PHYSICAL REVIEW C VOLUME 9, NUMBER 4 APRIL 1974

Electromagnetic corrections to allowed nuclear beta decay*

Barry R. Holstein
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002
(Received 26 October 1973)

Dominant Coulomb corrections to the spectral functions for allowed nuclear β transitions are calculated using the elementary particle approach.

[RADIOACTIVITY Calculated electromagnetic corrections. Allowed β decay.]

I. INTRODUCTION

In a recent series of papers[1] we have given model-independent forms for the spectra in various experiments utilizing allowed nuclear β decay. The forms given were correct to first order in the recoil parameter of smallness E/M, where E is the electron energy and M is the nuclear mass. Electromagnetic effects were neglected except for the dominant Coulomb interaction included in the standard Fermi approximation. We suggested careful examination of recoil terms in order to gain information concerning the possible existence of second class currents,[2] the validity of the conserved vector current (CVC) hypothesis,[3] the presence of possible T violation,[4] and other effects.

However, in order to properly assess the result of an experiment in recoil order, it is essential to know the electromagnetic corrections, since they can simulate a *bona fide* recoil term. In this note we calculate the electromagnetic corrections to spectra based on the elementary particle treatment.

Section II defines notation and sketches the derivation of final-state Coulomb interactions due to Armstrong and Kim.[5] In Sec. III the effect of these Coulomb corrections on the decay spectra is evaluated, and in Sec. IV, in order to understand some of the approximations involved in these expressions, we discuss an alternate procedure correct to order $Z\alpha$ based on single photon exchange.

II. COULOMB WAVE FUNCTION

Here as in Ref. 1 we shall assume the validity of the CVC hypothesis and of the usual $V+A$ form of the weak interaction. We deal temporarily with electron decay—modifications appropriate to positron decay will be discussed at a later stage. Let p_1, p_2, p, l denote the four-momenta of parent nucleus, daughter nucleus, electron, neutrino, and define

$$P=p_1+p_2\,, \quad q=p_1-p_2\,,$$
$$M=\tfrac{1}{2}(M_1+M_2)\,, \quad \Delta=M_1-M_2\,,$$

where M_1 and M_2 are the masses of parent and daughter.

An arbitrary allowed nuclear β decay can, to first order in recoil, be described in terms of four nuclear form factors. If

$$L^\mu=\bar{u}(p)\gamma^\mu(1+\gamma_5)v(l)$$

represents the lepton current then the hadronic current matrix element can be written as

$$L^\mu\langle\beta|V_\mu(0)+A_\mu(0)|\alpha\rangle=\delta_{JJ'}\delta_{MM'}\frac{a(q^2)}{2M}P_\mu L^\mu-i\epsilon_{ijk}C^{M'k;M}_{J'1;J}\frac{1}{4M}\{2b(q^2)L_i q_j+i\epsilon_{ij\lambda\eta}L^\lambda[c(q^2)P^\eta-d(q^2)q^\eta]\}, \tag{1}$$

where J, J' represent the spin of parent and daughter nucleus and M, M' its projection on some axis of quantization. Here Latin indices are summed from 1 to 3 while repeated Greek indices imply a four-vector contraction. The coefficients a, b, c, d are the conventional Fermi, weak magnetism, Gamow-Teller, and induced-tensor form factors and are discussed more completely in Ref. 1.

Armstrong and Kim[5] studied the problem of electromagnetic corrections from an elementary particle viewpoint and showed that the appropriate expression for the β-decay matrix element in the presence of the Coulomb interaction is (for electron decay)

$$\int d^3r\, e^{-i\vec{l}\cdot\vec{r}}\, \bar{\psi}(p,\vec{r})\gamma_\lambda(1+\gamma_5)v(l) \times\left[\delta_{JJ'}\delta_{MM'}\frac{P^\lambda}{2M}a(0)\rho_V(r) - i\,\epsilon_{ijk}C^{M';k;M}_{J';1;J}\,\epsilon_{ij\lambda\eta}\frac{P^\eta}{4M}c(0)\rho_A(\vec{r})\right], \qquad (2)$$

where

$$\rho_V(r)=\frac{1}{a(0)}\int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}a(q^2)$$

and

$$\rho_A(r)=\frac{1}{c(0)}\int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}c(q^2)$$

are the vector and axial "weak charge" densities, and $\psi(p,\vec{r})$ is the solution to the Dirac equation in the presence of the nuclear Coulomb potential $V(r)$ which reduces as $Z\to 0$ to

$$\psi(p,\vec{r})\underset{Z\to 0}{\longrightarrow}u(p)e^{i\vec{p}\cdot\vec{r}}.$$

In standard notation[6]

$$\psi(p,\vec{r})=\left(\frac{(2\pi)^3}{m|\vec{p}|}\right)^{1/2} \times\sum_{\kappa,\mu} i^l C^{\mu-\rho}_{l}{}^{\rho}_{\frac{1}{2}}{}^{;\,\mu}_{;\,j}\, Y_l^{\mu-\rho *}(\hat{p})e^{-i\sigma_\kappa}\psi_{\kappa\mu}(r), \qquad (3)$$

where

$$\sigma_\kappa=\tfrac{1}{2}\pi[l(\kappa)+1-\gamma]+\eta_\kappa-\arg\Gamma(\gamma+i\nu),$$

$$\psi_{\kappa\mu}(r)=\begin{pmatrix} g_\kappa(r)X_{\kappa,\mu}(r)\\ if_\kappa(r)X_{-\kappa,\mu}(r)\end{pmatrix},\quad \gamma=(\kappa^2-\alpha^2Z^2)^{1/2},$$

$$\nu=\frac{\alpha ZE}{p},\quad \exp(2i\eta_\kappa)=\frac{-\kappa+i\alpha Z(m/p)}{\gamma+i\nu}, \qquad (4)$$

$$\kappa>0,\quad 0<\eta_\kappa\le\tfrac{1}{2}\pi;\quad \kappa<0,\quad 0\ge\eta_\kappa>-\tfrac{1}{2}\pi.$$

For example, in the case of a point nucleus $V(r) = -\alpha Z/r$ and

$$\begin{pmatrix} g_\kappa(r)\\ f_\kappa(r)\end{pmatrix}=\left(\frac{p(E+m)}{\pi}\right)^{1/2}\begin{pmatrix}\mathrm{Re}\\ \mathrm{Im}\end{pmatrix}Q^*_\kappa(p,r) \qquad (5)$$

with

$$Q_\kappa(p,r)=2e^{1/2(\pi\nu)}\frac{|\Gamma(\gamma+i\nu)|}{\Gamma(2\gamma+1)}(\gamma+i\nu)(2pr)^{\gamma-1} \times e^{-ipr+i\eta_\kappa}F(\gamma+1+i\nu;2\gamma+1;2ipr).$$

In Eq. (2) we have included only the Fermi and Gamow-Teller matrix elements, as the additional (weak magnetism and the induced tensor) are already small corrections $O(q/m)$. Since the leading terms result only in final electron states with total angular momentum $\frac{1}{2}$, we project out $|\kappa|=1$ terms and neglect all others, yielding:

$$P_{j=1/2}\psi(p,r)=N^*(\tilde{a}+\tilde{b}\gamma^0+\tilde{c}\vec{\gamma}\cdot\hat{r}+\tilde{d}\vec{\gamma}\cdot\hat{r}\gamma^0)u(p), \qquad (6)$$

where

$$\begin{aligned} \tilde{a}&=\frac{1}{2}\left(\frac{2m}{E+m}\right)^{1/2}\left[g_{-1}(r)+\frac{E+m}{p}e^{-i\delta}f_{-1}(r)\right],\\ \tilde{b}&=\frac{1}{2}\left(\frac{2m}{E+m}\right)^{1/2}\left[g_{-1}(r)-\frac{E+m}{p}e^{-i\delta}f_{1}(r)\right],\\ \tilde{c}&=\frac{i}{2}\left(\frac{2m}{E+m}\right)^{1/2}\left[f_{-1}(r)+\frac{E+m}{p}e^{-i\delta}g_{1}(r)\right],\\ \tilde{d}&=\frac{i}{2}\left(\frac{2m}{E+m}\right)^{1/2}\left[f_{-1}(r)-\frac{E+m}{p}e^{-i\delta}g_{1}(r)\right], \end{aligned} \qquad (7)$$

and $\delta=\frac{1}{2}\pi+\eta_1-\eta_{-1}$. Assuming for simplicity that $\rho_V(r)=\rho_A(r)\equiv\rho(r)$, the decay amplitude becomes[5]:

$$\bar{u}(p)(\not{t}+\gamma_0\not{s})\gamma_\lambda(1+\gamma_5)v(l)M^\lambda(p_1,p_2), \qquad (8)$$

where

$$M^\lambda(p_1,p_2)=\left[\delta_{JJ'}\delta_{MM'}\frac{P^\lambda}{2M}a(0) +C^{M';k;M}_{J';1;J}\,\epsilon_{ijk}\,\epsilon_{ij\lambda\eta}\frac{P^\eta}{4M}c(0)\right],$$

$$t=(B,-C\vec{l}),\qquad s=(A,-D\vec{l}) \qquad (9)$$

with

$$N=\frac{1}{4\pi}\left(\frac{(2\pi)^3}{mp}\right)^{1/2}\exp i\left[\tfrac{1}{2}\pi(1-\gamma)+\eta_{-1}-\arg\Gamma(\gamma+i\nu)\right],$$

$$\begin{aligned} A&=\int d^3r\,\tilde{a}^*(r)e^{-i\vec{l}\cdot\vec{r}}N\rho(r),\\ B&=\int d^3r\,\tilde{b}^*(r)e^{-i\vec{l}\cdot\vec{r}}N\rho(r),\\ C&=\int d^3r\,\tilde{c}^*(r)\,\hat{l}\cdot\hat{r}\,e^{-i\vec{l}\cdot\vec{r}}N\rho(r),\\ D&=\int d^3r\,\tilde{d}^*(r)\,\hat{l}\cdot\hat{r}\,e^{-i\vec{l}\cdot\vec{r}}N\rho(r). \end{aligned} \qquad (10)$$

Given a model for $\rho(r)$ we can calculate the decay spectra.

III. DECAY SPECTRA

Suppose the parent nucleus has net polarization $\mathcal{P}=\langle M\rangle/J$ and orientation parameter $\Lambda_J=1-3\langle M^2\rangle/J(J+1)$. If

$$E_0=\Delta\,\frac{1+m_e^2/2M\Delta}{1+\Delta/2M}$$

represents the maximum electron energy, the spectrum in electron and neutrino variables is defined as

$$\begin{aligned}
d\omega=\frac{G_V^2\cos\theta_c}{(2\pi)^5}(E_0-E)^2pE\,dE\,d\Omega_{\hat{p}}\,d\Omega_{\hat{l}}\Bigg(&h_1(E)+\frac{\vec{p}\cdot\hat{l}}{E}h_2(E)+\left[\left(\frac{\vec{p}\cdot\hat{l}}{E}\right)^2-\frac{1}{3}\frac{p^2}{E^2}\right]h_3(E)\\
&+\mathcal{P}\left[\frac{\hat{n}\cdot\vec{p}}{E}h_4(E)+\frac{\hat{n}\cdot\vec{p}\,\vec{p}\cdot\hat{l}}{E^2}h_5(E)+\hat{n}\cdot\hat{l}\,h_6(E)+\hat{n}\cdot\hat{l}\,\frac{\vec{p}\cdot\hat{l}}{E}h_7(E)\right]\\
&-\Lambda_J\Bigg\{\left(\frac{\hat{n}\cdot\vec{p}}{E}\hat{n}\cdot\hat{l}-\frac{1}{3}\frac{\vec{p}\cdot\hat{l}}{E}\right)h_{10}(E)+\left(\frac{\hat{n}\cdot\vec{p}\hat{n}\cdot\hat{l}}{E}-\frac{1}{3}\frac{\vec{p}\cdot\hat{l}}{E}\right)\frac{\vec{p}\cdot\hat{l}}{E}h_{11}(E)\\
&+\left(\hat{n}\cdot\hat{l}\,\hat{n}\cdot\hat{l}-\frac{1}{3}\right)h_{12}(E)+\left[\left(\frac{\hat{n}\cdot\vec{p}}{E}\right)^2-\frac{1}{3}\frac{p^2}{E^2}\right]h_{13}(E)+\left(\hat{n}\cdot\hat{l}\,\hat{n}\cdot\hat{l}-\frac{1}{3}\right)\frac{\vec{p}\cdot\hat{l}}{E}h_{16}(E)\Bigg\}\Bigg),
\end{aligned}\tag{11}$$

where $h_i(E)$ are given in Ref. 2 in terms of a, b, c, d.[7] The dominant electromagnetic corrections (those proportional to $|a|^2$, $|c|^2$, $\mathrm{Re}a^*c$) alter the form of the spectral functions to

$$h_i(E)=[F(Z,E)][h_i^{(0)}(E)+\Delta h_i(E)]\ ,$$

where

$$F(Z,E)=2(1+\gamma)e^{\pi\nu}(2pR)^{2(\gamma-1)}\,|\Gamma(\gamma+i\nu)|^2/[\Gamma(2\gamma+1)]^2$$

is the conventional Fermi function, $h_i^{(0)}(E)$ is given in Ref. 2, and

$$\begin{aligned}
\Delta h_1(E)[F(Z,E)]=|a|^2\Big\{&|A|^2+|B|^2+|C|^2+|D|^2-2\,\mathrm{Re}(A^*D+B^*C)\\
&+2\frac{m_e}{E}\mathrm{Re}(A^*B+C^*D-A^*C-B^*D)-[F(Z,E)]\Big\}\\
+|c|^2\Big\{&|A|^2+|B|^2+|C|^2+|D|^2+\tfrac{2}{3}\,\mathrm{Re}(A^*D+B^*C)\\
&+2\frac{m_e}{E}\mathrm{Re}(A^*B+C^*D+\tfrac{1}{3}A^*C+\tfrac{1}{3}B^*D)-[F(Z,E)]\Big\},\\
\Delta h_2(E)[F(Z,E)]=|a|^2\{&|A|^2-|B|^2-|C|^2+|D|^2+2\,\mathrm{Re}(B^*C-A^*D)-[F(Z,E)]\}\\
-\tfrac{1}{3}|c|^2\{&|A|^2-|B|^2-|C|^2+|D|^2+6\,\mathrm{Re}(A^*D-B^*C)-[F(Z,E)]\},\\
\Delta h_4(E)[F(Z,E)]=\left(\frac{J}{J+1}\right)^{1/2}&2\,\mathrm{Re}a^*c\{|A|^2-|B|^2+|C|^2-|D|^2-[F(Z,E)]\}\\
\mp\frac{\gamma_{JJ'}}{J+1}|c|^2\{&|A|^2-|B|^2+|C|^2-|D|^2-[F(Z,E)]\},\\
\Delta h_6(E)[F(Z,E)]=\left(\frac{J}{J+1}\right)^{1/2}&2\,\mathrm{Re}a^*c\Big\{|A|^2+|B|^2+|C|^2+|D|^2-2\,\mathrm{Re}(A^*D+B^*C)\\
&+2\frac{m_e}{E}\mathrm{Re}(A^*B+C^*D-A^*C-B^*D)-[F(Z,E)]\Big\}\\
\pm\frac{\gamma_{JJ'}}{J+1}|c|^2\Big\{&|A|^2+|B|^2+|C|^2+|D|^2+2\,\mathrm{Re}(A^*D+B^*C)\\
&+2\frac{m_e}{E}\mathrm{Re}(A^*B+C^*D+A^*C+B^*D)-[F(Z,E)]\Big\},\\
\Delta h_7(E)[F(Z,E)]=\left(\frac{J}{J+1}\right)^{1/2}&2\,\mathrm{Re}a^*c[-2|C|^2+2|D|^2+2\,\mathrm{Re}(B^*C-A^*D)]\\
\pm\frac{\gamma_{JJ'}}{J+1}&|c|^2[2|C|^2-2|D|^2+2\,\mathrm{Re}(B^*C-A^*D)],
\end{aligned}\tag{12}$$

$$\Delta h_{10}(E)[F(Z,E)] = \theta_{JJ'}\,|c|^2\{|A|^2 - |B|^2 + |C|^2 - |D|^2 - [F(Z,E)]\},$$

$$\Delta h_{12}(E)[F(Z,E)] = -\theta_{JJ'}|c|^2\left[2\,\mathrm{Re}(A^*D + B^*C) + 2\,\frac{m_e}{E}\,\mathrm{Re}(A^*C + B^*D)\right],$$

$$\Delta h_{16}(E)[F(Z,E)] = -\theta_{JJ'}\,|c|^2[2|C|^2 - 2|D|^2],$$

where $\gamma_{JJ'}$, $\theta_{JJ'}$ are defined in Ref. 7.

For positron decay the lepton matrix element is replaced by

$$\bar{u}(l)\gamma_\lambda(1+\gamma_5)(\not{t}' + \not{s}'\gamma_0)v(p),$$

where

$$t' = (-B', C'\vec{1}), \quad s' = (A', -D'\vec{1}), \tag{13}$$

and A', B', C', D' are defined as in Eq. (10) except that $\tilde{a}^*(Z,r) \to \tilde{a}(-Z,r)$, $N(Z) \to N^*(-Z)$, etc., and η_κ is subject to the restriction

$$\kappa > 0, \quad \tfrac{1}{2}\pi \le \eta_\kappa < \pi,$$

$$\kappa < 0, \quad 0 \le \eta_\kappa < \tfrac{1}{2}\pi.$$

The changes in the spectral functions are then found by replacing $A \to A'^*$, $B \to B'^*$, etc. in Eq. (12) and using the lower sign.

IV. PHOTON EXCHANGE

If one needs only the lowest-order Coulomb corrections, as would be appropriate for light nuclei, we may employ an alternate elementary particle approach taken by Bottino and Ciocchetti.[8] These authors worked with the electron wave function as modified to first order in $Z\alpha$ by the Coulomb potential. However, it is useful to start with the entire correction to the decay matrix element due to the one photon exchange term in order to understand the approximations involved in going to the potential form. In the following discussion we shall assume that the initial and final nuclei have spin $\frac{1}{2}$ but it should be clear that a similar derivation obtains for arbitrary initial or final spins. Also, in order to simplify the discussion we shall assume that

$$\frac{g_V(q^2)}{g_V(0)} = \frac{g_A(q^2)}{g_A(0)} \equiv G(q^2).$$

We have then

$$M = M^{(0)} + \delta M,$$

where

$$M^{(0)} = (2\pi)^4\delta^4(p_1 - p_2 - p - l)\bar{u}(p)\gamma_\lambda(1+\gamma_5)v(l)G(q^2)\bar{u}(p_2)\gamma^\lambda[g_V(0) + g_A(0)\gamma_5]u(p_1) \tag{14}$$

and

$$\delta M = \alpha\int d^4x\int d^4z\int \frac{d^4k}{(2\pi)^4}\, e^{i(l+k+p_2-p_1)\cdot x}\,\frac{g^{\mu\nu}}{k^2+i\epsilon}\, e^{-i(k-p)\cdot z}\, G[(k-q)^2]\,\bar{u}(p)\gamma_\nu S_F(z-x)\gamma_\lambda(1+\gamma_5)v(l)$$
$$\times\left\{\bar{u}(p_2)\gamma^\lambda[g_V(0)+g_A(0)\gamma_5]\,\frac{1}{\not{p}_1 - \not{k} - M_1 + i\epsilon}\,\gamma_\mu Z_B F_B(k^2)u(p_1) + \bar{u}(p_2)\gamma_\mu Z_A F_A(k^2)\right.$$
$$\left.\times\frac{1}{\not{p}_2 + \not{k} - M_2 + i\epsilon}\,\gamma^\lambda[g_V(0) + g_A(0)\gamma_5]u(p_1)\right\}. \tag{15}$$

Here $F_A(k^2)$ and $F_B(k^2)$ are the charge form factors for the initial and final nuclei.[9] If we take $F_A(k^2) = F_B(k^2) \equiv F(k^2)$ and neglect $O(\alpha)$ terms with respect to terms in $Z\alpha$ we may write

$$\delta M = Z_B\alpha\int d^4x\int d^4z\int\frac{d^4k}{(2\pi)^4}\, e^{i(l+k+p_2-p_1)\cdot x}\,\frac{g^{\mu\nu}}{k^2+i\epsilon}\, e^{-i(k-p)\cdot z}\, G[(k-q)^2]\,F(k^2)\bar{u}(p)\gamma_\nu S_F(z-x)\gamma_\lambda(1+\gamma_5)$$
$$\times v(l)\left\{\bar{u}(p_2)\gamma^\lambda[g_V(0)+g_A(0)\gamma_5]\,\frac{2p_{1\mu} - \not{k}\gamma_\mu}{k^2 - 2p_1\cdot k + i\epsilon}u(p_1) + \bar{u}(p_2)\,\frac{2p_{2\mu} + \gamma_\mu\not{k}}{k^2 + 2p_2\cdot k + i\epsilon}\right.$$
$$\left.\times\gamma^\lambda[g_V(0) + g_A(0)\gamma_5]u(p_1)\right\}. \tag{16}$$

We now make the replacement $p_2 \to p_1$, dropping corrections of $O(q/M)$ and due to the presence of the nuclear form factors, we have $k/p \sim 1/MR \ll 1$. Thus we drop $\not k \gamma_\mu$ with respect to p_μ and write[10]

$$\left(\frac{1}{k^2 - 2k \cdot p + i\epsilon} + \frac{1}{k^2 + 2k \cdot p + i\epsilon}\right) \approx -2\pi i\, \delta(2k \cdot p). \tag{17}$$

Defining the nuclear potential by

$$V(r) = -8\pi Z_B \alpha \int \frac{d^4k}{(2\pi)^4} \frac{\delta(k_0)}{k^2 + i\epsilon} e^{ik\cdot r} F(k^2) \tag{18}$$

and making a change of variables we find the Armstrong-Kim result

$$\delta M = (2\pi)^4 \delta^4(p_1 - p_2 - p - l) \int d^3r\, e^{-i\vec{l}\cdot\vec{r}} \rho(r) \overline{\psi}(p, r) \gamma_\lambda (1 + \gamma_5) v(l) \overline{u}(p_2) \gamma^\lambda [g_V(0) + g_A(0)\gamma_5] u(p_1), \tag{19}$$

where

$$\overline{\psi}(p, r) = \overline{u}(p) e^{-i\vec{p}\cdot\vec{r}} - i \int d^4z\, \overline{u}(p) e^{ip\cdot z} \gamma_0 V(r) S_F(z - r)$$

is the modified electron wave function correct to first order in the nuclear potential.

In order to evaluate the effect on the decay spectra we apply a Fourier transform to the propagator yielding

$$\delta M = (2\pi)^4 \delta^4(p_1 - p_2 - p - l) \frac{\alpha Z}{2\pi^2} \int d^4k \frac{F(k^2) G[(k-q)^2]\, \delta(k_0)}{(k^2 + i\epsilon)(k^2 - 2p\cdot k + i\epsilon)} \overline{u}(p)(2p^0 - \vec{k}\cdot\vec{\gamma}\gamma_0)\gamma_\lambda(1+\gamma_5) v(l) \times \overline{u}(p_2)\gamma^\lambda [g_V(0) + g_A(0)\gamma_5] u(p_1). \tag{20}$$

For a point nucleus $\{F(k^2) = G[(k-q)^2] = 1\}$

$$\mathrm{Re} \int d^4k\, \delta(k^0) \frac{[1, \vec{k}]}{(k^2 + i\epsilon)(k^2 - 2p\cdot k + i\epsilon)} \approx \frac{\pi^3}{2|\vec{p}|} [1; 0] \tag{21}$$

so that

$$M^{(0)} + \delta M = M^{(0)} \left[1 + \frac{\pi\alpha Z E}{|\vec{p}|} + \cdots\right], \tag{22}$$

which is just the αZ part of the usual Fermi factor modification.

The Coulomb effects not included in the Fermi function are given by

$$\delta M_{\text{non-Fermi}} = \frac{\alpha Z}{2\pi^2} \int d^4k\, \delta(k^0) \frac{[F(k^2) G((k-q)^2) - 1]}{(k^2 + i\epsilon)(k^2 - 2k\cdot p + i\epsilon)} \overline{u}(p)(2p^0 - \vec{k}\cdot\vec{\gamma}\gamma^0)\gamma_\lambda(1+\gamma_5) v(l) M^\lambda(p_1, p_2). \tag{23}$$

If we retain only terms of first order in p, l this becomes[11]

$$\delta M_{\text{non-Fermi}} = \alpha Z \frac{4}{3\pi} \overline{u}(p)[4p^0(X+Y) + q^0 X + \gamma^0 \not p (X + 2Y) - \gamma^0 \not l X] \gamma_\lambda (1+\gamma_5) v(l) M^\lambda(p_1, p_2), \tag{24}$$

where

$$X = \int_0^\infty dk\, F(k^2) G'(k^2), \quad Y = \int_0^\infty dk\, G(k^2) F'(k^2).$$

The corrections to the spectral functions are

$$\Delta h_1(E) = \mp \frac{8\alpha Z}{3\pi} \left\{ |a|^2 \left[4E(X+Y) + E_0 X + \frac{m_e^2}{E}(X + 2Y)\right] + |c|^2 \left[E(\tfrac{16}{3}X + 4Y) - \tfrac{1}{3} E_0 X + \frac{m_e^2}{E}(X+2Y)\right] \right\},$$

$$\Delta h_2(E) = \mp \frac{8\alpha Z}{3\pi} \{ |a|^2 [4E(X+Y) + E_0 X] - |c|^2 [\tfrac{4}{3} E(2X + Y) - E_0 X] \},$$

$$\Delta h_4(E) = \mp \frac{8\alpha Z}{3\pi} \left\{ \left[\left(\frac{J}{J+1}\right)^{1/2} 2\,\mathrm{Re}\, a^* c \mp \frac{\gamma_{JJ'}}{J+1} |c|^2 \right] E(5X + 4Y) \right\}, \tag{25}$$

$$\Delta h_6(E) = \mp\frac{8\alpha Z}{3\pi}\left\{\left(\frac{J}{J+1}\right)^{1/2} 2\,\mathrm{Re}\,a^*c\left[4E(X+Y)+E_0X+\frac{m_e^{\,2}}{E}(X+2Y)\right]\right.$$
$$\left.\pm\frac{\gamma_{JJ'}}{J+1}\,|c|^2\left[E(6X+4Y)-E_0X+\frac{m_e^{\,2}}{E}(X+2Y)\right]\right\},$$

$$\Delta h_7(E) = \mp\frac{8\alpha Z}{3\pi}\left\{\left(\frac{J}{J+1}\right)^{1/2} 2\,\mathrm{Re}\,a^*c \pm \frac{\gamma_{JJ'}}{J+1}\,|c|^2\right\}(E_0-E)X,$$

$$\Delta h_{10}(E) = \mp\frac{8\alpha Z}{3\pi}\,\theta_{JJ'}\,|c|^2E(5X+4Y),$$

$$\Delta h_{12}(E) = \mp\frac{8\alpha Z}{3\pi}\,\theta_{JJ'}\,|c|^2(E_0-E)X,$$

where the upper (lower) sign is for electron (positron) decay.

For a uniform charge and "weak charge" density

$$X = Y = \frac{9\pi R}{140}$$

while for a surface charge distribution

$$X = Y = \frac{\pi R}{12},$$

where R is the nuclear radius.

We can check consistency with the Armstrong-Kim procedure by expanding $A, B, C, D,$ to order $Z\alpha$ for the case of a point charge electrostatic distribution but a uniform "weak charge" density

$$|A|^2 \approx (1-\tfrac{5}{4}\alpha ZER)F(Z,E),$$
$$2\,\mathrm{Re}A^*B \approx -\tfrac{1}{4}\alpha Zm_e RF(Z,E), \qquad (26)$$
$$2\,\mathrm{Re}A^*D \approx \tfrac{1}{4}\alpha Z(E_0-E)RF(Z,E).$$

Substitution in Eq. (12) reproduces the results given in Eq. (25) for the case

$$X = \frac{3\pi R}{32}, \quad Y = 0.$$

---

*Supported in part by the National Science Foundation.

[1]B. R. Holstein and S. B. Treiman, Phys. Rev. C 3, 1921 (1971); B. R. Holstein, W. Shanahan, and S. B. Treiman, Phys. Rev. C 5, 1849 (1972).

[2]S. Weinberg, Phys. Rev. 112, 1375 (1958); B. R. Holstein, Phys. Rev. C 4, 764 (1971).

[3]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958); Y. K. Lee, L. W. Mo, and C. S. Wu, Phys. Rev. Lett. 10, 253 (1963).

[4]C. W. Kim and H. Primakoff, Phys. Rev. 180, 1502 (1969); B. R. Holstein, Phys. Rev. C 5, 1529 (1972).

[5]L. Armstrong and C. W. Kim, Phys. Rev. C 5, 672 (1972).

[6]E. J. Konopinski, *The Theory of Beta Radioactivity* (Clarendon Press, Oxford, England, 1966).

[7]The forms given for the spectral functions in Ref. 2 are valid only for allowed transitions with $J=J'$. In order to treat arbitrary allowed decays $h_i(E)$ must be multiplied by $\gamma_{JJ'}$ (i=4, 5, 6, 7) and by $\theta_{JJ'}$ (i=10, 11, 12, 13, 16) where

$$\gamma_{JJ'} = \begin{cases} J+1 & J=J'+1 \\ 1 & J=J' \\ -J & J=J'-1\,, \end{cases}$$

$$\theta_{JJ'} = \begin{cases} -\dfrac{J+1}{2J-1} & J=J'+1 \\ 1 & J=J' \\ -\dfrac{J}{2J+3} & J=J'-1\,. \end{cases}$$

$h_i(E)$ (i=1, 2, 3) are unchanged. We have deleted T-violating correlations.

[8]A. Bottino and G. Ciocchetti, Phys. Lett. 43B, 170 (1973).

[9]We do not include the magnetic moments, as they are of $O(k/M)$ with respect to the charge terms.

[10]M. A. B. Beq, J. Bernstein, and A. Sirlin, Phys. Rev. D 6, 2597 (1972).

[11]D. A. Dicus and R. E. Norton, Phys. Rev. D 1, 1360 (1970).