{\langle T_{2b}a^\dagger_d a_4\rangle}{\langle N_d(1-N_4)\rangle}\right]\langle\phi|a^\dagger_4 a_d|0\rangle=\sum_{2b}V_{2b;4d}\langle\phi|a^\dagger_2 a_b+a^\dagger_{-b}a_{-2}|0\rangle \quad (13)$$

and

$$\left[-(\epsilon_4-\epsilon_d)-\omega+2\sum_{2b}V_{2b;4d}\frac{\langle T_{2b}a^\dagger_{-4}a_{-d}\rangle}{\langle N_{-4}(1-N_{-d})\rangle}\right]\langle\phi|a^\dagger_{-d}a_{-4}|0\rangle=\sum_{2b}V_{2b;4d}\langle\phi|a^\dagger_2 a_b+a^\dagger_{-b}a_{-2}|0\rangle\,. \quad (14)$$

If the interaction is separable, i.e.,

$$V_{2b;4d}=V_{2b}V_{4d}\,, \quad (15)$$

Eqs. (13) and (14) can be combined to give the dispersion relation

$$1=\sum_{4d}V_{4d}{}^2\left[(\epsilon_4-\epsilon_d)-\omega+2V_{4d}\sum_{2b}V_{2b}\frac{\langle T_{2b}a^\dagger_d a_4\rangle}{\langle N_d(1-N_4)\rangle}\right]^{-1}+\sum_{4d}V_{4d}{}^2\left[-(\epsilon_4-\epsilon_d)-\omega+2V_{4d}\sum_{2b}V_{2b}\frac{\langle T_{2b}a^\dagger_{-4}a_{-d}\rangle}{\langle N_{-4}(1-N_{-d})\rangle}\right]^{-1}. \quad (16)$$

In the model systems that we have considered, we find that the final summation in Eq. (16) is larger than $2(\epsilon_4 - \epsilon_d)$ and this dispersion relation does not give spurious eigenvalues, in contradistinction to the RPA.

Whether or not the interaction is separable, we can determine the eigenvalues ω and the relative magnitudes of the amplitudes $\langle \phi | a_d^\dagger a_4 | 0\rangle$ and $\langle \phi | a_{-d}^\dagger a_{-4} | 0\rangle$ once we have values for the ground-state expectation values $\langle T_{2b} a_d^\dagger a_4\rangle$, $\langle N_d(1-N_4)\rangle$, etc., that appear in Eqs. (13) and (14). In order to determine these quantities, we must construct a correlated ground state. To this end, we apply the method of correlated quasiparticles.[4,5] The basic idea in this method is to replace the creation and annihilation operators that appear in the residual interaction by products of number operators. We set

$$\langle a_i^\dagger a_j a_k^\dagger a_l\rangle = \langle N_i(1-N_j)N_k(1-N_l)\rangle^{1/2} \langle N_l(1-N_k)N_j(1-N_i)\rangle^{1/2}. \tag{17}$$

The residual interaction that we are studying here is of the same coherent form as the particle-particle interactions for which Eq. (17) was developed. Accordingly, this approximation should be quite accurate here. For the problem considered here, we note

$$N_2(1-N_b) \equiv N_2 \equiv 1 - N_b \tag{18}$$

and Eq. (17) is considerably simplified. Making use of Eq. (18), we have

$$\langle a_2^\dagger a_b a_{-3}^\dagger a_{-c}\rangle = \langle N_2 N_{-3}\rangle^{1/2} \langle (1-N_2)(1-N_{-3})\rangle^{1/2}\ , \tag{19}$$

and similar relations apply for all other combinations of creation and annihilation operators that appear in the residual interaction. Substituting Eq. (19) into Eq. (1) and taking ground-state expectation values we find

$$\langle H\rangle = \sum_\alpha \epsilon_\alpha \langle N_\alpha\rangle - \sum_{2b;3c} V_{2b;3c}(S_{2,-3} + S_{2,c} + S_{-b,c} + S_{-b,-3}) \tag{20}$$

with the quantity $S_{\alpha,\beta}$ defined as

$$S_{\alpha,\beta} = \langle N_\alpha N_\beta\rangle^{1/2} \langle (1-N_\alpha)(1-N_\beta)\rangle^{1/2}. \tag{21}$$

The ground-state properties of the system are determined by solving the set of algebraic equations

$$\frac{\partial\langle H\rangle}{\partial\langle N_\alpha\rangle} = 0 \tag{22}$$

after decomposing the expectation values that appear in $S_{\alpha,\beta}$. The decomposition of $S_{\alpha,\beta}$ is carried out in the manner developed for particle-particle residual interactions. In the interaction we consider here, orbitals are either correlated or anticorrelated with each other but never both. In such a case, we decompose the products $\langle N_\alpha N_\beta\rangle$ by satisfying sum-rule relations of the form

$$\sum_\beta{}' \langle N_\alpha N_\beta\rangle = \langle N_\alpha\rangle \sum_\beta{}' \langle N_\beta\rangle + \langle N_\alpha\rangle\langle 1 - N_\alpha\rangle, \tag{23}$$

where the prime on the summation indicates that the summation is just over the orbitals β with which orbital α is correlated by the residual interaction. Next, we define

$$R_{\alpha,\beta} \equiv R_{\beta,\alpha} \equiv \min(\langle N_\alpha\rangle\langle 1-N_\beta\rangle \text{ and } \langle N_\beta\rangle\langle 1-N_\alpha\rangle). \tag{24}$$

The quantity $R_{\alpha,\beta}$ is the maximum correlation enhancement that is possible for the number operators N_α and N_β. If the correlation enhancement were any larger than $R_{\alpha,\beta}$ we would have the unphysical situation

$$\langle N_\alpha N_\beta\rangle > \langle N_\alpha\rangle \quad \text{or} \quad \langle N_\alpha N_\beta\rangle > \langle N_\beta\rangle. \tag{25}$$

Also the correlation enhancement between number operators depends on the magnitude of the relevant matrix element in the residual interaction. When the matrix element is large, the correlation enhancement is large. With all of these features in mind, we set[5]

$$\langle N_2 N_{-3}\rangle = \langle N_2\rangle\langle N_{-3}\rangle + \frac{\langle N_2\rangle\langle 1-N_2\rangle}{2} \frac{V_{2b;3c}R_{2,3}}{\sum_{4d} V_{2b;4d}(R_{2,4}+R_{2,d})} + \frac{\langle N_3\rangle\langle 1-N_3\rangle}{2} \frac{V_{2b;3c}R_{2,3}}{\sum_{4d} V_{3c;4d}(R_{3,4}+R_{3,d})} \tag{26}$$

and equivalent relations for other products of number operators.

Making use of Eq. (18) to eliminate half of the number operators, and Eq. (26), we can now solve Eq. (22) to determine the ground-state expectation values for particle-state occupation probabilities; $\langle N_2\rangle, \langle N_3\rangle, \ldots$. With these expectation values and Eq. (26) we can evaluate the ground-state expectation values that appear in Eqs. (13) and (14), allowing us to determine the properties of the excited state $|\phi\rangle$.

The one remaining problem at this point is the determination of the absolute magnitudes of the amplitudes $\langle\phi|a_4^\dagger a_d|0\rangle$ and $\langle\phi|a_{-d}^\dagger a_{-4}|0\rangle$. The excited state of interest $|\phi\rangle$ is of the form

$$|\phi\rangle=\sum_{2b} C_{2b}a_2^\dagger a_b|0\rangle+\sum_{2b} D_{2b}a_{-b}^\dagger a_{-2}|0\rangle \tag{27}$$

and the quantity $\langle\phi|a_4^\dagger a_d|0\rangle$ is then just

$$\langle\phi|a_4^\dagger a_d|0\rangle=\sum_{2b} C_{2b}\langle a_b^\dagger a_2 a_4^\dagger a_d\rangle+D_{2b}\langle a_{-2}^\dagger a_{-b} a_4^\dagger a_d\rangle \tag{28}$$

or

$$\langle\phi|a_4^\dagger a_d|0\rangle=\sum_{2b\neq 4d} C_{2b}S_{4,b}+\sum_{2b} D_{2b}S_{4,-2}+C_{4d}\langle N_d\rangle, \tag{29}$$

where the quantities $S_{4,b}$ and $S_{4,-2}$ have already been determined. From Eq. (29) we see that a knowledge of the relative values of the amplitudes $\langle\phi|a_\alpha^\dagger a_\beta|0\rangle$ is easily converted to a knowledge of the relative magnitudes of the coefficients C_{2b} and D_{2b}. From Eq. (29) we also see that the determination of the absolute values of the amplitudes $\langle\phi|a_\alpha^\dagger a_\beta|0\rangle$ can be made via the normalization of $|\phi\rangle$. We have

$$\begin{aligned}\langle\phi|\phi\rangle=1=&\sum_{2b} C_{2b}{}^2\langle N_b\rangle+D_{2b}{}^2\langle N_2\rangle+2C_{2b}D_{2b}S_{2,-2}\\&+\sum_{2b}\sum_{4d\neq 2b} C_{2b}C_{4d}S_{2,d}+D_{2b}D_{4d}S_{-4,b}\\&+C_{2b}D_{4d}S_{2,-4}+D_{2b}C_{4d}S_{-b,d}\,.\end{aligned} \tag{30}$$

This result completes our treatment of the particle-hole interaction.

III. APPLICATION TO MODEL SYSTEMS

In order to compare our method with other approaches to the problem of particle-hole interactions, we have carried out some calculations using simplified models of the residual interaction. The first such model that we consider is the schematic model of Brown.[1] The simplifications introduced here are

$$\begin{aligned}&V_{2b;3c}=V^2 \text{ for all matrix elements,}\\&\epsilon_2=\epsilon \text{ for all particle orbitals,}\\&\epsilon_b=0 \text{ for all hole orbitals.}\end{aligned} \tag{31}$$

Making use of Eq. (26), we obtain the correlations between number operators in this model. These relations are

$$\langle N_2N_{-3}\rangle=\langle N_2\rangle^2+\frac{1}{\sigma-\langle N_2\rangle}\langle N_2\rangle\langle 1-N_2\rangle^2, \tag{32}$$

$$\langle N_2N_c\rangle=\langle N_2\rangle\langle 1-N_2\rangle\left(1+\frac{1}{\sigma-\langle N_2\rangle}\langle N_2\rangle\right), \tag{33}$$

$$\langle N_{-b}N_{-3}\rangle=\langle N_2N_c\rangle, \tag{34}$$

and

$$\langle N_{-b}N_c\rangle=\langle 1-N_2\rangle^2\left(1+\frac{1}{\sigma-\langle N_2\rangle}\langle N_2\rangle\right), \tag{35}$$

where σ is the number of terms in the summation $\sum_{2b}T_{2b}$. The factor $\sigma-\langle N_2\rangle$ rather than σ appears in the denominators because we have taken into account the fact that N_2 is not correlated with N_b. Using Eqs. (32) through (35) we find that

$$\begin{aligned}S_{2,-3}=S_{-b,c}=&\langle 1-N_2\rangle\left(1+\frac{\langle N_2\rangle}{\sigma-\langle N_2\rangle}\right)^{1/2}\\&\times\left(\langle N_2\rangle^2+\frac{1}{\sigma-\langle N_2\rangle}\langle N_2\rangle\langle 1-N_2\rangle^2\right)^{1/2}\end{aligned} \tag{36}$$

and

$$S_{2,c}=S_{-b,-3}=\langle N_2\rangle\langle 1-N_2\rangle\left(1+\frac{\langle N_2\rangle}{\sigma-\langle N_2\rangle}\right). \tag{37}$$

The single remaining expectation value $\langle N_2\rangle$ is determined with Eq. (22). For small values of $\sigma V^2/\epsilon$, we find

$$\langle N_2\rangle\approx\frac{\sigma V^4}{4\epsilon^2}, \tag{38}$$

i.e., the method of correlated quasiparticles gives positive occupation probabilities for the particle

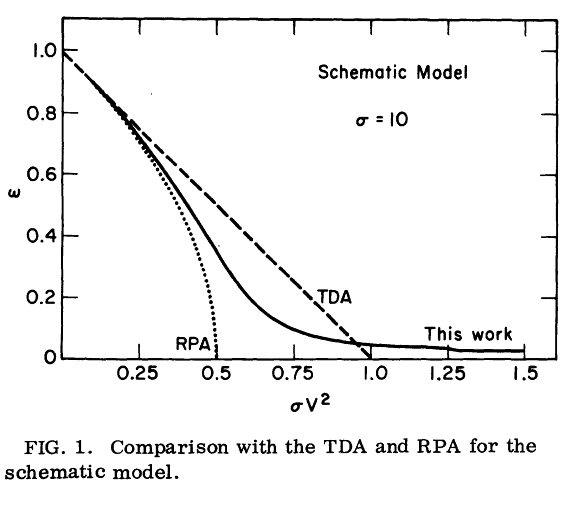

FIG. 1. Comparison with the TDA and RPA for the schematic model.

orbitals for all interaction strengths. If we had ignored correlations and set

$$S_{\alpha,\beta}=(\langle N_\alpha\rangle\langle 1-N_\alpha\rangle\langle N_\beta\rangle\langle 1-N_\beta\rangle)^{1/2}, \tag{39}$$

this would not be the case. Such a procedure would give negative values of $\langle N_2\rangle$ when $2\sigma V^2$ is less than ϵ. The important correlation relation is Eq. (36). In the limit of weak interactions we see that

$$S_{2,-3}\to(\langle N_2\rangle/\sigma)^{1/2} \text{ as } N_2\to 0. \tag{40}$$

Having determined $\langle N_2\rangle$, we solve Eq. (16) to obtain the excitation energy of the vibrational state $|\phi\rangle$.

In Fig. 1 we present the results of our calculations together with RPA and TDA results for $\sigma=10$; i.e., 20 particles. In calculating the excitation energy we have taken into account the features discussed after Eq. (3). The excitation energy of the vibrational state, as calculated with our method, is positive everywhere. It approaches zero asymptotically. From Fig. 1 it appears that there is very little advantage in calculating excitation energies with the RPA rather than the TDA.

Making use of Eqs. (29) and (30), we can calculate the absolute magnitudes of the amplitudes $\langle\phi|a_\alpha^\dagger a_\beta|0\rangle$. It is interesting to consider these quantities in the limits $V^2\to 0$ and $V^2\to\infty$. In Eqs. (29) and (30) we replace C_{2b} by C and D_{2b} by D. In the limit $V^2\to 0$, we note

$$D\to 0,\quad \langle N_2\rangle\to 0,\quad S_{2,c}\to 0,\quad S_{2,-3}\to 0; \tag{41}$$

hence

$$C=(\sigma)^{-1/2} \tag{42}$$

and

$$\langle\phi|a_4^\dagger a_d|0\rangle=(\sigma)^{-1/2} \tag{43}$$

and

$$\langle\phi|a_{-d}^\dagger a_{-4}|0\rangle=0. \tag{44}$$

In the limit $V^2\to\infty$ we set

$$C=D, \tag{45}$$

$$\langle N_2\rangle=\langle 1-N_2\rangle=\tfrac{1}{2}, \tag{46}$$

$$S_{2,-3}=S_{2,c}=\frac{1}{2}\frac{\sigma}{2\sigma-1}. \tag{47}$$

Using Eqs. (45) through (47), we obtain

$$C=D=\frac{1}{[\sigma(\sigma+1)]^{1/2}} \tag{48}$$

and

$$\langle\phi|a_4^\dagger a_d|0\rangle=\langle\phi|a_{-d}^\dagger a_{-4}|0\rangle=\frac{1}{2}\left(\frac{\sigma+1}{\sigma}\right)^{1/2}\approx\frac{1}{2}. \tag{49}$$

We next consider the monopole model[6] of Lipkin-Meshkov and Glick (LMG). This model is similar to the schematic model of Brown. Both exact and approximate solutions for this model have been presented in the literature.[7,8] In order to test the accuracy of our method, we have carried out some calculations for this model.

The Hamiltonian for the LMG model is

$$H=\sum_\alpha \epsilon_\alpha N_\alpha-\frac{V^2}{2}\sum_{2b}\sum_{3c}(P_{2b}P_{3c}+P_{b2}P_{c3}) \tag{50}$$

with the quantity $P_{\alpha\beta}$ defined as

$$P_{\alpha\beta}=a_\alpha^\dagger a_\beta. \tag{51}$$

In this model each level has a degeneracy of one and partners are unique in the residual interaction term. Also, the conditions of Eq. (31) apply here. This model differs from those that we have been discussing in that particle orbitals are always filled or emptied pairwise by the residual interaction. This simplified correlation pattern gives simpler ground-state correlations. Applying Eq. (23) to this model, we find

$$\langle N_2N_3\rangle=\langle N_2\rangle^2+\frac{1}{\sigma-1}\langle N_2\rangle\langle 1-N_2\rangle, \tag{52}$$

$$\langle N_bN_c\rangle=\langle 1-N_2\rangle^2+\frac{1}{\sigma-1}\langle N_2\rangle\langle 1-N_2\rangle, \tag{53}$$

and

$$S_{2,3}=S_{b,c}=\left(\langle N_2\rangle^2+\frac{1}{\sigma-1}\langle N_2\rangle\langle 1-N_2\rangle\right)^{1/2}\times\left(\langle 1-N_2\rangle^2+\frac{1}{\sigma-1}\langle N_2\rangle\langle 1-N_2\rangle\right)^{1/2}. \tag{54}$$

We note that

$$\langle a_2^\dagger a_b a_c^\dagger a_3\rangle=\langle a_3^\dagger a_c a_b^\dagger a_2\rangle=0 \tag{55}$$

in the LMG model. To determine $\langle N_2\rangle$ we substitute Eq. (54) into Eq. (50) and use Eq. (22). To compute the eigenvalues, Eqs. (13) and (14) are modified somewhat for this model. The relations of interest are easily obtained and found to be

$$\left[\epsilon-\omega+2V^2(\sigma-1)\frac{S_{2,3}}{\langle 1-N_2\rangle}\right]\langle\phi|a_2^\dagger a_b|0\rangle=(\sigma-1)V^2\langle\phi|a_b^\dagger a_2|0\rangle \tag{56}$$

and

$$\left[-\epsilon-\omega+2V^2(\sigma-1)\frac{S_{b,c}}{\langle N_2\rangle}\right]\langle\phi|a_b^\dagger a_2|0\rangle=(\sigma-1)V^2\langle\phi|a_2^\dagger a_b|0\rangle. \tag{57}$$

Combining Eqs. (56) and (57), we have

$$\left[\epsilon-\omega+2V^2(\sigma-1)\frac{S_{2,3}}{\langle 1-N_2\rangle}\right]\left[-\epsilon-\omega+2V^2(\sigma-1)\frac{S_{b,c}}{\langle N_2\rangle}\right]=(\sigma-1)^2V^4 \tag{58}$$

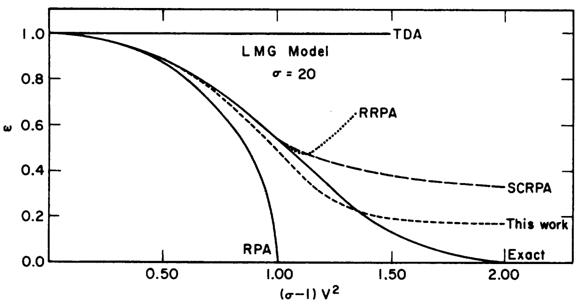

FIG. 2. Comparison with exact and approximate solutions of the LMG model. The RRPA is the most accurate approximation given in Ref. 7 and the SCRPA is the most accurate approximation given in Ref. 8.

which allows us to calculate ω.

In Fig. 2 we compare our result with the exact solution and some approximate solutions[7,8] for $\sigma = 20$. We see that the method developed here is quite accurate. We note also that Eq. (58) gives a considerably better approximation to the exact solution for the case $\sigma = 4$ than all approximations shown in Ref. 8. This indicates that there are no problems for small numbers of particles.

To calculate absolute magnitudes of amplitudes in this model we note

$$|\phi\rangle = C\sum_{2b} a_2^\dagger a_b|0\rangle + D\sum_{2b} a_b^\dagger a_2|0\rangle, \tag{59}$$

giving

$$\langle\phi|a_4^\dagger a_d|0\rangle = C\langle N_d\rangle + D(\sigma-1)S_{2,3} \tag{60}$$

and

$$\langle\phi|a_d^\dagger a_4|0\rangle = C(\sigma-1)S_{2,3} + D\langle N_4\rangle, \tag{61}$$

remembering Eq. (55).

The normalization condition for $|\phi\rangle$ gives us

$$1 = \langle\phi|\phi\rangle = \sigma(C^2\langle N_b\rangle + D^2\langle 1-N_b\rangle) + 2CD\sigma(\sigma-1)S_{2,3}. \tag{62}$$

From these relations we find for $V^2 \to 0$,

$$\langle\phi|a_4^\dagger a_d|0\rangle = \frac{1}{\sqrt{\sigma}}$$

and

$$\langle\phi|a_d^\dagger a_4|0\rangle = 0. \tag{63}$$

In the limit $V^2 \to \infty$, we find

$$\langle\phi|a_4^\dagger a_d|0\rangle = \langle\phi|a_d^\dagger a_4|0\rangle = \frac{1}{2}\left(\frac{\sigma+2}{2\sigma}\right)^{1/2} \approx \frac{1}{2\sqrt{2}}. \tag{64}$$

IV. SUMMARY

In this paper, we have developed an approach to particle-hole interactions based on our treatment of particle-particle interactions. This work provides a basis for a unified approach to the general problem of coherent residual interactions. Our method was applied to the schematic model of Brown and the monopole model of LMG. Comparison with exact solutions of the LMG model indicates that the method is quite accurate.

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