Partial conservation of axial-vector current and nuclear beta decay*

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The disagreement pointed out by Armstrong and Kim between the elementary particle approach and the impulse-approximation predictions for $\Delta J^P = 0^-$ nuclear β decay is examined from the point of view of the axial current operator. Naive theoretical predictions are found to be in disagreement with present data.

[RADIOACTIVITY Discuss PCAC in β decay; calculated spectra for 0^+-0^- transition.]

In a recent paper Armstrong and Kim[1] pointed out that the conventional impulse-approximation approach and the elementary particle treatment[2] of axial-current matrix elements, while consistent in most cases, disagree strongly for 0^+–0^- transitions. Their conclusion was based upon calculation of specific matrix elements via the two approaches and comparison for various spins and parities.

We wish to demonstrate that this result may be understood by examining a simple model of current operators, and we comment on the 0^+–0^- case.

If the absence of second-class currents is assumed,[3] then for neutron β decay we can write

$$\langle p_{p_2}|A_\mu(0)|n_{p_1}\rangle = \bar{u}(p_2)[g_A(q^2)\gamma_\mu + g_P(q^2)q_\mu]\gamma_5 u(p_1)\,, \qquad (1)$$

where $P=p_1+p_2$ and $q=p_1-p_2$. Strict partial conservation of axial-vector current (PCAC)[4] implies that

$$g_A(0) = \frac{F_\pi g_r(0)}{2m}\,, \qquad (2)$$

i.e., the Goldberger-Treiman relation,[5] and

$$g_P(q^2) = \frac{1}{q^2 - m_\pi^2}\left\{2mg_A(q^2) - \frac{m_\pi^2}{q^2}[2mg_A(q^2) - F_\pi g_r(q^2)]\right\}\,, \qquad (3)$$

Here m is the nucleon mass, and g_r is defined by

$$\langle p_{p_2}|j_\pi(0)|n_{p_1}\rangle = i\, g_r(q^2)\bar{u}(p_2)\gamma_5 u(p_1)\,.$$

Kim and Mintz[6] have argued that

$$\frac{m_\pi^2}{q^2}[2m\, g_A(q^2) - F_\pi g_r(q^2)] \ll 2mg_A(q^2)\,, \qquad |q^2| \le m_\mu^2\,,$$

so we shall take

$$g_P(q^2) = -\frac{2mg_A(q^2)}{m_\pi^2 - q^2}\,. \qquad (4)$$

Passing now to the impulse-approximation current via the Foldy-Wouthuysen transformation we obtain[7]

$$\begin{aligned} A_0 &= \frac{i}{2m} g_A(q^2) \sum_{i=1}^{A} \tau_i^- \{e^{-i\vec{q}\cdot\vec{r}_i}, \vec{\sigma}_i\cdot\vec{\nabla}_i\} - \frac{q_0}{2m} g_P(q^2) \sum_{i=1}^{A} \tau_i^- e^{-i\vec{q}\cdot\vec{r}_i}\vec{\sigma}_i\cdot\vec{q}\,, \\ \vec{A} &= -g_A(q^2) \sum_{i=1}^{A} \tau_i^- \vec{\sigma}_i e^{-i\vec{q}\cdot\vec{r}_i} - \frac{\vec{q}}{2m} g_P(q^2) \sum_{i=1}^{A} \tau_i^- e^{-i\vec{q}\cdot\vec{r}_i}\vec{\sigma}_i\cdot\vec{q}\,. \end{aligned} \qquad (5)$$

Evaluation of the divergence gives[8]

$$\partial^\mu A_\mu = +\frac{q_0}{2m} g_A(q^2) \sum_{i=1}^{A} \tau_i^- \{e^{-i\vec{q}\cdot\vec{r}_i}, \vec{\sigma}_i\cdot\vec{\nabla}_i\} - i\left(g_A(q^2) - \frac{q^2}{2m} g_P(q^2)\right) \sum_{i=1}^{A} \tau_i^- e^{-i\vec{q}\cdot\vec{r}_i}\vec{\sigma}_i\cdot\vec{q}\,. \qquad (6)$$

Neglecting two-body operators[9]

$$\phi_\pi = -i\, g_r(q^2)\frac{1}{m_\pi^2 - q^2} \sum_{i=1}^{A} \tau_i^- e^{-i\vec{q}\cdot\vec{r}_i}\vec{\sigma}\cdot\vec{q}/2m\,, \qquad (7)$$

so that

$$\partial^\mu A_\mu \approx F_\pi m_\pi^2 \phi_\pi + \frac{q_0}{2m} g_A(q^2) \sum_{i=1}^{A} \tau_i^- \{e^{-i\vec{q}\cdot\vec{r}_i}, \vec{\sigma}_i\cdot\vec{\nabla}_i\}\,. \qquad (8)$$

Thus in this simple model it is the presence of the term

$$\frac{i}{2m} g_A(q^2) \sum_{i=1}^{A} \tau_i^- \{e^{-i\vec{q}\cdot\vec{r}_i}, \vec{\sigma}_i \cdot \vec{\nabla}_i\}$$

in A_0 that violates the PCAC condition.

For decays with $\Delta J^P = 1^+, 2^-, 3^+, \ldots$ we find for the leading contribution to the axial matrix elements (neglecting the pseudoscalar)

$$\langle\beta|A_0|\alpha\rangle \approx \langle\beta|\sum_{i=1}^{A} \tau_i^-(-i\vec{q}\cdot\vec{r}_i)^{\Delta J}\vec{\sigma}_i\cdot\vec{\nabla}_i|\alpha\rangle \frac{g_A(q^2)}{2m}, \quad (9)$$
$$\langle\beta|\vec{A}|\alpha\rangle \approx \langle\beta|\sum_{i=1}^{A} \tau_i^-(-i\vec{q}\cdot\vec{r})^{\Delta J-1}\vec{\sigma}_i|\alpha\rangle g_A(q^2).$$

Thus

$$\left|\frac{\langle\beta|A_0|\alpha\rangle}{\langle\beta|\vec{A}|\alpha\rangle}\right| = \mathcal{O}\left(\frac{|\vec{q}|}{2m}\right) \ll 1,$$

so that violations of PCAC are guaranteed small when compared to the dominant Gamow-Teller term. Also, for $\Delta J^P = 1^-, 2^+, 3^-, \ldots$ we find (neglecting the pseudoscalar)

$$\langle\beta|A_0|\alpha\rangle = 0, \quad (10)$$
$$\langle\beta|\vec{A}|\alpha\rangle \approx \langle\beta|\sum_{i=1}^{A}\tau_i^-(-i\vec{q}\cdot\vec{r})^{\Delta J}\vec{\sigma}_i|\alpha\rangle,$$

so that no PCAC violation should occur. Finally, for $\Delta J^P = 0^-$ we have (neglecting the pseudoscalar)

$$\langle\beta|A_0|\alpha\rangle \approx \langle\beta|\sum_{i=1}^{A}\tau_i^-\vec{\sigma}_i\cdot\vec{\nabla}_i|\alpha\rangle\frac{g_A(q^2)}{2m}, \quad (11)$$
$$\langle\beta|\vec{A}|\alpha\rangle \approx \langle\beta|\sum_{i=1}^{A}\tau_i^-(-i\vec{q}\cdot\vec{r}_i)\vec{\sigma}_i|\alpha\rangle g_A(q^2),$$

so that

$$\left|\frac{\langle\beta|A_0|\alpha\rangle}{\langle\beta|\vec{A}|\alpha\rangle}\right| = \mathcal{O}(1)$$

and a large PCAC violation is expected. This is the result of Armstrong and Kim.[1]

Such $\Delta J^P = 0^-$ decays appear to offer an excellent arena in which to test the PCAC hypothesis for nuclei. In all other β decays, the induced pseudoscalar, already small, is obscured by a plethora of additional form factors. Even in μ-capture reactions, although $q \sim m_\mu$, enhancing the pseudoscalar contribution, there is still no reliable check of the Goldberger-Treiman prediction.[10]

For $0^\pm$-$0^\mp$ decay, Lorentz invariance and the standard weak Hamiltonian require a description in terms of two independent form factors,

$$\langle 0^\mp_{p_2}|A_\mu(0)|0^\pm_{p_1}\rangle = F_1(q^2)P_\mu + F_2(q^2)q_\mu, \quad (12)$$

and we find for the electron-energy spectrum

$$d\omega \approx F_-(Z,E)\frac{G_V^2\cos^2\theta_c}{(2\pi)^3}(E_0-E)^2pEdE16M^2 \times\left[|F_1(q^2)|^2\left(1+\frac{2E}{M}\right)+\mathrm{Re}F_1^*(q^2)F_2(q^2)\frac{m_e^2}{ME}\right], \quad (13)$$

where $E(p)$ is the electron energy (momentum),

$$M = \tfrac{1}{2}(M_1+M_2), \quad \Delta = M_1 - M_2,$$

and

$$E_0 = \Delta\frac{1+m_e^2/2M\Delta}{1+\Delta/2M}$$

is the maximum electron energy permitted by kinematics. The Fermi function $F_-(Z,E)$ includes the finite size corrections and hopefully accounts for all dominant Coulomb effects.[11]

The impulse-approximation prediction for the form factors which follows from Eq. (5) is

$$F_1(0) \simeq i\frac{g_A(0)}{2M}\left(-\frac{\Delta}{3}\langle\vec{\sigma}\cdot\vec{r}\rangle+\frac{1}{m}\langle\vec{\sigma}\cdot\vec{\nabla}\rangle\right), \quad (14)$$
$$F_2(0) \simeq i g_A(0)\left(\frac{m_\pi^2-\Delta^2}{3m_\pi^2}\langle\vec{\sigma}\cdot\vec{r}\rangle+\frac{1}{2Mm}\langle\vec{\sigma}\cdot\vec{\nabla}\rangle\right)$$

If we assume the nuclear-force operator to be velocity-independent, which is plausible as a first approximation, we may replace

$$\vec{\nabla} \approx m[\vec{r}, H_{\mathrm{nuc}}] + m\delta\cdot\vec{r}$$

with

$$\delta = m_n - m_p. \quad (15)$$

Then

$$\frac{F_2(0)}{F_1(0)} \approx \frac{M}{\Delta+(3/2)\delta}. \quad (16)$$

On the other hand, the PCAC assumption applied to Eq. (12) predicts[12]

$$f_\pi(0) = \frac{2M\Delta}{F_\pi}F_1(0) \quad (17)$$
$$F_2(0) = \frac{2M\Delta}{m_\pi^2}\left\{F_1(0)+m_\pi^2\left[\frac{F_\pi}{2M\Delta}f'_\pi(0)-F'_1(0)\right]\right\},$$

where $\langle 0^\mp_{p_2}|j_\pi(0)|0^\pm_{p_1}\rangle \equiv -if_\pi(q^2)$. If we assume, following Armstrong and Kim, that

$$F_1(q^2)=F_1(0)(1+\alpha q^2R^2+\cdots)$$
$$f_\pi(q^2)=f_\pi(0)(1+\beta q^2R^2+\cdots)\,, \tag{18}$$

then

$$F_2(0)=\frac{2M\Delta}{m_\pi^{\,2}}F_1(0)[1+m_\pi^{\,2}R^2(\beta-\alpha)]\,. \tag{19}$$

Since we expect $\alpha-\beta=\mathcal{O}(1)$, the impulse approximation predicts a value for $F_2(0)/F_1(0)$ much larger than does the PCAC assumption, given $\Delta\lesssim 5$ MeV and $A\lesssim 200$.

Looking at the experimental situation, shape factors are characterized by four parameters

$$\frac{f(E)}{f(0)}\equiv 1+a\frac{E}{m_e}+b\frac{m_e}{E}+c\frac{E^2}{m_e^{\,2}}+d\frac{E^3}{m_e^{\,3}}\,. \tag{20}$$

The measured values of these quantities are given in Table I. Our predictions are

$$a=\frac{2m_e}{M}(1+\tfrac{4}{3}\alpha ME_0R^2)\,,$$
$$b=\frac{m_e}{M}\frac{F_2(0)}{F_1(0)}+\tfrac{4}{3}\alpha m_eE_0R^2\,,\quad c=-\tfrac{8}{3}\alpha m_e^{\,2}R^2\,,\quad d\approx 0\,. \tag{21}$$

Clearly the experimental numbers for a, c, and d cannot be understood at all on this basis, and the prediction for b, while consistent in size at least for the impulse-approximation prediction, is of the wrong sign.

Our theoretical picture appears very inconsistent then with experiment. However, the experiments are very difficult, and, in addition, there are obvious weaknesses in our theoretical approach. One problem is that our one-body operator model is much too simplistic. Meson exchange forces[13] are expected to modify these predictions, perhaps appreciably. In particular, if PCAC is valid, meson exchange must restore agreement between the elementary particle result for F_2/F_1 and the operator calculation. Another source of disagreement may arise if the Fermi function does not include all Coulomb effects. These can be quite large, since $Z\alpha\gtrsim\frac{1}{2}$. Finally, the momentum dependence of the form factors is open to question. In our simple model, the assumption $\beta=\mathcal{O}(1)$ is unjustified since the one-body part of the p-wave pion field yields

$$f_\pi(q^2)=-ig_r(q^2)\,\frac{\Delta^2-q^2}{6m}\,\langle\vec{\sigma}\cdot\vec{r}\rangle+\cdots$$
$$\approx f_\pi(0)\left[1-\frac{q^2}{\Delta^2}+\cdots\right]. \tag{22}$$

Thus $-\beta\approx 1/\Delta^2R^2\gg 1$ so that if a considerable part of the pion coupling arises from this one-body term, the PCAC prediction becomes[14]

$$\frac{F_2(0)}{F_1(0)}\simeq-\frac{2M}{\Delta}\,.$$

Then

$$b\simeq-2\frac{m_e}{\Delta}\,,$$

which is of the proper order of magnitude and sign in order to explain the experimental result. The difficulty in explaining the size of the a, c, and d coefficients remains, however, and it is clear that both the theoretical and experimental situations demand further study.

TABLE I. Experimental measurements of the shape factor

$$S(E)=1+a\frac{E}{m_e}+b\frac{m_e}{E}+c\frac{E^2}{m_e^{\,2}}+d\frac{E^3}{m_e^{\,3}}$$

for $0^\pm\to 0^\mp$ transitions in nuclear β decay.

Parent nucleus	Ref.	E_0 (MeV)	a	b	c	d
$^{144}_{58}$Ce	a	0.32	−0.342	⋯	⋯	⋯
$^{144}_{59}$Pr	a	3.00	0.0376	−0.118	−0.0077	⋯
	b		⋯	−0.0977	⋯	⋯
$^{166}_{67}$Ho	a	1.85	−0.87	−1.03	0.225	−0.021
$^{206}_{81}$Tl	c	1.57	−0.154	−0.484	⋯	⋯

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$$A_\mu \to A_\mu + q_\mu g_p(q^2)\frac{\Delta}{8m^2}\left\{\vec{\sigma}\cdot\vec{p}, e^{-i\vec{q}\cdot\vec{r}}\right\}$$

$$-g_A(q^2)\frac{1}{4m^2}\,\vec{\sigma}\cdot\vec{p}\,e^{-i\vec{q}\cdot\vec{r}}\,\vec{\sigma}\,\vec{\sigma}\cdot\vec{p},$$

$$\phi_\pi \to \phi_\pi - g_r(q^2)\frac{1}{m_\pi^2-q^2}\,\frac{\Delta}{8m^2}\left\{\vec{\sigma}\cdot\vec{p}, e^{-i\vec{q}\cdot\vec{r}}\right\},$$

and the PCAC relation becomes

$$\partial^\mu A_\mu - F_\pi m_\pi^2\phi_\pi = i\,\frac{g_A(q^2)}{4m}\left\{\vec{\sigma}\cdot\vec{p}, e^{-i\vec{q}\cdot\vec{r}}\right\}$$

$$\times\left[q_0 + \frac{1}{2m}\,\vec{q}\cdot(\vec{q}-2\vec{p})\right].$$

Krmpotić and Tadić argue that $\vec{q}\cdot(\vec{q}-2\vec{p})\approx p_f^2 - p_i^2 \cong -2mq_0$ so that the PCAC condition is satisfied. However, $\vec{\sigma}\cdot\vec{p}\,q^2$ is an odd-parity operator while $\vec{\sigma}\cdot\vec{p}\,\vec{q}\cdot\vec{p}$ has even parity so that both cannot contribute to odd-parity decays. Instead $\vec{\sigma}\cdot\vec{p}\,\vec{q}\cdot\vec{p}$ must be accompanied by $-i\,\vec{q}\cdot\vec{r}$ from the expansion of the exponential. Then both terms are $\mathcal{O}(|\vec{q}\,|/m)$ compared to the leading term, and we neglect them in our analysis.

[9]Neglect of two-body contributions allows study of a tractable model, which we feel offers insight into what is going on.

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$$f_\pi(0) = \frac{2M\Delta}{F_\pi}F_1(0)$$

are obtained if we set $\langle\vec{\sigma}\cdot\vec{\nabla}\rangle = 0$ so that exact PCAC obtains.