Separable representations of T matrices valid in the vicinity of off-shell points*

D. J. Ernst, C. M. Shakin, and R. M. Thaler
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
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The method developed in a previous paper for the construction of separable representations for T matrices is extended. It is shown that the separable representation we propose is valid *in the vicinity* of specified off-shell points. Since the off-shell points considered include those with arbitrary negative-energy parameters, the method presented here should be of value for the bound-state three-body problem.

I. SEPARABLE REPRESENTATIONS IN A NONORTHONORMAL BASIS

Following our previous study,[1] we consider the rank-N separable potential

$$\mathcal{V} = VP(PVP)^{-1(P)}PV, \tag{1.1}$$

where P is a projection operator onto a space spanned by a finite set of functions.[2] Previously we considered the set of N functions $\{|\psi_{k_i^2,k_i}\rangle, |\psi_B\rangle\}$. For the purposes of this work we wish to extend our definitions of the space spanned by P to include the functions[3] $\{|\psi_{k_i^2,k_i}\rangle, |\psi_B\rangle, |\psi_{s_j^2,k_j}\rangle\}$. The new feature[4] in this discussion is the inclusion of the off-shell functions $|\psi_{s_j^2,k_j}\rangle$. These off-shell state vectors are neither mutually orthogonal nor orthogonal to the mutually orthogonal states

$$\{|\psi_{k_i^2,k_i}\rangle, |\psi_B\rangle\}.$$

Thus we must consider how to work with a nonorthogonal set of state vectors.

We have previously discussed the operator given in Eq. (1.1) in the case that P is a projection operator defined such that

$$P = \sum |\Phi_i\rangle\langle\Phi_i|, \tag{1.2}$$

where the state vectors $|\Phi_i\rangle$ are orthonormal. The operator $(PVP)^{-1(P)}$ was defined as an inverse in the subspace, i.e.,

$$(PVP)(PVP)^{-1(P)} = (PVP)^{-1(P)}(PVP) = P. \tag{1.3}$$

We now define an operator Π such that

$$\Pi = \sum |\phi_i\rangle\langle\phi_i|. \tag{1.4}$$

The operator Π is not necessarily a projection operator, since we do not require that the $|\phi_i\rangle$ be either orthogonal or normalized. We do, however, require that the $|\phi_i\rangle$ be linearly independent, and that the N states $|\phi_i\rangle$ span the same subspace as do the N orthonormal states $|\Phi_i\rangle$.

We now wish to show that $\mathcal{V}'$, defined as

$$\mathcal{V}' = V\Pi(\Pi V\Pi)^{-1(P)}\Pi V, \tag{1.5}$$

is identical to the operator $\mathcal{V}$ defined in Eq. (1.1). Of course, the operator $(\Pi V\Pi)^{-1(P)}$ which appears in Eq. (1.5) is defined to be an inverse in the subspace, i.e.,

$$(\Pi V\Pi)^{-1(P)}(\Pi V\Pi) = (\Pi V\Pi)(\Pi V\Pi)^{-1(P)} = P. \tag{1.6}$$

The definition of Π implies that

$$\Pi = \Pi P = P\Pi = P\Pi P. \tag{1.7}$$

Thus Eq. (1.5) may be reexpressed as

$$\begin{aligned}\mathcal{V}' &= (VP)(P\Pi P)\{(P\Pi P)(PVP)(P\Pi P)\}^{-1(P)}(P\Pi P)(PV)\\ &= VP(PVP)^{-1(P)}PV = \mathcal{V}.\end{aligned} \tag{1.8}$$

In order to construct $\mathcal{V}$ from the relationship in Eq. (1.5), it is necessary to work in a basis which is not orthonormal. We define the matrix M by

$$(\Pi V\Pi)^{-1(P)} = \sum_{i,j} |\phi_i\rangle M_{ij}\langle\phi_j|, \tag{1.9}$$

and similarly define the matrix $\tilde{M}$ by

$$\begin{aligned}\Pi(\Pi V\Pi)^{-1(P)}\Pi &= \sum_{i,j,k,l} |\phi_i\rangle\tilde{M}_{ij}\langle\phi_j|\\ &= \sum_{i,j,k,l} |\phi_k\rangle\langle\phi_k|\phi_i\rangle M_{ij}\langle\phi_j|\phi_l\rangle\langle\phi_l|.\end{aligned} \tag{1.10}$$

The definition of $(\Pi V\Pi)^{-1(P)}$, Eq. (1.6), may then be written as

$$\sum_{i,j,k,l} |\phi_i\rangle M_{ij}\langle\phi_j|\phi_k\rangle\langle\phi_k|V|\phi_l\rangle\langle\phi_l| = P. \tag{1.11}$$

Multiplication from the left by Π in Eq. (1.11) gives

$$\sum_{i,j,k,l,p} |\phi_p\rangle\langle\phi_p|\phi_i\rangle M_{ij}\langle\phi_j|\phi_k\rangle\langle\phi_k|V|\phi_l\rangle\langle\phi_l| - \sum_p |\phi_p\rangle\langle\phi_p|$$

$$= \sum_{k,l,p} |\phi_p\rangle \tilde{M}_{pk}\langle\phi_k|V|\phi_l\rangle\langle\phi_l| - \sum_p |\phi_p\rangle\langle\phi_p| = 0. \tag{1.12}$$

Since the states $|\phi_i\rangle$ are linearly independent, it necessarily follows that

$$\sum_k \tilde{M}_{pk}\langle\phi_k|V|\phi_l\rangle = \delta_{pl}. \tag{1.13}$$

The potential $\mathcal{V}$ is then given by

$$\mathcal{V} = \sum_{ij} V|\phi_j\rangle\tilde{M}_{ij}\langle\phi_j|V, \tag{1.14}$$

with $\tilde{M}_{ij}$ given by Eq. (1.13). Thus Eqs. (1.13) and (1.14) provide the explicit formulas necessary to construct $\mathcal{V}$ for the case that the states $|\phi_i\rangle$ are neither normalized nor orthogonal.

It then immediately follows from Eq. (1.1) that

$$\mathcal{V}|\xi_i\rangle = V|\xi_i\rangle \tag{1.15}$$

and likewise

$$\langle\xi_i|\mathcal{V} = \langle\xi_i|V, \tag{1.16}$$

if $|\xi_i\rangle$ is a state vector in the subspace defined by P, i.e., if

$$|\xi_i\rangle = P|\xi_i\rangle. \tag{1.17}$$

Thus we have shown that any vector $|\xi_i\rangle$ in the subspace has the desired property given in Eqs. (1.9) and (1.10). We also note that even if the state vectors $|\phi_i\rangle$ are not normalizable this result holds provided only that the operator V is such that $\langle\phi_i|V|\phi_j\rangle$ exists.

II. SEPARABLE REPRESENTATIONS OF THE POTENTIAL OPERATOR

We now return to the problem posed at the beginning of Sec. I: We consider the set of N state vectors[3] $\{|\psi_{k_i^2,k_i}\rangle, |\psi_B\rangle, |\psi_{s_j^2,k_j}\rangle\}$. It is useful to define the various solutions of the off-shell equation for $\mathcal{V}$,

$$|\hat{\psi}^{(0)}_{s^2,k}\rangle = |k\rangle + G_0^{(0)}(s^2)\mathcal{V}|\hat{\psi}^{(0)}_{s^2,k}\rangle, \tag{2.1}$$

where the superscript zero identifies the principal value prescription. The corresponding K matrix $\hat{K}(s^2)$,

$$\langle p|\hat{K}(s^2)|q\rangle = \langle p|\mathcal{V}|\hat{\psi}^{(0)}_{s^2,q}\rangle, \tag{2.2}$$

satisfies

$$\hat{K}(s^2) = \mathcal{V} + \mathcal{V}G_0^{(0)}(s^2)\hat{K}(s^2). \tag{2.3}$$

It is now easily shown that

$$|\hat{\psi}^{(0)}_{s_j^2,k_j}\rangle = |\psi^{(0)}_{s_j^2,k_j}\rangle, \tag{2.4}$$

$$|\hat{\psi}^{(0)}_{k_i^2,k_i}\rangle = |\psi^{(0)}_{k_i^2,k_i}\rangle, \tag{2.5}$$

and

$$|\hat{\psi}_B\rangle = |\psi_B\rangle, \tag{2.6}$$

where the functions on the right of Eqs. (2.4)–(2.6) are those solutions which are used to define the P space. Therefore, it follows immediately from Eq. (1.1) that[3]

$$\langle p|\hat{K}(s_j^2)|k_j\rangle = \langle p|K(s_j^2)|k_j\rangle \tag{2.7}$$

and

$$\langle p|\hat{K}(k_i^2)|k_i\rangle = \langle p|K(k_i^2)|k_i\rangle. \tag{2.8}$$

It is clear, therefore, from the two-potential formula for K that

$$\langle p|K(s^2)|q\rangle = \langle p|\mathcal{V}|\hat{\psi}^{(0)}_{s^2,q}\rangle + \langle\hat{\psi}^{(0)}_{s^2,p}|(V-\mathcal{V})|\psi^{(0)}_{s^2,q}\rangle$$

$$= \langle p|\hat{K}(s^2)|q\rangle + \langle p|R(s^2)|q\rangle. \tag{2.9}$$

The symmetric function $\langle p|R(s^2)|q\rangle$ satisfies the relation

$$\langle p|R(s_j^2)|k_j\rangle = \langle p|R(k_i^2)|k_i\rangle = 0. \tag{2.10}$$

For simplicity let us consider a rank-one potential $\mathcal{V}$ based on a single off-shell solution[3] $|\psi^{(0)}_{s_j^2,k_j}\rangle$:

$$\mathcal{V} = \frac{V|\psi^{(0)}_{s_j^2,k_j}\rangle\langle\psi^{(0)}_{s_j^2,k_j}|V}{\langle\psi^{(0)}_{s_j^2,k_j}|V|\psi^{(0)}_{s_j^2,k_j}\rangle}. \tag{2.11}$$

The K matrix for this choice is found to be

$$\langle p|\hat{K}(s^2)|q\rangle = \frac{\langle p|V|\psi^{(0)}_{s_j^2,k_j}\rangle\langle\psi^{(0)}_{s_j^2,k_j}|V|q\rangle}{\langle\psi^{(0)}_{s_j^2,k_j}|V - VG_0^{(0)}(s^2)V|\psi^{(0)}_{s_j^2,k_j}\rangle} \tag{2.12}$$

For the case $s^2 < 0$, this expression also yields the T matrix

$$\langle p|\hat{t}(s^2)|q\rangle \equiv \langle p|\mathcal{V}|\psi^{(+)}_{s^2,q}\rangle, \tag{2.13}$$

since in this case all the wave functions $|\psi^{(\pm)}_{s^2,q}\rangle$ and $|\psi^{(0)}_{s^2,q}\rangle$ are identical. For $s^2 > 0$ we may find $\hat{t}(s^2)$ from the standard relation between the T matrix and the K matrix.

One may also construct a separable potential $\mathfrak{V}$ from the off-shell wave functions which satisfy outgoing boundary conditions $|\psi^{(+)}_{s_j^2,k_j}\rangle$. In this case, one would have[3]

$$\langle p|\hat{t}(s_j^2)|k_j\rangle = \langle p|t(s_j^2)|k_j\rangle, \tag{2.14}$$

i.e., the matrix $\hat{t}$ would be equal to t along a single line in the (p, s, q) space. One finds, however, that $\langle k_j|\hat{t}(s_j^2)|q\rangle$ and $\langle k_j|t(s_j^2)|q\rangle$ are not necessarily equal. In the case of the K matrix this equality follows immediately from the Hermiticity of the K matrix. For the T matrix, the additional equality can be achieved if one also includes in the P space the functions $|\psi^{(-)}_{s_j^2,k_j}\rangle$. In our previous work[1] where only on-shell wave functions $|\psi^{(+)}_{k_i^2,k_i}\rangle$ were employed this point did not arise. This is because in that case we may generate the P space with the set of functions $\{|\psi^{(+)}_{k_i^2,k_i}\rangle\}$ or $\{|\psi^{(-)}_{k_i^2,k_i}\rangle\}$ or $\{|\psi^{(0)}_{k_i^2,k_i}\rangle\}$, since these functions are not linearly independent.

It is, perhaps, important to note that the relationship between t and K is such that the identity $\langle q|\hat{K}(s_j^2)|k_j\rangle = \langle q|K(s_j^2)|k_j\rangle$ *does not imply* $\langle q|\hat{t}(s_j^2)|k_j\rangle = \langle q|t(s_j^2)|k_j\rangle$. If however, we include both $|\psi^{(0)}_{s_j^2,k_j}\rangle$ *and* $|\psi^{(0)}_{k_j^2,k_j}\rangle$ among the states which define the P space, we have

$$\begin{aligned}\langle p|t(s_j^2)|k_j\rangle &= \langle p|K(s_j^2)|k_j\rangle - i\tfrac{1}{2}\pi\,\frac{\langle p|K(s_j^2)|s_j\rangle s_j\langle s_j|K(s_j^2)|k_j\rangle}{1-i\frac{1}{2}\pi s_j\langle s_j|K(s_j^2)|s_j\rangle}\\ &= \langle p|\hat{K}(s_j^2)|k_j\rangle - i\tfrac{1}{2}\pi\,\frac{\langle p|\hat{K}(s_j^2)|s_j\rangle s_j\langle s_j|\hat{K}(s_j^2)|k_j\rangle}{1-i\frac{1}{2}\pi s_j\langle s_j|\hat{K}(s_j^2)|s_j\rangle}\\ &= \langle p|\hat{t}(s_j^2)|k_j\rangle,\end{aligned} \tag{2.15}$$

and similarly

$$\langle k_j|t(s_j^2)|q\rangle = \langle k_j|\hat{t}(s_j^2)|q\rangle. \tag{2.16}$$

Equivalently, we may define the P space through the functions $|\psi^{(+)}_{s_j^2,k_j}\rangle$ and $|\psi^{(-)}_{s_j^2,k_j}\rangle$ to obtain the relations given in Eqs. (2.15) and (2.16).

III. CONCLUSIONS

We have shown how to construct a *separable* potential $\mathfrak{V}$ from the potential V, such that at N points in the (p, q, s) space the K matrix (or the T matrix) for $\mathfrak{V}$ and the K matrix (T matrix) for V are identical. These N points are given by $s=s_j$, $p=q=k_j$. These points may be either off-shell points $(s_j\neq k_j)$ or on-shell points $(s_j=k_j)$. Furthermore, we have seen that this construction necessarily has the further property that

$$\langle p|\hat{K}(s_j)|k_j\rangle = \langle p|K(s_j)|k_j\rangle \tag{3.1}$$

and

$$\langle k_j|\hat{K}(s_j)|q\rangle = \langle k_j|K(s_j)|q\rangle. \tag{3.2}$$

Thus we see that the K matrix (T matrix) for $\mathfrak{V}$ and the K matrix (T matrix) for V are identical on two intersecting lines, intersecting at the point $s=s_j$, $p=q=k_j$. This implies that at the point $s=s_j$, $p=q=k_j$ any derivative in the plane given by the intersection of the two lines referred to above must be the same for K and $\hat{K}$. Now it is also clear that the derivatives

$$\frac{\partial}{\partial s}\langle k_j|\hat{K}(s)|k_j\rangle\Big|_{s=s_j} = \frac{\partial}{\partial s}\langle k_j|K(s)|k_j\rangle\Big|_{s=s_j} \tag{3.3}$$

must be equal as well if Eqs. (3.1) and (3.2) obtain. This is very easy to prove.

However, we shall not pursue this argument because this is but a special case of the generalized effective range expansion which will be discussed elsewhere. The derivative in any direction at the point $s=s_j$, $p=q=k_j$ is given uniquely if the state vector $|\psi^{(0)}_{s_j^2,k_j}\rangle$ is known. The construction under discussion is one which has the property that $|\psi^{(0)}_{s_j^2,k_j}\rangle = |\hat{\psi}^{(0)}_{s_j^2,k_j}\rangle$. In fact Eqs. (3.1) and (3.2) are another expression of the property. Thus it follows immediately that

$$\langle p|\hat{K}(s^2)|q\rangle \cong \langle p|K(s^2)|q\rangle \tag{3.4}$$

in a volume containing the point $s=s_j$, $p=q=k_j$. This property is now established whether or not $s_j=k_j$, i.e., this property can hold for off-shell as well as on-shell points. Hence it follows that we have the latitude to choose to fit the approximate separable potential off shell as readily as on shell. Similar remarks hold for the T matrix subject to the relations discussed in Sec. II. These are the constructions we sought to establish.

Recently Pieper[5] has constructed a separable nucleon-nucleon potential based on the method given in Ref. 1. This potential provides a more accurate representation of the data than any other separable potential now available in the literature. This success motivates the present extension. The T matrices in the bound-state three-body problem, for example, are far off shell and evaluated at negative energies. Therefore it may prove advantageous to compromise the on-shell fit in favor of greater accuracy in the off-shell region of greatest importance in the many-body problem.

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[1]D. J. Ernst, C. M. Shakin, and R. M. Thaler, Phys. Rev. C 8, 46 (1973).

[2]The finite number of orthogonal continuum states $|\psi_i\rangle$ do not define a well behaved projection operator $P=\sum_i|\psi_i\rangle\langle\psi_i|$. This is a consequence of the fact that continuum state vectors are not normalizable. However, the expression in Eq. (1.1) is independent of the norm of the state vectors $|\psi_i\rangle$. This property leads to a proper definition of the operator $\mathcal{V}$ in Eq. (1.1).

[3]The functions $|\psi_{s_j{}^2,k_j}\rangle$ are defined in Eqs. (4.9)–(4.11) of Ref. 1. They satisfy $|\psi^{(0)}_{s^2,k}\rangle=|k\rangle+G^{(0)}(s^2)V|\psi^{(0)}_{s^2,k}\rangle$ where $G^{(0)}(s^2)$ is a principal value Green's function. The corresponding K matrix is $\langle p|K(s^2)|q\rangle=\langle p|V|\psi^{(0)}_{s^2,q}\rangle$ and satisfies $K(s^2)=V+VG^{(0)}(s^2)K(s^2)$. Similarly the T-matrix equation is $t(s^2)=V+VG^{(+)}(s^2)t(s^2)$.

[4]Separable approximations to the T matrix for negative energies have been discussed by K. L. Kowalski, Nuovo Cimento 45, 769 (1966).

[5]S. Pieper, Phys. Rev. C 9, 883 (1974).