PHYSICAL REVIEW C VOLUME 9, NUMBER 4 APRIL 1974

Deuteron stripping: A study of the sudden-approximation theory within the context of a soluble model containing an excitable nuclear core*

K. King
Department of Theoretical Physics, University of Sydney, Sydney, New South Wales, Australia 2006

Bruce H. J. McKellar
Department of Theoretical Physics, University of Sydney, Sydney, New South Wales, Australia 2006, and Department of Physics, University of Melbourne, Parkville, Victoria, Australia 3052†

(Received 12 September 1973)

We test the validity of the sudden-approximation theory of deuteron stripping of Butler, Hewitt, McKeller, and May (BHMM) by testing its predictions of final-state angular distributions and spectroscopic factors against those of an essentially soluble model of stripping reactions. This model has been constructed with an excitable nuclear core which therefore allows a calculation of the final-state spectroscopic factor to be made in advance. it is based on the solution of the Faddeev equation for multichannel separable s-wave two-body interactions between proton and neutron, proton and core, and neutron and core. There are two features which characterize BHMM; one is the sudden approximation and the other is use of intermediate neutron states of a special sort. Both these features give rise to difficulties.

I. INTRODUCTION

The use of the sudden approximation for calculations of stripping yields was suggested by Tanifuji and independently by Butler.[1] In this account the neutron-proton interaction is neglected at the instant of stripping: This approximation is valid for processes taking place during times that are less than the characteristic time of the internal motion of the deuteron and so is expected to have validity at moderately high energies. In distorted-wave theories the deuteron-nucleus interaction is all important although there is some uncertainty about its correct form. By contrast the Butler, Hewitt, McKellar, and May (BHMM) sudden-approximation theory transfers the emphasis onto a complete set of states associated with the neutron in the continuum. Optical-model wave functions are used for these which form part of matrix elements which (apart from energy conservation) represent deuteron breakup.

Empirical studies[2,3] have been conducted with the BHMM theory by applying it to (d,p) stripping reactions involving doubly magic target nuclei. The theoretical cross sections were generated from optical-model approximations for scattered neutron and proton wave functions at the required energies. It was already known *a priori* that the sudden approximation would not be valid at low incident deuteron energies. Empirically, these studies established that at energies above the target nucleus' Coulomb barrier the predicted angular distributions for the final proton were in good agreement with published experimental distributions. A comparison of theoretical and experimental amplitude normalizations provided an estimate of the spectroscopic factor (S) for the final-state nucleus. Unfortunately, the extracted value of S was found to be energy-dependent, even for energies above the Coulomb barrier. This was illustrated in the study[3] done on the ^{208}Pb$(d,p)^{209}$Pb(g.s.) reaction. Here the predicted value of S steadily rose from 0.50 at 14.8 MeV incident energy to 0.73 at 27.5 MeV. (The Coulomb barrier height is nearly 16 MeV.) In addition, the BHMM spectroscopic estimates were characteristically lower than those of distorted-wave theory.

It thus became desirable to test the features of the theory in an ideal situation for which the spectroscopic factor is known exactly by other means. A solvable three-body model which was constructed in order to do this is presented in this paper. The distinguishing feature of this model is that it contains one particle (the infinitely heavy nuclear core) which is excitable. The nucleon-core interactions are thus multichannel. Coupling the channels together produces ground-state spectroscopic factors of less than unity. These can be calculated in advance from a knowledge of the two-body parameters. The remaining two particles are light and identical and so the model can simulate nuclear reactions involving an excitable nucleus and two nucleons. Three-body forces, antisymmetrization. and breakup are not included and all parti-

cles are assumed to be spinless. The nucleons can bind together to form a "deuteron" and either one of them can bind separately to the core.

If stripping theories are cast with the same limitations, they can be tested against the essentially exact model calculations. This is what was done. The sudden approximation was examined thoroughly and distorted-wave calculations were sometimes included for comparison.

The model itself is based on the Faddeev equations. These were solved using separable two-channel two-body interactions from which the corresponding optical potentials were calculated exactly. (Optical potentials are required as data for BHMM calculations, although in realistic cases one has to rely on nonunique phenomenological parametrizations.)

The BHMM and distorted-wave Born-approximation (DWBA) calculations that were performed within the restrictions on the model revealed the same general features that characterized the more realistic calculations[2,3] mentioned earlier. It was found that the BHMM estimate of the spectroscopic factor was a monotonically increasing function of the incident-deuteron energy. This estimate then leveled off at a value near that of the DWBA estimate which appeared to be essentially energy-independent. At this point however, *both* theories overestimated the spectroscopic factor. Further calculations have revealed the cause of the energy dependence of the BHMM spectroscopic factor to be, not the sudden approximation, but the erroneous assumption that the optical wave functions together with the bound-state wave function form a complete set of neutron wave functions.

In Sec. II we give a schematic discussion of BHMM in a notation which is used later. The model is described in terms of the interaction used (Sec. III), the solution of the three-body equations (Sec. IV), and the casting of the approximative theories BHMM and DWBA into the same form (Sec. V). The remaining sections are devoted to discussing the results and to a conclusion.

II. SUDDEN-APPROXIMATION (BHMM) THEORY OF DEUTERON STRIPPING

This theory has been discussed elsewhere.[2,4] We repeat here the essential ideas in the same notation that we later use to discuss the theory and the three-body model.

The formal many-body matrix element[5] for deuteron stripping is given in its unsymmetrized form by

$$M = \langle \underline{\phi}_n \psi_p^{(-)} | \, V^{(12)} | \, \underline{\psi}_d^{(+)} \rangle , \tag{1}$$

where $\underline{\psi}_d^{(+)}$ is the incident-deuteron wave function; $V^{(12)}$ is the neutron-proton interaction; $\underline{\psi}_p^{(-)}$ is the final-state proton scattered from the target, with boundary conditions for incoming waves; and $\underline{\phi}_n$ is the wave function of the final-state neutron bound to the core.

The deuteron and bound-neutron wave functions are n-dimensional vectors whose components represent the amplitudes in each of the n states we assume for the core.

The spectroscopic amplitude $S^{1/2}$ for the final-state neutron is given by projection of the ground-state core out of the final state

$$\underline{\alpha}_1^\dagger \, | \, \underline{\phi}_n \rangle = S^{1/2} | \, \underline{\phi}_n \rangle , \tag{2}$$

where we have chosen the particular n-dimensional normalized vector

$$\underline{\alpha}_1^\dagger = (1, 0, \ldots, 0) \tag{3}$$

as the state vector of the ground state of the core and $\underline{\phi}_n$ is the bound-state neutron wave function normalized to 1.

In the BHMM approach it is assumed, in common with most direct-reaction theories, that the major contribution to the stripping cross section comes from the deuteron incident on the core ground state. The matrix element, with this restriction imposed, is referred to as M_C, where

$$M \simeq M_C = \langle \underline{\phi}_n \psi_p^{(-)} | \, V^{(12)} | \, \underline{\alpha}_1 \underline{\alpha}_1^\dagger \underline{\psi}_d^{(+)} \rangle . \tag{4}$$

Here the ground-state projection operator is just $\underline{\alpha}_1 \underline{\alpha}_1^\dagger$. Hence, applying the relationship for the spectroscopic amplitude (2),

$$M_C = S^{1/2} \langle \phi_n \psi_p^{(-)} \underline{\alpha}_1 | \, V^{(12)} | \, \underline{\alpha}_1^\dagger \psi_{d(1)}^{(+)} \rangle , \tag{5}$$

where $\psi_{d(1)}^{(+)}$ is the first term of the expansion of the full deuteron wave function into terms involving the various core states $\underline{\alpha}_i$; viz.

$$\underline{\psi}_d^{(+)} = \sum_{i=1}^{n} \underline{\alpha}_i \psi_{d(i)}^{(+)} . \tag{6}$$

That is $\psi_{d(1)}^{(+)}$ represents that part of the deuteron wave function which does not involve core excitation.

If a complete set of neutron states $\underline{\psi}_{n;j}^{(+)}$ is inserted into the right-hand side of Eq. (5) the matrix element M_C can be transformed without approximation into one which will render the central features of BHMM explicit.

$$M_C = \sum_j \int d\vec{k}_n \int d\xi_p S^{1/2} \langle \phi_n \psi_p^{(-)} \underline{\alpha}_1 | \, \underline{\psi}_{n;j}^{(+)} \xi_p \rangle \times \langle \underline{\psi}_{n;j}^{(+)} \xi_p | \, V^{(12)} | \, \psi_{d(1)}^{(+)} \underline{\alpha}_1 \rangle , \tag{7}$$

where the summation is over the core state j upon which the intermediate neutron is incident and the integration includes all the momentum states of

the intermediate neutron, both bound and scattered. The proton coordinates are grouped in ξ_p which is formally summed. The $\underline{\psi}^{(+)}_{n;j}$ have to be complete for both neutron and core in the sense that, if i' is the core index and $\vec{k}'_n$ the neutron coordinate

$$\int d\vec{k}_n \sum_j \underline{\psi}^{(+)}_{n;j}(\vec{k}'_n, i')\underline{\psi}^{(+)\dagger}_{n;j}(\vec{k}''_n, i'') = \delta_{i'i''}\delta(\vec{k}'_n - \vec{k}''_n)\,. \tag{8}$$

The term involving an intermediate bound neutron which is the same as the final-state neutron can be made explicit by substitution of $\underline{\phi}_n$ and use of Eq. (2). This term is simply SM_C as can be seen by comparing it to the expression for M_C given in Eq. (5). The other bound-state overlaps vanish. Hence

$$M_C = SM_C + S^{1/2}M_S\,, \tag{9}$$

where

$$M_S = \sum_{j=1}^{n} \int^{1} d\vec{k}_n \int d\xi_p \langle \phi_n \psi_p^{(-)} \underline{\alpha}_1 | \underline{\psi}^{(+)}_{n;j} \xi_p \rangle \times \langle \underline{\psi}^{(+)}_{n;j} \xi_p | V^{(12)} | \psi^{(+)}_{d(1)} \underline{\alpha}_1 \rangle \tag{10}$$

the dash indicating that the bound states are not included in the integration.

Therefore the stripping cross section can be written in the form

$$\frac{d\sigma}{d\Omega} \propto \frac{S}{(1-S)^2} |M_S|^2\,, \quad S \neq 1\,. \tag{11}$$

This expression is correct to the extent that the full matrix element M is given by M_C. No further approximations have been made.

The central feature of the BHMM theory is the use of the sudden approximation to describe the incident-deuteron wave function in terms of the wave functions for the neutron and proton. In this way it differs from DWBA where the emphasis is on the deuteron-core interaction, about which there is some uncertainty. In BHMM the emphasis (and the uncertainty) is transferred onto a complete set of states associated with the neutron in the continuum. Equation (10) can be written

$$M_S = \sum_{j=1}^{n} \int d\vec{k}_n \int d\vec{k}_p \langle \phi_n \psi_p^{(-)} \underline{\alpha}_1 | \underline{\psi}^{(+)}_{np;j}(\vec{k}_n, \vec{k}_p) \rangle \times \langle \underline{\psi}^{(+)}_{np;j}(\vec{k}_n, \vec{k}_p) | V^{(12)} | \psi^{(+)}_{d(1)} \underline{\alpha}_1 \rangle\,. \tag{12}$$

Here $\underline{\psi}^{(+)}_{np;j}(\vec{k}_n, \vec{k}_p)$ is the state consisting of isolated core and scattered neutron and proton with asymptotic momenta $\vec{k}_n$ and $\vec{k}_p$, respectively. Although this involves a mixture of core states it is specified by the core state j upon which the neutron and proton are incident.

The sudden-approximation deuteron transform is used to evaluate the last factor which includes the neutron-proton interaction $V^{(12)}$.

The other feature of BHMM is the assumption that the only core state which contributes significantly to the intermediate sum is the ground state $j=1$. Thereafter a product of optical wave functions can be used to describe $\underline{\psi}_{np}$, i.e.

$$\underline{\alpha}_1^{\dagger} | \underline{\psi}^{(+)}_{np;1}(\vec{k}_n, \vec{k}_p) \rangle = | \psi^{n(+)}_{\vec{k}_n} \psi^{p(+)}_{\vec{k}_p} \rangle\,, \tag{13}$$

where $\psi^{n(+)}_{\vec{k}_n}$ and $\psi^{p(+)}_{\vec{k}_p}$ are generated by the optical potential which describes elastic scattering on the ground-state core.

III. TWO-BODY SEPARABLE INTERACTION

A. One-term single-channel separable interaction

Over the last two decades, a lot of work[6] has been done on the properties and uses of an interesting class of nonlocal interactions known as "separable" or sometimes as "factorable." For the simplest such case, the one-term single-channel separable interaction, the potential operator factorizes. In the momentum representation it is written

$$V(\vec{p}, \vec{p}') \equiv \langle \vec{p} | V | \vec{p}' \rangle = -\lambda g(\vec{p}) g(\vec{p}')\,, \tag{14}$$

where λ is a parameter measuring the strength of the interaction and g (which is a function of the relative momentum $\vec{p}$) is the form factor providing the "shape" of the interaction. This can be written alternatively in projection-operator notation

$$V = -\lambda |g\rangle\langle g|\,, \tag{15}$$

where

$$\langle \vec{p} | g \rangle \equiv g(\vec{p})\,. \tag{16}$$

One reason this potential has gained so much attention is the fact that the Schrödinger equation is easily solved for both bound and continuum states. Usually this can be done in closed form, depending on the analytic properties of the form factor. Separable interactions can thus be pedagogical devices which illustrate in a clear way many aspects of collision theory. But they are also used in model calculations to elucidate aspects of more realistic calculations as in this paper and elsewhere.[7-10]

A second reason for using such potentials is that they may in fact represent the "real world" in certain instances. Thus, a potential due to Yamaguchi[11] reproduces the low-energy neutron-proton scattering parameters and leads to the Hulthén form of the deuteron bound state. However, one characteristic of this type of potential is that it generates at most one bound state, which seriously limits its application to physical problems. This problem can be circumvented through introducing

a sum of separable terms or a multichannel separable interaction.

There are additional difficulties in using separable potentials for the nucleon-nucleus interaction. Since the nucleus is a relatively large object, even at low energies a large number of partial waves contribute. This can in principle be taken into account by a multiterm separable interaction but at a great cost in simplicity. For this reason the exact formulation described in Sec. IV can be regarded only as a crude model of the realistic (d,p) reaction.

The Schrödinger equation for a pair of particles interacting through the potential (14) or (15) is

$$(z-T)|\phi\rangle = V|\phi\rangle\,, \tag{17}$$

where $H_0 = T$ is the kinetic energy of the relative motion. For the bound state $|\phi_\alpha\rangle$ at energy $z=-\alpha^2$ (in units of energy defined by requiring $\hbar^2/2m=1$) this becomes

$$|\phi_\alpha\rangle = -G_0(-\alpha^2)\lambda|g\rangle N_\alpha\,, \tag{18}$$

where G_0 is the Green function $(E-H_0)^{-1}$ and we have defined

$$N_\alpha \equiv \langle g|\phi_\alpha\rangle\,. \tag{19}$$

Hence, projecting out the form factor from the wave equation (18)

$$[1+\langle g|G_0(-\alpha^2)|g\rangle\lambda]N_\alpha = 0\,.$$

The condition for a nontrivial solution N_α is the condition for a bound state to exist. This is

$$D_+(-\alpha^2)=0\,, \tag{20}$$

where by definition

$$D_\pm(z) \equiv 1+\langle g|G_0^\pm(z)|g\rangle\lambda\,. \tag{21}$$

The normalization constant N_α is determinable to within a phase factor since

$$1=\langle\phi_\alpha|\phi_\alpha\rangle = |\lambda N_\alpha|^2\langle g|G_0(-\alpha^2)G_0(-\alpha^2)|g\rangle\,. \tag{22}$$

It can be demonstrated that there can exist at most one bound state and that this state, together with the solutions for positive energy, forms a complete orthonormal set.

B. One-term multichannel separable interaction

The potential operator describing the interaction between a light projectile (nucleon) and a fixed target (infinitely heavy excitable nucleus) can be represented by an $n\times n$ matrix, where n is the number of states of the target. Each element of the matrix is an operator acting on the coordinates of the projectile.

A particularly simple case, which we consider henceforth, is that in which the matrix elements are each factorable in the form (15) with different constants λ. Thus we can write for this case

$$V = -\underline{\Lambda}|g\rangle\langle g|\,, \tag{23}$$

where the diagonal elements of the strength matrix Λ refer to the different channels (ground and excited core states) and the off-diagonal terms to the coupling between them. [In the numerical work described later, this strength matrix is a real 2×2 matrix

$$\underline{\Lambda} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{pmatrix}. \tag{24}$$

Λ_{11} refers to the strength in the ground channel where the core is unexcited. Λ_{22} refers to the excited channel whilst Λ_{12} $(=\Lambda_{21})$ is a measure of the strength of the coupling between the two.]

The unperturbed Hamiltonian is given by

$$\underline{H}_0 = T + \underline{H}_C\,, \tag{25}$$

where T is the kinetic energy operator of the projectile and $\underline{H}_C$ (an $n\times n$ matrix) is the core Hamiltonian defined by

$$\underline{H}_C\underline{\alpha}_i = \eta_i\underline{\alpha}_i\,, \tag{26}$$

where $\underline{\alpha}_i$ is the wave function of the core in the ith state and η_i is its excitation relative to ground. Thus $\eta_1=0$.

The perturbed states are solutions of the Schrödinger equation

$$(z-\underline{H}_0)|\underline{\phi}(z)\rangle = \underline{V}|\phi(z)\rangle\,. \tag{27}$$

As before the bound and continuous states, solutions of the Schrödinger equation, can be shown to form a complete orthonormal set.

Bound states in the potential-well limit. The multichannel potential is required for a description of the nucleon-core interaction where the core, for simplicity of later calculations, is assumed to be infinitely massive. The bound states in this limit are given by a formula similar in structure to that used for the ordinary one-channel case:

$$|\underline{\phi}_\alpha\rangle = -\underline{G}_0(-\alpha^2)|g\rangle\underline{\Lambda}\underline{N}_\alpha\,, \tag{28}$$

where the bound-state energy is $-\alpha^2$ and by definition

$$\underline{N}_\alpha \equiv \langle g|\underline{\phi}_\alpha\rangle\,. \tag{29}$$

So if we define

$$\underline{D}_+(z) \equiv 1+\langle g|\underline{G}_0^{(+)}(z)|g\rangle \tag{30}$$

we can determine $\underline{N}_\alpha$ but only to within a constant multiplicative factor, by

$$\underline{D}_+(-\alpha^2)\underline{N}_\alpha = 0\,. \tag{31}$$

The over-all constant is found by normalizing the bound-state wave function

$$1 = \langle \underline{\phi}_\alpha | \underline{\phi}_\alpha \rangle = \underline{N}_\alpha^\dagger \underline{\Lambda} \langle g | \underline{G}_0(-\alpha^2)^2 | g \rangle \underline{\Lambda} \underline{N}_\alpha . \quad (32)$$

From (31) we see that binding occurs for $z = -\alpha^2$ when

$$\det \underline{D}(-\alpha^2) = 0 . \quad (33)$$

Spectroscopic factor. The operator $\underline{\alpha}_k \underline{\alpha}_k^\dagger$, with matrix elements $(\underline{\alpha}_k \underline{\alpha}_k^\dagger)_{ij} = \delta_{ik}\delta_{jk}$, projects onto the kth state of the core and so the spectroscopic strength of the bound state at $z = -\alpha^2$ in this channel is given by

$$S_k = \langle \underline{\phi}_\alpha | \underline{\alpha}_k \underline{\alpha}_k^\dagger | \underline{\phi}_\alpha \rangle . \quad (34)$$

Using the explicit form of the bound-state wave function

$$\langle \underline{k}_i \underline{\alpha}_k | \underline{\phi}_\alpha \rangle = (\alpha^2 + k_i^2 + \eta_k)^{-1} g(\underline{k}_i) \underline{\alpha}_k^\dagger \underline{\Lambda} \underline{N}_\alpha \quad (35)$$

we find

$$S_k = |\underline{\alpha}_k \underline{\Lambda} \underline{N}_\alpha|^2 \langle g | \underline{G}_0(-\alpha^2 - \eta_k)^2 | g \rangle . \quad (36)$$

It can be verified directly that

$$\sum_k S_k = 1 \quad (37)$$

as is to be expected.

Thus it is possible to calculate the wave functions and the spectroscopic factor from a knowledge of the two-body parameters. The form factors can be chosen to enable these calculations to be performed analytically.

T matrix. It can be shown by using the Lippman-Schwinger equation and summing the Born series that

$$\underline{T}(z) = -|g\rangle \underline{\Lambda} \underline{D}_+(z)^{-1} \langle g| \quad (38)$$

from which the two-body scattering cross sections are obtainable.

From this T matrix we can construct optical potentials which reproduce the scattering in the incident channel. The potentials again are separable. As we see below in Sec. V, this potential for particles incident on the ground state of the core is

$$V_{\text{opt}} = -\lambda_{\text{opt}}(z) |g\rangle \langle g| , \quad (39a)$$

where the strength parameter λ_{opt} is a complex scalar function of the energy z. The form factor g is unchanged.

IV. THREE-BODY PROBLEM

Nuclear reactions are by their very nature many-body and multichannel problems. A complete description of them is beyond the scope of current theory because formal and numerical problems abound. This has generally necessitated the introduction of approximation methods such as the optical model, distorted wave, and impulse approximation.

Scattering can be reduced to a simple two-body problem through the introduction of phenomenological potentials. Genuine rearrangements, on the other hand (such as stripping and pickup) must be regarded as at least three-body problems, and then only if the core is assumed to be inert.

Even in this simplified task one is confronted by a number of difficulties. One can use little of the intuition or methods developed for two-body cases. For instance, when two of the three particles can bind, the Born-series expansion for the three-particle collision matrix does not converge at any energy.[12,13] Furthermore, the Lippmann-Schwinger equations for three-particle systems are not solvable by finite-matrix methods.

Faddeev[12] developed equations which solved all the formal mathematical difficulties. There were two major difficulties associated with the Lippmann-Schwinger approach: The integral equations were homogeneous for cases where two of the three particles are asymptotically bound, and the equations contained disconnected diagrams. The first was solved by using coupled-integral equations and the second by having the potentials occur only in the three-body T matrices for two interacting and one free particle. However, practical difficulties remained. The three-body solution required the *complete* solution to the two-body problem (on-shell and off-shell) and also involved, in general, integral equations in six integration variables.

A. Solutions to the three-body problem involving separable interactions

There are, however, some limited formulations of the three-body equations which are tractable because they reduce the number of coordinates which are needed to specify intermediate states. This treatment is exact if we use separable interactions, but can be generalized since local interactions can be expanded to arbitrary accuracy as a finite sum of separable interactions.

These models thus allow exact treatment of the specifically three-body effects but the price paid is the inexact treatment of the two-body interaction, particularly the high-energy behavior. However, our work is motivated by the desire to test approximation schemes for bound states and low-energy rearrangement reactions, and the procedure is reasonable in the present case.

Mitra devised an exactly soluble three-body model[7,14] in which he solved the Schrödinger equation. He used s-state interactions in the nucleon-nucleon and two nucleon-core pairs, and assumed iden-

tical spinless nucleons and a static spinless core. His solutions involved single-parameter functions which obeyed coupled-integral equations. (This feature is common to all such models.) He cast DWBA into the form required by his model and, on testing, found that it was valid within the model.

A similar model[15] was derived by Amado which was based on field-theoretic considerations. This approach is equivalent to that based on the Faddeev formalism if separable two-body interactions are used.[16] It has been generalized by Shanley[9] to include spin and isospin variables. Shanley tested DWBA using the same phenomenology as is used in realistic cases; the optical-model parameters were derived for a potential of the Woods-Saxon type and the stripping amplitude was evaluated in the zero-range approximation. Whilst it was possible to reproduce scattering in this way (albeit with rather unphysical parameters) the stripping cross sections showed less agreement with the exact calculations than that which characterizes the comparison of realistic calculations to experiment.

Beregi, Lovas, and Revai[10] proposed a model based on the Faddeev equations. This model contained a sum of separable interactions describing the two-body forces and thus allowed them to construct a potential which could bind either of the two identical fermions in one of two possible energy states, thereby providing in the elastic scattering channel a compound-type resonance. This model, whilst suffering from the usual two-body restrictiveness, included all the essential features of nuclear reactions. Like the others it could be solved exactly. It also allowed the application of approximation schemes which could thus be compared directly. In this way the authors treated the isolated-resonance approximation and the coupled-channels approximation with and without exchange. The approximate calculations differed markedly from the exact results.

In the model presented here the three-body equations used are of the Faddeev type. The two-body interactions are multichannel s-wave separable potentials from which one can calculate the exact optical-model potentials for neutron-core and proton-core scattering and the exact spectroscopic factor. The sudden and distorted-wave theories of deuteron stripping have been formulated within the model in an effort to test not only the resulting angular distributions but also the extracted spectroscopic factors by comparing the shapes and normalizations against those of the essentially exact treatment.

TABLE I. The three-body energy spectrum.

Three-body energy	Rearrangment threshold
0	n-p continuum (breakup threshold)
$\eta-\alpha_d^2$	d continuum, excited core
$-\alpha_d^2$	d continuum (pickup threshold)
$-\alpha_n'^2$	p continuum, excited neutron-core bound state
$-\alpha_p'^2$	n continuum, excited proton-core bound state
$-\alpha_n^2$	p continuum [(d,p) stripping threshold]
$-\alpha_p^2$	n continuum [(d,n) stripping threshold]

B. Three-body formalism using multichannel separable interactions

In the model developed here it is assumed: (i) that only single-term multichannel s-wave separable interactions operate in the three two-body subsystems; (ii) that the core (A) is infinitely massive, spinless, and has internal degrees of freedom that lead to a ground ($E=0$) and at least one excited state ($E=\eta_i$; $i>1$); (iii) that the two nucleons ($n;p$) are equally massive (mass m) and spinless. (As before we use units defined by $\hbar^2/2m=1$.); and (iv) that nucleon-exchange and many-body forces do not operate. Breakup is not calculated explicitly.

These are the same conditions under which the amplitudes in the approximation schemes, considered in Sec. V, are derived.

Hence, it is possible to describe the two-body interactions: $n+p \rightarrow n+p$; $n+A \rightarrow n+A$; $n+A \rightarrow n+A^*$; $p+A \rightarrow p+A$; $p+A \rightarrow p+A^*$.

We will label the proton as particle 1, the neutron as particle 2, and the core as particle 3.

None of the scatterings have angular structure since the interaction is in the $l=0$ partial wave only. Also obtainable are the bound states for d, B, or B^* (neutron core), and C or C^* (proton core).

In the three-body system the Faddeev equations are used to relate the amplitudes of the processes which lead finally to any of the six states:

$$d+A, \quad d+A^*, \quad p+B, \quad p+B^*, \quad n+C, \quad n+C^*$$

from any of these states given initially. That is, the following reactions can be studied: pickup: (p,d), (n,d); stripping: (d,p), (d,n); exchange scattering: (p,n), (n,p); elastic and inelastic scattering: (p,p), (n,n), (d,d), for ground or excited targets and final-state nuclei.

Calculations based on this model, as well as those based on approximative schemes, are presented for various parametrizations in Sec. VI. The structure of the model is now discussed in more detail.

Two-body interactions. In the soluble model used, only two-body forces are assumed to operate. The interaction between the nucleons and the heavy excitable core were taken to be of the form

$$\underline{V}^{(23)} = -\underline{\Lambda}^{(23)} |g^{(23)}\rangle\langle g^{(23)}| \tag{39b}$$

for the neutron, labeled as "2," and the core "3." The proton-core interaction is likewise $\underline{V}^{(31)}$.

The neutron-proton interaction is taken to be the s-state single-channel interaction

$$V^{(12)} = -\lambda^{(12)} |g^{(12)}\rangle\langle g^{(12)}| . \tag{40}$$

Sometimes, for formal convenience, we write

$$\underline{\Lambda}^{(12)} = \underline{1}\lambda^{(12)} \tag{41}$$

and

$$\underline{V}^{(12)} = \underline{1}V^{(12)} , \tag{42}$$

where $\underline{1}$ is the unit $n\times n$ matrix, n being the number of channels employed (in our case $n=2$).

Three-body spectrum. Table I shows a schematic threshold spectrum for the case where there are two-channel potentials each of which is capable of binding the nucleon to the core twice. Note that this implies two states for the core. The core excitation energy is taken to be $\eta_2 = \eta$. For instance $\underline{V}^{(23)}$ binds the neutron at energies $-\alpha_n^2$ and $-\alpha_n'^2$. The second channel energy (core excitation) relative to ground is η.

Coordinates. The three-body states are labeled as either $|\vec{k}_1, \vec{k}_2\rangle$ or $|\vec{p}, \vec{k}\rangle$, where $\vec{k}_1$ is the momentum of the proton referred to the core, $\vec{k}_2$ is the momentum of the neutron referred to the core,

$$\vec{p} = \vec{k}_1 + \vec{k}_2 , \tag{43}$$

$$\vec{k} = \tfrac{1}{2}(\vec{k}_1 + \vec{k}_2) . \tag{44}$$

Transition operator. The two-body transition operators embedded in the three-body space become

$$\langle \vec{p}', \vec{k}' | T^{(12)}(z) | \vec{p}, \vec{k}\rangle = -\delta(\vec{p}' - \vec{p}')\lambda^{(12)} g^{(12)}(\vec{k}') \times g^{(12)}(k)\underline{D}_+^{(12)}(z - p^2/2)^{-1} \tag{45}$$

and

$$\langle \vec{k}_1', \vec{k}_2' | \underline{T}^{(23)}(z) | \vec{k}_1, \vec{k}_2\rangle = -\delta(\vec{k}_1' - \vec{k}_1) g^{(23)}(\vec{k}_2') g^{(23)}(\vec{k}_2) \underline{\Lambda}^{(23)} \underline{D}_+^{(23)}(z - k_1^2)^{-1} . \tag{46}$$

The matrix element for $\underline{T}^{(31)}$ is formally similar to that for $\underline{T}^{(23)}$.

Calculating the three-body rearrangement cross sections. The following is an outline of the scheme used to obtain expressions of these cross sections. The Faddeev equations are written out in the form of three coupled-integral equations which relate the wave functions of the three allowed configurations (each consisting of two bound particles and one free particle). However, if an n-channel interaction is used, each wave function is itself an n vector and the Faddeev equations are a set of $3n$ coupled-integral equations.

The number of integrations normally required is six because the intermediate states are three-body states. These integrations however can be reduced through the use of s-state separable interactions. At this stage, for convenience, the equations are recast in terms of coefficients $\underline{W}$ defined in terms of the configuration wave functions and the various form factors $\underline{V}$ of the constituent two-body interactions. A separation in partial waves is performed because the nature of the interactions allows the partial-wave contributions to uncouple. That is, the integral equations can be solved separately for each partial wave and this reduces the integrations required by intermediate states. The contributions are added up later for some finite number of contributing partial waves. The transition amplitudes for the allowed reactions are related simply to the coefficients $\underline{W}$, which in turn are the direct solutions to the simplified integral equations.

Faddeev equations. The two-body bound states within the three-body system are given asymptotically by

$$\langle \vec{p}, \vec{k} | \underline{\Phi}_i^{(12)}(\vec{p}_d)\rangle = \phi_d(\vec{k})\underline{\alpha}_i \delta(\vec{p} - \vec{p}_d) \tag{47}$$

for an incoming deuteron. Here $\phi_d(\vec{k})$ is the wave function of the internal motion of the deuteron, $\delta(\vec{p} - \vec{p}_d)$ describes its center-of-mass motion, and $\underline{\alpha}_i$ describes the state of the core upon which it is incident.

In a similar way, for an incoming proton

$$\langle \vec{k}_1, \vec{k}_2 | \underline{\Phi}_\mu^{(23)}(\vec{k}_p)\rangle = \underline{\phi}_\mu^{(23)}(\vec{k}_2)\delta(\vec{k} - \vec{k}_p) , \tag{48}$$

where $\underline{\phi}_\mu^{(23)}(k_2)$ describes the neutron-core state and $\delta(\vec{k}_1, \vec{k}_p)$ the proton. A similar equation holds for the incoming neutron.

The Faddeev equations yield the full wave function which is given as the sum of three configuration wave functions:

$$\underline{\Psi}^+ = \underline{\psi}_{12} + \underline{\psi}_{23} + \underline{\psi}_{31} . \tag{49}$$

If the particles i and j are initially bound then the wave function is calculated by solving the $3n$ coupled-integral equations for the configuration wave functions as follows:

$$\begin{pmatrix} \underline{\psi}_{jk} \\ \underline{\psi}_{ki} \\ \underline{\psi}_{ij} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{0} \\ \underline{\Phi}_\mu^{(ij)} \end{pmatrix} + \underline{G}_0(z) \begin{pmatrix} \underline{0} & \underline{T}^{(jk)} & \underline{T}^{(jk)} \\ \underline{T}^{(ki)} & \underline{0} & \underline{T}^{(ki)} \\ \underline{T}^{(ij)} & \underline{T}^{(ij)} & \underline{0} \end{pmatrix} \begin{pmatrix} \underline{\psi}_{jk} \\ \underline{\psi}_{ki} \\ \underline{\psi}_{ij} \end{pmatrix} . \tag{50}$$

In this equation the Green function is given in terms of the Hamiltonian $\underline{H}_0$ [see Eq. (25)]. This is the sum of a core Hamiltonian and the kinetic operator. The transition matrices are the two-body operators which are now embedded in the three-body space. The operators for the single-channel neutron-proton interaction are embedded in the vector space of the core. That is, $\underline{T}^{(12)}$ and $\underline{D}^{(12)}$ are now diagonal matrices given by

$$[\underline{T}^{(12)}(z)]_{ij} = \delta_{ij} T^{(12)}(z - \eta_j) \tag{51}$$

and

$$[\underline{D}^{(12)}_{\pm}(z)]_{ij} = \delta_{ij} D^{(12)}_{\pm}(z - \eta_j) . \tag{52}$$

Reduction of the number of integrations required. This is achieved in the first instance by the use of separable interactions and later by restricting their operation to the s state only.

To formulate the Faddeev equations in a way which emphasizes the separability of the interactions, it is convenient to define operators $\underline{W}$ as follows:

$$\underline{W}^{(ij)}(\vec{q}_k) \equiv \int d\vec{p}_k g^{(ij)}(\vec{p}_k) \langle \vec{p}_k \vec{q}_k | \underline{\psi}^{(lm)}_{ki}(\vec{k}_v) + \underline{\psi}^{(lm)}_{jk}(\vec{k}_v) \rangle . \tag{53}$$

The momentum variables span the full three-body space: $\vec{p}_k$ is the relative momentum for particles i and j and $\vec{q}_k$ is the center-of-mass momentum of the pair. The momentum of the incident particle is $\vec{k}_v$ asymptotically. The i, j, and k are cyclic indices. The initial configuration is described by the superscript $(lm) = (12)$, (23), or (31). Henceforth, the label (lm) and the parameter $\vec{k}_v$ will often be regarded as implicit since both characterize the given initial state. They are not variables to be summed over.

Thus terms used in the Faddeev equations (50) may be expressed in terms of the coefficients $\underline{W}$:

$$\langle \vec{p}_k, \vec{q}_k | \underline{T}^{(ij)}(z) \underline{\psi}_{jk} + \underline{\psi}_{ki} \rangle = -\underline{\Lambda}^{(ij)} \underline{D}^{(ij)}_{+}(z - q_k^2)^{-1} g^{(ij)}(\vec{p}_k) \underline{W}^{(ij)}(\vec{q}_k) . \tag{54}$$

Also needed are the operators, defined by

$$\underline{A}^{(j)}_{\pm}(\vec{q}_k, \vec{q}_i) \equiv \langle \vec{q}_k g^{(ij)} | \underline{G}_0^{\pm} | \vec{q}_i g^{(jk)} \rangle , \tag{55}$$

and the source terms with the form factors projected out. These are defined for a given initial configuration (lm) to be

$$\underline{F}^{(lm)}_{l}(\vec{q}_l) = \langle \vec{q}_l g^{(mn)} | \underline{\Phi}^{(lm)}_{\mu} \rangle , \tag{56}$$

where μ labels the initial bound-state energy and l, m, n are cyclic. With superscripts denoting the initial state omitted, the Faddeev equations become:

$$\underline{W}^{(12)}(\vec{x}) = -\int d\vec{y} \, [\underline{A}^{(2)}_{+}(\vec{x}, \vec{y}) \underline{\Lambda}^{(23)} \underline{D}^{(23)}_{+}(z - y^2)^{-1} \underline{W}^{(23)}(\vec{y}) + \underline{A}^{(1)}_{-}(\vec{y}, \vec{x})^* \underline{\Lambda}^{(31)} \underline{D}^{(31)}_{+}(z - y^2)^{-1} \underline{W}^{(31)}(\vec{y})] + \underline{F}_3(\vec{x}) \tag{57}$$

and similarly, by cyclic permutation, the equations for $\underline{W}^{(23)}$ and $\underline{W}^{(31)}$ are obtained. The six integration variables which are required in the general case have been reduced to three. These can be reduced still further.

Uncoupling the partial waves. The various partial waves of the Faddeev equations uncouple because the potentials act only in the s state.

The matrix elements of $\underline{A}$ are dependent only on the magnitudes x, y and the angle between the vectors $\vec{x}, \vec{y}$. This is because the form factors depend only on the magnitude of their parameters for s-state interactions

$$g^{(ij)}(\vec{k}) = g^{(ij)}(\kappa) , \quad \text{where } \kappa = |\vec{k}| . \tag{58}$$

Hence, the lth partial wave of $\underline{A}$ is given by

$$\underline{A}^{(j)l}_{+}(x, y) = \iint d\hat{x} d\hat{y} \, Y_l^{m*}(\hat{x}) Y_l^m(\hat{y}) \underline{A}^{(j)}_{+}(\vec{x}, \vec{y}) . \tag{59}$$

In addition, because the incident beam is chosen to lie along the z axis, we may partial-wave analyze $\underline{F}_k$ and $\underline{W}^{(ij)}$ in terms of the coefficients

$$\underline{F}^{l}_{k}(x) = \int d\hat{x} Y_l^0(\hat{x}) \underline{F}_k(\vec{x}) , \tag{60}$$

$$\underline{W}^{(ij)l}(x) = \int d\hat{x} Y_l^0(\hat{x}) \underline{W}^{(ij)}(x) \tag{61}$$

and obtain equations for the partial-wave components. These are

$$\underline{W}^{(12)l}(x) = -\int y^2 dy[\underline{A}_+^{(2)l}(x,y)\underline{\Lambda}^{(23)}\underline{D}_+^{(23)}(z-y^2)^{-1}\underline{W}^{(23)l}(y) + \underline{A}_-^{(1)l}(y,x)^*\underline{\Lambda}^{(31)}\underline{D}_+^{(31)}(z-y^2)\underline{W}^{(31)l}(y)] + \underline{F}_3^l(x) \quad (62)$$

and similarly for the (23) and (31) configurations. These equations do not couple different partial waves.

Transition amplitudes. Using the identity

$$\underline{V}^{(ij)}|\underline{\Psi}^+\rangle = (z-\underline{H}_0)|\underline{\psi}_{ij}\rangle \quad \text{for all } k, \quad (63)$$

the transition amplitude for the (n,j) process becomes

$$T_{jn} = \langle \underline{\Phi}_\mu^{(ki)}(\vec{k}_\mu)|(z-\underline{H}_0)|\underline{\psi}_{ij}^{(lm)}(\vec{k}_\nu) + \underline{\psi}_{jk}^{(lm)}(\vec{k}_\nu)\rangle$$
$$= -\underline{N}_\mu^{(ki)\dagger}\underline{\Lambda}^{(ki)}\underline{W}^{(ki)}(\vec{k}_\mu, \vec{k}_\nu; lm) \quad (64)$$

(here by definition $\underline{N}_\mu^{(12)} \equiv N^{(12)}\underline{\alpha}_\mu$, $\underline{\alpha}_\mu$ being the core state).

As before (lm) is the initial bound pair and (ki) the final.

In practice the coefficients $\underline{W}$ are directly calculated, not the wave functions. From these the theoretical cross sections can be simply derived.

V. APPROXIMATION SCHEMES

This section details the description of the sudden-approximation and distorted-wave theories of stripping. It begins with the derivation of the optical potentials because these are required as data for the theories.

The optical model provides a means of describing the scattering of a particle from a nuclear core whose degrees of freedom are more than simply those of center-of-mass translation. It may, for instance, have internal degrees of freedom associated with its deformations and vibrations. In these cases the incident particle could excite one of the core energy levels and scatter inelastically. The elastic beam is thus attenuated and this effect can be simulated by the use of a single-channel potential which contains an imaginary part.

Optical potentials can be calculated exactly from the separable multichannel interactions that the model uses. Thus, errors due to the uncertainty of phenomenological derivations can be eliminated and the physical approximations of different reaction theories can be tested without these complications.

Historically the first theory to deal with direct nuclear reactions was one due to Butler. This is a plane-wave Born approximation with a radial integration cutoff designed to simulate the effects of absorption in the nuclear interior. Although no more than qualitatively correct, it did provide some information on angular momentum transfer and spectroscopic parameters.

A modification of this theory, DWBA,[17] takes into account the effect of distorting the incident and final waves. The assumption involved here is that deuteron scattering from the core dominates and so the incident wave function may be approximated by the product of the internal deuteron wave function and a "distorted" wave function representing the scattering of the deuteron from the core. It is usually assumed too, that the core is inert and that the scattered wave function can be derived from an optical-model potential fitted phenomenologically to (d,d) data.

Although DWBA works remarkably well, little attempt has been made to justify its basic assumptions from a purely theoretical viewpoint, since it is an approximation to a completely incalculable many-body amplitude. Its success in predicting angular distributions is certainly surprising considering the apparent unreasonableness of using deuterons which suffer no internal distortion. It is precisely for these reasons that the sudden approximation was investigated as an alternative.

The BHMM theory[4] provides a different physical picture of the direct process but still gives reasonably good agreement with experimental angular distributions.[2] It differs markedly in its predictions of the spectroscopic factor over those of DWBA. Hence, this attempt to determine the validity of these theories within our model.

A. Optical potential derived from multichannel separable interactions

An optical potential[18] provides the same elastic scattering in the ground channel and has an imaginary part describing the loss of flux to the excited channels. The ground channel is labeled 1; the remaining vector of channels is labeled I.

$$V_{\text{opt}} = V_{11} + \underline{V}_{1\text{I}}(z - T - \underline{U}_{\text{II}} + i\epsilon)^{-1}\underline{V}_{\text{I}1}, \quad (65)$$

where

$$\underline{V} \equiv \begin{pmatrix} V_{11} & \underline{V}_{1\text{I}}^\dagger \\ \underline{V}_{\text{I}1} & \underline{V}_{\text{II}} \end{pmatrix}$$

and

$$U_{ij} = V_{ij} + \delta_{ij}\eta_j. \quad (66)$$

If, as assumed earlier, the potential has the form $\underline{V} = -\underline{\Lambda}|g\rangle\langle g|$, then the optical potential assumes the shape

$$V_{\text{opt}} = -\lambda_{\text{opt}}(z)|g\rangle\langle g|, \quad (67)$$

where $\lambda_{opt}(z)$ is a complex number dependent on the energy z.

Calculation of V_{opt} from the T matrix. Optical potentials are calculated by requiring that they provide the same scattering as the elastic channel of the multichannel potential. The one-channel optical T matrix is

$$T_{opt}(z) = -|g\rangle\lambda_{opt}(z)D_{+opt}(z)^{-1}\langle g|, \tag{68}$$

where

$$D_{+opt}(z) = 1 + \langle g|G_0(z)|g\rangle\lambda_{opt}(z). \tag{69}$$

From Eq. (38) for the multichannel case, the elastic T matrix becomes

$$\begin{aligned} T_{11}(z) &= \underline{\alpha}_1^{\dagger}\underline{T}(z)\underline{\alpha}_1 \\ &= -|g\rangle[\underline{\Lambda}\underline{D}_+^{-1}(z)]_{11}\langle g|. \end{aligned} \tag{70}$$

Comparing Eq. (68) with Eq. (70) the requirement becomes

$$\lambda_{opt}(z)D_{+opt}(z)^{-1} = [\underline{\Lambda}\underline{D}_+^{-1}(z)]_{11}. \tag{71}$$

From a two-channel potential we obtain the optical-model parameter

$$\lambda_{opt}(z) = \frac{\Lambda_{11} + I_2\Lambda}{1 + I_2\Lambda_{22}}, \tag{72}$$

where in this case

$$\underline{\Lambda} \equiv \begin{bmatrix} \Lambda_{11}\Lambda_{12} \\ \Lambda_{21}\Lambda_{22} \end{bmatrix}, \tag{73}$$

and we have written

$$\Lambda \equiv \det\underline{\Lambda} \tag{74}$$

and

$$I_i \equiv \langle g|G_0^{+}(z - \eta_i)|g\rangle. \tag{75}$$

Note that I_2 and hence $\lambda_{opt}(z)$ are real unless $z \geq \eta_2$. This is because the only resonances to which the potential gives rise involve the excitation of the core whose second energy level (there are only two) lies at an energy η_2 above ground. Thus no averaging is needed for $z < \eta_2$ and the potential is real in this region.

Also note that

$$\lambda_{opt}(z) \xrightarrow[z\to\infty]{} \Lambda_{11}. \tag{76}$$

In practice, λ_{opt} is very close to Λ_{11} at energies greater than about twice the inelastic threshold energy η_2.

B. Sudden approximation (BHMM)

The BHMM cross section for (d, p) stripping is given by

$$\frac{d\sigma}{d\Omega} = \frac{\frac{1}{2}m_p m_d}{(2\pi\hbar^2)^2}\frac{k_p}{p_d}\frac{S}{(1-S)^2}(2\pi)^6|M_S|^2, \quad (S \neq 1), \tag{77}$$

where for given incident-deuteron momentum $\vec{p}_d$ and final proton momentum $\vec{k}_p$

$$\begin{aligned} M_S = \int\int d\vec{k}_p' d\vec{k}_n' \langle \phi_n \psi_{\vec{k}_p}^{(-)} | \psi_{\vec{k}_n'}^{n(+)} \psi_{\vec{k}_p'}^{p(+)} \rangle \\ \times \langle \psi_{\vec{k}_n'}^{n(+)} \psi_{\vec{k}_p'}^{p(+)} | V^{(12)} | \psi_{\vec{p}_d}^{d(+)} \rangle. \end{aligned} \tag{78}$$

In Eq. (78) ϕ_n is the final bound state of the neutron and the core which is found by solving the Schrödinger equation using the form of the neutron-core optical potential but with a real strength parameter $\lambda^{(23)}$ fitted to reproduce the appropriate binding energy. (This is called the "well-depth prescription.") The n and p scattered states are calculated using the optical potentials whilst that for the deuteron is approximated to by using the sudden approximation.

The numerical procedures involved are:

(i) Calculation of the neutron overlap

$$\langle \phi_n | \psi_{\vec{k}_n'}^{n(+)} \rangle;$$

(ii) Calculation of the proton overlap (or proton S matrix)

$$\langle \psi_{\vec{k}_p}^{p(-)} | \psi_{\vec{k}_p'}^{p(+)} \rangle;$$

(iii) Evaluation of the transition amplitude

$$\langle \psi_{\vec{k}_n'}^{n(+)} \psi_{\vec{k}_p'}^{p(+)} | V^{(12)} | \psi_{\vec{p}_d}^{d(+)} \rangle$$

using the sudden-approximation deuteron transform; and

(iv) Integration over $\vec{k}_n'$ and $\vec{k}_p'$.

Yamaguchi forms are used throughout.

The neutron bound state, as mentioned, is generated by the "well-depth prescription." This is equivalent to a one-channel calculation using the same form factor as the optical potential.

$$|\phi_n\rangle = -G(-\alpha_{23}{}^2)\lambda^{(23)}|g^{(23)}\rangle N^{(23)}, \tag{79}$$

where for the single-channel case the strength is

$$\lambda^{(23)} = \frac{4}{\pi}\beta_{23}(\alpha_{23} + \beta_{23})^2 \tag{80}$$

and the normalization is

$$N^{(23)} = \left[\frac{\pi\alpha_{23}}{4\beta_{23}(\alpha_{23} + \beta_{23})}\right]^{1/2}. \tag{81}$$

The continuum wave function for the neutron is generated by the full optical potential

$$|\psi_{\vec{k}_n'}^{(+)}\rangle = [1 - \lambda_{opt}G_0^{+}(k_n'^2)|g\rangle D_{+opt}(k_n'^2)^{-2}\langle g|]|\vec{k}_n'\rangle \tag{82}$$

and a similar expression pertains to the wave function for the proton.

The deuteron transform is used to evaluate the last term in Eq. (78). The transform is evaluated

for the interaction $V^{(12)}$ described in terms of the separable interaction whose form factor is $g^{(12)}$. This interaction gives rise to the Hulthén form of the internal wave function for the deuteron:

$$\phi_{\alpha_{12}}(\vec{r}) = \frac{N}{(4\pi)^{1/2}} \frac{e^{-\gamma_{12} r} - e^{-\beta_{12} r}}{r}, \tag{83}$$

where

$$N^{-2} = \frac{1}{2\gamma_{12}} - \frac{2}{\gamma_{12}+\beta_{12}} - \frac{1}{2\beta_{12}} \tag{84}$$

and

$$\gamma_{12}^{\,2} = \alpha_{12}^{\,2}/2 . \tag{85}$$

The net result is an expression for the BHMM cross section for (d, p) stripping.

$$\frac{d\sigma}{d\Omega} = \frac{S}{(1-S)^2} 2\pi \frac{k_p}{p_d} (N^{(23)})^2 N^2 (\beta_{12}^{\,2} - \gamma_{12}^{\,2}) |T_1 + T_2|^2 , \tag{86}$$

where

$$T_1 = \frac{[\lambda^{(23)} - \lambda^{(23)}_{\rm opt}(k_n^{\,2})][D^{(23)}_{+\rm opt}(k_n^{\,2})]^{-1}}{(\alpha_{23}^{\,2}+k_n^{\,2})(\beta_{23}^{\,2}+k_n^{\,2})(p_d^{\,2}/4 + k_p^{\,2} - p_d k_p \cos\theta + \beta_{12}^{\,2})} , \quad \vec{k}_n \equiv \vec{p}_d - \vec{k}_p \tag{87}$$

and θ is the angle between $\vec{k}_p$ and $\vec{p}_d$ and

$$T_2 = \frac{i\pi}{2} \int_{-1}^{+1} \frac{d\cos\theta' [\lambda^{(23)} - \lambda^{(23)}_{\rm opt}(q^2)][D^{(23)}_{+\rm opt}(q^2)]^{-1}}{(\alpha_{23}^{\,2}+q^2)(\beta_{23}^{\,2}+q^2)(p_d^{\,2}/4 + k_p^{\,2} - p_d k_p \cos\theta' + \beta_{12}^{\,2})} \frac{k_p \lambda^{(31)}_{\rm opt}(k_p^{\,2})[D^{(31)}_{+\rm opt}(k_p^{\,2})]^{-1}}{(\beta_{31}^{\,2}+k_p^{\,2})} , \tag{88}$$

$$\vec{q} \equiv \vec{p}_d - |\vec{k}_p| \hat{\vec{k}}'_p$$

and θ is the angle between $\vec{k}_p$ and $\vec{p}_d$. These terms correspond, respectively, to the momentum and energy δ functions in the proton S matrix.

C. Distorted-wave Born approximation (DWBA)

The cross section for the DWBA formulation of the stripping problem is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{\hbar}\right)^4 \frac{k_p}{p_d} m_p m_d |T^{\rm DW}_{f'i}|^2 , \tag{89}$$

where

$$T^{\rm DW}_{f'i} = \langle \underline{\phi}_n \chi^{(-)}_f | V^{(12)} | \underline{\alpha} \chi^{(+)}_i \rangle , \tag{90}$$

$\underline{\alpha}\chi^{(+)}_i$ is the distorted incoming deuteron incident on the core state $\underline{\alpha}$, and $\underline{\phi}_n \chi^{(-)}_f$ is the distorted outgoing nucleon incident on the neutron-core bound state $\underline{\phi}_n$.

This form of the stripping amplitude can be considered as the result of three distinct approximations to the exact amplitude, which is

$$T_{f'i} = \langle \underline{\phi}_n \vec{k}_p | V^{(12)} + V^{(31)} | \underline{\Psi}^{(+)}_d (\vec{p}_d) \rangle . \tag{91}$$

Here (d, p) stripping is to the final state described by the neutron bound to the core $\underline{\phi}_n$ with the proton free, having asymptotic momentum $\vec{k}_p$. The full solution for scattered deuteron is expanded by inserting a complete set of deuteron and core states:

$$|\underline{\Psi}^{(+)}_d\rangle = \sum_{ij} |\phi_{d(i)} \underline{\alpha}_j\rangle \langle \underline{\alpha}_j \phi_{d(i)} | \underline{\Psi}^{(+)}_d \rangle . \tag{92}$$

The first and most important approximation[19] in DWBA consists of taking only the first term of the expansion, consisting of bound deuteron and ground-state core:

$$|\underline{\Psi}^{(+)}_d\rangle \simeq \int d\vec{p} \, |\phi_d \vec{p} \underline{\alpha}_1\rangle \langle \underline{\alpha}_1 \vec{p} \phi_d | \underline{\Psi}^{(+)}_d \rangle . \tag{93}$$

This is the many-body approximation employed in most direct-reaction theories and it is thought that the basic direct-reaction characteristics are embodied in it.

The second approximation, as in BHMM, consists in calculating the final-state scattered proton from the optical potential.

The third approximation consists of calculating the incident scattered deuteron wave function from the optical-model potential for deuteron-core scattering V_d. That is

$$\langle \underline{\alpha}_1 \vec{p} \phi_d | \underline{\Psi}^{(+)}_d \rangle \simeq \langle \vec{p} | \chi^{(+)}_d \rangle , \tag{94}$$

where

$$(E - T - V_d) |\chi^{(+)}_d\rangle = 0 . \tag{95}$$

From the formalism it is possible to make the first two approximations. The third approximation cannot be made unless further assumptions are made concerning the shape of the deuteron-core optical potential. This potential cannot be a single s-state potential such as that assumed for the proton-core and neutron-core scattering cases because these could never preserve the calculated structure of the exact (d, d) calculations. Thus this approximation can only be made by using the same phenomenology as is used in realistic DWBA calculations—using the calculated (d, d) curves as the "empirical" data and fitting the parameters on the basis of either a potential

well formula or a formula consisting of the sum of separable interactions.

Shanley[9] in fact worked with a fitted Woods-Saxon-type well but this is in no clear way related to the constituent separable interaction in the neutron-core subsystem.

By contrast, the method employed here is to make the first two approximations of DWBA. This is done by projecting out the ground-state core and the internal (undistorted) deuteron wave function from the exactly calculated wave function $\underline{\Psi}_d^{(+)}$. The optical-model wave function for proton scattering is also used for the final state.

$$T_{\mathrm{DW}} = \int d\vec{p}\langle\phi_n \chi_p^{(-)} | V^{(12)} | \vec{p}\phi_d \underline{\alpha}_1\rangle \langle \underline{\alpha}_1 \phi_d \vec{p} | \underline{\Psi}_d^{(+)}\rangle$$
$$= -\lambda^{(12)} N^{(12)} S^{1/2} \int d\vec{p}\langle\phi_n \chi_p^{(-)} | g^{(12)} \vec{p}\rangle$$
$$\times \langle \underline{\alpha}_1 \phi_d \vec{p} | \underline{\Psi}_d^{(+)}\rangle . \quad (96)$$

The results should show DWBA in its best possible light since this theory usually makes further approximations to obtain the deuteron wave function for the calculation. Our approximations here are similar to those made by Johnson and Soper,[20] who go on to derive different equations for the projection of $\underline{\Psi}_d^{(+)}$.

VI. RESULTS

This section details the comparison of exact to approximate calculations. It includes the parameters used, an outline of the numerical solution, and a discussion of the results obtained.

A. Parameters of the neutron-proton interaction

It is assumed that the separable interactions act only through the s state and that they have the Yamaguchi form, i.e.:

$$\langle \vec{k} | g\rangle = g(\vec{k}) = Y_0^0(\hat{\vec{k}}) h(k) = (4\pi)^{-1/2}(\beta^2 + k^2)^{-1}, \quad (97)$$

where

$$\langle \vec{k}' | V | \vec{k}\rangle = -\lambda \langle \vec{k}' | g\rangle \langle g | \vec{k}\rangle .$$

The neutron-proton interaction (one channel) is chosen[11] so as to reproduce the low-energy data for the triplet case; namely binding energy $\alpha_{12}^2 = 2.225$ MeV and triplet scattering length $a_T = 5.38$ fm.

The parameters are listed in Table II. This potential leads to the Hulthén form of the internal deuteron wave function [see Eqs. (83)–(85)].

TABLE II. The standard neutron-proton potential.

Binding energy	α_{12}^2	2.225 MeV
Shape parameter	β_{12}^2	43.527 MeV
Strength parameter	λ_{12}	216.226 MeV/fm
Overlap	N_{12}	0.864 fm
Total zero-energy triplet-scattering cross section	σ_{tot}	3.63 b

B. Parameters of the nucleon-core interaction

These were chosen to be s-wave Yamaguchi interactions acting in two channels which could be coupled. The parameters were similar to those employed by Shanley[9] except that allowance was made for a finite final-state proton-core interaction. (Identical neutron-core and proton-core parameters were used throughout.) The only restriction was that $\beta > \alpha$, for otherwise the bound-state wave function would not have had the appropriate asymptotic form which is

$$\phi_\alpha(r) \sim \frac{e^{-\alpha r}}{r} . \quad (98)$$

The parameters could be found numerically on the basis of four pieces of data: α^2 is the binding energy of ground-state nucleon-core combination; η is the second-channel excitation energy, i.e., core excitation energy relative to ground; σ_{tot} is the total zero-energy elastic scattering cross section; and c is the percentage coupling of the core-ground and core-excited states through the interaction. In general this is defined, for core states i and j as:

$$c_{ij} = \left| \frac{\Lambda_{ij}\Lambda_{ji}}{\Lambda_{ii}\Lambda_{jj}} \right| \times 100\% . \quad (99)$$

In the case of the real symmetric matrix used here, where the elastic scattering in both channels is identical ($\Lambda_{11} = \Lambda_{22}$),

$$c = c_{12} = \left| \frac{\Lambda_{12}}{\Lambda_{11}} \right|^2 \times 100\% . \quad (100)$$

The main calculations were based on a standard

TABLE III. The standard nucleon-core potential V_1.

Binding energy	α^2	7.0 MeV
Shape parameter	β^2	14.914 MeV
Strength matrix	$\underline{\Lambda}$	$\begin{pmatrix} 36.860 & 11.656 \\ 11.656 & 36.860 \end{pmatrix}$ MeV/fm
Overlap	$\underline{N}$	$\begin{pmatrix} 1.566 \\ 1.285 \end{pmatrix}$ fm
Coupling	c	10%
Total zero-energy elastic scattering cross section	σ_{tot}	1.0 b
Core excitation	η	2.5 MeV
Spectroscopic factor for ground state	S	0.651

potential V_1 whose parameters are listed in Table III. This potential imitated the neutron-^{208}Pb interaction in some respects. Notice the following: ^{208}Pb has a core state (3^-) at 2.614 MeV; a ground neutron-bound state at 3.93 MeV; an imaginary component in the optical potential which is approximately 10% of the real strength; and a BHMM-estimated spectroscopic factor of 0.65. The corresponding values for this separable potential are 2.5 MeV, 7.0 MeV, 10%, and 0.651, respectively.

Other potentials were found as variants of V_1 derived by altering one input datum at a time. One interesting feature of the model is that the spectroscopic factor is dependent almost entirely upon only two of the variables—the ground-state binding energy and the core excitation energy. It is virtually constant (±5%) over three orders of magnitude of both the assumed zero-energy scattering cross section (0.1 to 100 b) and the assumed percentage coupling (0.1 to 100%).

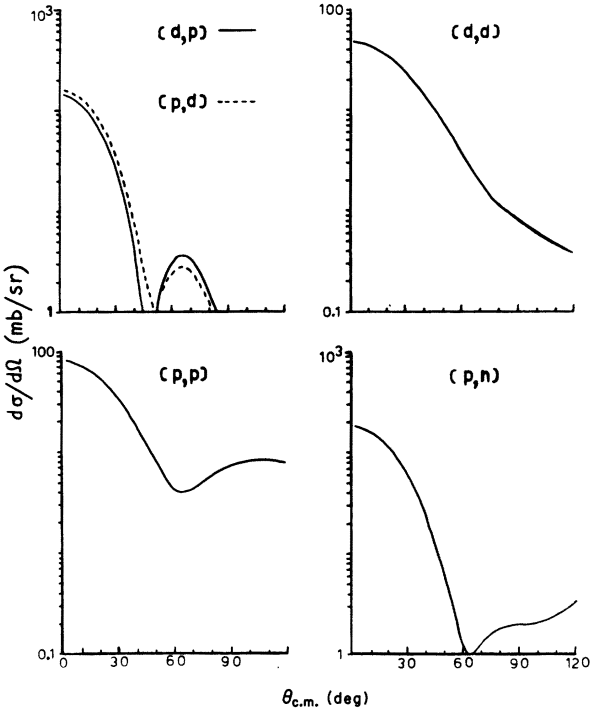

FIG. 1. The calculated differential cross sections for three-body rearrangements using potential V_1 and a center-of-mass energy of 25 MeV. The solid curve represents the direct (d,p) calculation whilst the dotted curve is the (d,p) cross section as calculated from the (p,d) cross section. This is an indication of the numerical accuracy of solution. The remaining rearrangement cross sections are also shown except those which are obtainable by time reversal and charge conjugation (neutron and proton parameters being the same).

C. Optical-model parameters

These parameters were calculated along the lines set out in Sec. V and checked by reproducing the elastic scattering of the full potential. Potential V_1 binds a neutron only once although it is

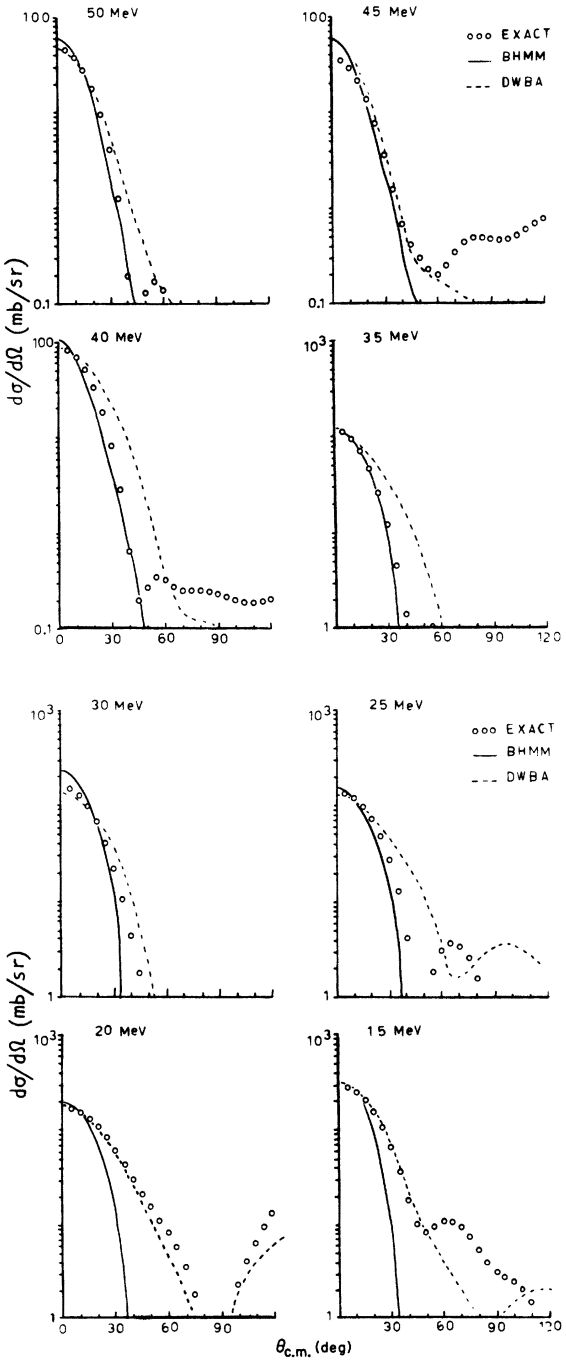

FIG. 2. The comparison of the exact and approximate solutions for (d,p) stripping at various energies and assuming nucleon-core interaction V_1.

possible to derive potentials which have another bound or quasibound state. The potentials were required by the sudden-approximation calculations. For a (d, p) reaction the proton potential was required at the energy of the final proton and the neutron and the neutron potential at a range of energies (approximately 0–100 MeV).

D. Numerical solution of the three-body equations

The equations could be written such that the different angular momentum contributions uncoupled as shown in Sec. IV. These coupled equations could be solved by matrix inversion provided the integrals were replaced by finite sums. This was made difficult by singularities which occur in the $\underline{A}$ matrices and the propagators $\underline{D}^{-1}$.

The bound-state singularities could be made explicit by using the pole-dominance form of the propagator for energies near the bound states:

$$\det \underline{D}_+(-\gamma^2) \sim (\gamma - \alpha - i\epsilon)^{-1} f(\alpha), \tag{101}$$

where $z \equiv -\gamma^2$ and α^2 is the energy of the bound state and parametrizes the remaining function f. The logarithmic principal parts and the residues were also functions of α but could be distributed as the weighted sum of terms each of which depended on one of the mesh points of the matrix. (The mesh points of the matrix had been determined by a quadrature formula which replaced the well-behaved part of the integral by a finite sum.) In this way the singularities could be included within the finite matrix.

The calculations were performed on the IBM 7040 and the English Electric KDF 9 computers belonging to the Basser Computing Department of the University of Sydney. The calculations were checked in various ways:

Convergence in l. The sum of the contributions from each of the angular momentum states converged. Contributions from

$$l > l_{\max} \simeq k r_{\text{rms}}$$

amounted to less than one part in 10^6. In practice $l_{\max}$ was small (around 5 or 6) and could be reliably estimated from the incident momentum k and the rms radius of the bound-state wave function

$$r_{\text{rms}}^2 = 4\pi \int_0^\infty r^4 |\phi(\vec{r})|^2 dr. \tag{102}$$

Stability of numerical integration and interpolation procedures. A priori reasonable alterations in the integration mesh or interpolation procedure did not alter the cross sections by more than about ±1%.

Time-reversal invariance. The calculations for the forward reaction were compared to those for the backward reaction which are based on an entirely different source vector and matrix. (That is, the calculations were not trivially identical.) This comparison, carried out for a number of data sets, yielded an over-all estimate of the accuracy of the numerical procedure of ±3%. See Fig. 1 for an example.

It should be remembered that normalization of

TABLE IV. Potentials used to test BHMM.

Data set	Differing datum [a]	Shape parameter β^2/(MeV)	Strength parameter λ_{11}/(MeV/fm)	Overlap $\underline{N}$/fm	Spectroscopic factor
V_1	•••	14.914	36.860	$\begin{pmatrix}1.566\\1.285\end{pmatrix}$	0.651
V_2	$\alpha^2 = 10$ MeV	31.546	95.685	$\begin{pmatrix}1.175\\1.033\end{pmatrix}$	0.602
V_3	$\alpha^2 = 5$ MeV	5.436	11.288	$\begin{pmatrix}2.328\\1.711\end{pmatrix}$	0.723
V_4	$c = 5\%$	42.572	130.841	$\begin{pmatrix}1.012\\0.827\end{pmatrix}$	0.648
V_5	$c = 2\%$	137.013	607.118	$\begin{pmatrix}0.603\\0.490\end{pmatrix}$	0.646
V_6	$c = 0\%$	41.955	150.868	$\begin{pmatrix}1.265\\0.000\end{pmatrix}$	1.000
				$\begin{pmatrix}0.773\\0.000\end{pmatrix}$ [b]	1.000
V_7	$\eta = 1.0$ MeV	33.646	89.819	$\begin{pmatrix}1.035\\0.968\end{pmatrix}$	0.556

[a] The potentials are the same as V_1 (see Table III) with respect to three of the four input parameters, α^2, η, σ_{tot}, and c.

[b] These values are for the second bound state at $\alpha^2 = 4.5$ MeV.

real experimental data could be uncertain to at least ±5%. Small normalization errors in this work will not significantly affect the BHMM spectroscopic estimates because of the relatively weak dependence this has on the ratio of BHMM to "exact" amplitudes, nor will they affect the DWBA estimates very much since any error will be communicated to the DWBA in the same way through the use there of the "exact" deuteron wave function.

The calculations were also done in the single-channel limit using the parameters of Shanley.[9] The distributions were in good agreement. As also noticed by Shanley and by Noble,[8] the effect of the proton-core interaction in the final state is very small.

The results of calculations for all the forward (incident-deuteron) and all the backward (incident-proton[21]) reactions for the standard potential at 25 MeV are shown graphically in Fig. 1. Note the fairly unstructured distribution for proton scattering. (For protons scattering directly off the core there is no structure because the interaction is in the s state.)

E. Comparison of the approximate with the exact calculations

The results for various center-of-mass energies in the stripping reaction using potential V_1 are shown in Fig. 2. Typical results for various other potentials (which are listed in Table IV) are displayed in Fig. 3. The extracted spectroscopic factors are given in Fig. 4(a) for both DWBA and BHMM.

Energy dependence of spectroscopic factors. The BHMM distributions improved with energy although the spectroscopic factor overshot the exact value. In fact there was only a very small region around 25-MeV center-of-mass energy where the extracted S was close to the real S. (See Fig. 4.) Calculations using potentials with different spectroscopic factors also yielded fairly good predictions. (See Fig. 3 and Table V.) This problem is discussed more fully in Sec. VIII.

Spectroscopic factors near unity. A test was made of a speculation[22] that BHMM would fail for potentials with spectroscopic factors close to unity because of compound-nuclear contributions. This

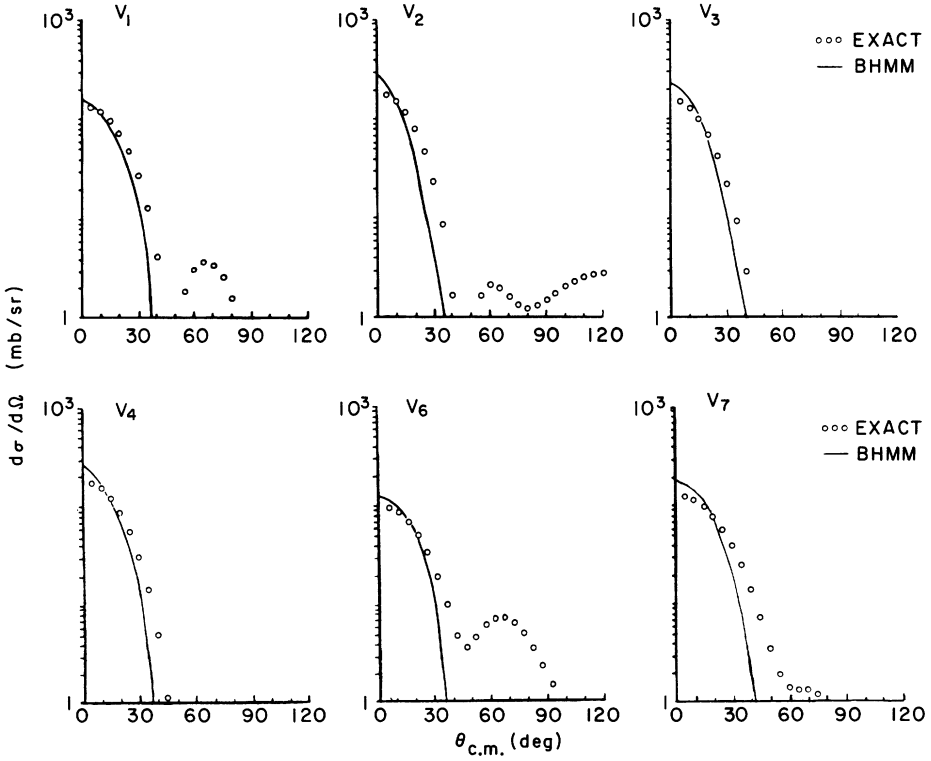

FIG. 3. The (d,p) stripping cross sections for various potentials (listed in Table IV). The potentials were constructed with different assumptions as to the binding energy, core excitation, or channel coupling. The spectroscopic factors (exact and BHMM extracted) are listed in Table V. Total energy is 25 MeV for each reaction, and the calculations are done exactly and also in the sudden approximation (BHMM).

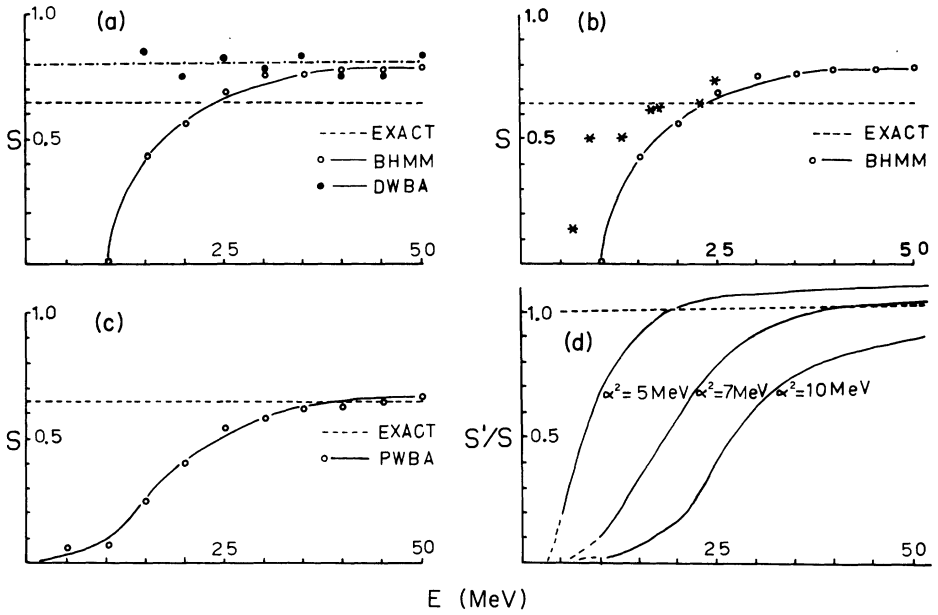

FIG. 4. These figures illustrate the point that the energy dependence of the BHMM S is characteristic of the theory and due to the use of optical-model neutron wave functions as described in the text. The first three figures show (1) the exact spectroscopic factor for potential V_1 calculated directly from the two-body data; $S=0.651$, and (2) the spectroscopic factors extracted by BHMM from a comparison of the approximate and exact (d,p) calculations worked out with V_1. The points are at 5-MeV intervals and the curve is fitted by inspection. The corresponding angular distributions are given in Fig. 2. The energy E is the reaction center-of-mass energy. (a) BHMM and DWBA spectroscopic factors are compared for the same potential V_1 and the same energies 15(5)50 MeV. (b) BHMM spectroscopic factors extracted from real experimental (d,p) cross sections (marked *) for a ^{208}Pb target show a similar energy dependence to that found in the model calculations. (c) The spectroscopic factors extracted from manipulated plane-wave calculations are compared to the exact spectroscopic factor, $S=0.651$ for V_1. These plane-wave calculations use the BHMM manipulation but not the sudden approximation. Refer to Eq. (107). (d) The ratio of the extracted spectroscopic factor S' to the exact spectroscopic factor S as a function of three-body energy for various potentials which bind the nucleon to the core by different amounts. The calculations were done with a plane-wave matrix element as in Eq. (107) and as depicted also (for V_1) in Fig. 4(c). The binding energy is the dominant parameter determining the approach to the symptotic value.

was found to be true only for those potentials which provided a quasibound state for the neutron at some positive energy (that is, where there was a singularity in the analytically derived optical potential for neutron scattering). No reasonable attempt to treat this singularity exactly or phenomenologically could bring the BHMM shape predictions into agreement with the exact calculations. However, the forward peak, essentially unaffected by the pole term, still provided a spectroscopic estimate which was close to the real value, provided this was done for energies close to 25 MeV.

The potential V_6 which has two bound state, has $S=1$ because of zero coupling (see Table IV). The BHMM curves are still comparitively good and the spectroscopic factor extracted is also unity. The sudden approximation factors into a shape term and a normalization term which, because it depends on the difference between the optical and single-channel strength parameters, vanishes for zero coupling. The BHMM can thus be calculated from the shape term alone or by taking the limit of the full calculations as the coupling tends to zero.

Stability of BHMM calculations. The sensitivity of the BHMM calculations to changes in the parameters was tested. Changes in the Hulthén parameter for the neutron-proton interaction produced negligible difference in both the shape and normalization of the BHMM predictions for the

TABLE V. Spectroscopic factors predicted by BHMM at $E_d=25$ MeV.

Data set	S (exact)	S (BHMM)
V_1	0.651	0.68
V_2	0.602	0.47
V_3	0.723	0.82
V_4	0.648	0.70
V_6	1	1
V_7	0.556	0.56

range $5 < \beta_{12}/\gamma_{12} < 7$. The actual value used in the exact calculations was $\beta_{12}/\gamma_{12} = 6.258$. The sensitivity to the optical potential was tested by using various two-body potentials which reproduced the same low-energy data (binding energy, core excitation, and scattering cross section). These potentials could be generated by changing the assumed coupling between ground and excited channels. The shape and strength of the potentials were then adjusted to preserve the low-energy two-body predictions. The change in the actual spectroscopic factor (that is, as calculated analytically) was slight, although the change in the strength and shape parameters was large, producing correspondingly large changes in the optical potentials. The BHMM (d, p) predictions using these potentials differed negligibly in shape from those using the "correct" potential and changes in the extracted spectroscopic factor were slight. See Table VI.

VII. CRITIQUE OF THE BHMM APPROACH

The inadequacy of BHMM can be shown empirically in the energy dependence of results obtained for spectroscopic estimates. The reason for its failure is the incorrect assumption that the bound state and the optical-model scattered wave functions for the neutron incident on the core are complete. Moreover the essence of BHMM cannot be preserved as will be demonstrated.

A. Energy dependence of BHMM spectroscopic factors

The energy dependence of the spectroscopic factor extracted by the sudden approximation has been the subject of some discussion.[23] The dependence derived by comparing extracted to exact spectroscopic factors is illustrated for our standard "^{208}Pb-like" potential V_1 in Fig. 4(a). This behavior is very similar to that found with realistic calculations[24] on ^{208}Pb as shown in Fig. 4(b). Thus the inaccuracies in the spectroscopic-factor extraction cannot be simply ascribed to errors in experimental normalizations, but reflect a very real weakness in the BHMM approach. The DWBA calculations are also shown in Fig. 4(a) for comparison. It should be remembered however that these latter calculations were based on the "real wave function," not one derived from deuteron scattering and thus involved one approximation less than realistic calculations. This may not necessarily have been "advantageous" as it meant that there was no ambiguity in the wave function and the fits could not have been improved by simply opting for another deuteron potential which satisfied the scattering data. On the other hand, the calculations presented were less reliable (±15%) than either the "exact" ones (±3%) or the BHMM (essentially analytic) because they involved many meshed numerical integrations. No valid inference may be derived concerning the extraction of the spectroscopic factor by this method unless a phenomenological deuteron-core interaction is derived or unless the integration mesh is made finer. (The mesh in the DWBA calculations was limited to that of the exact calculations which in turn was severely limited by considerations of space and time.)

B. Intermediate neutron states

It was possible to show that the energy dependence of the BHMM extraction of S was linked to the use of intermediate neutron states generated by the optical potential rather than to the use of the deuteron transform. This was done by inserting the neutron states into the plane-wave matrix element and subtracting the bound-state term in the manner of BHMM.

The plane-wave matrix element is

$$T^{\mathrm{PW}} = \langle \vec{k}_p \underline{\phi}_n \mid V^{(12)} \mid \phi_d \vec{p}_d \underline{\alpha}_1 \rangle \ . \tag{103}$$

It contains no compound-nuclear terms (that is ground-core projection on initial or final states does not alter its value). Its value upon insertion of complete intermediate neutron states and the ground-state projection operator $\underline{\alpha}_1 \underline{\alpha}_1^{\dagger}$ becomes

$$T^{\mathrm{PW}} = \sum_{\vec{k}_n; i} \langle \vec{k}_p \underline{\phi}_n \mid \underline{\alpha}_1 \underline{\alpha}_1^{\dagger} \underline{\psi}^{+}_{\vec{k}_n; i} \rangle \langle \underline{\psi}^{+}_{\vec{k}_n; i} \mid V^{(12)} \mid \underline{\alpha}_1 \phi_d \vec{p}_d \rangle \ . \tag{104}$$

The ground-state projection of the final state can be done without approximation since $V^{(12)}$ does not couple core states and because the deuteron is initially incident on the ground-state core $\underline{\alpha}_1$. In the above expression the integral over $\vec{k}_n$ includes all continuum states and the bound state $\underline{\phi}_n$. The sum over i refers to the incident core state on which the neutrons are scattered. It is

TABLE VI. Sensitivity of BHMM to optical potentials.

Data set	S (exact) [a]	S (BHMM) [b]
V_5	0.646	0.68
V_4	0.648	0.67
V_1	0.651	0.68

[a] The spectroscopic factor calculated exactly for the optical potential used in the BHMM calculations. The values for the exact (comparison) curves, which are generated by the Faddeev equations using V_1 are all $S = 0.651$.

[b] The spectroscopic factor extracted by comparing the BHMM (which used the altered potential) to the exact three-body calculations.

only if incident cores that are not in their ground states are neglected that the BHMM approximation is obtained. For then, using optical wave functions (which are found by projecting out the ground-state core from the scattered wave function for the neutron incident on the ground-state core)

$$|\psi^{\pm(\mathrm{opt})}_{\vec{k}_n}\rangle = \underline{\alpha}_1^{\dagger}\,|\underline{\psi}^{\pm}_{\vec{k}_n;1}\rangle \tag{105}$$

and the expression

$$\underline{\alpha}_1^{\dagger}\,|\underline{\phi}_n\rangle = S^{1/2}\,|\phi_n\rangle \tag{2}$$

(relating the single-channel "well-depth prescription" neutron bound state to the multichannel bound state) we obtain the BHMM assertion that the approximate equality (below) holds:

$$T^{\mathrm{PW}} \simeq S^{1/2}\int d\vec{k}_n \langle \vec{k}_p\,\phi_n\,|\psi^{\pm(\mathrm{opt})}_{\vec{k}_n}\rangle \times \langle \psi^{\pm(\mathrm{opt})}_{\vec{k}_n}\,|\,V^{(12)}\,|\,\phi_d\,\vec{p}_d\rangle + S\,T^{\mathrm{PW}}. \tag{106}$$

The last term results from the insertion of the neutron bound state $\underline{\phi}_n$, which is one of the intermediate states required in Eq. (104).

Thus the manipulation, if valid, implies that

$$T^{\mathrm{PW}} \simeq \frac{S^{1/2}}{1-S}\,T^{\mathrm{BHMM}}, \tag{107}$$

where

$$T^{\mathrm{BHMM}} \equiv \int d\vec{k}_n \langle \vec{k}_p\,\phi_n\,|\psi^{\pm(\mathrm{opt})}_{\vec{k}_n}\rangle\langle\psi^{\pm(\mathrm{opt})}_{\vec{k}_n}\,|\,V^{(12)}\,|\,\phi_d\,\vec{p}_d\rangle. \tag{108}$$

The cross sections derived from the two matrix elements T^{PW} and T^{BHMM} were compared in order to estimate S which in turn was compared, for various energies, to the known value [Fig. 4(c)]. This procedure was analogous to the extraction of S in realistic cases where, instead of a cross section derived from T^{PW}, the experimental one was used. Thus T^{PW} was calculated from Eq. (103), which was equivalent to using the exact spectroscopic amplitude, in order to preserve the analog. The comparison of $|T^{\mathrm{PW}}|^2$ to $|T^{\mathrm{BHMM}}|^2$ via Eq. (107) yielded an *extracted* estimate for S whose degree of closeness to the exactly calculated value provided, in turn, an estimate for the validity of the BHMM manipulation [Eq. (106)].

The parameter which largely determine the energy at which the BHMM value of S approaches the exact value is α^2. This is shown in Fig. 4(d) for potentials yielding different binding energies.

Thus, it is the failure of the optical-model wave functions to provide complete intermediate states for the neutron in the continuum that leads to the failure of BHMM to predict the spectroscopic factor at low incident energies. This is in addition to the fact that the sudden approximation is a high-energy approximation and so cannot be expected to predict low-energy proton distributions from stripping reactions.

C. Conclusion

This work highlights a major difficulty associated with the BHMM theory, which manifests itself in two ways.

(1) Empirically it has been shown that although BHMM can give good predictions as to the shape of the resulting proton angular distribution, the estimates it yields for the spectroscopic factor of the final state are energy-dependent to a significant extent. BHMM depends on the sudden approximation (a high-energy approximation) and so it is not expected to work at low energies; however, one can find an energy above which the shape predictions are fairly acceptable. The same is not true of the spectroscopic estimates which vary from "too low" to "too high" and do not approach the correct value asymptotically with increasing energy. Moreover it is not known how to select that narrow range of energy within which the spectroscopic-factor extraction could be expected to be accurate.

(2) Theoretically it has been demonstrated that the intermediate neutron wave functions used by BHMM are inappropriate. The theory relies on the sudden approximation and thus requires intermediate neutron and proton states as described in Sec. II.

The matrix element M_c is divided into two by separating the sum over intermediate neutron states. In the first part only the neutron bound states are retained whilst the second part contains the remaining (scattered) states.

The theory therefore requires that these intermediate neutron states be chosen so that the overlaps of the bound and scattered states with the final bound state can be calculated in a straightforward manner.

In practice, the intermediate and final neutron bound states are assumed to have the same form and the overlap is just the spectroscopic amplitude in the case where both the wave functions represent the same state, and zero otherwise. Optical potentials are used to generate scattered intermediate neutron states. However, the intermediate states so generated do not form a complete set. States representing the effect of neutrons being scattered off an excited core have been omitted and this leads, as shown above, to the energy dependence of the spectroscopic factor.

If the prescription used to generate the bound state is retained then there is at present no adequate way of generating the continuous states needed for completeness and vice versa.

A good theory of deuteron stripping would provide us with reliable spectroscopic information. If the shapes of the predicted angular distributions are in accordance with experiment they give us confidence in the theory but no new information. BHMM will not be able to provide unambiguous spectroscopic information until a theoretically sound and numerically feasible method can be devised for including the effect of the neglected neutron states or a procedure can be developed for determining the energy range over which good values are obtained. At present we can only say "use high energies and expect to overestimate S." In both realistic and model calculations, the angular predictions of DWBA are more acceptable and the extracted spectroscopic factor displays no great energy dependence. However, this does not mean that the theory is any more useful since the spectroscopic estimates can still vary by up to a factor of 2 depending on the parameters used.[25] This work suggests that DWBA overestimates the spectroscopic factor and that the good agreement with experiment which is often reported may be due in part to ambiguities in the deuteron wave function. It may well be advantageous to explore DWBA more thoroughly using a model similar to that used here in the hope that these problems concerning spectroscopic information can be clarified. Solvable models which do not include an excitable core are not able to throw light on this aspect of the theory and are thus inconclusive.

The fact that high-energy BHMM and DWBA are more or less in agreement and both overestimate S suggests that two-step processes may be responsible for the discrepancy. They are clearly contained in the exact solution, but further work is required to extract their explicit contribution and decide this point.

ACKNOWLEDGMENTS

This work was supported in part by the Science Foundation for Physics within the University of Sydney. It is a pleasure to acknowledge the support of its director, Professor H. Messel. One of us (K.K.) was supported by the Australian people on a Commonwealth Postgraduate Award and is grateful for this help. We also wish to thank Professor N. Austern at the University of Pittsburgh and Dr. R. G. L. Hewitt at the University of Sydney for helpful comments on this work.

---

*The results of this work were briefly communicated in a letter by the authors in Phys. Rev. Lett. 30, 562 (1973).

†Present address.

[1]M. Tanifuji, Nucl. Phys. 58, 81 (1964); S. T. Butler, Nature 207, 1346 (1965).

[2]S. T. Butler, R. G. L. Hewitt, B. H. J. McKellar, and R. M. May, Ann. Phys. (N. Y.) 43, 282 (1967); R. M. May and J. S. Truelove, *ibid.* 43, 322 (1967); S. T. Butler, R. G. L. Hewitt, and J. S. Truelove, Phys. Rev. 162, 1061 (1967); K. King and B. H. J. McKellar, Aust. J. Phys. 23, 453, 641 (1970).

[3]K. King and B. H. J. McKellar, Phys. Rev. C 1, 1538 (1970).

[4]Further clarifications have been provided by: C. F. Clement, Phys. Rev. Lett. 20, 22 (1968), on neutron states; B. H. J. McKellar, Phys. Rev. 171, 1137 (1968), on polarization; 181, 1502 (1969), on antisymmetrization. Some early objections were raised by: C. A. Pearson and M. Coz, Nucl. Phys. 82, 533 (1966); J. M. Bang and C. A. Pearson, *ibid.* 100, 1 (1967); L. S. Rodberg, Ann. Phys. (N. Y.) 48, 254 (1968); and J. D. Garcia and C. A. Pearson, Phys. Rev. Lett. 21, 301 (1968). However these have been countered by: S. T. Butler, Ann. Phys. (N. Y.) 48, 258 (1968); and S. T. Butler, and B. H. J. McKellar, Phys. Rev. Lett. 21, 1533 (1968).

[5]See for example M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 833.

[6]These potentials were originally discussed by Y. Yamaguchi, Phys. Rev. 95, 1628 (1954). More recent analytic work has been developed by: G. C. Ghirardi and A. Rimini, J. Math. Phys. 5, 722 (1964); and D. Gutkowski and A. Scalia, *ibid.* 9, 588 (1968). A pedagogical review was given by W. H. Nichols, Am. J. Phys. 33, 474 (1965). Phase shifts for elastic scattering have been computed to all orders by R. L. Cassola and R. D. Koshel, Proc. Phys. Soc. Lond. A 1, 224 (1968), series 2. The inverse problem was discussed by: M. Gourdin and A. Martin, Nuovo Cimento 8, 699 (1958); and Kh. Chadan, *ibid.* 10, 892 (1958). For a long bibliography see C. Lovelace, Phys. Rev. 135, B1225 (1964). A. N. Kamal and H. J. Kreuzer, Phys. Rev. D 2, 2033 (1970) discuss solvable two-channel problems.

[7]A. N. Mitra, Phys. Rev. 139, B1472 (1965).

[8]J. V. Noble, Phys. Rev. 157, 939 (1966).

[9]P. E. Shanley, Ann. Phys. (N. Y.) 44, 363 (1966).

[10]P. Beregi, I. Lovas, and J. Revai, Ann. Phys. (N. Y.) 61, 57 (1970).

[11]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).

[12]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [transl.: Sov. Phys. JETP 12, 1014 (1960)].

[13]R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. 121, 319 (1961).

[14]A. N. Mitra, Nucl. Phys. 32, 529 (1962).

[15]R. D. Amado, Phys. Rev. 132, 485 (1963).

[16]L. Rosenberg, Phys. Rev. 134, B937 (1964).

[17]W. Tobocman, Phys. Rev. 94, 1655 (1954).
[18]The relation between the multichannel and optical potentials is derived quite generally in, for example I. E. McCarthy, *Introduction to Nuclear Theory* (Wiley, New York, 1968), p. 326ff.
[19]For a discussion of the approximations involved in DWBA see N. Austern, *Direct Nuclear Reactions* (Wiley, New York, 1969); or D. P. Bouldin and F. S. Levin, Nucl. Phys. A189, 449 (1972).
[20]R. C. Johnson and P. J. R. Soper, Phys. Rev. C 1, 976 (1970).
[21]The neutron has the same interaction as the proton in this model, so results for incident and unbound final neutron states have not been shown as they duplicate the results using proton states.
[22]J. Dobes, Phys. Lett. 30B, 404 (1969).
[23]T. F. Baker and L. J. B. Goldfarb, Nucl. Phys. A146, 577 (1970).
[24]Reference for the experimental distributions and the results of the BHMM calculations are given for the points in Ref. 3 excepting the first one which is at $E_d = 8.7$ MeV. This experiment was done by J. J. Van der Merwe and G. Heymann, Z. Phys. 220, 130 (1969).
[24]V. Scheib, A. Hofman, G. Philip, and F. Vogler, Nucl. Phys. A203, 177 (1973).