PHYSICAL REVIEW C VOLUME 9, NUMBER 4 APRIL 1974

Intermediate-energy pion-nucleus scattering assuming a separable fixed-scatterer pion-nucleon interaction*

M. G. Piepho and G. E. Walker
Department of Physics, Indiana University, Bloomington, Indiana 47401
(Received 27 September 1973)

A formal discussion of the optical potential due to Foldy and Walecka is extended to include spin and isospin degrees of freedom and projectile relativistic kinematics. The formalism is applied to intermediate-energy pion-nucleus elastic scattering. Assuming an infinitely heavy nucleon mass, we obtain a fixed-scatterer separable-potential parametrization of the two-body pion-nucleon scattering data in the laboratory system. This effective two-body potential is used in a consistent manner as microscopic input in the many-body pion-nucleus elastic scattering problem. Using the approximations adopted by Foldy and Walecka, we obtain a pion-nucleus optical potential which is then applied to study pion elastic scattering from ^{12}C, ^{16}O, ^{28}Si, ^{32}S, and ^{40}Ca. Total cross-section and angular-distribution predictions for pion kinetic energies $\sim$70–280 MeV are presented. Good agreement is obtained with the available experimental data on ^{12}C and ^{32}S.

[NUCLEAR REACTIONS ^{12}C, ^{16}O, ^{28}Si, ^{32}S, ^{40}Ca; calculated pion-nucleus elastic and total reaction σ, elastic $\sigma(\theta)$. Separable pion fixed-nucleon interaction. Extended Foldy-Walecka formalism.]

1. INTRODUCTION

The current theoretical interest in intermediate-energy pion-nucleus interactions is in anticipation of the high-quality data to be forthcoming from the nearly completed high beam current and excellent energy resolution medium-energy facilities such as the Los Alamos Meson Physics Facility (LAMPF) and Tri-University Meson Facility (TRIUMF).[1] The experimental programs at these pion factories will offer new horizons in both the quality and type of pion-nucleus data available to the theorist for interpretation.

The pion is an alternative to the nucleon for studying the nucleus via an elementary strongly interacting probe. Thus, a theoretical investigation of the scattering of a pion from a complex nucleus (with its associated intrinsic pion field) and its relation to pion-nucleon scattering is of considerable importance. In this first paper we devote our attention to the description of pion-nucleus elastic scattering from light closed-shell nuclei.

The development of well understood pion-nucleus optical potentials that correctly describe elastic scattering is important for several reasons. Given the usual situation of an optical potential whose parameters have been fitted to properly describe elastic scattering from a given nucleus, one can relatively easily make predictions regarding elastic scattering from other nuclei. Moreover, the distorted waves associated with a given optical potential provide valuable input to currently fashionable inelastic scattering and reaction theories such as the distorted-wave Born approximation (DWBA). Thus, to describe the inelastic scattering of pions leaving the nucleus in a particular final state or to study pion production (above or below threshold) from nuclei by protons or light ions, one must first have a reasonably well understood description of elastic scattering. In order for the optical potential to be "well understood" or "trustworthy," one has in mind, for example, a clear relation between the pion-nucleus optical potential and the microscopic pion-nucleon interaction. Usually this connection is made via some multiple-scattering theory which after certain simplifying approximations allows one to employ the on-shell pion-nucleon data (the π-nucleon t matrix or a derived π-nucleon potential) in obtaining the equivalent one-body or optical potential. Thus, if one is to have some confidence in the distorted waves generated from a given optical potential (or if an understanding of the many-body problem in terms of more microscopic interactions is desired), it is clearly necessary that a given optical potential not only fit the elastic data, but the validity of the simplifying assumptions attendant to the particular "derivation" of the optical potential be clearly stated and insofar as is possible be well understood. This is particularly important in the case of pion-nucleus scattering where the $(\frac{3}{2},\frac{3}{2})$ π-nucleon resonance probably plays a dominant role at energies near the resonance (~200 MeV). The resonance could be shifted, or otherwise modified in the many-body environment and thus some detailed studies of the

many-body effects seem desirable.

Our approach[2] to this problem has been influenced by an investigation due to Foldy and Walecka[3] (FW) on the theory of the optical potential. Assuming a separable microscopic interaction, FW have considered the scattering of a projectile from a many-body system of *fixed* scatterers. One of the basic approximations in their theory is that the projectile energy is high enough so that closure may be used on the target. Our procedure in studying pion-nucleus scattering has been to generalize the technique used by FW to include spin and isospin degrees of freedom and complex separable potentials and to use the correct relativistic energy-momentum relationship for the pion. We have worked backwards from the π-nucleon laboratory phase shifts to obtain the π-nucleon microscopic separable potential assuming an infinitely heavy nucleon. Details of the procedure used are presented in the next section.

In Sec. 3 we outline the multiple-scattering formalism used to generate the equivalent one-body potential or optical potential for describing elastic scattering. In Sec. 4 we present and discuss our results. Appendixes A and B contain a further elaboration of some details of the calculations.

2. PARAMETRIZATION OF PION-NUCLEON ELASTIC SCATTERING

In this section we parametrize the pion-nucleon elastic scattering data in a manner appropriate for the two-body input needed in the extended FW formalism. The basic requirements are (1) the two-body elastic data be represented by a separable potential and (2) the data be interpreted as though the pion scatters from a fixed (i.e., infinitely heavy) nucleon. In the formalism applied herein one considers the nucleon fixed in *both* the two-body and many-body problem. This approximation as well as the more complicated formulas obtained when target recoil is included has been discussed in detail elsewhere.[4]

If the nucleon is assumed infinitely heavy, the two-body lab, $(\pi\text{-}n)_{\text{lab}}$, and center-of-mass $(\pi\text{-}n)_{\text{c.m.}}$ systems become identical (similarly the π- nucleus lab and c.m. systems coincide). Of course, in reality the lab and c.m. systems do not coincide and it *does* make a difference from which frame one takes the data to obtain the basic two-body "fixed-scatterer" potential or T matrix. A detailed discussion is presented in Appendix A; however, we note here that we have used the pion-nucleon *laboratory* differential-cross-section data. Assuming this laboratory data resulted from a pion scattering from a fixed nucleon reduces the situation to a one-body problem and "laboratory" elastic scattering phase shifts may be determined. In the solution of the inverse scattering problem for separable potentials it is the "laboratory" phase shifts which the "fixed-scatterer" potential is required to fit. We view this procedure as a technique for obtaining a "pseudopotential" or definite prescription for going off-shell in the many-body problem that is consistent with the basic assumptions in the FW formalism.

We start by assuming that the following relativistic Schrödinger equation describes the interaction of the pion with an infinitely heavy (i.e., fixed or recoiless) nucleon in the laboratory system (spin and isospin variables are momentarily suppressed).

$$(-\hbar^2\nabla^2 + m_\pi{}^2)^{1/2}\psi_k(\vec{r}) + \int v(\vec{r},\vec{r}\,')\psi_k(\vec{r}\,')d\vec{r}\,' = E(k)\psi_k(\vec{r}), \tag{2.1}$$

where m_π is the pion rest mass, $\hbar k$ is the pion laboratory momentum, E is the pion laboratory energy, $v(\vec{r},\vec{r}\,')$ is a general nonlocal, spin- and isospin-dependent potential which is assumed to describe the pion-nucleon interaction, and $\vec{r}$ is the relative coordinate vector between the pion and fixed nucleon. In the case of a free pion, we have the usual relativistic energy-momentum relationship

$$E^2(k) = \hbar^2k^2 + m_\pi{}^2. \tag{2.2}$$

We assume the pion-nucleon interaction may be written as a particular type of spin- and isospin-dependent nonlocal separable potential

$$v(\vec{r},\vec{r}\,') = \sum_{\substack{ljT\\ j_z T_z}} 4\pi\lambda_{ljT}\, v_{ljT}(r)v_{ljT}(r')\,|\,ljj_zTT_z,\vec{r}\rangle\langle ljj_zTT_z,\vec{r}\,'|, \tag{2.3}$$

where

$$|\,ljj_zTT_z,\vec{r}\rangle \equiv \sum_{\substack{m_l m_s\\ m_t m_\tau}} (lm_l\tfrac{1}{2}m_s|\,l\tfrac{1}{2}jj_z)(lm_\tau\tfrac{1}{2}m_t|\,1\tfrac{1}{2}TT_z)Y_{lm_l}(\Omega_r)u_{m_s}u_{m_t}\eta_{m_\tau}, \tag{2.4}$$

$$\langle ljj_zTT_z,\vec{r}\,'| \equiv \sum_{\substack{m_l' m_s'\\ m_t' m_\tau'}} (lm_l'\tfrac{1}{2}m_s'|\,l\tfrac{1}{2}jj_z)(lm_\tau'\tfrac{1}{2}m_t'|\,1\tfrac{1}{2}TT_z)Y^*_{lm_l'}(\Omega_{r'})\eta^\dagger_{m_\tau'}u^\dagger_{m_t'}u^\dagger_{m_s'}. \tag{2.5}$$

In addition to a potential that depends on the relative coordinate vector between the pion and nucleon, the potential, Eq. (2.3), includes operators (u, η) that can change the nucleon's spin projection (s_z) and isospin projection (t_z) as well as the pion's isospin projection (τ_z). In the decomposition of the potential, operators in configuration space of rank l (l is the usual relative orbital angular momentum) are coupled to the nucleon spin operator u_{m_s}, to form an operator of rank j. Similarly, the isospin-dependent part of the potential is written, with the aid of a Clebsch-Gordan coefficient, as the vector coupling of the pion (η_{m_τ}) and nucleon (u_{m_t}) isospin operators to form an operator of rank T in isospace. The rank j and T operators are then employed in the total separable potential [see Eq. (2.5)] in a manner so that total isospin and total angular momentum are conserved in the two-body problem.

The *separable* form chosen for the spin and isospin operators is to write them as the product of column and row vectors defined in the appropriate space. Thus, for the nucleon u $(u^\dagger)$ is a two-component column (row) spinor and the spin operator is written as a combination of terms $u_{m_s} u^\dagger_{m_s}$, (note that the order cannot be changed). For the pion, η $(\eta^\dagger)$ is a three-component column (row) vector and the pion-isospin operator is a linear combination of terms $\eta_{m_\tau}\eta^\dagger_{m'_\tau}$. In general we have

$$u^\dagger_{m_s} u_{m_s'} = \delta_{m_s, m_s'}\,, \qquad \eta^\dagger_{m_\tau}\eta_{m_\tau'} = \delta_{m_\tau, m_\tau'}\,. \tag{2.6}$$

The u and η column vectors are, of course, also the usual functions designating the appropriate z projections of the pion and nucleon spin and isospin degrees of freedom. Thus, for a noninteracting system consisting of a pion with energy $E(k)$ and isospin projection τ_z, and a nucleon at rest with spin and isospin projections s_z and t_z, respectively, the appropriate wave function would be

$$\frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} u_{s_z} u_{t_z} \eta_{\tau_z}\,. \tag{2.7}$$

[In order to avoid confusion, whenever a spin space spinor is associated with the separable *operator* the subscripts m_s or m'_s are adopted (i.e., u_{m_s}) and when we refer to the actual two-component spin wave function associated with a target nucleon we employ the subscript s_z or s'_z (i.e., u_{s_z}). Similarly, when the column vector η (spinor u) defined in the *isospin space* of the pion (nucleon) is associated with the *operator* given in Eqs. (2.3)–(2.5), we use the subscript m_τ (m_t), while if it refers to the actual pion (nucleon) isospin wave function, the subscript τ_z (t_z) is adopted.]

We wish the scattering equation to reduce to independent equations for each total angular momentum (j) and isospin (T) channel. Furthermore, we want to recover, in each ljT channel, the simple relation between the potential v_{ljT} and the phase shift δ_{ljT} [see Eq. (2.21)]. To accomplish this latter reduction it is not sufficient to assume the potential is separable in configuration space only (and multiply the separable configuration space potential by variables such as $\vec{l}\cdot\vec{s}$) but, in fact, one must adopt a form [i.e., Eqs. (2.3)–(2.5)] which is also appropriately separable in the angular momentum and isospin operators.

The symbol λ_{ljT} appearing in Eq. (2.3) is a constant depending on the particular ljT channel. The configuration space operator $v_{ljT}(r)$ is in general complex and thus the potential given by Eq. (2.3) is not, in general, Hermitian. This is exactly what is desired since the pion-nucleon phase shifts will be complex in the medium-energy region and the non-Hermiticity of the potential simply insures the appropriate loss of flux from the elastic channel. When the energy of the pion-nucleon system is such that only the elastic channel is open (real phase shift) there will be no loss of flux caused by the potential (2.3).

The integral equation corresponding to Eq. (2.1), incorporating outgoing-wave boundary conditions and adopting Eq. (2.3) for the pion-nucleon potential, may be written ($\alpha \equiv s_z, t_z, \tau_z$)

$$\psi^\alpha_k(\vec{r}) = \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} u_{s_z} u_{t_z} \eta_{\tau_z} \sum_{s'_z t'_z \tau'_z} \int\int\int \frac{d\vec{k}}{(2\pi)^3}\, d\vec{r}\,'\, d\vec{r}\,'' \, u_{s'_z} u_{t'_z} \eta_{\tau'_z} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{E(t) - E(k) - i\epsilon} u^\dagger_{s'_z} u^\dagger_{t'_z} \eta^\dagger_{\tau'_z}$$

$$\times \sum_{\substack{ljT \\ j_z T_z}} 4\pi\lambda_{ljT} v_{ljT}(r') v_{ljT}(r'') \sum_{\substack{m_l m_s \\ m_t m_\tau}} (l m_l \tfrac{1}{2} m_s | l \tfrac{1}{2} j j_z)(1 m_\tau \tfrac{1}{2} m_t | 1 \tfrac{1}{2} T T_z) Y_{lm_l}(\Omega_{r'}) u_{m_s} u_{m_t} \eta_{m_\tau}$$

$$\times \sum_{\substack{m'_l m'_s \\ m'_t m'_\tau}} (l m'_l \tfrac{1}{2} m'_s | l \tfrac{1}{2} j j_z)(1 m'_\tau \tfrac{1}{2} m'_t | 1 \tfrac{1}{2} T T_z) Y^*_{lm'_l}(\Omega_{r''}) \eta^\dagger_{m'_\tau} u^\dagger_{m'_t} u^\dagger_{m'_s} \psi^\alpha_k(\vec{r}\,'')\,. \tag{2.8}$$

Now we define the T matrix as[5]

$$T^{\alpha'\alpha}_{\vec{k}'\vec{k}} \equiv \langle s_z' t_z' \tau_z' \vec{k}' | T | \vec{k}, s_z t_z \tau_z \rangle = \int\int d\vec{r}\, d\vec{r}\,' u^\dagger_{s_z'} u^\dagger_{t_z'} \eta^\dagger_{\tau_z'} \frac{e^{-i\vec{k}'\cdot\vec{r}}}{(2\pi)^{3/2}} v(\vec{r}, \vec{r}\,') \psi_k^{(+)\alpha}(\vec{r}\,'). \tag{2.9}$$

Now projecting (we follow here a simple extension of a standard procedure—see for example Ref. 3)

$$\psi^{TT_z\alpha}_{ljj_z}(\vec{k}) \equiv (4\pi)^{1/2} \int v_{ljT}(r) \langle l j j_z TT_z, \vec{r} | \psi_k^\alpha(\vec{r}) d\vec{r} \tag{2.10}$$

in Eq. (2.8) allows the result

$$\psi^{TT_z\alpha}_{ljj_z}(\vec{k}) = \frac{v^\alpha(\vec{k})_{[ljj_zTT_z]}(2\pi)^{-3/2}}{1 + [\lambda_{ljT}/(2\pi)^3] \int v_{ljT}^2(t) d\vec{t}/[E(t) - E(k) - i\epsilon]} , \tag{2.11}$$

where

$$v^\alpha(\vec{k})_{[ljj_zTT_z]} \equiv (4\pi)^{1/2} i^l Y^*_{l(j_z - s_z)}(\Omega_k) (l[j_z - s_z]\tfrac{1}{2} s_z | l\tfrac{1}{2} j j_z)(1\tau_z \tfrac{1}{2} t_z | 1\tfrac{1}{2} TT_z) v_{ljT}(k) \tag{2.12}$$

and

$$v_{ljT}(k) \equiv 4\pi \int v_{ljT}(r) j_l(kr) r^2 dr. \tag{2.13}$$

In order to obtain Eq. (2.11) use has been made of the usual partial-wave decomposition of a plane wave

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{lm} 4\pi i^l j_l(kr) Y^*_{lm}(\Omega_k) Y_{lm}(\Omega_r), \tag{2.14}$$

as well as the orthogonality properties of spherical harmonics and of the u and η functions [see Eq. (2.6)].

If Eq. (2.3) for the potential and subsequently Eq. (2.10) are substituted into Eq. (2.9) we obtain the following expression for the T matrix:

$$\langle s_z' t_z' \tau_z' \vec{k}' | T | \vec{k} s_z t_z \tau_z \rangle = \frac{1}{2\pi^2} \sum_{\substack{ljT \\ j_z T_z}} \frac{\lambda_{ljT} \sum_{mm'} (lm\tfrac{1}{2} s_z | l\tfrac{1}{2} j j_z)(lm'\tfrac{1}{2} s_z' | l\tfrac{1}{2} j j_z)}{1 + [\lambda_{ljT}/(2\pi)^3] \int v_{ljT}^2(t) d\vec{t}/[E(t) - E(k) - i\epsilon]}$$

$$\times (1\tau_z \tfrac{1}{2} t_z | 1\tfrac{1}{2} TT_z)(1\tau_z' \tfrac{1}{2} t_z' | 1\tfrac{1}{2} TT_z) v_{ljT}(k) v_{ljT}(k') Y_{lm'}(\Omega_{k'}) Y^*_{lm}(\Omega_k). \tag{2.15}$$

The asymptotic form of the wave function is given by

$$\psi^\alpha_{\vec{k}}(\vec{r}) = (2\pi)^{-3/2} \left[e^{i\vec{k}\cdot\vec{r}} u_{s_z} u_{t_z} \eta_{\tau_z} + \frac{e^{ikr}}{r} \sum_{s_z' t_z' \tau_z'} \langle s_z' t_z' \tau_z' | f(\vec{k}', \vec{k}) | s_z t_z \tau_z \rangle u_{s_z'} u_{t_z'} \eta_{\tau_z'} \right], \tag{2.16}$$

where the generalized scattering amplitude is defined by[5]

$$\langle s_z' t_z' \tau_z' | f(\vec{k}', \vec{k}) | s_z t_z \tau_z \rangle = [-(2\pi)^2/\hbar^2] E(k) \langle s_z' t_z' \tau_z' \vec{k}' | T | \vec{k} s_z t_z \tau_z \rangle. \tag{2.17}$$

The differential cross section is related to the T matrix via

$$\frac{d\sigma}{d\Omega}[\vec{k} s_z t_z \tau_z \to \vec{k}' s_z' t_z' \tau_z'] = [(2\pi)^4/\hbar^4] E^2(k) |\langle s_z' t_z' \tau_z' \vec{k}' | T | \vec{k} s_z t_z \tau_z \rangle|^2. \tag{2.18}$$

It is conventional in pion-nucleon scattering to express the phase shifts in the (ljT) basis, i.e., the T matrix is expanded as[5]

$$\langle s_z' t_z' \tau_z' \vec{k}' | T | \vec{k} s_z t_z \tau_z \rangle = -\sum_{\substack{lm'm \\ jj_z TT_z}} Y_{lm'}(\Omega_{k'}) Y^*_{lm}(\Omega_k) (lm\tfrac{1}{2} s_z | l\tfrac{1}{2} j j_z)(lm'\tfrac{1}{2} s_z' | l\tfrac{1}{2} j j_z)$$

$$\times (1\tau_z \tfrac{1}{2} t_z | 1\tfrac{1}{2} TT_z)(1\tau_z' \tfrac{1}{2} t_z' | 1\tfrac{1}{2} TT_z) \left[\frac{\hbar^2 e^{i\delta_{ljT}} \sin\delta_{ljT}}{\pi k E(k)} \right]. \tag{2.19}$$

Comparing Eqs. (2.15) and (2.19) allows us to identify

$$T_{ljT}(k) \equiv \frac{(\lambda_{ljT}/2\pi)v_{ljT}^{2}(k)}{1+[\lambda_{ljT}/(2\pi)^3]\int v_{ljT}^{2}(t)d\vec{t}/[E(t)-E(k)-i\epsilon]} = \frac{-\hbar^2 e^{i\delta_{ljT}(k)}\sin\delta_{ljT}(k)}{kE(k)} \tag{2.20}$$

or

$$\frac{e^{i\delta_{ljT}(k)}\sin\delta_{ljT}(k)}{k} = \frac{(-\lambda_{ljT}/4\pi)v_{ljT}^{2}(k)2E(k)/\hbar^2}{1+[\lambda_{ljT}/(2\pi)^3]\int v_{ljT}^{2}(t)d\vec{t}/[E(t)-E(k)-i\epsilon]} \ . \tag{2.21}$$

The inverse problem for separable potentials has been the subject of considerable study. We assume the laboratory "fixed-scatter" phase shifts obey the following conditions[6,7]

$$\delta_{ljT}(0) - \delta_{ljT}(\infty) = 0 \ , \tag{2.22a}$$

$$\delta_{ljT} \to 0 \quad \text{as } k \to \infty. \tag{2.22b}$$

Under these conditions one can invert Eq. (2.21) and determine the potential $v_{ljT}^{2}(k)$ from a knowledge of the complex phase shift, $\delta_{ljT}(k)$, at all energies via the relation

$$\lambda_{ljT}v_{ljT}^{2}(k) = \frac{-4\pi\hbar^2\sin\delta_{ljT}(k)}{2kE(k)}\exp\left[\frac{-1}{\pi}P\int_0^\infty \frac{\delta_{ljT}(t)t\,dt}{E(t)-E(k)}\frac{\hbar^2}{E(t)}\right] \ . \tag{2.23}$$

We refer the reader to the literature[6,7] for a discussion of the derivation of Eq. (2.23) and the more complicated expressions required when condition (2.22a) is not satisfied. We have used the convention

$$\lambda_{ljT} = \pm 1 \quad \text{if } \mathrm{Re}\delta_{ljT} \lesssim 0.$$

The principal value integral in Eq. (2.23) extends to infinity while the experimental phase shifts[8] are given up to a finite value. Thus, some assumption is required regarding the asymptotic behavior of the phase shifts. Our procedure has been to study the influence of different asymptotic forms for the phase shift on (1) the derived pion-nucleon "fixed-scatterer" potential and (2) the predicted pion-nucleus elastic scattering. The two types of asymptotic phase-shift behavior considered were an exponential decay

$$\alpha e^{-\beta\hbar k} \tag{2.24a}$$

and a less rapid falloff

$$\frac{\alpha\hbar k}{(\hbar k)^2+\gamma^2} \tag{2.24b}$$

behavior. The constants in Eq. (2.24) were chosen so that the asymptotic phase shifts would join continuously to the known experimental phase shifts. Of course, with two constants, there was still freedom to vary the rate of falloff and thus study the influence of varying the phase-shift decay rate for a particular analytic form. We apply the potential in the next section in the energy region below 300 MeV. One might hope that the behavior of the phase shifts at very high energy would not alter the derived medium-energy potential. This

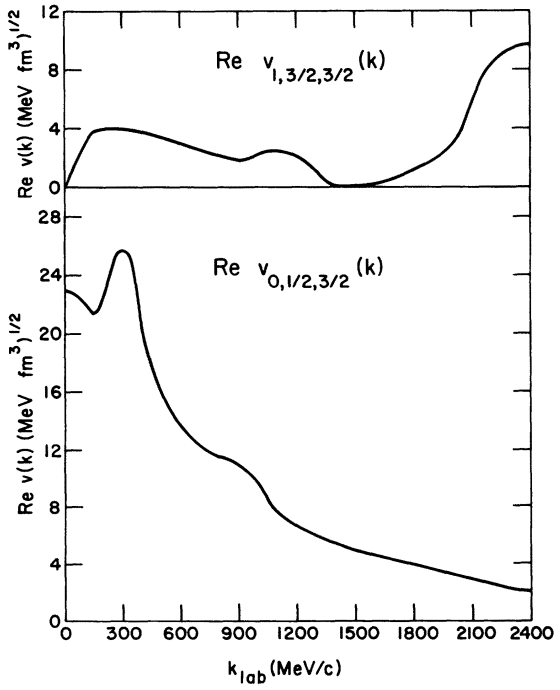

FIG. 1. Real part of the "fixed scatterer" potential $v(k)$ for the $s(j=\frac{1}{2}, t=\frac{3}{2})$ and $p(j=t=\frac{3}{2})$ partial waves plotted versus the pion laboratory momentum.

is in fact not the case and, in our studies, we find considerable difference in the over-all strength of the medium-energy potential for different asymptotic phase-shift falloff rates. The changes in the momentum dependence of the potential at medium energies is *much* less dramatic. From our perspective the important conclusion is that while the potential at medium energies is affected by the asymptotic phase-shift behavior, the predicted pion-*nucleus* elastic scattering is not. As we shall see in the next section, the potential enters only in the ratio (resulting from off-energy-shell scattering of the pion in the many-body nuclear environment)

$$\frac{v_{ljT}(p)v_{ljT}(q)}{v_{ljT}^{2}(k)} \tag{2.25}$$

with p and q not very different from k. Thus, for example, any constant multiplicative factors in v_{ljT} disappear in expression (2.25). In the many-body problem, the nuclear form factor and the appropriate propagator weight heavily the region $q \approx p \approx k$.

We illustrate in Fig. 1 the $v_{ljT}(k)$ obtained when the asymptotic form

$$\alpha e^{-\beta \hbar k}, \quad \beta = 0.14 \times 10^{-2} (\mathrm{MeV}/c)^{-1} \tag{2.26}$$

is chosen for the phase shifts. We have checked, using Eq. (2.21) to insure that the derived potential does indeed reproduce the "fixed-scatterer" laboratory phase shifts.

We note that the fixed-scatterer potentials we obtain and the center-of-mass potentials obtained by Landau and Tabakin[9] are not identical but in several cases are similar due to the fact that the nucleon is a factor of 7 more massive than the pion projectile (so that c.m. and lab systems are not widely different).

3. PION-NUCLEUS ELASTIC SCATTERING

The integral equation describing the scattering in the laboratory system of a pion of momentum k and isospin projection τ_z from a many-body nuclear target containing A nucleons is assumed to take the form of a Lippmann-Schwinger equation

$$\psi^{(+)}_{\vec{k},\tau_z}(\vec{x}^0_\pi, \vec{x}^0_1, \vec{x}^0_2, \ldots, \vec{x}^0_A) = \Phi_0(\vec{x}^0_1, \ldots, \vec{x}^0_A)\frac{e^{i\vec{k}\cdot\vec{x}^0_\pi}\eta_{\tau_z}}{(2\pi)^{3/2}}$$

$$-\sum_n \Phi_n(\vec{x}^0_1 \ldots \vec{x}^0_A)\sum_{\tau_{z'}} \eta_{\tau_{z'}} \int \cdots \int d\vec{x}^1_\pi \ldots d\vec{x}^1_A \int \frac{d\vec{t}}{(2\pi)^3} \frac{e^{i\vec{t}\cdot(\vec{x}^0_\pi - \vec{x}^1_\pi)}\eta^+_{\tau_z}\Phi^*_n(\vec{x}^1_1 \ldots \vec{x}^1_A)}{E_n - E_0 + E(t) - E(k) - i\epsilon}$$

$$\times \sum_{i=1}^{A} \sum_{\substack{ljj_z \\ TT_z}} 4\pi\lambda_{ljT} v_{ljT}(r^1_i) v_{ljT}(r^2_i) \left|\tfrac{1}{2} ljj_z TT_z, \vec{r}^1_i\right\rangle \left\langle \tfrac{1}{2} ljj_z TT_z, \vec{r}^2_i\right|$$

$$\times \delta\left[\frac{1}{m_\pi + m_i}(m_\pi \vec{x}^1_\pi + m_i \vec{x}^1_i) - \frac{1}{m_\pi + m_i}(m_\pi \vec{x}^2_\pi + m_i \vec{x}^2_i)\right] \psi^{(+)}_{\vec{k},\tau_z}(\vec{x}^2_\pi, \vec{x}^1_1, \ldots \vec{x}^2_i, \ldots \vec{x}^1_A) d\vec{x}^2_i\, d\vec{x}^2_\pi . \tag{3.1}$$

In Eq. (3.1) we use relativistic free-particle kinematics for the pion, i.e.,

$$E^2(t) = \hbar^2 t^2 + m_\pi{}^2 . \tag{3.2}$$

The nuclear many-particle states Φ_n represent a set of energy eigenstates of the A nucleon target satisfying

$$H_n|\Phi_n\rangle = E_n|\Phi_n\rangle \ (E_0 \equiv \text{nuclear ground-state energy}), \tag{3.3}$$

where H_n is the free nuclear Hamiltonian. In Eq. (3.1) all variables referring to the pion have a subscript π (i.e., $m_\pi, \vec{x}_\pi$) while variables which refer to the ith target nucleon have a subscript i (i.e., m_i, x_i). Different dummy integration variables referring to the same particle are distinguished by different superscripts. The pion-ith nucleon interaction appearing in Eq. (3.1) is the spin- and isospin-dependent nonlocal separable potential discussed in Sec. 2. The coordinate $\vec{r}_i$ is defined as the relative coordinate between the pion and the ith nucleon:

$$\vec{r}_i \equiv \vec{x}_\pi - \vec{x}_i . \tag{3.4}$$

We shall treat the nucleus as infinitely heavy so the pion-nucleus c.m. and lab systems coincide and the origin is taken at the center of mass of the nucleus. The δ function involving the two-particle (pion-nucleon) center of mass appearing in Eq. (3.1) is included here only for discussion purposes. In general, in the two-body problem, such a δ function is required to reduce the two-

body problem to a one-body problem in the center-of-mass system. [Its presence is, of course, related to Galilean invariance in the two-body problem.] However, in Sec. 2 we have derived a potential for a pion interacting with a fixed nucleon, $m_i = \infty$, thus the δ function trivially disappears in the two-body problem and simply reduces to the requirement $\vec{x}_i^1 = \vec{x}_i^2$ in Eq. (3.1). This means that instead of the potential being nonlocal in the relative coordinate form

$$v_{ljT}(\vec{x}_0^1 - \vec{x}_i^1) v_{ljT}(\vec{x}_0^2 - \vec{x}_i^2)\delta\,, \tag{3.5}$$

we have a potential (essentially nonlocal in only the pion coordinate) which is of the form

$$v_{ljT}(\vec{x}_\pi^1 - \vec{x}_i^1) v_{ljT}(\vec{x}_\pi^2 - \vec{x}_i^1)\,. \tag{3.6}$$

This important simplification is a basic starting point in the FW formalism and is the reason that the pion-nucleon interaction was parametrized by a fixed-scatter potential in Sec. 2.

A discussion of the situation where one keeps m_i finite and the resulting optical potential obtained is given in Ref. 4. To reiterate: for our fixed-scatterer potential the δ function in Eq. (3.1) disappears and the nonlocal separable potential reduces to the type given in Eq. (3.6) which is nonlocal only in the pion configuration space coordinate.

We now assume that the excitation energies of the "important" intermediate nuclear states E_n in the iteration of the integral equation (3.1) are low compared to the initial pion energy of 100–400 MeV.

Thus

$$E(k) \gg E_n - E_0\,, \tag{3.7}$$

and we ignore the n dependence in the denominator thereby allowing a trivial evaluation of the sum $\sum_n$ over intermediate nuclear states in Eq. (3.1) since

$$\sum_n \Phi_n(\vec{x}_1^0 \ldots \vec{x}_A^0)\Phi_n^*(\vec{x}_1^1 \ldots \vec{x}_A^1) = \prod_{j=1}^{A} \delta(\vec{x}_j^0 - \vec{x}_j^1)\,. \tag{3.8}$$

Using the closure approximation discussed above and the fixed-scatterer potential derived in Sec. 2 allows us to reduce the integral equation (3.1) to the form

$$\begin{aligned}\psi^{(+)}_{k,\tau_z}(\vec{x}_\pi^0, \vec{x}_1^0, \vec{x}_2^0, \ldots \vec{x}_A^0) = \Phi_0(\vec{x}_1^0, \ldots \vec{x}_A^0)\frac{e^{i\vec{k}\cdot\vec{x}_\pi^0}\eta_{\tau_z}}{(2\pi)^{3/2}} \\ - \sum_{\tau_z'} \eta_{\tau_z'} \int\int d\vec{x}_\pi^1 \frac{d\vec{t}}{(2\pi)^3} \frac{e^{i\vec{t}\cdot(\vec{x}_\pi^0 - \vec{x}_\pi^1)}}{E(t) - E_{\text{eff}}(k) - i\epsilon}\, \eta^\dagger_{\tau_z'} \sum_{i=1}^{A} \sum_{\substack{ljj_z \\ TT_z}} 4\pi\lambda_{ljT} v_{ljT}(|\vec{x}_i^0 - \vec{x}_\pi^1|) \\ \times v_{ljT}(|\vec{x}_i^0 - \vec{x}_\pi^2|)\,|\,ljj_z TT_z, \vec{x}_i^0 - \vec{x}_\pi^1\rangle_i \\ \times \langle ljj_z TT_z, \vec{x}_i^0 - \vec{x}_\pi^2\,|_i\, \psi^{(+)}_{k,\tau_z}(\vec{x}_\pi^2, \vec{x}_1^0, \ldots, \vec{x}_i^0, \ldots \vec{x}_A^0)\,.\end{aligned} \tag{3.9}$$

The $E_{\text{eff}}(k)$ appearing in Eq. (3.9) will be taken to be $E(k)$ in our work. Actually, one may wish (to obtain a better fit to elastic scattering) to take into account some "average" intermediate nuclear excitation by replacing $E(k) - (E_n - E_0)$ by the adjustable parameter $E_{\text{eff}}(k)$ instead of simply dropping $E_n - E_0$. The subscript i on the bra and ket in Eq. (3.9) is a reminder that the nucleon spin and isospin operators refer to the ith nucleon.

The elastic scattering T matrix is given by

$$\begin{aligned}\langle \Phi_0^{J_z', \mathcal{T}_z}\vec{k}', \tau_z'|\, T\,|\Phi_0^{J_z, \mathcal{T}_z}\vec{k}, \tau_z\rangle = \int\cdots\int d\vec{x}_\pi^0, d\vec{x}_\pi^1, d\vec{x}_1^0 \cdots d\vec{x}_A^0 \Phi_0^{\dagger(J_z', \mathcal{T}_z)}(\vec{x}_1^0 \cdots \vec{x}_A^0)\frac{e^{-i\vec{k}'\cdot\vec{x}_\pi^0}\eta^\dagger_{\tau_z'}}{(2\pi)^{3/2}} \\ \times \sum_{i=1}^{A} \sum_{\substack{ljj_z \\ TT_z}} \lambda_{ljT} v_{ljT}(|\vec{x}_i^0 - \vec{x}_\pi^0|) v_{ljT}(|\vec{x}_i^0 - \vec{x}_\pi^1|)\,|\,ljj_z TT_z, \vec{x}_i^0 - \vec{x}_\pi^0\rangle_i \\ \times \langle ljj_z TT_z, \vec{x}_i^0 - \vec{x}_\pi^1\,|_i\, \psi^{(+)}_{k,\tau_z}(\vec{x}_\pi^1, \vec{x}_1^0, \ldots \vec{x}_A^0)\,,\end{aligned} \tag{3.10}$$

where $\psi^{(+)}$ may be obtained by iterating Eq. (3.9). We shall present results only for $J = \mathcal{T} = 0$ "closed-shell" nuclei, and in this case the differential cross section for elastic scattering is obtained from

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}(J, \mathcal{T} = 0, \vec{k} \to \vec{k}') = [(2\pi)^4/\hbar^4]E^2(k)\,|\langle \Phi_0, \vec{k}', \tau_z|\, T\,|\Phi_0, \vec{k}, \tau_z\rangle|^2 \quad (\mathcal{T} \equiv \text{isotopic spin of nucleus})\,. \tag{3.11}$$

Actually in Eq. (3.9) only the pion coordinates appear as integration variables; thus, Eq. (3.9) is an integral equation involving only the projectile in a fundamental way. This, as FW demonstrate, allows one to break up the elastic-scattering problem into two parts: (1) finding a solution for a many-body scattering amplitude for fixed-scattering centers and (2) averaging this A-body operator over the nuclear configuration space probability distribution $\Phi_0^*\Phi_0$. We shall outline a slightly different derivation below which adopts exactly the same approximations as FW and obtains the same results, extended to include spin and isospin degrees of freedom and projectile relativistic kinematics.

The basic procedure is to iterate Eq. (3.9) for $\psi^{(+)}$ and substitute the resulting expansion into Eq. (3.10). With each iteration of Eq. (3.9) one obtains a pion-nucleon potential interaction denoted v_i with the ith target nucleon; thus we may write the expanded equation (3.10) symbolically as

$$\langle\Phi_0(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})|\sum_{i=1}^{A} v_i+\sum_{i,j=1}^{A} v_iGv_j + \sum_{i,j,k=1}^{A} v_iGv_jGv_k\cdots|\Phi_0(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})\rangle . \tag{3.12}$$

The first step is to reorder the expansion in Eq. (3.12) so that one replaces v_i, a pion-ith nucleon potential interaction, with T_i, a pion-ith nucleon T-matrix interaction where

$$T_i=v_i+v_iGv_i+v_iGv_iGv_i+\cdots . \tag{3.13}$$

This is easily accomplished and Eq. (3.12) takes the form

$$\langle\Phi_0(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})|\sum_{i=1}^{A} T_i+\sum_{\substack{i,j=1\\ i\neq j}}^{A} T_iGT_j + \sum_{\substack{i,j,k=1\\ i\neq j\\ j\neq k}}^{A} T_iGT_jGT_k\cdots|\Phi_0(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})\rangle . \tag{3.14}$$

Now two approximations are made by FW that allow expression (3.14) to be reduced from a correlated many-body problem to a product of one-body integrals.

(1) The never-come-back approximation· Although the pion may have any number of potential interactions with a given target nucleon, once it has interacted with the ith nucleon and subsequently interacted with another target particle the pion never interacts again with the ith nucleon. With this approximation Eq. (3.14) may be rewritten

$$\langle\Phi_0(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})|\sum_{i=1}^{A} T_i+\sum_{\substack{i,j=1\\ i\neq j}}^{A} T_iGT_j+\sum_{\substack{i,j,k=1\\ i\neq j,i\neq k\\ j\neq k}}^{A} T_iGT_jGT_k\cdots|\Phi_0(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})\rangle . \tag{3.15}$$

(2) Single-particle-density assumption on the nuclear-matter distribution, i.e., that the nuclear many-particle ground-state density may be written as a product of single-particle densities

$$\Phi_0^*(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})\Phi_0(\vec{x}_1^{\,0}\cdots\vec{x}_A^{\,0})=\prod_{i=1}^{A}\rho(\vec{x}_i)+\left[\begin{array}{l}\text{terms neglected in obtaining}\\ \text{zero-order optical potential}\end{array}\right]. \tag{3.16}$$

If one were evaluating only sums of single-nucleon operators such as $\sum_i T_i$ then this approximation (2) would yield the same result as using the full many-nucleon wave function. Because of the appearance of multinucleon operators in Eq. (3.15) such as T_iT_j, which depends on the coordinates of the two nucleons i and j, Eq. (3.16) must be regarded as an approximation ignoring two- and higher-order nucleon correlations, leading to a zero-order optical potential.[3]

For definiteness in this paper we present the following model for obtaining the single-nucleon density. Assume the nuclear many-particle wave function Φ_0 is given by a Slater determinant of normalized independent-particle-model j-j coupled orbitals. Then after integrating over all but the ith nucleon's coordinates we obtain an expression of the following form for single-particle matrix elements:

$$\begin{aligned}\int\rho(\vec{x}_i)O(\vec{x}_i)d\vec{x}_i&=\sum_{t_{zi}}\sum_{\substack{j_ij_{zi}\\ l_i}}A^{l_i}_{j_ij_{zi}t_{zi}}\int\varphi^{+t_{zi}}_{l_ij_{zi}}(\vec{x}_i)O_i\,\varphi^{t_{zi}}_{l_ij_{zi}}(\vec{x}_i)d\vec{x}_i\\ &=\sum_{t_{zi}}\sum_{\substack{j_ij_{zi}\\ l_i}}A^{l_i}_{j_ij_{zi}t_{zi}}\sum_{l_{zi}s_{zi}}(l_il_{zi}\tfrac{1}{2}s_{zi}|l_i\tfrac{1}{2}j_ij_{zi})\sum_{l'_{zi}s'_{zi}}(l_il'_{zi}\tfrac{1}{2}s'_{zi}|l_i\tfrac{1}{2}j_ij_{zi})\\ &\quad\times\int\varphi^{\dagger}_{nl_i}(x_i)Y^{\dagger}_{l_il'_{zi}}(\Omega_{\vec{x}_i})u^{\dagger}_{s'_{zi}}u^{\dagger}_{t_{zi}}O_i\,\varphi_{nl_i}(x_i)Y_{l_il_{zi}}(\Omega_{\vec{x}_i})u_{s_{zi}}u_{t_{zi}}d\vec{x}_i ,\end{aligned} \tag{3.17}$$

where the constant $A^{l_i j_i}_{j_{zi} t_{zi}}$ is zero or one depending on what orbitals are assumed present in the original Slater determinant. Three interesting special cases are (1) a major closed-shell nucleus such as ^{16}O ($1s_{1/2}$, $1p_{3/2}$, and $1p_{1/2}$ j-j shells filled) or ^{40}Ca ($1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$ j-j shells filled) where, for occupied l_i states, both the $j_i = l_i + \frac{1}{2}$ and $j_i = l_i - \frac{1}{2}$ shells are assumed filled; (2) minor closed-shell nuclei such as ^{12}C, ^{28}Si, or ^{32}S where for the "highest" filled l_i states only the $j_i = l_i + \frac{1}{2}$ shell is filled, and (3) a nucleus with a single valence particle outside a major closed shell (such as ^{17}O) or a minor closed shell (such as ^{13}C). In case (3) the nucleus no longer has $J = \mathcal{T} = 0$ and this situation will not be treated in this paper on elastic scattering from closed-shell nuclei. In order to treat case (3) properly a more complicated spin-dependent optical potential is required. The single-nucleon operator O_i is in general spin- and isospin-dependent, and thus one must be careful of the order in which factors appear in Eq. (3.17). Note that even assuming a diagonal single-particle j-j density there is still in general a spin-flip contribution, since s_{zi} does not have to equal s'_{zi} in Eq. (3.17). There is no isospin-flip (charge-exchange) contribution in Eq. (3.17) since we are adopting the product single-particle density approximation. One of the major differences occurring when one considers inelastic scattering or reactions is that terms of the form of Eq. (3.17) appear where the initial and final Slater determinants are different and one is dealing with single-particle transition densities.

With the never-come-back approximation, the single-particle density approximation, and the above j-j coupling model of the nucleus, the expanded right-hand side of Eq. (3.10) may be written

$$\sum_{n=1}^{A} T_n^k(\vec{k}', \vec{k}) - \sum_{n \neq n'=1}^{A} \int d\vec{t}_1 T_n^k(\vec{k}', \vec{t}_1) \frac{1}{E(t_1) - E(k) - i\epsilon} T_{n'}^k(\vec{t}_1, \vec{k}) + \sum_{\substack{n \neq n' \neq n''=1 \\ n \neq n''}}^{A} \int\int d\vec{t}_1 d\vec{t}_2 \frac{T_n^k(\vec{k}\vec{t}_2) T_{n'}^k(\vec{t}_2, \vec{t}_1) T_{n''}^k(\vec{t}_1, \vec{k})}{[E(t_2) - E(k) - i\epsilon][E(t_1) - E(k) - i\epsilon]} + \cdots, \quad (3.18)$$

where

$$(2\pi)^3 T_n^k(\vec{p}, \vec{q}) \equiv \sum_{t_{zi}} \sum_{l_i j_i j_{zi}} A^{l_i j_i}_{j_{zi} t_{zi}} \sum_{\substack{l_{zi} s_{zi} \\ l'_{zi} s'_{zi}}} (l_i l_{zi} \tfrac{1}{2} s_{zi} | l_i \tfrac{1}{2} j_i j_{zi})(l_i l'_{zi} \tfrac{1}{2} s'_{zi} | l_i \tfrac{1}{2} j_i j_{zi})$$

$$\times \int \varphi^{\dagger}_{l_i l'_{zi}}(\vec{k}_n) \varphi_{l_i l_{zi}}(\vec{k}_n + \vec{p} - \vec{q}) d\vec{k}_n \sum_{\substack{j j_z T T_z \\ l m m'}} (l m \tfrac{1}{2} s_{zi} | l \tfrac{1}{2} j j_z)(l m' \tfrac{1}{2} s'_{zi} | l \tfrac{1}{2} j j_z)$$

$$\times \frac{(1 \tau_z \tfrac{1}{2} t_{zi} | 1 \tfrac{1}{2} T T_z)(1 \tau'_z \tfrac{1}{2} t_{zi} | 1 \tfrac{1}{2} T T_z) 4\pi \lambda_{ljT} v_{ljT}(p) v_{ljT}(q) Y_{lm'}(\hat{p}) Y^*_{lm}(\hat{q})}{1 + [\lambda_{ljT}/(2\pi)^3] \int [d\vec{t}\, v_{ljT}^2(t)/[E(t) - E(k) - i\epsilon]} . \quad (3.19)$$

The Fourier transform of the bound-nucleon orbital is defined by

$$\varphi(\vec{r}) = \int \frac{d\vec{k}_n}{(2\pi)^{3/2}} e^{-i\vec{k}_n \cdot \vec{r}} \varphi(\vec{k}_n) . \quad (3.20)$$

We note that the isospin Clebsch-Gordan coefficients in Eq. (3.19) require that $\tau_z = \tau'_z$ and so, for elastic scattering, there are no *intermediate* isospin-flip scattering contributions.

We may identify Eq. (3.19) as the T matrix for pion-nucleon scattering in the nuclear many-body environment. We shall see that Eq. (3.19) is of central importance both in elastic and inelastic scattering.

It is interesting to compare expression (3.18) with the series resulting from considering a one-body problem containing a nonlocal optical potential U which when inserted into the "relativistic" equivalent one-body Schrödinger equation for the pion

$$\psi^{(+)}_{k, \tau_z}(r) = \frac{e^{i\vec{k}\cdot\vec{r}} \eta_{\tau_z}}{(2\pi)^{3/2}} - \sum_{\tau'_z} \int\int\int \frac{d\vec{t}}{(2\pi)^3} \eta_{\tau'_z} \frac{e^{i\vec{t}\cdot(\vec{r}-\vec{r}')}}{E(t) - E(k) - i\epsilon} \eta^{\dagger}_{\tau'_z} U(\vec{r}', \vec{r}'') \psi^{\dagger}_{k, \tau_z}(\vec{r}'') d\vec{r}' d\vec{r}'' \quad (3.21)$$

will yield a T matrix in agreement with the T matrix calculated from Eq. (3.18). The T matrix associated with Eq. (3.21) is given by

$$T_{\tau'_z \tau_z}(\vec{k}', \vec{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{k}'\cdot\vec{r}} \eta^{\dagger}_{\tau'_z} U(\vec{r}, \vec{r}') \psi^{(+)}_{k, \tau_z}(\vec{r}') d\vec{r} d\vec{r}' . \quad (3.22)$$

Substituting Eq. (3.21) into Eq. (3.22) and iterating yields

$$T_{\tau_z'\tau_z}(\vec{k}',\vec{k}) = U_{\tau_z'\tau_z}(\vec{k}',\vec{k}) - \sum_{\tau_z''} \int d\vec{t} U_{\tau_z'\tau_z''}(\vec{k}',\vec{t}) \frac{1}{E(t)-E(k)-i\epsilon} U_{\tau_z''\tau_z}(\vec{t},\vec{k})$$
$$+ \sum_{\tau_z''\tau_z'''} \int\int d\vec{t}_1 d\vec{t}_2 U_{\tau_z'\tau_z''}(\vec{k},\vec{t}_2) \frac{1}{E(t_2)-E(k)-i\epsilon} U_{\tau_z''\tau_z'''}(\vec{t}_2,\vec{t}_1) \frac{1}{E(t_1)-E(k)-i\epsilon} U_{\tau_z''\tau_z}(\vec{t}_1,\vec{k}), \quad (3.23)$$

where

$$U_{\tau_z'\tau_z}(\vec{p},\vec{q}) \equiv \frac{1}{(2\pi)^3} \int\int e^{-i\vec{p}\cdot\vec{x}} \eta^+_{\tau_z'} U(\vec{x},\vec{y}) \eta_{\tau_z} e^{i\vec{q}\cdot\vec{y}} d\vec{x}\, d\vec{y} . \quad (3.24)$$

Treating the target nucleons equally in Eq. (3.18) allows that expression to be rewritten [since all $T_n(k,k)$ are equivalent, we carry out the sum over n and drop the subscript on T]

$$AT^k(\vec{k}',\vec{k}) - A(A-1)\int d\vec{t}_1 \frac{T^k(\vec{k}',\vec{t}_1)T^k(\vec{t}_1,\vec{k})}{E(t_1)-E(k)-i\epsilon} + A(A-1)(A-2)\int d\vec{t}_1 d\vec{t}_2 \frac{T^k(\vec{k}',\vec{t}_2)}{E(t_2)-E(k)-i\epsilon} T^k(\vec{t}_2,\vec{t}_1) \frac{T^k(\vec{t}_1,\vec{k})}{E(t_1)-E(k)-i\epsilon} + \cdots . \quad (3.25)$$

If we follow FW and assume, for the degree of accuracy required, that the number of iterations of the multiple-scattering series (3.25) needed, is small compared to A, the number of target nucleons (see later discussion), then we can replace $A(A-1)(A-2)\cdots(A-n+1)$ by A^n in the nth order term of Eq. (3.25). With this additional assumption, if we compare the multiple-scattering series, Eq. (3.25), with the expansion of the T matrix involving the optical potential, Eq. (3.23), we may identify the optical potential as

$$U_{\tau_z'\tau_z}(\vec{p},\vec{q}) = AT^k(\vec{p},\vec{q})\delta_{\tau_z'\tau_z}, \quad (3.26)$$

where the Kronecker δ results from the fact that the T^k derived here [see Eq. (3.19) and the discussion following Eq. (3.20)] does not permit isospin flip.

Equation (3.26) represents a first approximation to the optical potential using a simple product of single-particle densities and the never-come-back approximation. Foldy and Walecka discuss in a systematic manner the relaxation of the approximations used above, and thus formally obtain expressions for the second- and higher-order corrections to the optical potential. They discuss the relaxation of the single-particle density assumption (introduction of two-nucleon correlations) and the never-come-back approximation (local field corrections). In evaluating the local field correction it is important to note that in replacing $A(A-1)(A-2)\cdots(A-n+1)$ by A^n[10] one has included terms that are not actually present in the multiple-scattering series. These terms are just of the type (in each order) which would arise if the never-come-back approximation were *dropped* and one assumed the appropriate target nucleon integrations still factored instead of being correlated as they, in fact, are. Thus when considering the corrections due to dropping the never-come-back approximation, one is actually looking at the difference between a certain number of included uncorrelated integrals and neglected correlated integrals. In this introductory paper we will make use only of the first approximation to the optical potential. However, we regard a quantitative discussion of the importance of the various corrections to the medium-energy pion-nucleus optical potential we have obtained to be a high-priority item for future research.

The optical potential given by Eqs. (3.26) and (3.19) can be considerably simplified for certain types of nuclei. If we ignore the small t_{zi} dependence of the orbital functions $\varphi_{l_i l_{zi}}$ and consider a nucleus with an equal number of neutrons and protons then the sum over t_{zi} and T_z may be carried out trivially in Eq. (3.19) with the result that the isospin Clebsch-Gordan coefficients are replaced by the single factor $\frac{1}{3}(2T+1)$.

The angular integrations involving the bound-state wave function may be carried out, yielding

$$\int \varphi^*_{l_i' l_{zi}'}(\vec{k}_n)\varphi_{l_i l_{zi}}(\vec{k}_n + \vec{p} - \vec{q})\, d\vec{k}_n = \sum_{\underline{l}\underline{m}} F^{\underline{l}}_{l_i' l_i}(|\vec{p}-\vec{q}|)(-i)^{\underline{l}}(-1)^{l'_{zi}}$$
$$\times \left(\frac{(2l_i+1)(2l_i'+1)(2\underline{l}+1)}{4\pi}\right)^{1/2} \begin{pmatrix} l_i' & \underline{l} & l_i \\ -l'_{zi} & \underline{m} & l_{zi} \end{pmatrix} \begin{pmatrix} l_i' & \underline{l} & l_i \\ 0 & 0 & 0 \end{pmatrix} Y^*_{\underline{l}\underline{m}}(\Omega_{\vec{p}-\vec{q}}), \quad (3.27)$$

where

$$F^{\underline{l}}_{l_i l_{i'}}(|\vec{p}-\vec{q}|)\equiv\frac{4\pi}{A}\int R^*_{l_i}(x)R_{l_i}(x)\,j_{\underline{l}}(|\vec{p}-\vec{q}|x)x^2dx\,. \tag{3.28}$$

The functions $R(x)$ appearing in Eq. (3.28) are normalized single-particle radial wave functions. From a purely technical point of view the optical potential given by Eqs. (3.19), (3.26), and (3.28) is difficult to use unless further reductions are possible. Further reductions are possible for the simple major closed-shell or spin-saturated systems we treat in this paper. However, for a study of nonclosed-shell nuclei or for a study of the importance of spin-flip contributions in elastic and inelastic scattering the more complicated optical potential must apparently be used. Computer programs for this purpose are currently being constructed and will play an important role in future research in this area.

If one is treating a major closed-shell nucleus such as ^{16}O or ^{40}Ca then the sum over j_i ($=l_i+\frac{1}{2}$ and $l_i-\frac{1}{2}$) and j_{zi} in Eq. (3.19) results in the condition $\delta_{lsi,l'zi}\delta_{szi,s'zi}$ [i.e., the spin-flip contribution vanishes]. Subsequently, summing over j_z and s_{zi} in Eq. (3.19) yields the expression $[(2j+1)/(2l+1)]\delta_{m,m'}$. Finally, summing over l_{zi} requires $\underline{l}=0$.

Combining all these reductions allows the optical potential to be written

$$U(\vec{p},\vec{q})=\frac{4\pi A}{(2\pi)^3}\sum_{l_i}\sum_{ljT}\frac{(2T+1)}{3}\frac{(2j+1)}{(2l+1)}F^0_{l_i l_i}(|\vec{p}-\vec{q}|)\frac{(2l_i+1)}{4\pi}2\pi T_{ljT}(k)\frac{v_{ljT}(p)v_{ljT}(q)}{v_{ljT}{}^2(k)}Y_{lm}(\vec{p})Y^*_{lm}(\vec{q})\,, \tag{3.29}$$

where

$$T_{ljT}(k)\equiv\frac{(\lambda_{ljT}/2\pi)v_{ljT}{}^2(k)}{1+[\lambda_{ljT}/(2\pi)^3]\int v_{ljT}{}^2(t)d\vec{t}/[E(t)-E(k)-i\epsilon]} \tag{3.30}$$

[see Eq. (2.20)]. An important feature of the optical potential given by Eqs. (3.29) and (3.30) is the appearance of the free two-particle fixed-scatterer pion-nucleon T matrix, $T_{ljT}(k)$. The "off-energy-shell" intermediate-scattering effect is contained in the pion-nucleon potential ratio

$$\frac{v_{ljT}(p)v_{ljT}(q)}{v_{ljT}{}^2(k)}\,.$$

As discussed in Sec. 2, following Eq. (2.24), it is important that only the potential ratio appear in the optical potential because of the ambiguity of the phase shifts as $k\to\infty$ which results in multiplicative-constant uncertainties in $v(p)$ in the intermediate-energy region.

Using the relation

$$\sum_m Y_{lm}(\vec{p})Y^*_{lm}(\vec{q})=\frac{2l+1}{4\pi}P_l(\cos\theta_{pq}) \tag{3.31}$$

and defining

$$\rho(\vec{p}-\vec{q})\equiv\sum_{l_i}\frac{4(2l_i+1)}{4\pi}F^0_{l_i l_i}$$

$$=\sum_{l_i}\frac{4(2l_i+1)}{4\pi A}\int e^{-i(\vec{p}-\vec{q})\cdot\vec{x}_i}|\varphi_{l_i}|^2d\vec{x}_i \tag{3.32}$$

allows the optical potential to be written in the form

$$U(\vec{p},\vec{q})=\frac{8\pi^2A}{(2\pi)^3}\sum_{ljT}\frac{(2T+1)}{12}(2j+1)\rho(\vec{p}-\vec{q})T_{ljT}(k)\times\frac{v_{ljT}(p)v_{ljT}(q)}{v_{ljT}{}^2(k)}P_l(\cos\theta_{pq})\,. \tag{3.33}$$

If one assumes $T(k)$ and v are independent of j and T, this optical potential reduces to that given by FW (except for some easily reconcilable normalization factors of 2π). The pion-nucleus elastic scattering T matrix is obtained from the integral equation (3.23) involving the optical potential. (In the actual numerical calculations the matrix inversion technique was used to solve the integral equation.) The differential cross section was determined from the T matrix via Eq. (3.11).

The final simplification of the optical potential, Eq. (3.33), does not strictly hold for minor closed-shell nuclei in the j-j coupling model such as ^{12}C, ^{28}Si, or ^{32}S (since the sum over j cannot include $j=l-\frac{1}{2}$ for the highest filled l orbit, there can now be some intermediate spin-flip contribution.) However, if one abandons the j-j coupling model and simply assumes a spin-saturated system then Eq. (3.33) can be recovered. In order to compare with other research that has used the simplification for ^{12}C and because the more complicated codes involving spin-flip are not yet completed, Eq.

(3.33) has been used for elastic scattering from the minor closed-shell nuclei treated in this paper.

The actual single-particle orbitals used for the bound nucleons were harmonic-oscillator eigenfunctions. Calculational details associated with this form of $\rho(\vec{p}-\vec{q})$ are presented in Appendix B. The results of the pion-nucleus elastic scattering calculations are presented in the next section.

4. RESULTS AND DISCUSSION

Using the pion-nucleon separable potential obtained in Sec. 2 as input in the pion-nucleus formalism developed in Sec. 3, we have calculated elastic total cross sections σ_{el} and differential cross sections $d\sigma/d\Omega$ for pion scattering from ^{12}C, ^{16}O, ^{28}Si, ^{32}S, and ^{40}Ca. By employing the familiar optical theorem

$$\sigma_{tot}=\frac{4\pi}{k}\,\mathrm{Im}f(\vec{k},\vec{k}) \tag{4.1}$$

total pion-nucleus cross sections have also been obtained for the five nuclei mentioned.

The basic two-body fixed-scatterer potential for s, p, and d waves in the lab system with all $j=l\pm\frac{1}{2}$, $t=\frac{3}{2},\frac{1}{2}$ channels is included in the calculations. The optical potential used in the calculations is given by Eq. (3.33). Harmonic-oscillator single-particle wave functions were adopted to obtain $\rho(\vec{p}-\vec{q})$. The oscillator parameters used

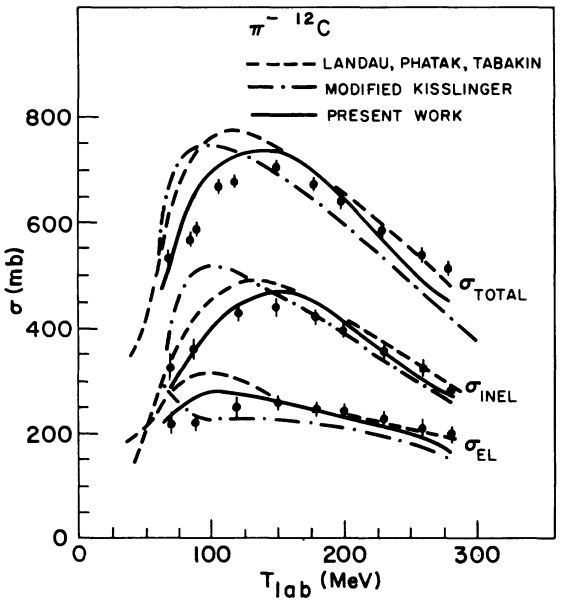

FIG. 2. Comparison of several theoretical predictions with experiment for π-^{12}C total cross sections. The experimental data is from Ref. 11, the theoretical calculations labeled Landau, Phatak, and Tabakin and modified Kisslinger are from Ref. 12.

are listed in Appendix B.

The results of the calculations for σ_{tot} and σ_{el} in the case of π-^{12}C scattering are shown in Fig. 2. The experimental data are from Binon *et al.*[11] Also shown for comparative purposes are the predictions of two other calculations[12]; one based on an approach by Landau, Phatak, and Tabakin (LPT)[12-14] that also uses a separable two-particle T matrix, the other prediction is based on using a Kisslinger-type optical potential[15] modified to include the angle transformation[12] from the π-nucleon c.m. to the π-nucleus c.m. In what follows we confine our remarks to the separable approaches to π-nucleus scattering. It is clear that both nonadjustable parameter separable theories (that of Ref. 12 and our fixed-scatterer approach) yield good fits to the data. Although the fixed-scatterer potential apparently yields a better fit to the data at lower energies, it is premature to draw strong conclusions since several simplifying assumptions have gone into obtaining the zero-order optical potential in both approaches. Thus, for example, corrections due to inclusion of Pauli principle effects may reduce the cross section at lower energies.[16]

It may be helpful if we distinguish between the approach adopted by LPT[12] and the procedure we follow. The familiar Kerman, McManus, and Thaler (KMT)[17] formalism has been used by LPT. The zero-order optical potential in the KMT formalism is obtained assuming the intermediate-nuclear states include the ground state only (we adopt closure). Under this assumption and adopting the impulse approximation the zero-order optical potential is found to be

$$U^0=(A-1)\tau(w), \tag{4.2}$$

where τ is the pion-nucleon collision matrix in the nuclear many-body environment. In actual application there is some ambiguity with regard to the value w to be used in the off-shell τ matrix. LPT make the reasonable assumption that the appropriate form for τ is given schematically by

$$\frac{v(p)v(q)}{1+\lambda\int v^2(t)d\vec{t}/(t^2-k^2-i\epsilon)} \quad (p=q=k \text{ not required}),$$

$$[p, q, \text{ and } k \text{ are } \pi\text{-}n \text{ c.m. variables}], \tag{4.3}$$

where the potential $v(p)$ is that obtained by solving the inverse-scattering problem using pion-nucleon c.m. phase shifts. (We use lab "phase shifts" assuming a fixed scatterer.) Thus LPT assume a form for the off-shell τ matrix in the pion-nucleon center-of-mass system. Then a form for the transformation of this off-shell quantity from the π-nucleon c.m. to the π-nucleus c.m. system must be assumed by LPT. So, in fact, one difference

between the optical potential we adopt and that used by LPT is that we include an angle transformation in obtaining the basic two-body interaction while LPT include an angle transformation when the two-body τ interaction is included in the optical potential. This could result in a different effect in the off-shell term given by the ratio

$$\frac{v_{ljT}(p)v_{ljT}(q)}{v_{ljT}^{2}(k)} . \tag{4.4}$$

Off-shell effects may not be expected to be as important in elastic scattering as in selected inelastic and reaction processes. However, it is still interesting to see the effect of using the purely on-shell π-n scattering amplitude in the optical potential. This is easily accomplished by setting the ratio given in Eq. (4.4) equal to one in the optical potential expression, Eq. (3.33). In Fig. 3 we show the effect of neglecting the off-shell behavior. It is seen that multiple scattering and simple nuclear size broadening effects lower the peak too far from the two-body resonance position to give good agreement with experiment unless the off-shell behavior of the two-body transition matrix is explicitly included in the optical potential.

The predictions obtained for π-nucleus scattering from ^{16}O, ^{28}Si, ^{32}S, and ^{40}Ca are shown in Fig. 4. The dotted line shown for ^{16}O indicates the σ_{tot} obtained when the ratio, Eq. (4.4) (off-shell contribution), is set equal to one. The effect of the off-shell contribution appears to be a slight broadening of the cross-section peak. The effect becomes less appreciable for the heavier nuclei. As can be observed from Fig. 5, the ^{16}O results are quite similar to those for ^{12}C scaled by $(\frac{16}{12})^{2/3}$. For heavier nuclei the total cross section increases more rapidly and the cross-section peak is broader than the $A^{2/3}$ scaling of the ^{12}C results.

The disagreement between the early ^{16}O data and the separable model predictions for the location of the total cross-section peak discussed in Ref. 14 also occurs in our model (theoretical peak position ~150 MeV, experimental peak position[18] ~180 MeV). Rather than a reflection on the adequacy of the models, this could serve as motivation for additional π-^{16}O scattering experiments to obtain σ_{tot}. Our theoretical predictions for ^{32}S are in very good agreement with recent data[19] on σ_{tot} for that nucleus where σ_{tot} peaks at approximately

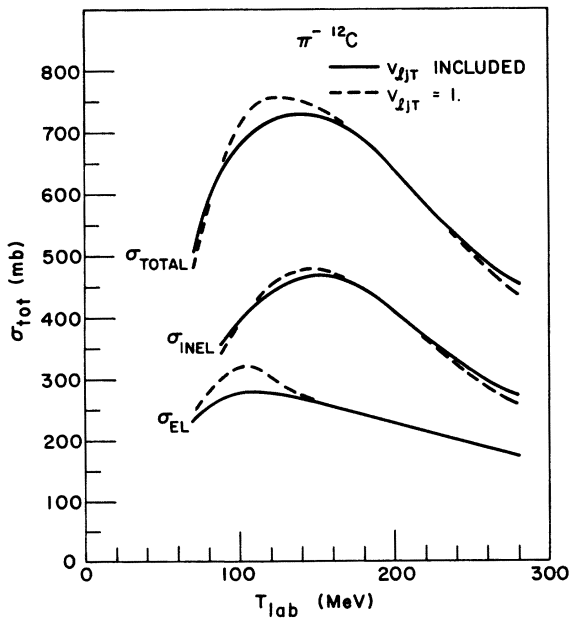

FIG. 3. Effect of neglecting the off-shell behavior of the two-body T Matrix in π-^{12}C scattering. The dotted curve is obtained by setting equal to one the ratio

$$\frac{v_{ljT}(p)v_{ljT}(q)}{v_{ljT}^{2}(k)}$$

appearing in the optical potential.

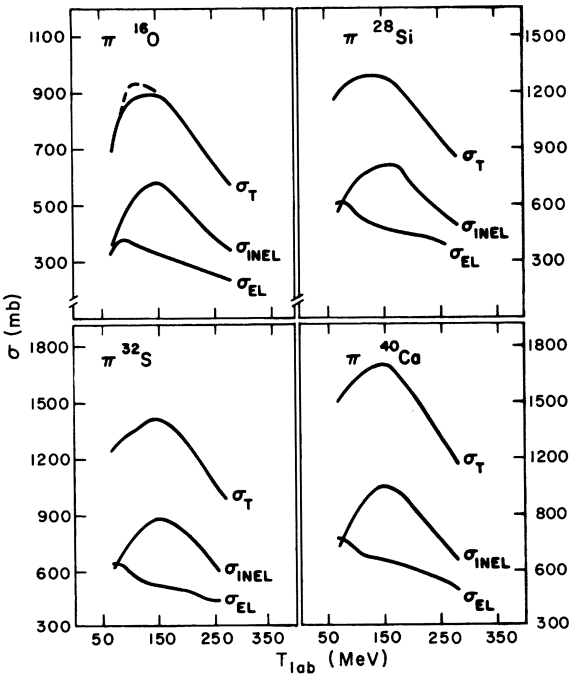

FIG. 4. Pion-nucleus cross sections using the fixed-scatterer two-body potential. Total cross sections for ^{16}O, ^{28}Si, ^{32}S, and ^{40}Ca are plotted versus the pion laboratory kinetic energy. The dotted curve for ^{16}O is obtained by setting to one the ratio

$$\frac{v_{ljT}(p)v_{ljT}(q)}{v_{ljT}^{2}(k)} .$$

140 MeV.

Differential cross sections for $^{12}C + \pi$ elastic scattering are shown in Fig. 6 for several pion laboratory kinetic energies. The dotted line indicates the effect of including a center-of-mass correction factor [see Appendix B and in particular Eq. (B.8)]. The correction factor is clearly more important at higher energies and larger angles and in some cases results in a substantial increase in the large-angle differential cross section. The c.m. correction factor is required because, as is often the case in calculations involving shell-model wave functions, the nuclear center-of-mass condition

$$\sum_i m_i r_i = 0 \tag{4.5}$$

has not been incorporated in the single-particle densities utilized in obtaining the optical potential.[20] The agreement with experiment is quite good except at the larger angles where we generally tend to underestimate $d\sigma/d\Omega$. It is interesting that the approach used by LPT generally yields better agreement with the large-angle ^{12}C experimental differential cross sections than we obtain. As in the case of the σ_{tot} comparison we feel that more investigation is required into the higher-order optical-potential corrections in each model before definitive conclusions can be drawn.

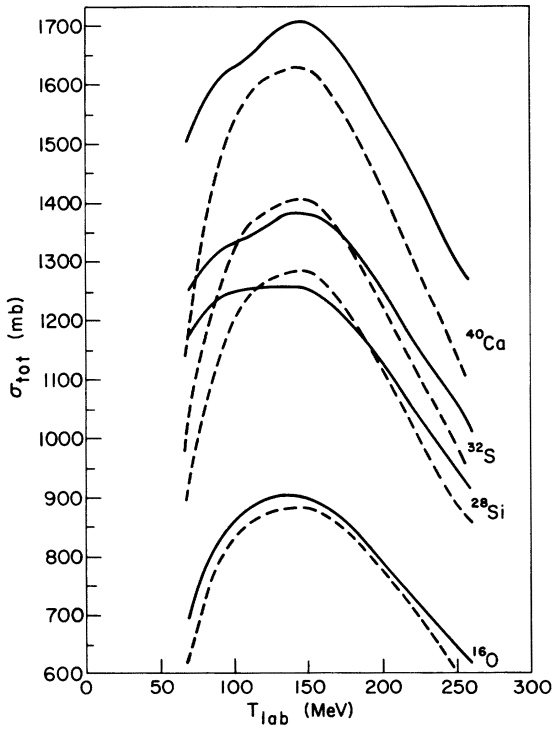

FIG. 5. Comparison of σ_{tot} calculated using the fixed-scatterer two-body potential for ^{16}O, ^{28}Si, ^{32}S, and ^{40}Ca (solid lines) with a "black disc" $A^{2/3}$ scaling (dotted lines) of the ^{12}C results as a function of the pion laboratory kinetic energy.

Finally, in Fig. 7 we exhibit typical differential-cross-section predictions for ^{16}O, ^{28}Si, ^{32}S, and ^{40}Ca and a T_π^{lab} of 150 MeV. As would be expected for diffractive scattering from an absorptive disk, the angular distribution is more rapidly oscillating (and forward peaked) as the radius of the disk (higher A) increases. Although the first two diffraction minima do seem to behave [as a function of k and $R = r_0(A)^{1/3}$] like the zeroes appearing in the black-disk scattering, the wide-angle predictions show strong deviation from the simple absorptive scatterer result. Clearly large-angle elastic differential-cross-section data will yield interesting and challenging tests for a given microscopic theory. Unfortunately, of course, it is just in this large-angle region that *small* corrections to the optical potential may manifest themselves. [In the forward direction different partial-wave scattering amplitudes tend to be in phase. At larger angles this coherence is lost and small relative errors at forward angles can become magnified.] However, as the various pion-nucleon microscopic models become more sophisticated one hopes, naturally, that the main features of large-angle scattering will be correctly predicted.

The main objective of this paper was to present a microscopic theory of the optical potential building on a previous discussion by Foldy and Walecka and to apply the formalism to make predictions for elastic scattering from selected light nuclei. Research in progress extends the predictions to a microscopic analysis of inelastic scattering.

The study of pion-nucleus scattering can be arbitrarily separated into two (overlapping) areas: (1) the details of the pion-nucleus reaction mechanism and (2) the response of the nucleus to the interaction. The particular states of the nucleus strongly excited by the probe (the nuclear response) depends, in addition to the reaction mechanism, on the details of nuclear structure. Thus, if the reaction mechanism is understood one has an additional tool for studying the nucleus. Optimistically, one may hope that for intermediate-energy pion-nucleus scattering the reaction mechanism will be greatly simplified because the impulse approximation will be valid.[17] In fact, even at intermediate energies the standard technique for the reduction of the many-body elastic scattering interaction to a problem involving two-body t matrices "off shell" (where one tries to relate the nuclear environment pion-nucleon t matrix

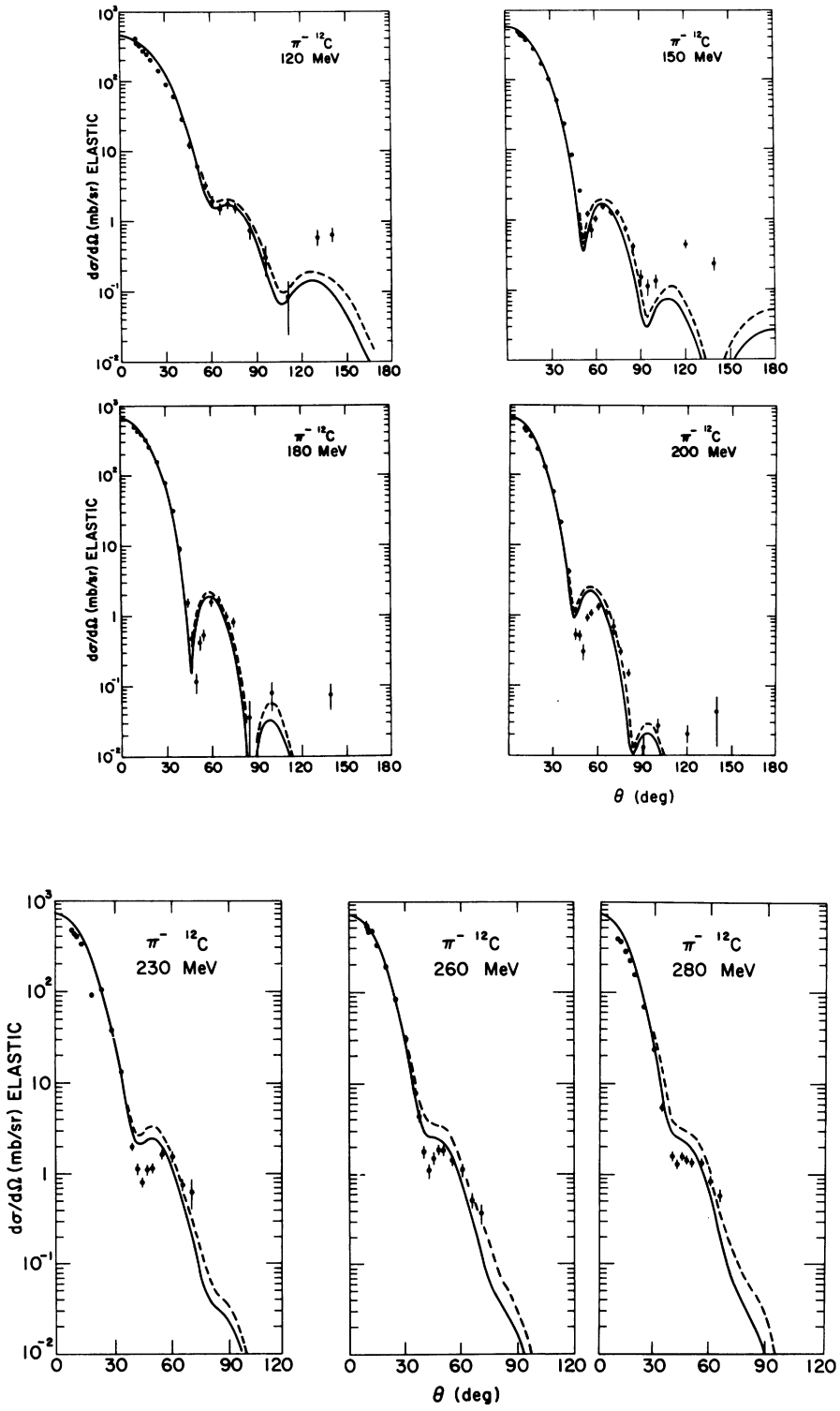

FIG. 6. Angular distributions for elastic π-^{12}C scattering for selected pion laboratory kinetic energies. The experimental data is from Ref. 11. The dotted line results from including a nuclear center-of-mass correction factor as discussed in Appendix B.

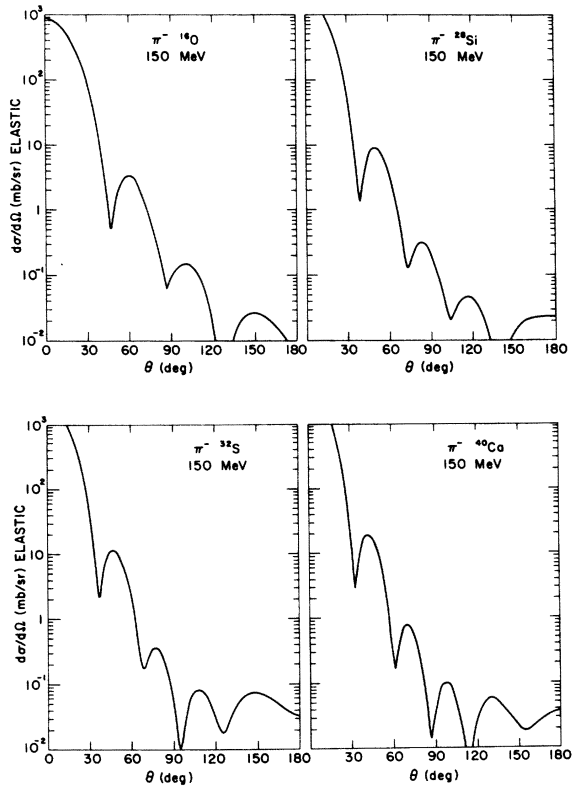

FIG. 7. Angular distributions for ^{16}O, ^{28}Si, ^{32}S, and ^{40}Ca. The center-of-mass correction factor discussed in Appendix B [see Eq. (B8)] has been included in the calculations.

to the free pion-nucleon t matrix) may not be a suitably fertile approach. One of the authors has suggested an extension of this approach in which one obtains a closed-form expression for an infinite iteration of a pion-nucleon Galilean invariant separable potential in the *many-body environment*.[4] This "effective" two-particle t matrix is then related to the optical potential. The more complicated optical potential explicitly shows the complicated intertwining between the nucleon momentum distribution in the nucleus and the "effective" pion-nucleon t matrix in the nuclear environment. From a study of this more complicated optical potential (in simple model problems) one might hope to motivate or better understand the validity of some of the important approximations made in the present paper and in Ref. 12. Such an investigation is in progress.

APPENDIX A: LABORATORY SCATTERING AMPLITUDE

We wish to obtain a partial-wave decomposition of the laboratory scattering amplitude. Assuming the laboratory data resulted from the scattering of a pion from an infinitely heavy nucleon allows one to define laboratory "fixed-scatterer" phase shifts that can then be employed in the inverse scattering problem to determine a "fixed-scatterer" pion-nucleon potential (as discussed in the main text). Our starting point is the two-body center-of-mass pion-nucleon data referred to in Ref. 8 as the CERN theoretical fit. We transform the resulting c.m. differential cross section back into the laboratory system using the standard transformation[21]

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = G^2(y)\left.\frac{d\sigma}{d\Omega}\right|_{\text{c.m.}}, \tag{A1}$$

where

$$G^2(y) = \frac{|E_\pi{}^2 + m_n{}^2 + k_{\text{c.m.}}{}^2 y^2 + 2E_\pi E_n y|^{3/2}}{m_n{}^2(E_n + E_\pi y)} \tag{A2}$$

$(y \equiv \cos\theta_{\text{c.m.}})$.

Of course, one would like to relate the scattering amplitudes in the two systems. This can be done by taking the square root of both sides of Eq. (A1). (The phases of both sides of the resulting equations may be taken to be the same with no loss of generality.) Thus we obtain

$$f^{\text{lab}}(\vec{k}'_{\text{lab}}, \vec{k}_{\text{lab}}) = G(y) f^{\text{c.m.}}(\vec{k}'_{\text{c.m.}}, \vec{k}_{\text{c.m.}}). \tag{A3}$$

For *each* total isospin channel we make an angular momentum decomposition of the laboratory and center-of-mass scattering amplitude

$$f_T^{\text{lab}} = \sum_{lJM} f_{lJT}^{\text{lab}}(k_{\text{lab}})\, Y_{l\frac{1}{2}}^{JM}(\hat{k}'_{\text{lab}}) Y_{l\frac{1}{2}}^{JM+}(\hat{k}_{\text{lab}}), \tag{A4}$$

$$f_T^{\text{c.m.}} = \sum_{l'J'M'} f_{l'J'T}^{\text{c.m.}}(k_{\text{c.m.}}) Y_{l'\frac{1}{2}}^{J'M'}(\hat{k}'_{\text{c.m.}}) Y_{l'\frac{1}{2}}^{J'M'+}(\hat{k}_{\text{c.m.}}), \tag{A5}$$

where

$$Y_{l\frac{1}{2}}^{JM}(\hat{k}) \equiv \sum_{m_l m_s} (lm_l \tfrac{1}{2} m_s \,|\, l\tfrac{1}{2} JM) Y_{lm_l}(\Omega_{\hat{k}}) u_{m_s}. \tag{A6}$$

We make a partial-wave decomposition of $G(y)$

$$G(y) = 4\pi \sum_{lm} G_l(k_{\text{c.m.}}) Y_{lm}(\Omega_{\hat{k}'_{\text{c.m.}}}) Y_{lm}^*(\Omega_{k_{\text{c.m.}}})$$

$$= \sum_l (2l+1) G_l(k_{\text{c.m.}}) P_l(y), \tag{A7}$$

where

$$G_l(k_{\text{c.m.}}) = \tfrac{1}{2}\int_{-1}^{1} \frac{(E_\pi{}^2 + m_n{}^2 + k_{\text{c.m.}}{}^2 y^2 + 2E_\pi E_n y)^{3/4}}{m_n(E_n + E_\pi y)^{1/2}} P_l(y)\, dy. \tag{A8}$$

[It has been found sufficient for our purposes in

this investigation to truncate the partial wave sum appearing in Eq. (A 7) after the $l=1$ term.] Finally, the procedure is to substitute Eqs. (A4), (A5), and (A7) into Eq. (A3), multiply by appropriate $Y_l^{JM}_{\frac{1}{2}}$ factors, carry out the appropriate spin sums, and perform the angular integration (numerically) in order to obtain a set of linear equations relating the $f^{\text{c.m.}}_{l'J'T'}$ to the f^{lab}_{ljT}. In carrying out the angular integrations numerically it is useful to note that

$$\varphi_{\text{lab}} = \varphi_{\text{c.m.}} \text{ (azimuthal angles)} \tag{A9}$$

and

$$\cos\theta_{\text{lab}} = \frac{E_\pi + E_n y}{(E_\pi{}^2 + m_n{}^2 + k_{\text{c.m.}}{}^2 y^2 + 2E_\pi E_n y)^{1/2}} . \tag{A10}$$

Scattering amplitudes for S, P, and D waves (with all possible spin and isospin combinations) have been utilized in both the two-particle laboratory and center-of-mass systems.

APPENDIX B: SINGLE-PARTICLE DENSITY

In order to keep the calculations tractable at this stage, we have adopted a single-particle density based on assuming harmonic-oscillator single-particle orbitals. The basic normalized radial-oscillator orbitals required for the five light closed-shell nuclei treated in this paper may be obtained from the formula[22]

$$\varphi_{n_i l_i}(x) = \left(\frac{2(n-1)!}{b^3[\Gamma(n+l+\frac{1}{2})]^3}\right)^{1/2} \left(\frac{r}{b}\right)^l e^{-\frac{1}{2}(r/b)^2} L_{n-1}^{l+\frac{1}{2}}(r^2/b^2), \tag{B1}$$

where the Laguerre polynomial $L_{n-1}^{l+\frac{1}{2}}$ is given by

$$L_p^a(z) = \frac{\Gamma(a+p+1)e^z}{\Gamma(p+1)z^a} \frac{d^p}{dz^p}[z^{a+p}e^{-z}] . \tag{B2}$$

The single-particle density $\rho(\vec{p}-\vec{q})$ [Eq. (3.32)] is normalized to unity for $\vec{p}=\vec{q}$.

In the actual calculations one needs the partial-wave decomposition of $\rho(\vec{p}-\vec{q})$ which is given by

$$\rho(\vec{p}-\vec{q}) = \sum_l \rho_l(p,q) P_l(\cos\theta_{pq}), \tag{B3}$$

where[23]

$$\rho_l(p,q) = \frac{4e^{-z}}{A(2l+3)(2l-1)} \big((2l+3)(2l-1)\{(2l+1)i_l(y)(\alpha-\beta' z+2\gamma z^2+\tfrac{2}{3}\gamma y^2)+y[(l+1)i_{l+1}(y)+li_{l-1}(y)](\beta'-4\gamma z)\}$$
$$+2\gamma y^2[(2l-1)(l+2)(l+1)i_{l+2}(y)+\tfrac{2}{3}l(l+1)(2l+1)i_l(y)+l(l-1)(2l+3)i_{l-2}(y)]\big) \tag{B4}$$

with[24]

$$\alpha = \tfrac{1}{4}A, \quad \beta' = \beta + 5\gamma,$$
$$y = pq\,b^2/2, \quad z = (p^2+q^2)b^4/4;$$

$$\begin{aligned}
&{}^{12}\text{C}: \ \beta=\tfrac{4}{3}, \ \gamma=0, \ b=1.64 \text{ fm},\\
&{}^{16}\text{O} \ \ \beta=2, \ \gamma=0, \ b=1.77 \text{ fm},\\
&{}^{28}\text{Si}: \ \beta=2, \ \gamma=\tfrac{4}{5}, \ b=1.80 \text{ fm},\\
&{}^{32}\text{S}: \ \beta=0, \ \gamma=\tfrac{22}{15}, \ b=1.84 \text{ fm},\\
&{}^{40}\text{Ca}: \ \beta=0, \ \gamma=2, \ b=2.03 \text{ fm}.
\end{aligned} \tag{B5}$$

Equations (B4) and (B5) are obtained assuming that the nucleons in a given nucleus occupy the lowest shell-model orbits in a j-j coupling harmonic-oscillator independent-particle model (including a single-particle spin-orbit potential that lowers $j=l+\frac{1}{2}$ states relative to otherwise identical $j=l-\frac{1}{2}$ states).

In treating pion-nucleus scattering we have assumed the nucleus is infinitely heavy so that the center of mass of the nucleus is taken as the origin of the coordinate system.

As is frequently done in shell-model calculations, we have assumed the A-nucleon coordinates are independent (in obtaining the pion-nucleus scattering amplitude). In fact, of course, the nucleon coordinates are related by the nucleus center-of-mass condition

$$\sum_{i=1}^{A} m_i r_i = 0. \tag{B6}$$

For the case of a many-particle system of equal mass constituents in a harmonic-oscillator potential and the system in the ground state, FW have pointed out that the simple and familiar center-of-mass correction given below should be used to renormalize the scattering amplitude[3]

$$f_{\text{correct}}(\vec{k}', \vec{k}) = e^{(1/A)(qb/2)^2} f_{\substack{\text{shell}\\\text{model}}}(\vec{k}', \vec{k}) . \tag{B7}$$

For the situation under consideration, involving relativistic kinematics for the pion, expression (B7) becomes, for the differential *cross section*,

$$\frac{d\sigma}{d\Omega} = \exp\left[\frac{2b^2}{A}\frac{(T^2+2m_\pi c^2 T)}{\hbar^2 c^2}\sin^2(\theta/2)\right]\left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{shell}\\\text{model}}}, \tag{B8}$$

where T is the laboratory kinetic energy of the pion.

The correction factor shown in Eq. (B8) has been incorporated in the calculations we report in this paper. The correction is negligible at forward angles for the nuclei and energies we consider (and thus does not affect total cross sections); however, at large angles the center-of-mass correction is important. For example, in ^{12}C at 100 MeV the cross section is increased ~27% at 100°; at 300 MeV the cross section is tripled at 100° due to the center-of-mass correction.

ACKNOWLEDGMENTS

We would like to acknowledge useful discussions and encouragement from F. Tabakin and R. H. Landau. One of the authors (G. E. Walker) is grateful to E. Moniz and J. D. Walecka for conversations and motivation helpful in the early stages of this investigation.

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*Work supported in part by the National Science Foundation.

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