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Unitarity and off-shell effects in the impulse approximation*

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For elastic scattering of a particle from a finite many-body target, the procedure of eliminating the two-body potential in favor of a two-body scattering matrix is examined. It is shown that the theory of Kerman, McManus, and Thaler leads to unitary relations which are not of a particularly convenient form. A simple procedure which identifies the appropriate off-shell scattering matrix to be used in the impulse approximation is presented. The projection operator technique is employed to define an optical potential and then, in a truncated Hilbert space, a relation similar to a two-body Low equation is used to identify the appropriate two-body T matrix. With this latter procedure the unitarity relations are maintained in a form which allows them to be used as a guide to the validity of the impulse approximation.

I. INTRODUCTION

In the theory of the elastic scattering of nucleons by nuclei, the two-body potential is frequently eliminated in favor of a two-body scattering matrix. In the theory of Kerman, McManus, and Thaler (KMT)[1] or in the multiple-scattering series of Watson,[2] this procedure requires the direct replacement of a many-body operator by a two-body operator. This scheme has the disadvantage, however, that it is difficult to know what off-shell effects must be included in the two-body operator so that it may "best" represent the many-body operator it is to replace. In the Watson multiple-scattering series, some of these effects may be included by performing the selective resummation as a three-body problem.[3]

In order to understand better the replacement of a two-body potential by a two-body scattering matrix, we reexamine here the KMT formalism. In Sec. II the theory of KMT is reviewed. In Sec. III we examine the relationships that unitarity implies for the T matrix. In Sec. IV we examine the case of elastic scattering of a projectile from a target composed of A nonidentical particles. We employ the projection-operator[4] technique to define an optical potential. By limiting the nonelastic space to single-hole states and employing an equation similar in form to a two-body Low equation, we demonstrate how to identify the two-body scattering matrix which occurs naturally in the many-body problem. This approach is closely related to treatments presented elsewhere.[5,6] However, the application of this approach to the simple problem considered here clarifies the procedure, and also demonstrates the origin of the effects which distinguish the free scattering matrix from the scattering matrix used in the impulse approximation.

The results are shown to satisfy unitarity relations. In the Appendix, these results are generalized to the case where the target particles are identical.

II. MULTIPLE-SCATTERING FORMALISM

In the theory of the scattering of nucleons by nuclei, where the nuclear force is considered to be singular, it is customary to eliminate the singular two-body potential in favor of a two-body scattering matrix. In the theory of KMT one writes a T matrix which satisfies an equation of the form

$$T(E)=Av_{01}+Av_{01}\,\frac{1}{E-h_0-H_A+i\epsilon}\,T(E)\,, \qquad (2.1)$$

where v_{01} is the potential between the incident particle, labeled zero, and particle one. The identity of the target particles has been employed to replace $V=\sum_i v_{0i}$ by Av_{01}. A two-body T matrix $t(\omega)$ is then introduced. This operator satisfies the equation

$$t_{01}(\omega)=v_{01}+v_{01}\,\frac{1}{\omega-h_0-h_1+i\epsilon}\,t_{01}(\omega)\,, \qquad (2.2)$$

or equivalently,

$$v_{01}=\frac{1}{1+t_{01}(\omega)(\omega-h_0-h_1+i\epsilon)^{-1}}\,t_{01}(\omega)\;. \qquad (2.3)$$

Substitution of Eq. (2.3) into Eq. (2.1) immediately yields

$$T(E)=At_{01}(\omega)+(A-1)t_{01}(\omega)\frac{1}{E-h_0-H_A+i\epsilon}\,T(E)$$
$$+\,t_{01}(\omega)\left[\frac{1}{E-h_0-H_A+i\epsilon}-\frac{1}{\omega-h_0-h_1+i\epsilon}\right]T(E) \qquad (2.4)$$

or

$$T(E) = A t_{01}(\omega) + (A-1) t_{01}(\omega) G_0^{(+)}(E) T(E) + t_{01}(\omega)[G_0^{(+)}(E) - g_{01}^{(+)}(\omega)] T(E), \tag{2.5}$$

where $G_0^{(+)}(E)$ and $g_{01}^{(+)}(\omega)$ have the obvious definitions. The standard approximation used with KMT consists in dropping the last term in Eq. (2.4) or Eq. (2.5) and setting $\omega = E$. In this approximation, we find that the approximate transition operator $\tilde{T}(E)$ satisfies the equation

$$\tilde{T}(E) = A t_{01}(E) + (A-1) t_{01}(E) G_0^{(+)}(E) \tilde{T}(E). \tag{2.6}$$

Since Eq. (2.6) is not a many-body Lippmann-Schwinger equation, it then proves convenient to define an operator $T'(E)$ such that

$$T'(E) = [(A-1)/A] \tilde{T}(E), \tag{2.7}$$

so that $T'(E)$ does satisfy a Lippmann-Schwinger equation, viz.,

$$T'(E) = (A-1) t_{01}(E) + (A-1) t_{01}(E) G_0^{(+)}(E) T'(E). \tag{2.8}$$

We now note that Eq. (2.8) is still a many-body equation. Again following KMT, we may reduce Eq. (2.8) to a one-body equation for elastic scattering by projective techniques. We define projection operators P and Q such that $P + Q = 1$, without at this point specifying P any further. We then find a relation for $PT'(E)$ from Eq. (2.8) by observing that

$$PT'(E) = P(A-1) t_{01}(E) + P(A-1) t_{01}(E) G_0^{(+)}(E) PT'(E) + P(A-1) t_{01}(E) G_0^{(+)}(E) QT'(E) \tag{2.9}$$

and

$$Q[1 - (A-1) t_{01}(E) G_0^{(+)}(E)] QT'(E) = Q(A-1) t_{01}(E) + Q(A-1) t_{01}(E) G_0^{(+)}(E) PT'(E). \tag{2.10}$$

By solving Eq. (2.10) for $QT'(E)$ and substituting that result in Eq. (2.9), we obtain

$$PT'(E) = PU'(E) + PU'(E) G_0^{(+)}(E) PT'(E), \tag{2.11}$$

with

$$U'(E) \equiv (A-1) t_{01}(E) + (A-1) t_{01}(E) \times \frac{Q}{Q[E - h_0 - H_A - (A-1) t_{01}(E)]Q} (A-1) t_{01}(E). \tag{2.12}$$

In order to obtain Eq. (2.12), it is necessary to assume that $[G_0^{(+)}(E), P] = 0$. If we take P to be

$$P \equiv \int |\vec{k}_0, \Phi_A\rangle \, d\vec{k}_0 \langle \vec{k}_0, \Phi_A|, \tag{2.13}$$

where $|\vec{k}_0, \Phi_A\rangle$ represents particle zero in a plane-wave state incident on the ground state $|\Phi_A\rangle$, then this immediately gives that $[G_0^{(+)}(E), P] = 0$.

The matrix elements of $T'(E)$ representing elastic scattering are $\langle \vec{k}_0', \Phi_A | T'(E) | \vec{k}_0, \Phi_A\rangle$. Thus, with the definition of P given by Eq. (2.13), we see that

$$PT'(E)P \equiv T'_{el}(E) \tag{2.14}$$

satisfies the equation

$$T'_{el}(E) = PU'(E)P + PU'(E) P G_0^{(+)}(E) T'_{el}(E). \tag{2.15}$$

The identifications

$$\langle \vec{k}' | \mathcal{T}'(E) | \vec{k} \rangle \equiv \langle \vec{k}_0' \Phi_A | T'(E) | \vec{k}_0 \Phi_A \rangle, \tag{2.16}$$

$$\langle \vec{k}' | \mathcal{V}'_{opt}(E) | \vec{k} \rangle \equiv \langle \vec{k}_0' \Phi_A | U'(E) | \vec{k}_0 \Phi_A \rangle, \tag{2.17}$$

and

$$\langle \vec{k}' | g_0^{(+)}(E) | \vec{k} \rangle \equiv \langle \vec{k}_0' \Phi_A | G_0^{(+)}(E) | \vec{k}_0 \Phi_A \rangle = \frac{\delta(\vec{k}' - \vec{k})}{E - k^2 + i\epsilon}, \tag{2.18}$$

transform Eq. (2.15) into the one-body equation for elastic scattering (units such that $\hbar^2 = 2m = 1$ are used),

$$\mathcal{T}'(E) = \mathcal{V}'_{opt}(E) + \mathcal{V}'_{opt}(E) g_0^{(+)}(E) \mathcal{T}'(E). \tag{2.19}$$

The optical potential $\mathcal{V}'_{opt}(E)$ is a complex energy-dependent one-body potential. Clearly Eq. (2.19) is a one-body Lippmann-Schwinger equation, with all the properties that implies. However, according to Eq. (2.7) the elastic scattering transition operator in the impulse approximation is given by $\tilde{\mathcal{T}}(E)$ which is related to $\mathcal{T}'(E)$ by

$$\tilde{\mathcal{T}}(E) = \left(\frac{A}{A-1}\right) \mathcal{T}'(E). \tag{2.20}$$

Since $\tilde{\mathcal{T}}$ rather than $\mathcal{T}'$ represents the elastic scattering operator, the unitarity relations of interest necessarily concern themselves with $\tilde{\mathcal{T}}$. However, we have noted that it is $\mathcal{T}'$ and not $\tilde{\mathcal{T}}$ that satisfies a one-body Lippmann-Schwinger equation. This fact leads to an inconvenient statement of the required current conservation. This question will be examined at the end of the next section.

III. UNITARITY RELATIONS

In this section we discuss the unitarity relations for both two-body potential scattering and also for the full many-body problem. We shall be particularly interested in the implications that the unitarity of the S matrix has for the T matrix.

We begin by considering a general transition operator $\mathcal{T}(E)$. We need not specify, at this time, whether $\mathcal{T}(E)$ is a one-body or many-body operator, as we are interested in formal manipulations which are valid for both the case when $\mathcal{T}(E)$ is a one-body operator and when $\mathcal{T}(E)$ is a many-body operator. We shall return later to the physical implications of a specific identification of $\mathcal{T}(E)$.

We assume that $\mathcal{T}(E)$ may be related to a potential $\mathcal{V}$ by a Moller wave operator $\Omega^{(+)}(E)$:

$$\mathcal{T}(E)=\mathcal{V}\,\Omega^{(+)}(E). \tag{3.1}$$

This immediately enables us to write

$$\Omega^{(+)\dagger}\mathcal{T}(E)=\Omega^{(+)\dagger}(E)\mathcal{V}\Omega^{(+)}(E). \tag{3.2}$$

The Hermitian conjugate of Eq. (3.2) is

$$\mathcal{T}^{\dagger}(E)\Omega^{(+)}(E)=\Omega^{(+)\dagger}(E)\mathcal{V}^{\dagger}\Omega^{(+)}(E), \tag{3.3}$$

and the difference between Eq. (3.3) and Eq. (3.2) is

$$\mathcal{T}^{\dagger}(E)\Omega^{(+)}(E)-\Omega^{(+)\dagger}(E)\mathcal{T}(E)=\Omega^{(+)\dagger}(E)[\mathcal{V}^{\dagger}-\mathcal{V}]\Omega^{(+)}(E). \tag{3.4}$$

The wave operator $\Omega^{(+)}(E)$ obeys the relation

$$\Omega^{(+)}(E)=1+\mathcal{G}_0^{(+)}(E)\mathcal{V}\,\Omega^{(+)}(E)$$
$$=1+\mathcal{G}_0^{(+)}(E)\mathcal{T}(E), \tag{3.5}$$

with $\mathcal{G}_0^{(+)}(E)$ defined by

$$\mathcal{G}_0^{(+)}(E)=(E-\mathcal{H}_0+i\epsilon)^{-1}, \tag{3.6}$$

where $\mathcal{H}_0$ is taken to be the difference between the full Hamiltonian, H, and the potential $\mathcal{V}$, i.e., $\mathcal{H}_0=H-\mathcal{V}$. This relation, together with its Hermitian conjugate, when inserted in Eq. (3.4) yields

$$\mathcal{T}(E)-\mathcal{T}^{\dagger}(E)=\mathcal{T}^{\dagger}(E)[\mathcal{G}_0^{(+)}(E)-\mathcal{G}_0^{(-)}(E)]\mathcal{T}(E)$$
$$+\Omega^{(+)\dagger}(E)(\mathcal{V}-\mathcal{V}^{\dagger})\Omega^{(+)}(E)$$
$$=-2\pi i\mathcal{T}^{\dagger}(E)\delta(E-\mathcal{H}_0)\mathcal{T}(E)$$
$$+\Omega^{(+)\dagger}(E)(\mathcal{V}-\mathcal{V}^{\dagger})\Omega^{(+)}(E). \tag{3.7}$$

If $\mathcal{V}$ is Hermitian this becomes the familiar relation

$$\mathcal{T}(E)-\mathcal{T}^{\dagger}(E)=-2\pi i\mathcal{T}^{\dagger}(E)\delta(E-\mathcal{H}_0)\mathcal{T}(E). \tag{3.8}$$

It is of consequence to note that the derivation of Eq. (3.7) requires only that the wave operator obey a Lippmann-Schwinger equation, Eq. (3.5), and that the transition operator be defined according to Eq. (3.1). We note that the $\mathcal{V}$ in Eq. (3.1) and that in Eq. (3.7) are necessarily identical. If we take that matrix element of Eq. (3.8) which corresponds to forward elastic scattering, we obtain the usual optical theorem.

The form of Eq. (3.7) is perhaps worthy of further note. Let us consider the specific case where $\mathcal{T}(E)$ in Eq. (3.1) is taken to be a projection of the *full many-body* T matrix [given in Eq. (2.1)] on a subspace of the full Hilbert space. We shall define this projection by $\hat{\mathcal{T}}(E)$,

$$\hat{\mathcal{T}}(E)\equiv\mathcal{P}T(E)\mathcal{P}. \tag{3.9}$$

The usual operator algebra indicates that $\hat{\mathcal{T}}(E)$ satisfies

$$\hat{\mathcal{T}}(E)=\hat{U}(E)+\hat{U}(E)G_0^{(+)}(E)\hat{\mathcal{T}}(E), \tag{3.10}$$

where $\hat{U}(E)$ is given by

$$\hat{U}(E)=\mathcal{P}\left[V+V\frac{\mathcal{Q}}{\mathcal{Q}(E-h_0-V-H_A+i\epsilon)\mathcal{Q}}V\right]\mathcal{P} \tag{3.11}$$

where, as usual, $\mathcal{P}+\mathcal{Q}=1$. We have also assumed that

$$[\mathcal{P},G_0^{(+)}(E)]=0. \tag{3.12}$$

In general, of course, $\hat{U}(E)$ is not Hermitian even though V may be a Hermitian operator.

In this case, Eq. (3.7) becomes

$$\hat{\mathcal{T}}(E)-\hat{\mathcal{T}}^{\dagger}(E)=-2\pi i\hat{\mathcal{T}}^{\dagger}(E)\delta(E-H_0)\hat{\mathcal{T}}(E)$$
$$+\hat{\mathcal{W}}^{\dagger}(E)[\hat{U}(E)-\hat{U}^{\dagger}(E)]\hat{\mathcal{W}}(E), \tag{3.13}$$

where we have made the identification $H_0=h_0+H_A$, and where $\hat{\mathcal{W}}$ satisfies the relation

$$\hat{\mathcal{W}}(E)=\mathcal{P}[1+G_0^{(+)}(E)\hat{U}(E)\hat{\mathcal{W}}(E)]\mathcal{P}. \tag{3.14}$$

It is convenient to add the condition that the incident state be included in $\mathcal{P}$.

As a second example of the implications of Eq. (3.7), we may identify $\mathcal{T}(E)$ with the many-body T matrix $T(E)$, but consider the Lippmann-Schwinger equation, Eq. (2.1), which relates $T(E)$ to V. If V is Hermitian, Eq. (3.7) implies

$$T(E)-T^{\dagger}(E)=-2\pi i\,T^{\dagger}(E)\delta(E-H_0)T(E), \tag{3.15}$$

so that

$$\mathcal{P}[T(E)-T^{\dagger}(E)]\mathcal{P}=-2\pi i\,\mathcal{P}T^{\dagger}(E)\mathcal{P}\delta(E-H_0)\mathcal{P}T(E)\mathcal{P}$$
$$-2\pi i\mathcal{P}T^{\dagger}(E)\mathcal{Q}\delta(E-H_0)\mathcal{Q}T(E)\mathcal{P}, \tag{3.16}$$

or

$$\hat{\mathcal{T}}(E)-\hat{\mathcal{T}}^{\dagger}(E)=-2\pi i\,\hat{\mathcal{T}}^{\dagger}(E)\delta(E-H_0)\hat{\mathcal{T}}(E)$$
$$-2\pi i\mathcal{P}T^{\dagger}(E)\mathcal{Q}\delta(E-H_0)\mathcal{Q}T(E)\mathcal{P}. \tag{3.17}$$

Comparison of Eq. (3.13) or Eq. (3.14) with Eq. (3.10) indicates that

$$\hat{\mathcal{W}}^{\dagger}(E)[\hat{U}(E)-\hat{U}^{\dagger}(E)]\hat{\mathcal{W}}(E)$$
$$=-2\pi i\mathcal{P}T^{\dagger}(E)\mathcal{Q}\delta(E-H_0)\mathcal{Q}T(E)\mathcal{P}. \tag{3.18}$$

If we now examine the diagonal matrix element of Eq. (3.15) corresponding to forward scattering, we obtain the optical theorem relating the imaginary part of the forward-scattering amplitude to the total cross section. The forward-scattering matrix element of Eq. (3.16) or Eq. (3.17) then yields the expected result, viz that the total cross section may be split into two parts, the first of which is manifestly the total cross section of scattering into all possible final states included in $\mathcal{P}$. The second term then necessarily represents the total cross section for all scatterings into states not included in $\mathcal{P}$. Thus it is obvious that the total cross section for scattering into the states in $\mathcal{Q}$ is given by

$$\sigma_{\rm tot}(\mathcal{Q}) = -(4\pi)^2\frac{\pi}{4ik_0}\langle \vec{k}_0\,\Phi_A|\hat{w}^\dagger[\hat{U}(E)-\hat{U}^\dagger(E)]\hat{w}(E)|\vec{k}_0\Phi_A\rangle$$
$$= (4\pi^2)^2\frac{1}{2k_0}\langle \vec{k}_0\;\Phi_A|\,T(E)^\dagger\mathcal{Q}\times\delta(E-H_0)\mathcal{Q}T(E)|\vec{k}_0\;\Phi_A\rangle\,. \quad (3.19)$$

This discussion is, of course, a mathematical statement of the conventional remark that lack of Hermiticity arises from the failure (by a truncation of the Hilbert space) to include all possible physically available states.

If we now specify $\mathcal{P}\equiv P$, where P is as given by Eq. (2.13), then we are dealing with the subspace in which the target nucleus remains in its ground state. In this case, we may define a one-body optical potential by

$$\langle \vec{k}_0'|\,V_{\rm opt}|\vec{k}_0\rangle \equiv \langle \vec{k}_0'\Phi_A|\,U(E)\,|\vec{k}_0\Phi_A\rangle\,, \quad (3.20)$$

where $U(E)$ is given by Eq. (3.11) with $\mathcal{P}$ and $\mathcal{Q}$ replaced by P and Q, respectively. The corresponding T matrix, we shall indicate as $T_{\rm opt}(E)$,

$$T_{\rm opt}(E) = V_{\rm opt}(E)\Omega^{(+)}_{\rm opt}(E)\,, \quad (3.21)$$

where $\Omega^{(+)}_{\rm opt}(E)$ is defined by

$$\Omega^{(+)}_{\rm opt}(E) = 1 + (E-h_0+i\epsilon)^{-1}V^{(+)}_{\rm opt}(E)\,. \quad (3.22)$$

The forward-scattering matrix element of Eq. (3.7) is [where now we identify $\mathcal{T}(E)$ with the one-body operator $T_{\rm opt}(E)$]

$$\langle \vec{k}_E|\,T_{\rm opt}(E) - T^\dagger_{\rm opt}(E)\,|\vec{k}_E\rangle$$
$$= -2\pi i\,\langle \vec{k}_E|\,T^\dagger_{\rm opt}(E)\delta(E-h_0)T_{\rm opt}(E)|\vec{k}_E\rangle + \langle\psi^{(+)}_{\vec{k}_E}|\,V_{\rm opt}(E) - V^\dagger_{\rm opt}(E)\,|\psi^{(+)}_{\vec{k}_E}\rangle\,, \quad (3.23)$$

where $|\psi^{(+)}_{\vec{k}_E}\rangle$ is the solution for energy E of the Hamiltonian $H(E) = h_0 + V_{\rm opt}(E)$. The first term on the right-hand side of Eq. (3.23) is proportional to the total elastic cross section, whereas the second term on the right-hand side is proportional to the total cross section for absorption. With these identifications we have

$$\frac{4\pi}{k_E}\,{\rm Im}\,f_{k_E}(0) = \sigma_{\rm elastic} + \sigma_{\rm nonelastic}\,, \quad (3.24)$$

where $f_{k_E}(0)$ is the forward-scattering amplitude.

We now return to the many-body problem and rewrite Eq. (3.15) as

$$T(E) - T^\dagger(E) = -2\pi i\,T^\dagger(E)P\delta(E-h_0-H_A)PT(E) - 2\pi i\,T^\dagger(E)Q\delta(E-h_0-H_A)QT(E)\,. \quad (3.25)$$

For forward-elastic scattering, we take the matrix elements of Eq. (3.25) with the state $|\vec{k}_0\;\Phi_A\rangle$,

$$\langle \vec{k}_0\;\Phi_A\,|\,T(E) - T^\dagger(E)|\vec{k}_0\;\Phi_A\rangle$$
$$= -2\pi i\langle \vec{k}_0\;\Phi_A\,|\,T^\dagger(E)P\delta(E-h_0-H_A)PT(E)\,|\vec{k}_0\;\Phi_A\rangle - 2\pi i\langle \vec{k}_0\;\Phi_A\,|\,T^\dagger(E)Q\delta(E-h_0-H_A)QT(E)\,|\vec{k}_0\;\Phi_A\rangle\,. \quad (3.26)$$

For the optical potential $V_{\rm opt}(E)$, which has been constructed to give the elastic scattering matrix elements of $T(E)$, one may identify, term by term, Eq. (3.23) with Eq. (3.26). Thus, Eq. (3.26) leads again to the unitarity relationship of Eq. (3.24), where now, however, each term in Eq. (3.24) can be identified with the corresponding many-body term in Eq. (3.26).

We now return to the theory of KMT which was reviewed in Sec. I. We recall that the physical elastic scattering is given by the matrix elements of $P\tilde{T}(E)P$ with $\tilde{T}(E)$ given in Eq. (2.6). On the other hand, it was the operator $T'(E)$ which was shown to satisfy an equation of the Lippmann-Schwinger form. Thus it is of interest to derive the unitarity relations implied by the structure of the KMT formulation of the multiple-scattering problem.

It is worthwhile to recall that the only approximation made in deriving Eq. (2.6) for $\tilde{T}(E)$ was the omission of the term proportional to $[G_0^{(+)}(E) - g_{01}^{(+)}(\omega)]$ in Eq. (2.5). If this omission is justified for the particular problem of interest, then $\tilde{T}(E)$ is a good approximation to the exact T matrix and thus must satisfy an optical theorem of the form

$$\frac{4\pi}{k_0}\,{\rm Im}\,f_{k_0}(0) = \sigma_{\rm total} = \sigma_{\rm elastic} + \sigma_{\rm nonelastic}$$
$$= -(4\pi)^2\frac{\pi}{2k_0}\,{\rm Im}\,\langle \vec{k}_0\Phi_A\,|\,T\,|\vec{k}_0\Phi_A\rangle\,. \quad (3.27)$$

A model in which the neglect of $[G_0^{(+)}(E) - g_{01}^{(+)}(\omega)]$ can be made exact, is one in which all of the eigenstates of H_A are taken to be degenerate. The many-body Green's function, $G_0^{(+)}(E)$, would then become

$$G_0^{(+)}(E) \approx \frac{1}{E - h_0 + i\epsilon}, \qquad (3.28)$$

which can clearly be exactly cancelled by redefining $g_{01}^{(+)}(E)$ as $G_{01}^{(+)}(E) = G_0^{(+)}(E)$. In that case of course, $t_{01}(E)$ represents the two-body T matrix for the scattering from an infinitely heavy-target particle, and is not therefore related directly to the observations. Such a model [Eq. (3.28)] derives from the use of the closure approximation in the many-body problem. There are, of course, other models in which the neglect of this term is exact. We mention this here only to demonstrate that there are models in which $\tilde{T}(E)$ may be the exact T matrix, and in these models the optical theorem given in Eq. (3.27) is also exact. Thus, we may conclude that the neglect of the term proportional to $[G_0^{(+)}(E) - g_{01}^{(+)}(\omega)]$ in Eq. (2.5) does not preclude the existence of unitarity relations.

A unitarity relation for the KMT theory can be derived immediately for the operator $PT'(E)P \equiv T'_{el}(E)$ which satisfies a Lippmann-Schwinger equation of the form

$$T'_{el}(E) = PU'(E)P + PU'(E)PG_0^{(+)}(E)T'_{el}(E), \qquad (3.29)$$

where $U'(E)$ is defined in Eq. (2.12). Equation (3.7) then implies that $T'_{el}(E)$ satisfies a unitarity relation of the form

$$T'_{el}(E) - T'^{\dagger}_{el}(E) = -2\pi i T'^{\dagger}_{el}(E)\delta(E - h_0)T'_{el}(E) + \bar{\Omega}^{(+)\dagger}(E)[U'(E) - U'^{\dagger}(E)]\bar{\Omega}^{(+)}(E), \qquad (3.30)$$

where $\bar{\Omega}^{(+)}(E)$ is defined by

$$\bar{\Omega}^{(+)}(E) = 1 + G_0^{(+)}(E)U'(E)\bar{\Omega}^{(+)}(E). \qquad (3.31)$$

The physical elastic scattering amplitude, however, is given in terms of $P\tilde{T}(E)P$ which is related to $T'_{el}(E)$ by

$$P\tilde{T}(E)P = \left(\frac{A}{A-1}\right) T'_{el}(E). \qquad (3.32)$$

If we multiply Eq. (3.30) by $[A/(A-1)]$, we have the unitarity relation for $P\tilde{T}(E)P$ given by

$$P\tilde{T}(E)P - P\tilde{T}^{\dagger}(E)P = -2\pi i \left(\frac{A-1}{A}\right) P\tilde{T}^{\dagger}(E)P\delta(E - h_0) \times P\tilde{T}(E)P + \left(\frac{A}{A-1}\right)\bar{\Omega}^{(+)\dagger}(E) \times [U'(E) - U'^{\dagger}(E)]\bar{\Omega}^{(+)}(E). \qquad (3.33)$$

If we now take the forward-scattering matrix element of this equation, we find the "optical theorem"

$$\sigma_{\text{total}} = \frac{4\pi}{k_0} \operatorname{Im} f_{k_0}(0) = -(4\pi)^2 \frac{\pi}{2k_0} \operatorname{Im}\langle \vec{k}_0 \Phi_A | \tilde{T}(E) | \vec{k}_0 \Phi_A \rangle = \frac{A-1}{A}\sigma_{\text{elastic}} + R(k_0), \qquad (3.34)$$

where $R(k)$ is given by

$$R(k_0) = -(4\pi)^2 \frac{\pi}{4ik_0}\frac{A}{A-1}\{\langle \vec{k}_0 \Phi_A | \bar{\Omega}^{(+)\dagger}(E)[U'(E) - U'^{\dagger}(E)] \times \bar{\Omega}^{(+)}(E) | \vec{k}_0 \Phi_A \rangle\}. \qquad (3.35)$$

This is not a particularly convenient form for an "optical theorem" to take. The difficulty is that one is *not* able separately to identify the two terms on the right-hand side of Eq. (3.33) with elastic scattering and with nonelastic scattering, respectively. It is clear that this came about because $T'_{el}(E)$ satisfied a Lippmann-Schwinger equation while the physical elastic scattering operator is given by $PT(E)P = [A/(A-1)]T'_{el}(E)$. Comparison of Eq. (3.34) with Eq. (3.27) allows one to make the identification of $R(k_0)$ as

$$R(k_0) = \frac{1}{A}\sigma_{\text{elastic}} + \sigma_{\text{nonelastic}}. \qquad (3.36)$$

We may pursue this point further by noting that the theory of KMT may be rewritten in a form which identifies the "true" optical potential, that is that potential which when inserted in a one-body Lippmann-Schwinger equation, will generate $\tilde{T}(E)$. This can be accomplished if we write Eq. (2.15) as

$$P\tilde{T}(E)P = P\tilde{U}(E)P + \frac{A-1}{A}P\tilde{U}(E)PG_0^{(+)}(E)P\tilde{T}(E)P, \qquad (3.37)$$

where $\tilde{U}(E)$ is defined by

$$\tilde{U}(E) = \frac{A}{A-1}U'(E), \qquad (3.38)$$

with $U'(E)$ given in Eq. (2.12). Equation (3.37) may be written

$$P[1 + \frac{1}{A}P\tilde{U}(E)PG_0^{(+)}(E)]P\tilde{T}(E)P = P\tilde{U}(E)P + P\tilde{U}(E)PG_0^{(+)}(E)P\tilde{T}(E)P. \qquad (3.39)$$

Thus Eq. (3.38) may now be rewritten as

$$P\tilde{T}(E)P = P\tilde{V}_{\text{eff}}P + P\tilde{V}_{\text{eff}}PG_0^{(+)}(E)P\tilde{T}(E)P, \qquad (3.40)$$

where the effective potential $P\tilde{V}_{\text{eff}}P$ is given by

$$P\tilde{V}_{\text{eff}}P = P\frac{1}{P[1 + (1/A)P\tilde{U}(E)PG_0^{(+)}(E)]P}P\tilde{U}(E)P. \qquad (3.41)$$

From this equation one may identify an optical potential $\langle \vec{k}' | \tilde{V}_{opt} | \vec{k} \rangle$ defined by

$$\langle \vec{k}' | \tilde{V}_{opt} | \vec{k} \rangle \equiv \langle \vec{k}' \Phi_A | \tilde{V}_{eff} | \vec{k} \Phi_A \rangle . \tag{3.42}$$

An integral equation which relates $\tilde{V}_{opt}$ to $\tilde{U}(E)$ can then be found from Eq. (3.42),

$$\langle \vec{k}' | \tilde{V}_{opt} | \vec{k} \rangle = \langle \vec{k}' \Phi_A | \tilde{U}(E) | \vec{k} \Phi_A \rangle - \frac{1}{A} \int d\vec{k}'' \langle \vec{k}' \Phi_A | \tilde{U}(E) | \vec{k}'' \Phi_A \rangle \times \frac{1}{E - E_{k''} + i\eta} \langle \vec{k}'' | \tilde{V}_{opt} | \vec{k} \rangle , \tag{3.43}$$

where $\tilde{U}(E)$ is defined in Eq. (3.38).

Since $P\tilde{T}(E)P$ is related to $P\tilde{V}_{eff}P$ by a Lippmann-Schwinger equation, we have immediately

$$P\tilde{T}(E)P - P\tilde{T}^{\dagger}(E)P = -2\pi i P\tilde{T}^{\dagger}(E)P\delta(E - h_0) \times P\tilde{T}(E)P + P\tilde{\Omega}^{(+)\dagger}(E)P(V_{eff} - V_{eff}^{\dagger})P\tilde{\Omega}^{(+)}(E)P, \tag{3.44}$$

where $\tilde{\Omega}^{(+)}(E)$ is given by

$$\tilde{\Omega}^{(+)}(E) = 1 + G_0^{(+)}(E)PV_{eff}P\Omega^{(+)}(E) . \tag{3.45}$$

The expression for $P\tilde{T}(E)P - P\tilde{T}^{\dagger}(E)P$ in Eq. (3.44) should be compared with the expression given in Eq. (3.33). In Eq. (3.44), one may now take the forward-scattering matrix element and derive the optical theorem

$$\begin{aligned} \sigma_{total} &= \frac{4\pi}{k_0} \operatorname{Im} f_{k_0}(0) \\ &= -(4\pi)^2 \frac{\pi}{2k_0} \operatorname{Im} \langle \vec{k}_0 \Phi_A | \tilde{T}(E) | \vec{k}_0 \Phi_A \rangle \\ &= \sigma_{elastic} - (4\pi)^2 \frac{\pi}{4ik_0} \\ &\quad \times \langle \vec{k}_0 \Phi_A | \tilde{\Omega}^{(+)\dagger}(E) [\tilde{V}_{eff} - \tilde{V}_{eff}^{\dagger}] \tilde{\Omega}^{(+)}(E) | \vec{k}_0 \Phi_A \rangle , \end{aligned} \tag{3.46}$$

from which we may make the identification

$$\sigma_{nonelastic} = (4\pi)^2 \frac{\pi}{4ik_0} \times \langle \vec{k}_0 \Phi_A | \tilde{\Omega}^{(+)\dagger}(E) [\tilde{V}_{eff} - \tilde{V}_{eff}^{\dagger}] \tilde{\Omega}^{(+)}(E) | \vec{k}_0 \Phi_A \rangle . \tag{3.47}$$

The unitarity relations given in Eqs. (3.33) and (3.46) are exact. One does not, however, wish to calculate $P\tilde{T}(E)P$ exactly, but rather to approximate the potential $U'(E)$ defined in Eq. (2.12) and then to calculate the corresponding approximation to $P\tilde{T}(E)P$. For example, in KMT it is noted that the leading term for $U'(E)$ at high energies arises from dropping the Q-space contributions to $U'(E)$, i.e.,

$$U'(E) \approx (A-1)t_{01}(E) . \tag{3.48}$$

It would be convenient if the unitarity relations could be used as a guide to the validity and applicability of an approximation to $U'(E)$. For example, one would like to know exactly which inelastic channels are being ignored by the approximation in Eq. (3.48). The form of Eq. (3.33), in which a piece of the elastic scattering cross section is contained in each of the two terms on the right-hand side, does not provide a convenient method for identifying the physical implications of an approximation to $U'(E)$. Furthermore, Eqs. (3.45) and (3.46) provide unitarity relations in terms of the potential $\tilde{V}_{eff}$ which is related to $U'(E)$ via an integral equation. This is also inconvenient. Thus one may conclude that although in the theory of KMT one may derive unitarity relations, the resulting relations are not especially helpful in understanding the physical implications of approximations to the theory.

In the next section, we shall examine an alternate approach to the replacement of a potential by a two-body scattering matrix in the many-body scattering theory. This approach relies on the truncation of the full Hilbert space to include the space of the elastic scattering states plus all single-particle single-hole excitations of the target. Since the problem may be treated exactly in this truncated space, unitarity is treated in a consistent manner. This approach has been applied to the problem of an incident nucleon scattering from a target which is composed of A correlated and identical nucleons in Ref. 5. In order to understand more clearly the treatment of unitarity in this approach, the simple-model problem of a distinguishable nucleon scattering from A distinguishable, uncorrelated nucleons is examined in the next section. It is shown that in this approach unitarity may be used as a guide to the validity of the impulse approximation.

IV. OFF-SHELL EFFECTS IN THE IMPULSE APPROXIMATION

In this section we shall present a simple derivation of the impulse approximation and discuss the implications which the unitarity relations, developed in the previous section, have for the resulting approximation. The t matrix which appears in the impulse approximation will be identified through the use of an equation which is formally similar to the Low equation. This t matrix will also be shown to be a modified Bethe-Goldstone reaction matrix. The procedure used here is that of Ref. 5. However, in Ref. 5 the inclusion of correlation effects and the need to maintain complete antisymmetrization necessarily leads to a complicated development. Here, we should like to concentrate our attention on the identification of the appropriate t matrix to be used in the impulse approximation. We shall thus examine the simplest case, the case of a distinguishable particle incident on A distinguishable particles bound in a "nucleus." An alternative derivation to the one presented here would be to generate the independent-particle approximation to the very general T matrix given in Ref. 6. The case where the target particles are treated as identical fermions is a simple generalization of the following development and is given in the Appendix.

We recall that the target ground state is represented by the A-body state vector $|\Phi_A\rangle$. The target is taken to be in an eigenstate of the Hamiltonian H_A with eigenvalue E_A,

$$H_A|\Phi_A\rangle = E_A|\Phi_A\rangle . \tag{4.1}$$

Here we consider the target to consist of A distinguishable particles occupying fully A states, $|b_i\rangle$. The state consisting of a particle in a plane-wave state (labeled by zero) incident upon the target will be represented by $|\vec{k}_0\ \Phi_A\rangle$. This state is clearly an eigenstate of the Hamiltonian $H_0 = h_0 + H_A$,

$$H_0|\vec{k}_0\ \Phi_A\rangle = (h_0+H_A)|\vec{k}_0\ \Phi_A\rangle$$
$$= (E_{\vec{k}_0}+E_A)|\vec{k}_0\ \Phi_A\rangle . \tag{4.2}$$

Use of the projection operator P, defined by

$$P \equiv \int d\vec{k}_0 |\vec{k}_0\ \Phi_A\rangle\langle\vec{k}_0\ \Phi_A| , \tag{4.3}$$

lead earlier to the optical potential,

$$\langle\vec{k}_0'|V_{\rm opt}(E)|\vec{k}_0\rangle = \left\langle\vec{k}_0'\ \Phi_A\right|\sum_i v_{0i}\left|\vec{k}_0\ \Phi_A\right\rangle + \left\langle\vec{k}_0'\ \Phi_A\right|\left(\sum_i v_{0i}\right)Q\frac{1}{E-QHQ+i\epsilon}Q\left(\sum_i v_{0i}\right)\left|\vec{k}_0\ \Phi_A\right\rangle. \tag{4.4}$$

In order to replace the potential operator in this equation by a T matrix, we must explicitly include certain of the eigenstates of QHQ. If we are interested in the leading term in a hole-line expansion, the appropriate states to consider are those which arise from a single-hole state being created in the target. We thus define our model problem as the truncation of the Q space to include only the space of the incident-particle and a single-particle-hole excitation of the nucleus. This approach is thus very similar to treating the Watson multiple-scattering series as a three-body problem.[3]

The $A-1$ particle state vector which corresponds to particle j being removed from the state $|b_j\rangle$ in the target, we shall denote as $|\Phi_{A-1}^j\rangle$. We may now define a projection operator Q_j which projects onto the space where the $A-1$ particles $i\neq j$ are in the state $|\Phi_{A-1}^j\rangle$ and particle j is *not* in the state $|b_j\rangle$,

$$Q_j = 1_0(q_j|\Phi_{A-1}^j\rangle\langle\Phi_{A-1}^j|q_j) , \tag{4.5}$$

where q_j is defined by

$$q_j = 1_j - |b_j\rangle\langle b_j| , \tag{4.6}$$

and 1_0 (or 1_j) stands for the identity operator in the space of particle zero (particle j).

We may then write

$$QHQ \approx \left(\sum_j Q_j\right)H\left(\sum_{j'} Q_{j'}\right) . \tag{4.7}$$

Moreover, H may be written as

$$H = h_0 + h_j + v_{0j} + \sum_{\substack{i=1\\ i\neq j}} v_{0i} + \sum_{i=1} v_{ji} + H_{A-1}^j , \tag{4.8}$$

where H_{A-1}^j is defined by

$$H_{A-1}^j \equiv \sum_{i\neq j}\frac{P_i^2}{2m} + \tfrac{1}{2}\sum_{i\neq j}\sum_{k\neq j} v_{ik} . \tag{4.9}$$

We now further assume that the matrix $\langle\Phi_{A-1}^{j'}|H|\Phi_{A-1}^j\rangle$ is diagonal in j' and j,

$$\langle\Phi_{A-1}^{j'}|H|\Phi_{A-1}^j\rangle = (E_A+E_{b_j})\delta_{j'j} . \tag{4.10}$$

With this assumption, only the diagonal terms in j and j' in Eq. (4.7) survive,

$$QHQ \approx \left(\sum_j Q_j\right)H\left(\sum_j Q_{j'}\right) = \sum_j Q_jHQ_j . \tag{4.11}$$

The off-diagonal terms represent higher-order effects (in a systematic hole-line expansion) which will not interest us here.

With this approximation for QHQ, Eq. (4.4) for the optical potential becomes

$$\langle\vec{k}_0'|V_{\rm opt}(E)|\vec{k}_0\rangle = \left\langle\vec{k}_0'\Phi_A\right|\sum_i v_{0i}\left|\vec{k}_0\Phi_A\right\rangle + \left\langle\vec{k}_0'\Phi_A\right|\left(\sum_i v_{0i}\right)\sum_j Q_j\frac{1}{E-Q_jHQ_j+i\eta}Q_j\left(\sum_i v_{0i}\right)\left|\vec{k}_0\Phi_A\right\rangle . \tag{4.12}$$

This formula may be reduced to the form of the impulse approximation if we explicitly construct in the Q_j space the eigensolutions of Q_jHQ_j. These eigenvectors we shall denote by $Q_j|\tilde{\psi}_E\rangle$,

$$(Q_jHQ_j)Q_j|\tilde{\psi}_E\rangle = EQ_j|\tilde{\psi}_E\rangle . \tag{4.13}$$

The vector $Q_j|\tilde{\psi}_E\rangle$ may be expanded as

$$Q_j|\tilde{\psi}_E\rangle = q_j|u_{j,E}\rangle|\Phi^j_{A-1}\rangle , \tag{4.14}$$

where, when the vector $q_j|u_{j,E}\rangle$ is expressed in coordinate space, it will be a function of $\vec{r}_0$ and $\vec{r}_j$. Thus, Eq. (4.13) implies the equation for $q_j|u_{j,E}\rangle$ given by

$$(E-E_A-E_{b_j}-h_0-h_j-v_{0j}-U_j-U_0)q_j|u_{j,E}\rangle = 0, \tag{4.15}$$

where the potential U_i is defined for $i=j$ and $i=0$ by

$$U_i(\vec{r}_i)=\int d^3r_1\cdots d^3r_{j-1}d^3r_{j+1}\cdots d^3r_A\sum_{k\neq j} v(\vec{r}_i-\vec{r}_k)|\langle\vec{r}_1\cdots\vec{r}_{j-1}\vec{r}_{j+1}\cdots\vec{r}_A|\Phi^j_{A-1}\rangle|^2 . \tag{4.16}$$

We may now rewrite the optical potential in Eq. (4.12) as

$$\langle\vec{k}_0'|v_{\rm opt}(E)|\vec{k}_0\rangle=\sum_j\int d^3k_jd^3k_j'\langle b_j|\vec{k}_j'\rangle\langle\vec{k}_j|b_j\rangle\left\{\langle\vec{k}_0'\vec{k}_j'|v_{0j}|\vec{k}_0\vec{k}_j\rangle + \int_{E_A+E_{b_j}}^{\infty} dE'\langle\vec{k}_0'\vec{k}_j'|v_{0j}\,q_j|u_{j,E'}\rangle\frac{1}{E-E'+i\epsilon}\langle u_{j,E'}|q_jv_{0j}|\vec{k}_0\vec{k}_j\rangle\right\} . \tag{4.17}$$

At this point, it is convenient to choose an energy scale such that $E_A=0$ and to introduce the states $q_j|\tilde{u}_{j,E}\rangle$ which are solutions to the equation

$$(E-h_0-h_j-v_{0j}-U_j-U_0)q_j|\tilde{u}_{j,E}\rangle=0 . \tag{4.18}$$

The optical potential will then become

$$\langle\vec{k}_0'|V_{\rm opt}(E)|\vec{k}_0\rangle=\sum_j\int d^3k_jd^3k_j'\langle b_j|\vec{k}_j'\rangle\langle\vec{k}_j|b_j\rangle\left\{\langle\vec{k}_0'\vec{k}_j'|v_{0j}|\vec{k}_0\vec{k}_j\rangle + \int_0^{\infty} dE'\langle\vec{k}_0'\vec{k}_j'|v_{0j}\,q_j|\tilde{u}_{j,E'}\rangle\frac{1}{E-E_{b_j}-E'+i\epsilon}\langle\tilde{u}_{j,E'}|q_jv_{0j}|\vec{k}_0\vec{k}_j\rangle\right\} . \tag{4.19}$$

In this form, the term in the curly brackets in Eq. (4.19) is similar in form to the two-body Low equation for a T matrix. We are thus led to define $t_j(E)$ by

$$\langle\vec{k}_0'\vec{k}_j'|t_j(E)|\vec{k}_0\vec{k}_j\rangle\equiv\langle\vec{k}_0'\vec{k}_j|v_{0j}|\vec{k}_0\vec{k}_j\rangle+\int_0^{\infty}dE'\langle\vec{k}_0'\vec{k}_j'|v_{0j}q_j|u_{j,E'}\rangle\frac{1}{E-E_{b_j}-E'+i\eta}\langle u_{j,E'}|v_{0j}q|\vec{k}_0\vec{k}_j\rangle . \tag{4.20}$$

The optical potential then may be written

$$\langle\vec{k}_0'|V_{\rm opt}(E)|\vec{k}_0\rangle=\sum_j\int d^3k_jd^3k_j'\langle b_j|\vec{k}_j'\rangle\langle\vec{k}_j|b_j\rangle\langle\vec{k}_0'\vec{k}_j|t_j(E)|\vec{k}_0\vec{k}_j\rangle . \tag{4.21}$$

The T matrix defined in Eq. (4.20) is the T matrix which occurs naturally in the reduction of the many-body problem to the form of the impulse approximation as given in Eq. (4.21).

The T matrix may be written in operator notation by

$$t_j(E)=v_{0j}+v_{0j}G^j(E)v_{0j} , \tag{4.20}$$

where $G^j(E)$ is defined by

$$G^j(E)\equiv q_j\frac{1}{E-q_j(h_0+h_j+U_j+U_0+v_{0j})q_j}q_j . \tag{4.22}$$

We may derive an integral equation for $t_j(E)$ if we note that $G^j(E)$ satisfies

$$G^j(E)=g^j(E)q_jv_{0j}q_jG^j(E) , \tag{4.23}$$

where $g^j(E)$ is defined by

$$g^j(E) = q_j[(E - q_j(h_0 + h_j + U_j + U_0)q_j]^{-1}q_j . \quad (4.24)$$

Substitution of Eq. (4.23) into Eq. (4.21) immediately yields an integral equation for $t_j(E)$,

$$\begin{aligned} t_j(E) &= v_{0j} + v_{0j}[g^j(E) + g^j(E)q_j v_{0j} q_j G^j(E)]v_{0j} \\ &= v_{0j} + v_{0j} g^j(E)[v_{0j} + v_{0j} G^J(E) v_{0j}] \\ &= v_{0j} + v_{0j} g^j(E) t_j(E) . \end{aligned} \quad (4.25)$$

From Eq. (4.25) one sees that $t_j(E)$ is a modified Bethe-Goldstone reaction matrix in which the Pauli principle is imposed by the operators q_j in Eq. (4.24) and the potentials U_0 and U_j distort the particle propagation in intermediate states.

Thus we have two expressions for $t_j(E)$, Eqs. (4.20) and (4.25). As we shall see, the form in Eq. (4.20) lends itself more readily to a discussion of unitarity. The integral equation Eq. (4.25) is, however, much more suitable to numerical analysis. It is also interesting to note that the solution for $t_j(E)$ involves the solution of a three-body problem. This is most readily seen in the equation for $q_j|u_{j,E}\rangle$, Eq. (4.15), where one sees that the problem is one of dealing with two mutually interacting particles both of which are in a potential field. The full implications of the three-body nature of this problem have not yet been investigated.

There are several effects which differentiate the T matrix defined in Eq. (4.22) from the free two-particle T matrix. The first of these effects is the replacement of the on-shell energy $\epsilon_{k_0} + \epsilon_{k_j}$ by the energy $\epsilon_{k_0} - E_{b_j}$. In a typical nucleus, this is a shift in energy of approximately 40 MeV. This effect can, of course, be ignored if the two-nucleon T matrix does not change substantially when the energy is varied by 40 MeV. This is certainly the case for a nucleon scattering from an uncorrelated nucleus above several hundred MeV. For elastic pion scattering from a nucleus in the region of the pion (3, 3) resonance, however, the T matrix is a rapidly varying function of the energy. In this case, one may not ignore this energy shift as is shown quantitatively in Ref. 7.

The second effect which distinguishes the T matrix of Eq. (4.20) is the presence of the operator q_j. For the case of an antisymmetrized target, q_j is replaced by an operator which excludes the recoil-target particle from all of the space which is occupied by the nucleons in the target (as is demonstrated in the Appendix). The Pauli principle has a well-known and important effect in nuclear structure calculations.[8] Its importance has also been studied in pion-nucleus elastic scattering.[9]

The third effect which distinguishes $t_j(E)$ from the free two-body amplitude is the presence of the distorting potentials $U_0(\vec{r}_0)$ and $U_j(\vec{r}_j)$. These potentials represent the fact that particles 0 and j are scattering in the presence of the remaining particles and thus must propagate in intermediate states which are distorted. This effect is also familiar in nuclear structure calculations where it appears as the potential which generates the appropriate intermediate state spectrum in a Brueckner-Hartree-Fock calculation. The sensitivity of the calculational results to the choice of the potential $U_i(r_i)$ if particle 0 and j are both nucleons has been studied.[10] The choice is not critical because the strong short-range repulsion of the nucleon-nucleon force is such that the important range of intermediate momenta in a nucleon-nucleon collision is quite high,[11] and the distortions due to $U_i(r_i)$ are thus not large. For the scattering of a pion from a nucleus in the region of the (3, 3) resonance, the intermediate pion and nucleon momenta are not high, and thus the effects of the distorting potentials could be large. It has been suggested[12] that because the pion-nucleus interaction is so absorptive in the region of the (3, 3) resonance, $U(r)$ for the pion should be treated self-consistently. One should notice that for a singular two-body interaction, the potentials $U_i(r_i)$ as defined in Eq. (4.16) are infinite. This infinity may be canceled by keeping certain two-hole terms in the expansion of Q.

Finally, we should like to discuss the implications of the unitarity relations of Sec. II for the impulse approximation as defined in Eqs. (4.19) and (4.20). According to Eq. (3.17) the anti-Hermitian part of the optical potential is given by

$$\langle\vec{k}_0'|[V_{\text{opt}}(E) - V_{\text{opt}}^\dagger(E)]|\vec{k}_0\rangle = -2\pi i\left\langle\vec{k}_0'\Phi_A\right|\left(\sum_i v_{0i}\right)\sum_j Q_j\delta(E - Q_jHQ_j)Q_j\left(\sum_n v_{0n}\right)\left|\vec{k}_0\Phi_A\right\rangle. \quad (4.26)$$

From our explicit construction of the eigenstates of Q_jHQ_j we may infer the types of states which may contribute to the anti-Hermitian part of $V_{\text{opt}}(E)$ and thus contribute to the absorption present in the impulse approximation. The complete set of states for $q_j|u_{j,E}\rangle$ may be characterized by the nature of the incident wave boundary condition imposed. These states are: (1) particle zero incident on particle j which is bound by potential U_j (not in the state $|b_j\rangle$, however, which is excluded by q_j),

(2) particle j incident on particle zero which is bound by potential U_0, (3) particle zero and particle j bound together by the potential v_{0j} and incident on the potentials U_0 and U_j, or (4) particles zero and j unbound and incident on the potentials U_0 and U_j. As these are the only inelasticities available in the truncated Hilbert space, and as the derivation of the impulse approximation requires this truncation, a necessary criterion for the validity of the impulse approximation is that the dominant inelastic channels at a given energy are reasonably included in the inelasticities which arise from the insertion of the above described solutions for $|u_{j,E}\rangle$ into Eq. (4.26). In particular, it is well known that the impulse approximation is valid in the region where quasielastic scattering is the dominant inelastic channel. We see here that the excitation of a single target particle to a bound excited state is also included in the inelasticities present in the impulse approximation. Most importantly, we have seen that the impulse approximation as defined in Eqs. (4.19) and (4.20) treats unitarity in a consistent way within the truncated Hilbert space.

APPENDIX

In this Appendix, the results of Sec. IV are generalized to the case where the target particles are identical, but the incident particle remains distinguishable. The results of this Appendix will thus be of particular interest to the study of pion-nucleus elastic scattering. The more general case of an incident particle which is also identical to the target particles has been discussed in Ref. 5. There, the target was also considered to contain correlations.

We begin by defining $\eta_{b_i}^\dagger$ as the fermion creation operator which creates a particle in the bound state b_i. The target wave function $|\Phi_A\rangle$ will be taken to be a single Slater determinant, which can be written

$$|\Phi_A\rangle=\prod_{i=1}\eta_{b_i}^\dagger|0\rangle\,, \tag{A1}$$

where $|0\rangle$ is the vacuum. The creation operator for the incident particle in a plane-wave state of momentum $\vec{k}$ we shall denote by $a_{\vec{k}}^\dagger$. We may again define the projection operator P as in Eq. (4.3) by

$$P\equiv\int d^3k a_{\vec{k}}^\dagger|\Phi_A\rangle\langle\Phi_A|a_{\vec{k}}\equiv\int d^3k|\vec{k}\ \Phi_A\rangle\langle\vec{k}\ \Phi_A|\,. \tag{A2}$$

If Q is then defined as the complement of P by

$$Q=1-P\,, \tag{A3}$$

the definition of the optical potential given in Eq. (4.4) still holds.

As before, we approximate Q by keeping only those states which are a single-particle-hole excitation of the target. We thus define Q_j by

$$Q_j=\int d^3kd^3k' a_{\vec{k}}^\dagger\eta_{\vec{k}'}^\dagger\eta_{b_j}|\Phi_A\rangle\langle\Phi_A|\eta_{b_j}^\dagger\eta_{\vec{k}'}a_{\vec{k}}\,. \tag{A4}$$

The fermion creation operator $\eta_{\vec{k}}^\dagger$ creates a target particle in the state $|\chi_{\vec{k}}\rangle$ which is orthogonal to the A bound states which are occupied in the target. The "orthogonality scattering" states of Ref. 5 represent an explicit construction of such states. The operator Q may then be approximated by

$$Q\approx\sum_j Q_j\,. \tag{A5}$$

As in Eq. (4.11), we now assume that QHQ is approximately diagonal in the hole index j,

$$QHQ=\Big(\sum_i Q_i\Big)H\Big(\sum_j Q_j\Big)\approx\sum_j Q_jHQ_j\,. \tag{A6}$$

The eigenstates of QHQ, denoted by $Q|\bar{\psi}_E\rangle$, may again be expanded in terms of the particle-hole states,

$$Q|\bar{\psi}_E\rangle=\sum_j\int d^3kd^3k'\langle\vec{k}\vec{k}'|u_{j,E}\rangle a_{\vec{k}}^\dagger\eta_{\vec{k}'}^\dagger\eta_{b_j}|\Phi_A\rangle\,. \tag{A7}$$

The equation

$$[QHQ-QEQ]Q|\bar{\psi}_E\rangle=0 \tag{A8}$$

then yields an equation for $\langle\vec{k}\vec{k}'|u_{j,E}\rangle$ given by

$$\begin{aligned}(E-\epsilon_k-\epsilon_{k'}-E_{b_j})\langle\vec{k}\vec{k}'|u_{j,E}\rangle&-\int d^3p\langle\vec{k}|U_0|\vec{p}\rangle\langle\vec{p}\vec{k}'|u_{j,E}\rangle\\&-\int d^3p'\langle\chi_{\vec{k}'}|U|\chi_{\vec{p}'}\rangle\langle\vec{k}\vec{p}'|u_{j,E}\rangle\\&-\int d^3pd^3p'\langle\vec{k}\chi_{\vec{k}'}|v_0|\vec{p}\chi_{\vec{p}'}\rangle\langle\vec{p}\vec{p}'|u_{j,E}\rangle=0\,,\end{aligned} \tag{A9}$$

where v_0 is the interaction between the incident particle and the target particles, and the following definitions have been used:

$$\langle\vec{k}|U_0|\vec{p}\rangle\equiv\sum_j\langle\vec{k}b_j|v_0|\vec{p}b_j\rangle\,, \tag{A10}$$

$$\langle\chi_{\vec{k}'}|U|\chi_{\vec{p}'}\rangle\equiv\langle\chi_{\vec{k}'}|h_0|\chi_{\vec{p}'}\rangle+\sum_j\langle\chi_{\vec{k}'}b_j|v|\chi_{\vec{p}'}b_j\rangle_A-\epsilon_{k'}\delta(\vec{k}'-\vec{p}') \tag{A11}$$

and

$$E_{b_j}\equiv\langle b_j|h_0|b_j\rangle+\sum_{l=1}^{A}\langle b_jb_l|v|b_jb_l\rangle_A\,. \tag{A12}$$

The matrix element $\langle b_j b_l | v | b_j b_l \rangle_A$ is the antisymmetrized matrix element of the two-body interaction between the target particles.

This equation for $|u_{j,E}\rangle$, Eq. (A9), is quite similar to Eq. (4.14). The only differences are: First, the Pauli principle restricts the target particles in $|u_{j,E}\rangle$ in Eq. (A7) to the space which is orthogonal to *all* of the states occupied in the target; and, secondly, the potential U in Eq. (A11) is more complicated than U_j in Eq. (4.15) due to the identity of particle j with the other target particles. The correct off-shell T matrix to be used in the impulse approximation is then given by Eq. (4.17) with the wave function $q_j|u_{j,E}\rangle$, as given by Eq. (4.14), replaced by $|u_{j,E}\rangle$, as given in Eq. (A9).

It is interesting to note that in the high-energy limit where one may drop the subscript j on the T matrix of Eq. (4.17), the identity of the target particles will allow one to replace $t(E)$ by $At(E)$ as was done in the KMT approach.

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