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Nuclidic mass relationships and mass equations

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Generalized nuclidic mass relationships represent a link between mass equations and mass relationships. They are partial difference equations which can be used recursively. The quality of mass predictions, particularly for very neutron-rich or proton-rich nuclei, depends essentially on our knowledge of the effective neutron-proton interaction I_{np}. The transverse and longitudinal mass relationships of Garvey and Kelson which are special cases contain small systematic errors due to the assumed independence of I_{np} on the neutron excess or the nucleon number. Information about the dependence of I_{np} on T and A has been extracted from the experimental masses by a variety of χ^2 tests and was found in partial disagreement with macroscopic and microscopic theories. Several procedure have been devised to test theoretical expressions for I_{np} for their compatibility and consistency with the experimental masses.

[NUCLEAR STRUCTURE Derived generalized nuclidic mass relationships with solutions; consistency tests.]

I. INTRODUCTION

The most commonly used procedures to estimate masses of unknown nuclei are based on mass equations[1] and on mass relationships. Nuclidic mass relationships constitute recursion formulas for estimating unknown masses from the masses of neighboring known nuclei.[2] Well-known examples are the transverse and longitudinal mass relationships GKT and GKL derived by Garvey and Kelson.[3-5] Mass relationships may also be considered as partial difference equations, and functional forms for the solutions have been derived.[5]

Several questions and problems exist. For example, no obvious general connection between ordinary mass equations and the above solutions has ever been established. The generalized mass relationships which are discussed in the present work represent a link.

The relationships GKT and GKL reproduce the known masses with a standard deviation of about ±200 keV. This result is better than for most mass equations, and it is particularly important for successfully predicting masses of nuclei adjacent to the known nuclei. However, it is not clear how the accuracy of reproducing the known masses, often achieved by introducing a great number of adjustable parameters, is related to the reliability of predicting unknown masses far away from the line of β stability. It will be shown in the present work that the latter depends entirely on the underlying physical assumptions, and the accuracy of reproducing the known masses is by no means a sufficient condition. This statement is substantiated by the result that any mass equation, given in analytical form or tabulated, can be used to construct a partial difference equation the solution of which will essentially preserve the (good or bad) characteristics of the original equation for nuclei far away but will reproduce the known masses with an accuracy comparable with that of GKT or GKL.

The relationships GKT and GKL are based on the same physical model, and they both reproduce the known masses about equally well.[5] Nevertheless, they are not compatible with each other[5] and lead to strongly diverging predictions[6] for the masses of very neutron-rich or proton-rich nuclei. The model does not explain this behavior, and it is not obvious which of the two relationships should be better far away from the line of β stability. In addition to these problems there is increasing evidence for systematic deviations between newly measured masses of neutron-rich and proton-rich nuclei (see for example Refs. 7–10) and the predictions from GKT.

These and other considerations point to the need for a reinvestigation of the physical assumptions underlying GKT and GKL and, if possible, for establishing new or modified procedures or techniques. Also, it was considered desirable to obtain additional quantitative evidence for systematic errors which arise in a single application of the relationships GKT or GKL and to determine their magnitude and characteristics.

Both objectives have been achieved. A variety of χ^2 tests of the experimental masses have been performed and small systematic errors superimposed on the more random fluctuations have been established. Information about the dependence on T and A of the effective neutron-proton interaction I_{np} could be extracted (GKT and GKL assume independence of I_{np} on T or A, respectively). Generalized nuclidic mass relationships which contain GKT and GKL as limiting cases have been derived. A necessary condition for applying the generalized relationships is certain knowledge about the effective neutron-proton interaction I_{np} which has been recognized as the quantity which determines the reliability of extrapolations based on mass relationships as well as mass equations. Details of phenomenological, macroscopic, and microscopic properties of I_{np} have been investigated and several tests have been devised to check expressions for I_{np} derived from mass equations or from other explicit theories for their compatibility and consistency with the experimental masses. The assumption that I_{np} is independent of T_z and of A (separately for even-A and odd-A) nuclei is not compatible with the experimental data. No numerical applications of the above results are included in the present work.

II. GENERALIZED NUCLIDIC MASS RELATIONSHIPS

Three nuclidic mass relationships have been reported by Garvey and Kelson[3-5] based on the assumption of a nuclear model with fourfold degenerate Hartree-Fock or Nilsson-model-like single-particle orbits. These and other mass relationships can best be formulated by defining

$$\begin{aligned}\Sigma_{\mathrm{T}}(A, T_z) \equiv{}& M(A, T_z+2) - M(A, T_z) + M(A-1, T_z+\tfrac{1}{2}) \\ & - M(A-1, T_z+\tfrac{3}{2}) + M(A+1, T_z+\tfrac{1}{2}) \\ & - M(A+1, T_z+\tfrac{3}{2})\end{aligned} \tag{1}$$

and

$$\begin{aligned}\Sigma_{\mathrm{L}}(A, T_z) \equiv{}& M(A+4, T_z) - M(A, T_z) + M(A+1, T_z+\tfrac{1}{2}) \\ & - M(A+3, T_z+\tfrac{1}{2}) + M(A+1, T_z-\tfrac{1}{2}) \\ & - M(A+3, T_z-\tfrac{1}{2}) .\end{aligned} \tag{2}$$

The quantities $\Sigma_{\mathrm{T}}(A, T_z)$ and $\Sigma_{\mathrm{L}}(A, T_z)$ represent sums over subsets of masses.

The transverse nuclidic mass relationships GKT states that

$$\Sigma_{\mathrm{T}}(A, T_z) \approx 0 \tag{3}$$

for $T_z > 0$ or $T_z = 0$, $A = 4n$ (n = integer). The longitudinal nuclidic mass relationship GKL states that

$$\Sigma_{\mathrm{L}}(A, T_z) \approx 0 \tag{4}$$

for $T_z > \frac{1}{2}$ or $T_z = 0$, $A = 4n$ (n = integer). The charge symmetric nuclidic mass relationship GKS states that

$$\Sigma_{\mathrm{T}}(A, T_z) \approx 0 \tag{5}$$

for $T_z = -1$.

The modified relationships[6] which have been reported recently can be written as

$$\Sigma_{\mathrm{T}}(A, T_z) = I_{np}(A+1, T_z+\tfrac{3}{2}) - I_{np}(A+1, T_z+\tfrac{1}{2}) \tag{6}$$

and

$$\Sigma_{\mathrm{L}}(A, T_z) = -I_{np}(A+4, T_z) + I_{np}(A+2, T_z) \tag{7}$$

with no restriction on T_z. In these equations the quantity $I_{np}(A, T_z)$ is defined by

$$\begin{aligned}I_{np}(A, T_z) &= B_p(A, T_z) + B_n(A, T_z) - B_{np}(A, T_z) \\ &= -M(A, T_z) + M(A-1, T_z-\tfrac{1}{2}) \\ &\quad + M(A-1, T_z+\tfrac{1}{2}) - M(A-2, T_z),\end{aligned} \tag{8}$$

where B_p, B_n, and B_{np} denote the binding energies for the last proton, neutron, or neutron and proton, respectively. It is the so-called effective neutron-proton interaction, which will be discussed below in more detail. In the application of these and the subsequent equations, phenomenological or theoretical expressions will be used for I_{np}.

From Eqs. (6) and (7) one can derive several generalized nuclidic mass relationship such as[6]

$$\begin{aligned}&\frac{1}{1+\alpha_1(A+1, T_z+\frac{3}{2})} \Sigma_{\mathrm{T}}(A, T_z) \\ &\qquad + \frac{\alpha_1(A+1, T_z+\frac{3}{2})}{1+\alpha_1(A+1, T_z+\frac{3}{2})} \Sigma_{\mathrm{L}}(A-1, T_z+\tfrac{3}{2}) = 0\end{aligned} \tag{9}$$

with

$$\alpha_1(A, T_z) = \frac{I_{np}(A, T_z) - I_{np}(A, T_z-1)}{I_{np}(A+2, T_z) - I_{np}(A, T_z)} . \tag{10}$$

The equation combines the masses of 10 neighboring nuclei. Three similar relationships can be obtained from Eqs. (9) and (10) by replacing in Σ_{L} and in the denominator of α_1 the arguments (A, T_z) by $(A-2, T_z)$ or (A, T_z-1) or $(A-2, T_z-1)$.

One additional generalized nuclidic mass relationship will be presented here because of its special symmetry. Based on the definitions

$$\Xi_{\mathrm{T}}(A, T_z) \equiv \Sigma_T(A, T_z) + \Sigma_T(A, T_z+1) \tag{11}$$

and

$$\Xi_L(A,T_z)\equiv\Sigma_L(A,T_z)+\Sigma_L(A+2,T_z) \tag{12}$$

one obtains

$$\Xi_T(A,T_z)=I_{np}(A+1,T_z+\tfrac{5}{2})-I_{np}(A+1,T_z+\tfrac{1}{2}) \tag{13}$$

and

$$\Xi_L(A,T_z)=-I_{np}(A+6,T_z)+I_{np}(A+2,T_z). \tag{14}$$

Equations (13) and (14) can now be used to obtain the generalized nuclidic mass relationship

$$\Xi_T(A,T_z)+\alpha(A+1,T_z+\tfrac{3}{2})\,\Xi_L(A-3,T_z+\tfrac{3}{2})=0 \tag{15}$$

with

$$\alpha(A,T_z)=\frac{I_{np}(A,T_z+1)-I_{np}(A,T_z-1)}{I_{np}(A+2,T_z)-I_{np}(A-2,T_z)}\ . \tag{16}$$

This equation combines the masses of 16 neighboring nuclei in a completely symmetric way. Equations (13)–(16) are also shown in Fig. 1 based on the schematic notation introduced earlier.[6]

It should be mentioned that Eqs. (6), (7), (13), and (14) as well as the generalized nuclidic mass relationships (9) and (15) represent only identities if the effective neutron-proton interaction I_{np}, its partial differences with respect to T_z and A, and the ratio thereof are obtained from the known masses. However, phenomenological or theoretical expressions will be used for I_{np}. Therefore, the generalized nuclidic mass relationships represent an intermediate step between a mass relationship (difference equation) and a mass equation.

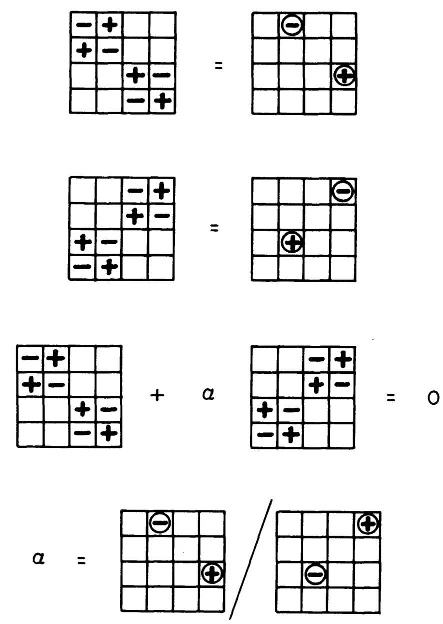

FIG. 1. Schematic representation of Eqs. (13)–(16). The fields represent nuclei from the nuclidic chart with N horizontal and Z vertical. The signs and encircled signs represent M and I_{np}, respectively. The third equation is a generalized nuclidic mass relationship.

III. MODIFIED TRANSVERSE AND LONGITUDINAL NUCLIDIC MASS RELATIONSHIPS

A seemingly sensitive check on the internal consistency of any theory for the effective neutron-proton interaction I_{np} is based on the modified transverse and longitudinal mass relationships (6) and (7). Given a phenomenological or theoretical expression for I_{np}, both equations can be used as recursion relationships to extrapolate from the region of known nuclei into the regions of unknown neutron-rich or proton-rich nuclei. Clearly, each extrapolation should lead to a prediction which is reasonably consistent with the other. This consistency test (test 1; two other tests will be presented later) constitutes a necessary condition for the quality of the expression for I_{np}.

The extrapolated mass values $M_T(A,T_z)$ and $M_L(A,T_z)$ can be expressed in terms of the predictions[5] $M(\mathrm{GKT};A,T_z)$ and $M(\mathrm{GKL};A,T_z)$ of the transverse and longitudinal Garvey-Kelson mass relationships in which the right-hand sides of Eqs. (6) and (7) are assumed to be zero. The result for neutron-rich nuclei is

$$M_T(A_0,T_{z0})=M(\mathrm{GKT};A_0,T_{z0})+\sum_{A=A_{\min}}^{A_0}\left[I_{np}(A,T_{z0}-1-\tfrac{1}{2}(A_0-A))-I_{np}(A,T_z^{\beta}(A))\right]$$
$$+\sum_{A=A_0+1}^{A_{\max}}\left[I_{np}(A,T_{z0}-\tfrac{1}{2}(A-A_0))-I_{np}(A,T_z^{\beta}(A))\right] \tag{17}$$

and

$$M_L(A_0,T_{z0})=M(\mathrm{GKL};A_0,T_{z0})+\sum_{T_z=T_{z\min}}^{T_{z\max}}\left[I_{np}(A_0-2(T_{z0}-T_z),T_z)-I_{np}(A^{\beta}(T_z),T_z)\right]. \tag{18}$$

Similar expressions exist for proton-rich nuclei. The correction terms of Eqs. (17) and (18) are shown schematically in Fig. 2. The quantities A and T_z are, of course, integer or half integer, respectively, and $\frac{1}{2}A_0 \pm T_{z0}$, $\frac{1}{2}A \pm T_z^\beta(A)$, and $\frac{1}{2}A^\beta(T_z) \pm T_z$ are integer. Furthermore: $T_z^\beta(A)$ = value of T_z nearest the line of β stability for a given A; $A^\beta(T_z)$ = value of A nearest the line of β stability for a given T_z; $A_{\min}$ = minimum value of A with $\frac{1}{2}A - T_z^\beta(A) \geq \frac{1}{2}A_0 - T_{z0} + 2$; $A_{\max}$ = maximum value of A with $\frac{1}{2}A + T_z^\beta(A) \leq \frac{1}{2}A_0 + T_{z0} - 1$; $T_{z\min}$ = minimum value of T_z with $\frac{1}{2}A^\beta(T_z) - T_z \geq \frac{1}{2}A_0 - T_{z0} + 2$; $T_{z\max} = T_{z0} - \frac{1}{2}$.

If the quantity I_{np} were constant or distributed randomly about a constant value (separately for even-A and odd-A), the correction terms of Eqs. (6), (7), (17), and (18) would not contain systematic contributions. In this case the extrapolated mass values M_T and M_L are, of course, those obtained from GKT and GKL. It has been shown earlier[6] that the two relationships lead to strongly diverging results and the above assumption about I_{np} must be rejected. However, test 1 does not exclude the possible independence of I_{np} on T_z *or* A which are the underlying assumptions for the validity of GKT or GKL, respectively.

It will be shown later that any mass equation given in analytical or tabular form can be used to obtain an expression for the effective neutron-proton interaction I_{np}. Thus, test 1 makes it possible to test an important aspect of any mass equation.

IV. CHARACTERISTICS OF THE EFFECTIVE NEUTRON-PROTON INTERACTION I_{np} DERIVED FROM EXPERIMENTAL MASSES

It has been shown earlier that certain knowledge about the effective neutron-proton interaction I_{np} is essential for an application of the generalized nuclidic mass relationships. Therefore, the behavior of I_{np} derived from the experimental masses according to Eq. (8) will be discussed first.

Figure 3 shows a plot of about 500 values for I_{np} calculated from Eq. (8) and the 1971 atomic mass evaluation.[11] The data have been plotted separately for even-A and odd-A nuclei. We find $I_{np}^{\text{even-}A} > I_{np}^{\text{odd-}A}$ and both exhibit a general decrease along the line of β stability. Thus, I_{np} must depend on the nucleon number and/or on the neutron excess.

Based on Fig. 3 or related plots,[12,13] a phenomenological study of the dependence of I_{np} on A and T_z becomes possible. Such a study has already been performed[14] for the isospin doublets and

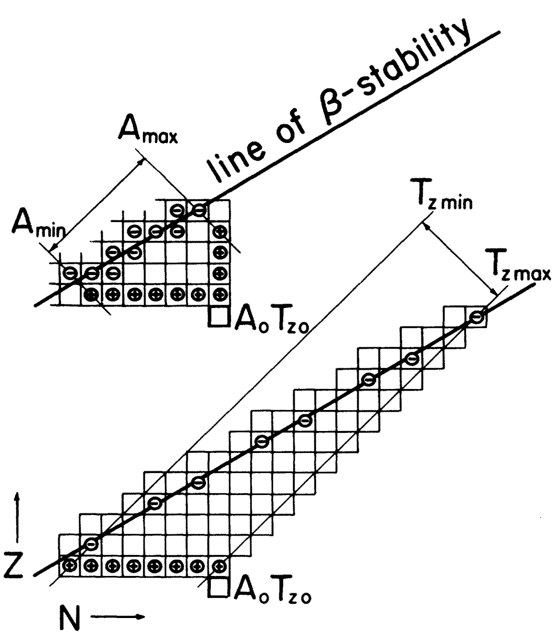

FIG. 2. Schematic representation of the correction terms in Eqs. (17) and (18). The fields represent nuclei from the nuclidic chart with N horizontal and Z vertical. The encircled signs represent I_{np} for the respective nuclei. The correction terms consist of certain I_{np} differences summed over ranges of A or T_z, respectively.

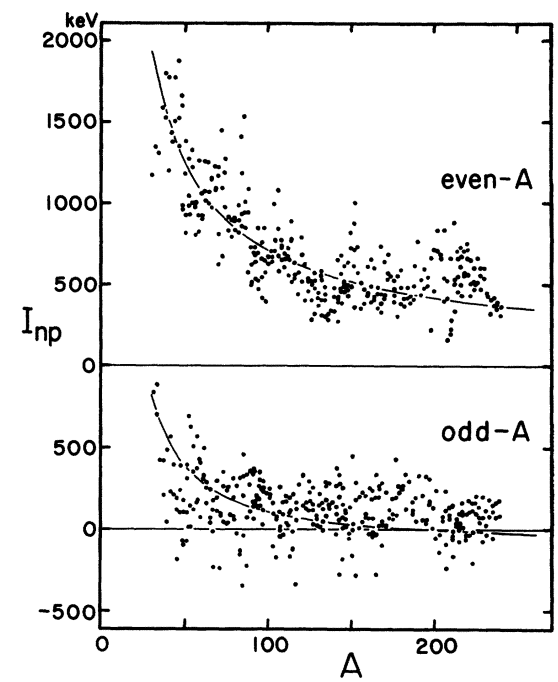

FIG. 3. Plot of the effective neutron-proton interaction I_{np} derived from the experimental masses as a function of A for even-A and odd-A nuclei. Only data with $A \geq 30$ and with experimental uncertainties less than 200 keV are included. The lines are calculated from the liquid-drop-model equation (29) for nuclei along the line of β stability.

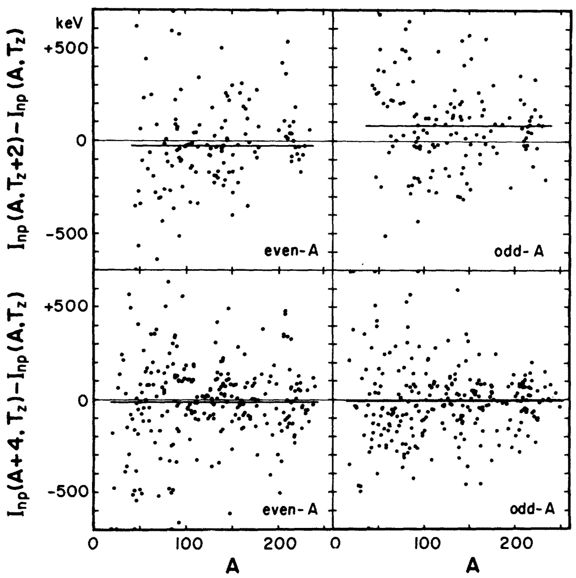

FIG. 4. Plot of the experimental differences $I_{np}(A, T_z+2) - I_{np}(A, T_z)$ and $I_{np}(A+4, T_z) - I_{np}(A, T_z)$ for even-A and odd-A nuclei as a function of A. Only data points with experimental uncertainties less than 200 keV are included. The horizontal lines represent the average values obtained from the χ^2 analyses.

triplets, i.e., for $T_z = \pm\frac{1}{2}$ and ± 1. It has been found empirically that at least for small values of T_z the quantity I_{np} depends on A and $|T_z|$ rather than T_z which shows that the charge-symmetric mass relationship GKS Eq. (5) is not, or not strongly, affected by the present considerations.

To show the dependence of I_{np} on A and T_z separately, a three-dimensional plot becomes necessary. However, the "fluctuations" make it practically impossible to separate the dependence on A and on T_z. Therefore, another approach has been taken.

Figure 4 shows a plot of the differences $I_{np}(A, T_z+2) - I_{np}(A, T_z)$ and $I_{np}(A+4, T_z) - I_{np}(A, T_z)$ for even-A and odd-A nuclei. There are no striking systematic deviations from zero; but nevertheless, a closer inspection of the strongly fluctuationg data points show a slight preference for negative values of $I_{np}(A, T_z+2) - I_{np}(A, T_z)$ for light even-A nuclei and a definite preference for positive values for odd-A nuclei, particularly light odd-A nuclei. It thus appears that I_{np} decreases with increasing T_z for light even-A nuclei but increases with increasing T_z for odd-A nuclei. To further substantiate these findings, the residuals of Fig. 4 as well as many other differences were subjected to a χ^2 analysis. The systematic contributions to the differences were assumed to be constant, and a search was made to find those constant values which minimize χ^2. The searches were performed separately for nuclei with $Z=N$ = even, $Z+1=N+1$ = even, $Z+1=N$ = even, and $Z=N+1$ = even. The light nuclei with $Z<8$ were excluded from the analysis as were all nuclei with $T_z<0$ or $T_z=0$, $N=Z$ = odd. The remaining 500–600 mass combinations were treated jointly or they were subdivided into three groups with increasing A and with an approximately equal number of combinations. It was hoped that from such a subdivision additional information about the A dependence of the various quantities could be obtained. By quadratically adding 100 keV to the experimental uncertainties of the individual masses, a more even distribution of the weight factors is ensured. The results differ only little from those where the 100-keV term was not added.

Table I shows the results of the χ^2 analyses. The results for the individual differences of Fig. 4 can be found in the lines labeled by (1C), (1F), (2C), and (2F). The average residuals of column (3) are displayed in the figure as horizontal lines. The differences of Table I are grouped into four categories according to their dependence on T_z, A, N, and Z. Many of the listed values are compatible with zero. Several, however, deviate markedly from zero such as the values in column (3) labeled (1d), (1f), (1E), (1F), (4E), and (4F).

The presence of small systematic deviations from zero is substantiated by the internal consistency of the results. For example, most of the values listed in the lines labeled (A) and (B) or (D) and (E) have the same sign and are of similar magnitude. Altogether there exist 20 equations which should hold such as (in an obvious notation) (1a) + (1b) = $\frac{1}{2}$(1c) = (1A) = (1B) = (1C), (1a) + (2b) = (3A), or −(1a) + (2a) = (4A). These equations are satisfied remarkably well and they are consistent with

$$I_{np}^{\text{even-}A}(N_0+\Delta N,\ Z_0+\Delta Z) = I_0^{\text{even-}A}(N_0, Z_0) + a_1[\tfrac{1}{2}(\Delta N+1)] + a_2[\tfrac{1}{2}(\Delta Z+1)] + a_3\delta_{\text{odd, odd}} \tag{19}$$

and

$$I_{np}^{\text{odd-}A}(N_0+1+\Delta N, Z_0+\Delta Z) = I_0^{\text{odd-}A}(N_0+1, Z_0) + b_1[\tfrac{1}{2}(\Delta N+1)] + b_2[\tfrac{1}{2}(\Delta Z+1)] + b_3\delta_{\text{even, odd}} \tag{20}$$

or

$$I_{np}^{\text{even-}A}(A_0+\Delta A,\ T_{z0}+\Delta T_z) = I_0^{\text{even-}A}(A_0,\ T_{z0}) + \tfrac{1}{4}(a_1+a_2)\Delta A + \tfrac{1}{2}(a_1-a_2)\Delta T_z + \{\tfrac{1}{2}(a_1+a_2)+a_3\}\,\delta_{\text{odd, odd}} \tag{21}$$

TABLE I. Residuals in keV for the partial I_{np} differences obtained from χ^2 minimizations. The number of mass combinations used in the calculations are given in parentheses. Column (3) shows the results for the complete range of nuclei, while columns (4)–(6) show the results for the subranges characterized by Z.

(1) No.	(2) Partial I_{np} difference	(3) $Z=8-100$	(4) $Z=8-40$	(5) $Z=41-62$	(6) $Z=63-100$
T_z dependence:					
1a	$I_{np}^{oo}(A,T_z+1) \quad -I_{np}^{ee}(A,T_z)$	−15 ± 19 (138)	−40 ± 51 (39)	−10 ± 26 (46)	+15 ± 26 (53)
1b	$I_{np}^{ee}(A,T_z+1) \quad -I_{np}^{oo}(A,T_z)$	−5 ± 21 (122)	−35 ± 56 (35)	+10 ± 27 (42)	+10 ± 27 (45)
1c	$I_{np}^{\text{even-}A}(A,T_z+1) \quad -I_{np}^{\text{even-}A}(A,T_z)$	−10 ± 14 (260)	−40 ± 37 (74)	−5 ± 18 (88)	+5 ± 19 (98)
1d	$I_{np}^{oe}(A,T_z+1) \quad -I_{np}^{eo}(A,T_z)$	+60 ± 14 (148)	+90 ± 32 (45)	+50 ± 26 (46)	+45 ± 17 (57)
1e	$I_{np}^{eo}(A,T_z+1) \quad -I_{np}^{oe}(A,T_z)$	+5 ± 19 (121)	0 ± 52 (31)	−10 ± 26 (42)	+15 ± 28 (48)
1f	$I_{np}^{\text{odd-}A}(A,T_z+1) \quad -I_{np}^{\text{odd-}A}(A,T_z)$	+35 ± 12 (269)	+60 ± 28 (76)	+25 ± 19 (88)	+35 ± 15 (105)
1A	$I_{np}^{ee}(A,T_z+2) \quad -I_{np}^{ee}(A,T_z)$	−20 ± 31 (73)	−35 ± 88 (20)	−55 ± 35 (28)	+30 ± 39 (25)
1B	$I_{np}^{oo}(A,T_z+2) \quad -I_{np}^{oo}(A,T_z)$	−35 ± 28 (77)	−170 ± 71 (20)	+10 ± 32 (31)	+20 ± 40 (26)
1C	$I_{np}^{\text{even-}A}(A,T_z+2) \quad -I_{np}^{\text{even-}A}(A,T_z)$	−25 ± 21 (150)	−105 ± 57 (40)	−20 ± 24 (59)	+25 ± 28 (51)
1D	$I_{np}^{eo}(A,T_z+2) \quad -I_{np}^{eo}(A,T_z)$	+65 ± 27 (79)	+150 ± 64 (23)	−25 ± 41 (26)	+70 ± 32 (30)
1E	$I_{np}^{oe}(A,T_z+2) \quad -I_{np}^{oe}(A,T_z)$	+95 ± 25 (75)	+205 ± 66 (18)	+20 ± 30 (32)	+100 ± 39 (25)
1F	$I_{np}^{\text{odd-}A}(A,T_z+2) \quad -I_{np}^{\text{odd-}A}(A,T_z)$	+80 ± 18 (154)	+175 ± 68 (41)	0 ± 24 (58)	+85 ± 25 (55)
A dependence:					
2a	$I_{np}^{oo}(A+2,T_z) \quad -I_{np}^{ee}(A,T_z)$	−30 ± 18 (159)	−85 ± 43 (54)	+10 ± 23 (52)	−10 ± 23 (53)
2b	$I_{np}^{ee}(A+2,T_z) \quad -I_{np}^{oo}(A,T_z)$	+10 ± 17 (180)	+15 ± 40 (64)	−25 ± 24 (52)	+25 ± 18 (64)
2c	$I_{np}^{\text{even-}A}(A+2,T_z) \quad -I_{np}^{\text{even-}A}(A,T_z)$	−10 ± 13 (339)	−30 ± 30 (118)	−10 ± 17 (104)	+10 ± 14 (117)
2d	$I_{np}^{oe}(A+2,T_z) \quad -I_{np}^{eo}(A,T_z)$	+20 ± 15 (158)	+25 ± 34 (49)	+10 ± 23 (53)	+20 ± 22 (56)
2e	$I_{np}^{eo}(A+2,T_z) \quad -I_{np}^{oe}(A,T_z)$	−45 ± 16 (165)	−80 ± 40 (52)	−25 ± 25 (55)	−25 ± 18 (58)
2f	$I_{np}^{\text{odd-}A}(A+2,T_z) \quad -I_{np}^{\text{odd-}A}(A,T_z)$	−15 ± 11 (323)	−30 ± 26 (101)	−10 ± 17 (108)	−5 ± 15 (114)
2A	$I_{np}^{ee}(A+4,T_z) \quad -I_{np}^{ee}(A,T_z)$	−15 ± 21 (137)	−55 ± 49 (46)	+15 ± 26 (42)	−5 ± 28 (49)
2B	$I_{np}^{oo}(A+4,T_z) \quad -I_{np}^{oo}(A,T_z)$	−15 ± 23 (142)	−40 ± 54 (52)	+20 ± 24 (47)	−15 ± 26 (43)
2C	$I_{np}^{\text{even-}A}(A+4,T_z) \quad -I_{np}^{\text{even-}A}(A,T_z)$	−15 ± 16 (279)	−45 ± 37 (98)	+15 ± 17 (89)	−10 ± 19 (92)
2D	$I_{np}^{eo}(A+4,T_z) \quad -I_{np}^{eo}(A,T_z)$	−15 ± 17 (151)	−15 ± 41 (54)	0 ± 24 (40)	−25 ± 17 (57)
2E	$I_{np}^{oe}(A+4,T_z) \quad -I_{np}^{oe}(A,T_z)$	−10 ± 18 (140)	0 ± 41 (50)	−20 ± 27 (47)	−5 ± 20 (43)
2F	$I_{np}^{\text{odd-}A}(A+4,T_z) \quad -I_{np}^{\text{odd-}A}(A,T_z)$	−10 ± 13 (291)	−10 ± 29 (104)	−10 ± 18 (87)	−15 ± 13 (100)
N dependence:					
3A	$I_{np}^{ee}(A+2,T_z+1) \quad -I_{np}^{ee}(A,T_z)$	−15 ± 19 (136)	−45 ± 47 (40)	−15 ± 23 (44)	+5 ± 26 (52)
3B	$I_{np}^{oo}(A+2,T_z+1) \quad -I_{np}^{oo}(A,T_z)$	−30 ± 18 (138)	−95 ± 50 (41)	−10 ± 21 (46)	+5 ± 20 (51)
3C	$I_{np}^{\text{even-}A}(A+2,T_z+1) \quad -I_{np}^{\text{even-}A}(A,T_z)$	−20 ± 13 (274)	−70 ± 34 (81)	−15 ± 16 (90)	+5 ± 17 (103)
3D	$I_{np}^{eo}(A+2,T_z+1) \quad -I_{np}^{eo}(A,T_z)$	+15 ± 16 (147)	+45 ± 42 (46)	−30 ± 25 (44)	+25 ± 16 (57)
3E	$I_{np}^{oe}(A+2,T_z+1) \quad -I_{np}^{oe}(A,T_z)$	+40 ± 14 (135)	+75 ± 32 (36)	+15 ± 22 (49)	+30 ± 19 (50)
3F	$I_{np}^{\text{odd-}A}(A+2,T_z+1) \quad -I_{np}^{\text{odd-}A}(A,T_z)$	+25 ± 11 (282)	+60 ± 27 (82)	−5 ± 16 (93)	+30 ± 12 (107)

TABLE I *(Continued)*

(1) No.	(2) Partial I_{np} difference		(3) $Z=8-100$	(4) $Z=8-40$	(5) $Z=41-62$	(6) $Z=63-100$
Z dependence:						
4A	$I_{np}^{ee}(A+2,T_z-1)$	$-I_{np}^{ee}(A,T_z)$	0 ± 24 (111)	-20 ± 58 (37)	$+45\pm25$ (37)	-25 ± 32 (37)
4B	$I_{np}^{oo}(A+2,T_z-1)$	$-I_{np}^{oo}(A,T_z)$	$+10\pm24$ (117)	$+40\pm55$ (42)	$+15\pm27$ (40)	-25 ± 34 (35)
4C	$I_{np}^{\text{even}-A}(A+2,T_z-1)$	$-I_{np}^{\text{even}-A}(A,T_z)$	$+5\pm17$ (228)	$+10\pm40$ (79)	$+25\pm19$ (77)	-25 ± 23 (72)
4D	$I_{np}^{eo}(A+2,T_z-1)$	$-I_{np}^{eo}(A,T_z)$	-40 ± 18 (120)	-80 ± 40 (42)	$+20\pm28$ (36)	-45 ± 18 (42)
4E	$I_{np}^{oe}(A+2,T_z-1)$	$-I_{np}^{oe}(A,T_z)$	-50 ± 17 (113)	-65 ± 36 (37)	-50 ± 27 (41)	-30 ± 25 (35)
4F	$I_{np}^{\text{odd}-A}(A+2,T_z-1)$	$-I_{np}^{\text{odd}-A}(A,T_z)$	-45 ± 12 (233)	-70 ± 27 (79)	-20 ± 19 (77)	-35 ± 15 (77)

and

$$I_{np}^{\text{odd}-A}(A_0+1+\Delta A,\,T_{z0}+\tfrac{1}{2}+\Delta T_z)=I_0^{\text{odd}-A}(A_0+1,\,T_{z0}+\tfrac{1}{2})+\tfrac{1}{4}(b_1+b_2)\Delta A+\tfrac{1}{2}(b_1-b_2)\Delta T_z+\{\tfrac{1}{2}(b_1+b_2)+b_3\}\delta_{\text{even},\text{odd}}\,. \tag{22}$$

Here, N_0 and Z_0 are assumed to be even and $A_0=N_0+Z_0$, $2T_{z0}=N_0-Z_0$. The square brackets $[x]$ denote the largest integer less or equal to x; and $\delta_{\text{odd},\text{odd}}=1$ for N and Z odd, and $=0$ otherwise. Other δ symbols are defined accordingly.

These equations represent the leading terms of a Taylor expansion with the major splitting between $I_{np}^{\text{even}-A}$ and $I_{np}^{\text{odd}-A}$, as evidenced by Fig. 1, given by the respective first terms. Table II contains the 6 quantities a_i and b_i obtained from a χ^2 adjustment to the respective 36 quantities of Table I. Both $I_{np}^{\text{even}-A}$ and $I_{np}^{\text{odd}-A}$ decrease with increasing A at about the same rate (a_1+a_2 and b_1+b_2). The dependence on T_z, however, differs considerably for even-A and odd-A nuclei (a_1-a_2 and b_1-b_2). While $I_{np}^{\text{even}-A}$ decreases, $I_{np}^{\text{odd}-A}$ *increases* with increasing T_z. This result explains the relatively fast decrease of $I_{np}^{\text{even}-A}$ along the stability line and the relatively slow decrease of $I_{np}^{\text{odd}-A}$ (see Fig. 1). The decrease of I_{np} is enhanced by the decrease with T_z for even-A nuclei, but it is slowed down by the increase with T_z for odd-A nuclei. Even though the dependence on A seems to be weaker than that on T_z, one has to remember that the range in A values is much bigger than the range in T_z values.

The quantity b_3 differs from zero. Oscillations in the T_z dependence of $I_{np}^{\text{odd}-A}$ are therefore indicated. Columns (4)–(6) of Table II show further that the deviations from zero are more pronounced in the light nuclei.

When averaged over all even-A and odd-A nuclei, the observed corrections to the nuclidic mass relationships GKT and GKL are $+(12\pm4.5)$ keV and $-(16\pm7)$ keV, respectively. These systematic contributions which appear in a *single* application of the relationships are indeed small compared to the "fluctuations" which are of the order of ±200 keV (see also Fig. 3). One might therefore ask what the significance of the systematic contributions is. There does indeed exist a major difference between the two. The "fluctuations" lead to contributions which are practically random in character. They should therefore remain essentially the same as the recursive process is applied in repeated steps away from the line of β stability. On the other hand, the systematic contributions lead to errors which propagate in a multiplicative way. It is the latter contributions which are responsible for the differences in the predictions between GKT and GKL. Since these differences amount to[6] about 11 MeV ($A=60$) and 4 MeV ($A=120$) for isobars which are located only five units in T_z beyond the most neutron-rich known isobar, one has to conclude that the systematic errors increase very strongly with neutron excess. A difference equation which approximately describes the increase of the systematic contributions suggests a dependence on ΔT_z of the third or perhaps even the fourth power.

Since I_{np} decreases with increasing A and since the increase with T_z of $I_{np}^{\text{odd}-A}$ seems to exceed the decrease with T_z of $I_{np}^{\text{even}-A}$, one is lead to the conclusion that GKL [Eq. (4)] *and* GKT [Eq. (3)] probably underestimate the masses of very neutron-rich nuclei. This preliminary conclusion is supported by new experimental results[8,9] on light neutron-rich nuclei.

V. MACROSCOPIC THEORIES OF I_{np}

Macroscopic theories of I_{np} are based on the liquid-drop or related models. Any mass equation derived from such a model contains explicitly or implicitly an expression for the effective neutron-proton interaction I_{np}. The theory underlying the expression is of course that used in the derivation of the respective mass equation. The quantity I_{np} can be calculated directly from the mass equation by using the defining equation (8). The correction terms in Eqs. (6), (7), (13), and (14) as well as the quantities $\alpha_1(A, T_z)$ and $\alpha(A, T_z)$ of Eqs. (10) and (16) can then be obtained from I_{np}. Thus, *any mass equation, whether given in analytical form or tabulated, provides us with a theoretical expression for* I_{np} to be used in the modified transverse and longitudinal mass relationships and in the generalized mass relationships.

It is obvious that the above method for obtaining I_{np} is applicable not only to mass equations based on macroscopic models but also to mass equations based on shell-model or other microscopic theories and on combinations thereof.

In a practical application of the above method it is advisable to use Eq. (8) together with a computer program that calculates $M(A, T_z)$ or $B(A, T_z)$. However, as an illustration and as basis for a discussion we have calculated explicit analytical expressions for I_{np} for a simple liquid-drop-

TABLE II. Coefficients of Eqs. (19)–(22) and of the equations in Fig. 7 in units of keV derived from the results of Table I.

(1) Coefficient	(2) Z = 8–100	(3) Z = 8–40	(4) Z = 41–62	(5) Z = 63–100
a_1	−20 ± 8	−70 ± 21	−8 ± 10	+7 ± 10
a_2	+4 ± 9	+17 ± 22	+17 ± 10	−12 ± 12
a_3	−5 ± 10	−5 ± 26	+1 ± 14	−15 ± 13
a_1+a_2	−16 ± 11	−53 ± 27	+9 ± 13	−6 ± 14
a_1-a_2	−23 ± 13	−87 ± 35	−25 ± 16	+19 ± 18
b_1	+28 ± 7	+55 ± 18	−1 ± 10	+29 ± 8
b_2	−44 ± 7	−72 ± 18	−14 ± 10	−41 ± 9
b_3	+65 ± 9	+107 ± 22	+29 ± 15	+52 ± 11
b_1+b_2	−16 ± 9	−17 ± 21	−15 ± 13	−11 ± 10
b_1-b_2	+72 ± 11	+127 ± 29	+13 ± 16	+70 ± 13
$\frac{1}{2}(a_1+b_1)$	+4 ± 5	−7 ± 14	−4 ± 7	+18 ± 7
$\frac{1}{2}(a_2+b_2)$	−20 ± 6	−27 ± 14	+2 ± 7	−26 ± 7
$\frac{1}{2}(a_1+a_2+b_1+b_2)$	−16 ± 7	−34 ± 17	−2 ± 9	−8 ± 9
$\frac{1}{2}(a_1-a_2+b_1-b_2)$	+24 ± 9	+20 ± 23	−6 ± 11	+44 ± 11

model mass equation. The calculations are simplified if, for not-too-small values of A and T_z, one separates $B(A, T_z)$ and $I_{np}(A, T_z)$ into continuous and discontinuous contributions (such as contributions from pairing energies) according to

$$B(A, T_z) = B^{\text{cont}}(A, T_z) + B^{\text{discont}}(A, T_z)\,, \tag{23}$$

$$I_{np}(A, T_z) = I_{np}^{\text{cont}}(A, T_z) + I_{np}^{\text{discont}}(A, T_z)\,. \tag{24}$$

Then

$$I_{np}^{\text{cont}}(A, T_z) \approx \frac{\partial^2}{\partial A^2} B^{\text{cont}}(A-1, T_z) - \tfrac{1}{4}\frac{\partial^2}{\partial T_z^{\,2}} B^{\text{cont}}(A-1, T_z) \tag{25}$$

and

$$I_{np}^{\text{cont}}(A, T_z+\tfrac{1}{2}) - I_{np}^{\text{cont}}(A, T_z-\tfrac{1}{2}) \approx \frac{\partial}{\partial T_z} I_{np}^{\text{cont}}(A, T_z)\,, \tag{26}$$

$$I_{np}^{\text{cont}}(A+1, T_z) - I_{np}^{\text{cont}}(A-1, T_z) \approx 2\frac{\partial}{\partial A} I_{np}^{\text{cont}}(A, T_z). \tag{27}$$

We recognize the quantities α_1 and α essentially as ratios of certain third partial derivatives of $B(A, T_z)$ with respect to A and T_z.

The Bethe-Weizsäcker semiempirical mass equation[15] can be written in a slightly modified form as

$$B(A, T_z) = a_v A - a_s A^{2/3} - 4a_{\text{sym}}\frac{|T_z|(|T_z|+1)}{A} - a_c\frac{Z(Z-1)}{A^{1/3}} + \begin{cases} +\dfrac{a_p^{(1)}+a_p^{(2)}}{A^{1/2}}, & \text{even-even} \\ +\dfrac{a_p^{(3)}}{A^{1/2}}, & \text{even-odd} \\ -\dfrac{a_p^{(3)}}{A^{1/2}}, & \text{odd-even} \\ -\dfrac{a_p^{(1)}-a_p^{(2)}}{A^{1/2}}, & \text{odd-odd}\,. \end{cases} \tag{28}$$

The symmetry energy is taken as proportional to $T(T+1)$ rather than T^2 to agree with various shell-model expressions obtained with the use of the isospin formalism. This modification eliminates the need for the so-called Wigner term. The pairing energy is also slightly modified to account for a finite neutron-proton pairing energy (a term with $a_p^{(2)}$) and to account for possible differences between the neutron and the proton pairing energies (a term with $a_p^{(3)}$). Generally, $a_p^{(2)}$ and $a_p^{(3)}$ are assumed to be zero. Using Eq. (28) with Eq.

(25) gives

$$I_{np}(A, T_z) = \frac{(2+4\delta_{T_z 0})a_{\text{sym}}}{A-1}\left(1 - 4\,\frac{|T_z|(|T_z|+1)}{(A-1)^2}\right) + (-1)^A \frac{2a_p^{(2)}}{(A-1)^{1/2}} - \frac{1}{2}\left\{(-1)^{A/2+T_z} + (-1)^{A/2-T_z}\right\}\frac{2a_p^{(1)}}{(A-1)^{3/2}}$$

$$+\frac{1}{2}\left\{(-1)^{A/2+T_z} - (-1)^{A/2-T_z}\right\}\frac{2a_p^{(3)}}{(A-1)^{3/2}} + \frac{2}{9}\,\frac{a_s}{(A-1)^{4/3}} + \frac{2}{9}\,\frac{a_c}{(A-1)^{1/3}}\left(1 - \frac{1}{2}\,\frac{2T_z+1}{A-1} - 2\,\frac{T_z(T_z+1)}{(A-1)^2}\right), \tag{29}$$

where $\delta_{T_z 0}$ is the Kronecker symbol. Equation (29) shows that only the symmetry and the pairing energies contribute significantly to I_{np}. The last two terms in Eq. (29) are very small corrections due to the curvature of the mass surface from the surface and Coulomb energies.

The pairing energy terms in the binding energy expression introduce an oscillatory behavior. With $a_p^{(1)} \approx 11$ MeV and $a_p^{(2)} = a_p^{(3)} = 0$, one obtains oscillations for I_{np} which are in total disagreement with the experimental I_{np} of Fig. 1. It is therefore concluded that the analytical form of the term $\pm a_p^{(1)}/A^{1/2}$ is not correct and should be replaced by two pairing terms which depend separately on the number of neutrons or protons. No contributions to I_{np} will result from such terms. They can be simulated in Eq. (29) by setting $a_p^{(1)} = 0$. (See also the discussion in Sec. VI.) One might expect that the same argument applies to the term with $a_p^{(3)}$. This is not quite correct though, since it is known theoretically and experimentally[16, 17] that the Coulomb pairing energy depends on T and thus on the number of protons *and* neutrons. Because of these complications but mostly because the term with $a_p^{(3)}$ in Eq. (29) is presumably quite small, it will be neglected. The only oscillating term which remains is that due to $a_p^{(2)}$. This term generates an oscillatory structure in good agreement with the observed splitting of I_{np} for even-A and odd-A nuclei. A value of $a_p^{(2)} \approx 1.5$ MeV is required. The quantity I_{np} from Eq. (29) for nuclei along the line of β stability is included in Fig. 3 as a solid line and compared to the experimental I_{np}. The general trend of the data and in particular the splitting for even-A and odd-A nuclei is well reproduced. The following coefficients have been used: $a_v = 14$ MeV, $a_s = 13$ MeV, $a_{\text{sym}} = 19$ MeV, $a_c = 0.6$ MeV, $a_p^{(1)} = 0$ MeV, $a_p^{(2)} = 1.5$ MeV, $a_p^{(3)} = 0$ MeV. Similar good agreement, except for the even-A/odd-A splitting, is obtained with other liquid-drop-type mass equations.[18-20]

The effective interaction I_{np} from Eq. (29) ($a_p^{(1)} = a_p^{(3)} = 0$ will be used from now on) decreases with increasing A approximately inversely with A. It also decreases with increasing $|T_z|$ and, except for the small Coulomb energy term, it depends on $|T_z|$ rather than T_z. The dependence on T_z is approximately quadratic. The quantity I_{np} approaches zero for $N \to 0$ or $Z \to 0$.

Based on the liquid-drop-model equation (29), one can now derive theoretical expressions (not given here) for the correction terms in the modified mass relationships (6), (7), (13), and (14) and for the coefficients α_1 and α in the generalized mass relationships (9) and (15).

The partial difference $I_{np}(A, T_z + \frac{1}{2}) - I_{np}(A, T_z - \frac{1}{2})$ is antisymmetric about $T_z = 0$ except for a small Coulomb energy term. This behavior is expected and necessary to satisfy charge symmetry of nuclear forces. The difference is negative for positive T_z, and shows no oscillations but instead a discontinuity near $T_z = 0$. The magnitude of the term is approximately proportional to T_z, a result which is particularly significant in connection with mass predictions for very neutron-rich nuclei. It decreases with A approximately as A^{-3}. The sign of this expression disagrees with the experimental results of Tables I and II. Also, the experimental results exhibit a very pronounced even-A/odd-A effect which is not reproduced.

The partial difference $I_{np}(A+1, T_z) - I_{np}(A-1, T_z)$ is negative. It decreases with increasing A approximately as A^{-2}. A weak oscillatory contribution for even-A and odd-A nuclei is indicated. Except for a small Coulomb energy term the difference is symmetric about $T_z = 0$, as is expected and necessary to satisfy charge symmetry of nuclear forces. A comparison of these predictions with the experimental results of Tables I and II shows basic agreement.

The two partial differences are plotted in Fig. 5 as a function of A for nuclei along the line of β stability. Also plotted in the figure are the same partial differences obtained from the liquid-drop-type mass equations of Cameron *et al.*,[18] Myers and Swiatecki,[19] and Seeger.[20] The use of slightly different parameters[21] in the equations for the droplet model of Myers and Swiatecki has only little effect. The curves represent calculated correction terms for the mass relationships GKT and GKL for nuclei along the line of β stability. The hatched areas represent the result for the A-averaged differences of Tables I and II obtained from the χ^2 analyses of the experimental mass combinations.

There exists good agreement between the experimental and calculated partial differences of I_{np} with respect to A as shown by the lower curves in Fig. 5. Poor agreement exists between the experimental and calculated partial differences I_{np} with respect to T_z as shown by the upper curves. None of the four curves describes the observed splitting for even-A and odd-A nuclei. Also, the calculated, mostly negative, values for this partial difference disagree with the slightly positive average experimental value.

The above comparison between the experimental and calculated partial I_{np} differences constitutes another test (test 2) of certain aspects of a given mass equation. Information about the T_z dependence is of particular importance since it strongly affects the extrapolations into the regions of unknown neutron-rich or proton-rich nuclei, particularly for the light nuclei.

The T_z dependence of I_{np} obtained from the mass equations is strongly affected by the presence of higher-order contributions to the symmetry energy. The leading volume symmetry energy term is proportional to I^2 [with $I=(N-Z)/A$];

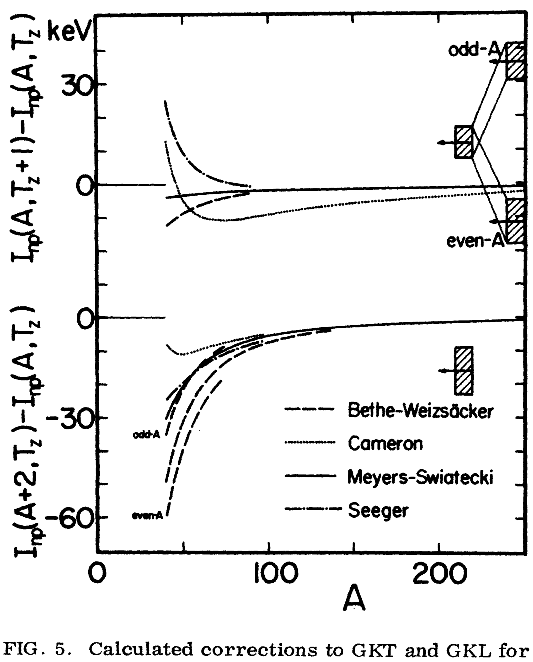

FIG. 5. Calculated corrections to GKT and GKL for nuclei along the line of β stability. The estimates were obtained from several liquid-drop-model mass equations, namely Bethe-Weizsäcker [Eq. (28)], Cameron (Ref. 18), Myers-Swiatecki (Ref. 19), and Seeger (Ref. 20). The hatched areas represent the A-averaged results of the χ^2 analysis of the experimental masses. The comparison between calculated and experimental values constitutes test 2.

higher-order terms are proportional to I^4. In the droplet model,[19] for example, the sign of the latter term is determined by $(MK-L^2)$, where K, L, and M are three expansion coefficients (K = compressibility coefficient; L = density-symmetry coefficient; M = symmetry anharmonicity coefficient). It requires a considerable change in the listed parameters L and/or M to change the sign of that term.

The charge-symmetric relationship (GKS) is obtained from Eq. (29) as a special case. Even for $T_z=0$, there remains a small residual due to the fact that the Coulomb interaction is not charge-independent. If instead of the simple Coulomb energy expression of Eq. (28) one uses the isobaric multiplet mass equation, the general result is

$$\begin{aligned} I_{np}(A, T_z=+T) - I_{np}(A, T_z=-T) &= \left\{-\frac{\partial^2}{\partial A^2} b(A-1, T) + \frac{1}{4}\frac{\partial^2}{\partial T^2} b(A-1, T)\right\} 2T + \frac{\partial}{\partial T} b(A-1, T) \\ &= \left\{+\frac{\partial^2}{\partial A^2} E^{(1)}(A-1, T) - \frac{1}{4}\frac{\partial^2}{\partial T^2} E^{(1)}(A-1, T)\right\} 2T - \frac{\partial}{\partial T} E^{(1)}(A-1, T) . \end{aligned} \tag{30}$$

Here $b(A, T)$ is the coefficient of the linear term and $E^{(1)}(A, T)$ is the vector Coulomb energy. Equation (30) shows that the charge-symmetric relationship GKS requires a small correction term. The modified relationship becomes

$$\Sigma_{\mathrm{T}}(A, -1) = \left(\frac{\partial^2}{\partial A^2} - \frac{1}{4}\frac{\partial^2}{\partial T^2} - \frac{\partial}{\partial T}\right) E^{(1)}(A-1, \tfrac{1}{2}). \tag{31}$$

Equation (31) predicts negative contributions of the order of a few keV for a homogenous charge distribution. Indeed, the analysis of the experimental I_{np} for the mirror nuclei[10] shows a slight preference for negative values. However, the presence of the Thomas-Ehrman shift and related effects probably precludes Eqs. (30) and (31) from becoming useful tools for studying finer details of the coefficients of the isobaric multiplet mass equation.

VI. MICROSCOPIC THEORIES OF I_{np}

Comments about three microscopic descriptions of I_{np} will be presented in this section. One approach involves the use of fourfold-degenerate Nilsson-like or Hartree-Fock single-particle orbits. This model was used originally by Garvey and Kelson[3-5]; it is related to the quartet model.[22]

The other approaches involve shell-model mass equations such as equations based on the seniority or supermultiplet scheme.[23,24]

Theoretical studies of the quantity I_{np} have been performed in the past on the basis of special assumptions or models. One of the earliest works was that of de-Shalit[25] who used a single particle shell model with an odd number of neutrons and of protons in the same shell. He derived the expression $I_{np}=I_0+(-1)^A I'$. Here, I_0 represents a spin-averaged interaction energy and I' accounts for the increased binding in the ground state of an odd-odd nucleus. The expression reproduces the observed splitting between the even-A and odd-A nuclei (see Fig. 1).

A. Fourfold-degenerate single-particle orbits

Based on the extreme single-particle model with fourfold-degenerate levels, the effective neutron-proton interaction I_{np} can be schematically represented as shown in Fig. 6. The distinction between even-A and odd-A nuclei can clearly be seen. A comparison between Eqs. (19)–(22) and the representations of I_{np} according to Fig. 6 leads to the equations schematically shown in Fig. 7. The numerical values of the various quantities are those of Table II. The figure displays the dependence on N, Z, A, and T_z of $I_{np}^{\text{even}-A}$, $I_{np}^{\text{odd}-A}$, and of $\frac{1}{2}(I_{np}^{\text{even}-A}+I_{np}^{\text{odd}-A})$. All three quantities decrease with increasing A at about the same rate. Such a behavior is reasonable because the radii are increasing with A, and I_{np} involves overlaps of neutron and proton radial wave functions.[26] The dependence on T_z is unexpected. The quantities

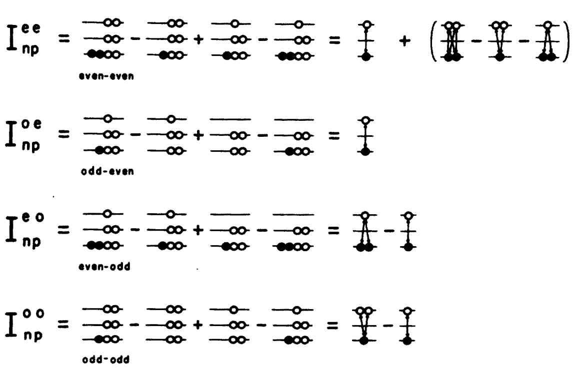

FIG. 6. Schematic representation of the effective neutron-proton interaction I_{np} using an extreme single-particle shell model with fourfold degenerate Hartree-Fock or Nilsson-model-like single-particle orbits. Four types of nuclei are considered with N and Z even or odd. The arrows indicate effective interactions between the respective nucleons. The first term for odd-A nuclei represents the sum of two interactions which are averaged over the respective spins. (The superscripts odd-even and odd-odd have to be exchanged.)

$I_{np}^{\text{even}-A}$ and $\frac{1}{2}(I_{np}^{\text{even}-A}+I_{np}^{\text{odd}-A})$ decrease/increase with T_z at roughly the same averaged rate. No explanation can be provided. The quantity a_3 is practically zero due to the cancellation of the spin-averaged interaction energies. Unexpectedly, however, the quantity b_3 is different from zero. This noncancellation is probably due to another inadequacy of the model. The Coulomb pairing energy between proton pairs, for example, shows a dependence on T of the form T^{-1} which is definitely not satisfied by the representations of I_{np} of Fig. 6. Other pairing contributions may also not be satisfied by the representation. The Nilsson model (see for example Brink and Kerman[27]) or the quartet model[22] provide natural extensions of the above considerations and may result in an improved description of the various findings.

B. Shell model with neutron and proton seniority

Other microscopic theories for the effective neutron-proton interaction I_{np} are based on the shell model. The procedure to obtain I_{np} from a mass equation is of course that used earlier in Sec. V. Assuming that proton seniority and neutron seniority are good quantum numbers, the binding energy of a nucleus with n neutrons and p protons outside a core (N_0, Z_0) is written as[1,5,28]

$$\begin{aligned} B(N_0+n, Z_0+p) = {} & B_0(N_0, Z_0) + n\alpha_n + \tfrac{1}{2}n(n-1)\beta_n + [\tfrac{1}{2}n]\pi_n \\ & + p\alpha_p + \tfrac{1}{2}p(p-1)\beta_p + [\tfrac{1}{2}p]\pi_p + npI_0 \\ & + I'\delta_{\text{odd, odd}} \,. \end{aligned} \tag{32}$$

The coefficients can be expressed in terms of two-body interaction matrix elements. The above

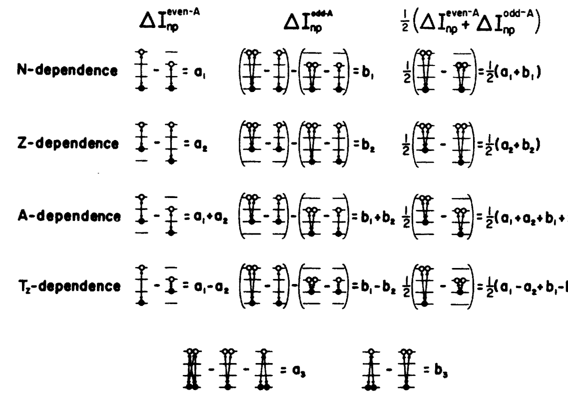

FIG. 7. Comparison between the difference of effective neutron-proton interactions I_{np} based on the model of fourfold degenerate single-particle orbits with the coefficients of Eqs. (19)–(22). The numerical values of the various coefficients which were obtained from the experimental masses are given in Table II.

equation is useful when neutrons and protons are in different shells. The term $I'\delta_{\text{odd, odd}}$ accounts for the energy difference in an odd-odd nucleus between the centroid of the low-lying states (coupled to the same seniorities as the ground state) and the energetically favored ground state (Nordheim rules[29]). The energy difference I' depends on J. If it is assumed that all coefficients in Eq. (32) are constant with a given shell, one obtains

$$I_{np}^{\text{even}-A} = I_0 + I' \qquad (33)$$
$$I_{np}^{\text{odd}-A} = I_0 - I' .$$

Thus, I_{np} would indeed be constant (separately for even-A and odd-A nuclei) and GKT and GKL would both hold. A comparison between Eq. (33) and Fig. 3 shows that I' is definitely finite and of the order of 200 keV.

Zeldes *et al.*[28,30] have shown long ago that I' in Eq. (33) should be written with an explicit dependence on n and p of the form

$$I'(n, p, J) = I''(J) + I'''(J)\left(1 - \frac{2(n-1)}{2j_n - 1}\right)\left(1 - \frac{2(p-1)}{2j_p - 1}\right). \qquad (34)$$

Equation (33) has to be modified accordingly and I_{np} becomes dependent on A and T_z. The experimentally observed over-all dependence on A and T_z is still not reproduced. However, the above modification leads to different signs for $\partial/\partial T_z(I_{np}^{\text{even}-A})$ and $\partial/\partial T_z(I_{np}^{\text{odd}-A})$ in agreement with the experimentally observed signs. An oscillatory dependence of $I_{np}^{\text{odd}-A}$ on T_z is also predicted but appears to be out of phase with the observed one. Despite these shortcomings, it is interesting to see that an explicit dependence on T_z or T of the term $I'\delta_{\text{odd, odd}}$ in Eq. (32) generates different signs in the T_z dependence for even-A and odd-A nuclei as well as oscillations.

C. Shell model with seniority and isospin

Another shell-model mass equation is[23]

$$\begin{aligned} B(A_0 + \Delta A, T) = B(A_0, 0) &+ \Delta A\alpha + \tfrac{1}{2}\Delta A(\Delta A - 1)\beta \\ &+ \{T(T+1) - \tfrac{3}{4}\Delta A\}\gamma \\ &+ [\tfrac{1}{2}\Delta A]\pi + I'\delta_{\text{odd, odd}} + \text{C.E.} \end{aligned} \qquad (35)$$

with $T = |T_z|$. Neutrons and protons are assumed to be in the same j^n configuration outside a doubly magic core with $N_0 = Z_0 = \frac{1}{2}A_0$. The seniority coupling scheme is assumed in the isospin formalism. The Coulomb energy is not included explicitly. Without the term $I'\delta_{\text{odd, odd}}$, Eq. (35) describes the ground-state energies of even-even nuclei ($v = 0$, $t = 0$) and the ground-state energies of odd-A nuclei ($v = 1$, $t = \frac{1}{2}$), but only the average or centroid energies of the low-lying states with $v = 2$, $t = 1$ ($T_z \neq 0$) in odd-odd nuclei. It is for this reason that the term $I'\delta_{\text{odd, odd}}$ is again added to describe the energy difference between the centroid and ground-state energies. If the coefficients in Eq. (35) are again assumed to be constant within a given shell, one obtains (except for a possible small Coulomb energy correction term C.E.)

$$I_{np}^{\text{even}-A}(A, T_z) = I_0 + (\pi + I'), \qquad (36)$$
$$I_{np}^{\text{odd}-A}(A, T_z) = I_0 - (\pi + I')$$

with

$$I_0 \equiv (\beta - \tfrac{1}{2}\gamma) = \tfrac{1}{2}(\overline{V}_1 + \overline{V}_2)$$
$$\pi = \frac{2j+2}{2j+1}(V_0 - \overline{V}_2) . \qquad (37)$$

Here, V_0, $\overline{V}_1$, and $\overline{V}_2$ are the two-body matrix elements for nucleons coupled to $J = 0$, averaged over odd J or even J ($\neq 0$), respectively. Again, I_{np} is constant (separately for even-A and odd-A nuclei). It is interesting to note, however, that the splitting between $I_{nb}^{\text{even}-A}$ and $I_{np}^{\text{odd}-A}$ is due to the above energy I' *and* to the nucleon pairing energy π.

It is an extreme simplification to assume that the coefficients in Eqs. (32) or (35) are constant. Zeldes and co-workers[1,12,28,31] found it necessary to introduce within major shells a linear dependence of all coefficients on n and p (for light nuclei even a quadratic dependence) to describe the experimental masses. More recently Liran and Zeldes[32] introduced a dependence of the form A^{-1}. Terms to account for configuration mixing and for deformations had to be added also. It is interesting to note that the use of Zeldes's earlier mass equation leads to constant values within major shells for the parameters α_1 and α in the generalized mass relationships (9) and (15).

Clearly, the above simple shell-model equations do not account for higher-order perturbations. While the true nucleon-nucleon interaction is presumably a two-body interaction, the effective interaction between nucleons outside a core includes three- and more-body interactions due to particle-hole excitations out of the core. Therefore, higher-order terms and corrections are generated.

The symmetry energy term with $T(T+1)$ may be considered as the leading term of a Taylor expansion. Hecht[33] has actually shown that seniority mixing will introduce a term with $T^2(T+1)^2$. We thus add a term $\lambda I_0 T^2(T+1)^2$ to Eq. (35). Similarly, the two terms with π and I' may have to be modified, the latter, according to Zeldes,[28,30] even in first order. To study the effect such perturbations may have on I_{np} and its dependence on

A and T_z, we substitute in Eq. (35)

$$I_0 \to I_0\{1+\lambda T(T+1)\}\,, \qquad \pi \to \pi\{1+\phi_1(\Delta A, T)\}\,, \qquad I' \to I'\{1+\phi_2(\Delta A, T)\}\,. \tag{38}$$

Here $T=|T_z|$, and ϕ_1 and ϕ_2 are as yet undetermined functions of ΔA and $|T_z|$. The result is

$$I_{np}^{ee}(A, T_z) = I_0\{1+\tfrac{5}{4}\lambda+6\lambda T(T+1)\} + \pi\{1+\phi_1(\Delta A, T)\} + I'\{1+\phi_2(\Delta A-2, T)\}\,, \tag{39}$$

$$I_{np}^{oo}(A, T_z) = I_0\{1+\tfrac{5}{4}\lambda+6\lambda T(T+1)\} + \pi\{1+\phi_1(\Delta A-2, T)\} + I'\{1+\phi_2(\Delta A, T)\}\,, \tag{40}$$

$$I_{np}^{eo}(A, T_z) = I_0\{1+\tfrac{5}{4}\lambda+6\lambda T(T+1)\} - \pi\{1+\phi_1(\Delta A-1, T-\tfrac{1}{2})\} - I'\{1+\phi_2(\Delta A-1, T+\tfrac{1}{2})\}\,, \tag{41}$$

$$I_{np}^{oe}(A, T_z) = I_0\{1+\tfrac{5}{4}\lambda+6\lambda T(T+1)\} - \pi\{1+\phi_1(\Delta A-1, T+\tfrac{1}{2})\} - I'\{1+\phi_2(\Delta A-1, T+\tfrac{1}{2})\}\,. \tag{42}$$

The average nonoscillating contributions from Eqs. (39)–(42) to the partial difference $I_{np}(A, |T_z|+\frac{1}{2}) - I_{np}(A, |T_z|-\frac{1}{2})$ become $12\lambda I_0(T+\frac{1}{2})$. Thus, the sign of λ determines whether there is an average increase or decrease of I_{np} with increasing T. The experimental data (see Sec. IV) require $\lambda > 0$ while essentially all mass equations predict $\lambda < 0$.

The observed differences for even-A and odd-A nuclei require $\partial\phi_1/\partial T < 0$. It appears that no such T dependence has ever been included in a mass equation, and a term I' has been included only occasionally.[12, 32]

Another required modification in Eqs. (36) and (38) is, of course, a decrease with increasing A of the coefficients assumed to be constant so far. All interaction energies involve overlap integrals for neutron and proton radial wave functions which decrease with increasing nuclear size. Ferguson[26] found decreases with $A^{-1/2}$ or A^{-1}.

The quantity I_{np} appears to present the key for obtaining reliable estimates for the masses of unknown neutron-rich or proton-rich nuclei. It is interesting to find that higher-order terms in shell-model mass equations play an important role. Such terms have not been included in the equations based on the seniority coupling scheme[23] or the Wigner supermultiplet coupling scheme.[24] It would be very desirable to have such terms included in shell-model mass equations or shell-model theories of the effective neutron-proton interaction I_{np}. The "fluctuations" in the experimental values of I_{np} which are of the order of ± 200 keV, however, cannot be reproduced unless much finer details of the wave functions are included in the calculations.

VII. SOLUTIONS OF THE GENERALIZED NUCLIDIC MASS RELATIONSHIPS

It has been pointed out earlier that any mass equation, whether given in analytical form or tabulated, can be used to obtain expressions for $I_{np}(A, T_z)$ and the parameters $\alpha_1(A, T_z)$ and $\alpha(A, T_z)$. With these, the modified transverse and longitudinal mass relationships and the generalized mass relationships can be used as recursion relationships and become tools for predicting masses of unknown nuclei. A more general approach will be presented below.

The generalized nuclidic mass relationships as well as the relationships GKT and GKL represent partial difference equations. The functional forms of the solutions constitute mass equations. Garvey *et al.*[5] have given the solutions for GKT and GKL. The relationship GKT is satisfied by any mass equation $M^*(A, T_z)$ of the form

$$M^*(A, T_z) = g_1(N) + g_2(Z) + g_3(A) \tag{43}$$

and GKL is satisfied by

$$M^*(A, T_z) = f_1(N) + f_2(Z) + f_3(T_z). \tag{44}$$

The authors constructed these functions in tabular form by minimizing the deviations from the known masses. The use of such tables is simpler than the use of relationships. Also, the accuracy of predictions of nearby masses should be improved because use is made of *all* known masses.

If the parameters $\alpha_1(A, T_z)$ and $\alpha(A, T_z)$ of the generalized relationships Eqs. (9) and (15) are derived from some given mass equation $M(A, T_z)$, this very same mass equation will, of course, satisfy the partial difference equations. However, the solutions $M^*(A, T_z)$ are more general than the original mass equation. A very general solution can be obtained with the ansatz

$$M^*(A, T_z) = M(A, T_z) + F_1(N) + F_2(Z) + F_3(T_z) + F_4(A)\,. \tag{45}$$

In addition to $M(A, T_z)$, arbitrary functions $F_1(N)$ and $F_2(Z)$ will always satisfy the difference equations. The functions $F_3(A)$ and $F_4(T_z)$, however, are not arbitrary because they have to satisfy a coupled difference equation. One can show that quadratic contributions as well as certain pairing expressions satisfy this equation independent of $\alpha_1(A, T_z)$. Thus

$$F_3(T_z) + F_4(A) = -2\eta_1 NZ + \eta_2\delta_{\text{even,even}} + \eta_3\delta_{\text{odd,odd}} + \phi_3(T_z) + \phi_4(A) \tag{46}$$

with $N=\frac{1}{2}A+T_z$ and $Z=\frac{1}{2}A-T_z$. The quantities η_1, η_2, and η_3 are constant. The functions $\phi_3(T_z)$ and $\phi_4(A)$ represent higher-order contributions which can only be determined if $\alpha_1(A, T_z)$ is given explicitly. It should be added that Eq. (45), with Eq. (46), satisfies *all* the generalized nuclidic mass relationships of Sec. III.

Two cases deserve special attention. If we assume a mass equation which can be represented as a sum of functions of N, Z, and A, it follows that $\alpha_1(A, T_z)=0$, $\phi_3(T_z)=0$, and $\phi_4(A)$ is arbitrary. Similarly, if a mass equation can be represented by a sum of functions of $N, Z,$ and T_z, it follows that $\alpha_1(A, T_z)=\infty$, $\phi_3(T_z)$ is arbitrary, and $\phi_4(A)=0$. We thus recognize the functional forms for GKT and GKL of Eqs. (43) and (44) as special cases of the more general solution, Eq. (45).

The above procedure of adding functions $F_1(N)$ and $F_2(Z)$ to a given mass equation had been used many years ago by Burbridge *et al.*[34] as a purely empirical method. The terms with η_1, η_2, and η_3 were not included. They used a simple liquid-drop-model mass equation. More recently, Cameron and co-workers[18] employed a more elaborate liquid-drop-model mass equation. They added empirical functions $F_1(N)$ and $F_2(Z)$ which they separated into shell and pairing contributions. The addition of these functions improved the standard deviation for reproducing the known masses from a few MeV to a few hundred keV.

The terms $F_3(T_z)$ and $F_4(A)$ in Eq. (45) represent contributions from the effective neutron-proton interaction I_{np}. However, the better the true I_{np} is represented by the respective terms in the mass equation $M(A, T_z)$, the smaller the contribution from $F_3(T_z)+F_4(A)$ will be. It should therefore generally suffice to use

$$M^*(A, T_z)=M(A, T_z)+F_1(N)+F_2(Z)+\eta_1(N-Z)^2+\eta_2\delta_{\text{even, even}}+\eta_3\delta_{\text{odd, odd}}\,. \tag{47}$$

Here, the term with η_1 has been rewritten and the functions $F_1(N)$ and $F_2(Z)$ have been redefined in an obvious way. The corrections to a given mass equation $M(A, T_z)$ can easily be constructed from the known masses by introducing about $150+100+1$ parameters which represent the functional values of $F_1(N)$ and $F_2(Z)$ and the three parameters η_1, η_2, and η_3. Actually, the number of parameters is reduced by two because instead of $F_1(N)$, $F_2(Z)$, η_1, η_2, η_3 any new set $F_1(N)+c_1+(-1)^N c_2$, $F_2(Z)-c_1+(-1)^Z c_2$, η_1, η_2-2c_2, η_3+2c_2 will lead to identical results for arbitrary c_1 and c_2. It is therefore practical to require $\eta_2=\eta_3$ and to require the two functions to agree for certain arguments, for example $F_1(20)=F_2(20)$. The approximately 250 parameters can be obtained for any given mass equation by minimizing

$$\chi^2=\sum^{1250}\left\{\frac{M^*(A, T_z)-M_{\exp}(A, T_z)}{\delta M_{\exp}(A, T_z)}\right\}^2 \tag{48}$$

This procedure results in a simple system of 250 linear equations for 250 unknowns.

There are two important aspects to this procedure: It provides a test for the quality of a given mass equation and it generates correction terms for a given mass equation.

The three parameters η_1, η_2, and η_3 should be zero if the effective neutron-proton interaction is well described by the mass equation $M(A, T_z)$. Thus, the smallness of η_1, η_2, and η_3 provides a test (test 3) for the quality of $M(A, T_z)$ in describing I_{np}. Furthermore, the functions $F_1(N)$ and $F_2(Z)$ should satisfy $F_1(i)\approx F_2(i)$. A strong deviation from this requirement indicates a misrepresentation of the electrostatic energy in the mass equation $M(A, T_z)$. Finally, the smallness of the ratio of $\chi^2_{\min}$ over the number of degrees of freedom characterizes the goodness of fit. The procedure can be applied to the region of all known nuclei or only to smaller regions.

The procedure based on Eqs. (47) and (48) generates additive correction terms for any mass equation $M(A, T_z)$. The terms due to the interaction among the neutrons or the protons need not be described well at all by $M(A, T_z)$. Terms such as the volume energy in a liquid-drop-model mass equation could actually be left out altogether. The better $M(A, T_z)$ with η_1, η_2, and η_3 describes the true effective n-p interaction, the better will $M(A, T_z)$ *with* $F_1(N)$ and $F_2(Z)$ describe the true effective n-n and p-p interactions.

The mass equation $M^*(A, T_z)$ combines the accuracy and reliability of the mass equation $M(A, T_z)$ and of a mass relationship in the following way: While $M^*(A, T_z)$ is expected to reproduce the known masses with a standard deviation similar to that of GKT or GKL, the reliability in predicting unknown masses far away from the line of β stability will still be mostly determined by the expression for I_{np} implicitly contained in the mass equation $M(A, T_z)$. The added correction terms in $M^*(A, T_z)$ may lead to some improvement. The important conclusion, however, is that the three tests presented earlier in this work make it possible to actually judge the reliability of the mass equations $M(A, T_z)$ and $M^*(A, T_z)$.

VIII. SUMMARY AND CONCLUSION

The Garvey-Kelson nuclidic mass relationships have been modified and generalized nuclidic mass relationships have been introduced. It has been shown that generalized nuclidic mass relationships represent a link between mass relationships and mass equations. They are partial difference equations and they can be used as recursion relationships to estimate masses of unknown nuclei. A necessary condition for the use of the modified or the generalized relationships is certain knowledge about the effective neutron-proton interaction I_{np}, particularly about its dependence on T_z or T and on A. Such information can, in conjunction with phenomenological studies of the existing data, be obtained from macroscopic theories, such as the liquid-drop model, or from microscopic theories, such as the shell model. It has been found that I_{np} depends on A and on T. Solutions of the partial difference equations have been obtained for the case where I_{np} is derived from any mass equation $M(A, T_z)$ given in analytical or tabular form. The result is a mass equation $M^*(A, T_z)$ consisting of the original mass equation (used in the derivation of I_{np}) and correction terms. The correction terms can be constructed from a χ^2 minimization to the known masses.

A variety of χ^2 tests of the experimental masses have been performed and information about the dependence of I_{np} on T_z and A has been obtained. The dependence on T_z was found to be in disagreement with theoretical expectations. It has been further established that the mass relationships GKT and GKL contain small systematic errors superimposed on the more random fluctuations which are of the order of ± 200 keV. The magnitude of the systematic deviations is generally quite small but reaches 200 keV for certain types of light nuclei. The deviations accumulate at a rapid rate in the repeated application of the relationships.

The reliability of any mass equation or relationship for predicting masses of nuclei far away from the line of β stability depends on the quality of the expression for I_{np}. Three tests have been devised to check expressions for I_{np} derived from a mass equation M or other explicit theories of I_{np} for their compatibility and consistency with the experimental masses.

Test 1 makes use of the modified transverse and longitudinal nuclidic mass relationships of Eqs. (6) and (7) or, in the more general form, of Eqs. (17) and (18). The two respective equations give two independent estimates of the mass of any unknown nucleus. The two extrapolations should be reasonably consistent. It is felt that this consistency requirement constitutes a very sensitive test of the expression for I_{np}.

Test 2 requires agreement between the calculated partial differences of I_{np} for nuclei along the line of β stability with the experimentally determined values of Tables I and II and of Fig. 5.

Test 3, finally, concerns the solutions $M^*(A, T_z)$ obtained from the procedure defined by Eqs. (47) and (48). The effective interaction I_{np} which is implicitly contained in a mass equation $M(A, T_z)$ depends essentially on the symmetry energy and on certain pairing energy terms. The smallness of the parameters η_1, η_2, and η_3 obtained from the above procedure is a necessary requirement for the true I_{np} to be well represented by $M(A, T_z)$. The smallness of $F_2(k) - F_1(k)$ reflects on the quality of the Coulomb energy terms in $M(A, T_z)$.

Many numerical applications of the ideas represented in this work become possible. The most important problem still to be solved is the apparent discrepancy between the experimental and theoretical T dependence of the effective interaction I_{np}. An application of the three tests to various mass equations should provide additional insight. Ultimately, of course, one wants to predict unknown masses near and far away from the line of β stability based on the solution of a generalized mass relationship which incorporates a theoretically sound expression for I_{np}. While this approach is not possible at the moment due to the above discrepancies, only phenomenological expressions can presently be used.

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