Level-level correlations in Hauser-Feshbach theory and Moldauer's sum rule for resonance reactions

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It is shown that Moldauer's sum rule is equivalent to analyticity and unitarity of the S matrix. Inconsistencies derived from the sum rule with the help of some approximation thus indicate that the approximation violates unitarity. It is also shown that a derivation of the Hauser-Feshbach formula beyond the regime of small absorption in all channels is possible only in the frame of a dynamical model for the compound nucleus.

1. PURPOSE

If level-level correlations are neglected, Hauser-Feshbach theory can consistently be derived only up to terms of lowest order in the eigenvalues p_a of Satchler's transmission matrix P_{ab}. Terms of higher order give rise to a[illegible]consistency. This result is based[1] upon Mold[illegible]ler's sum rule[2] for resonance reactions. It ra[illegible]ses the question: What is the domain of validity of the sum rule? By giving two derivations, we show that one does not have to invoke the statistical arguments used originally,[2] and that the rule applies whenever the averaging interval I can be chosen large in comparison with the total width of each of the resonances. At the same time, we show that the inclusion of level-level correlations removes the above-mentioned inconsistency without destroying the validity of the sum rule. This further illuminates the central role of the sum rule in Hauser-Feshbach theory, and reinforces the conclusions of Ref. 1.

2. NOTATION

For fixed values of spin and parity of the compound nucleus, we write the unitary and symmetric S matrix in the form

$$S_{ab}=S_{ab}^{(0)}-i\sum_{\mu=1}^{M} g_{\mu a}g_{\mu b}(E-\xi_\mu)^{-1}\ , \qquad (1)$$

where the background matrix $S^{(0)}$, the partial-width amplitudes $g_{\mu a}$, and the complex resonance energies $\xi_\mu=E_\mu-\frac{1}{2}i\Gamma_\mu$ are assumed constant; where E is the energy of the system; and where the number M of poles of S is assumed finite, although M may be very large. The indices $a, b, \ldots$ refer to the open channels. The constancy of the parameters $S_{ab}^{(0)}$, $g_{\mu a}$, and ξ_μ implies neglect of threshold effects in the energy interval of interest. This specific model for the nuclear S matrix is frequently used in the study of resonance reactions. The average over energy is performed with a Lorentzian weighting factor of width I centered at E_0. We assume that

$$I\gg\Gamma_\mu,\quad \text{for all } \mu=1,\ldots,M\ . \qquad (2)$$

This imposes a restriction on the distribution of the poles of S. Numerical studies indicate[3] that for sensible dynamical models, the poles of S tend to lie not too far below the real E axis, so that assumption (2) can consistently be made. We define

$$\frac{\pi}{D}\langle g_{\mu a}g_{\mu b}\rangle_\mu=\sum_\mu \frac{I g_{\mu a}g_{\mu b}}{(E_\mu-E_0)^2+I^2}\ , \qquad (3)$$

where D is the average level spacing. Satchler's transmission matrix[4] is given by

$$P_{ab}=\delta_{ab}-\sum_c \langle S_{ac}\rangle\langle S_{bc}^*\rangle\ . \qquad (4)$$

The symbol $\langle S_{ac}\rangle$ denotes the energy-averaged S matrix.

3. UNITARITY AND THE SUM RULE

The unitarity of the S matrix implies that for complex values of E, we have

$$\sum_c S_{ac}(E)S_{cb}^*(E^*)=\delta_{ab}\ . \qquad (5)$$

Equation (5) is the continuation of the unitarity relationship to complex E. It is, therefore, referred to as "analytic unitarity" in the sequel. To derive Moldauer's sum rule from Eq. (5), we observe that under the assumption (2), Eq. (3) can be written in the form:

$$\frac{2\pi}{D}\langle g_{\mu a}g_{\mu b}\rangle_\mu=S_{ab}(E_0-iI)-S_{ab}(E_0+iI). \qquad (6)$$

Equation (5) shows that $S_{ab}(E_0-iI)=[S^{*-1}(E_0+iI)]_{ab}$, and Eq. (1) implies $S_{ab}(E_0+iI)=\langle S_{ab}\rangle$. Inserting all this into Eq. (6), we obtain the sum rule[2]:

$$\frac{2\pi}{D}\langle g_{\mu a} g_{\mu b}\rangle_\mu = (\langle S^*\rangle^{-1})_{ab} - \langle S_{ab}\rangle \ . \tag{7}$$

This derivation shows that the sum rule can be obtained without the use of statistical assumptions on S. It is also evident that, aside from the assumptions made in the relations (1) and (2), *the sum rule embodies analytic unitarity as the essential ingredient*. Inconsistencies with the sum rule such as the ones found in Ref. 1 are, therefore, indications that the approximations employed violate unitarity.

4. INCLUSION OF LEVEL-LEVEL CORRELATIONS

We show that the inconsistency found in Ref. 1 disappears if level-level correlations are not neglected. We evaluate the unitarity relation (5) at a pole ξ_ν^* of S^*. Using Eq. (1), we find

$$\sum_b S_{ab}^{(0)} g_{\nu b}^* = i\sum_{b,\mu} g_{\mu a} g_{\mu b} g_{\nu b}^* \left[E_\nu - E_\mu + \frac{i}{2}(\Gamma_\mu + \Gamma_\nu)\right]^{-1}. \tag{8}$$

We see that analytic unitarity implies correlations between the partial-width amplitudes of different levels: It is impossible to consider the $g_{\mu a}, g_{\nu b}$ as *independent* random variables, since they must obey Eq. (8). With the help of the definition (3), we cast Eq. (8) into the form

$$\sum_b S_{ab}^{(0)} \frac{2\pi}{D}\langle g_{\nu b}^* g_{\nu c}^*\rangle_\nu = 2i\pi \sum_{b,\mu,\nu} \frac{I/\pi}{(E_\nu - E_0)^2 + I^2} \, \frac{g_{\mu a} g_{\mu b} g_{\nu b}^* g_{\nu c}^*}{E_\nu - E_\mu + \frac{1}{2}i(\Gamma_\mu + \Gamma_\nu)} \ . \tag{9}$$

Writing $S_{ab} = \langle S_{ab}\rangle + S_{ab}^{\text{fl}}$, averaging the relation (5) for real E over energy, and using the definition (4) we find, in obvious notation:

$$P_{ac} = \sum_b \langle S_{ab}^{\text{fl}} S_{bc}^{\text{fl}*}\rangle \ . \tag{10}$$

The calculation of the right-hand side yields[5] with the help of Eq. (1)

$$P_{ac} = 2i\pi \sum_{b,\mu,\nu} \frac{I/\pi}{(E_\nu - E_0)^2 + I^2} \, \frac{g_{\mu a} g_{\mu b} g_{\nu b}^* g_{\nu c}^*}{E_\nu - E_\mu + \frac{1}{2}i(\Gamma_\mu + \Gamma_\nu)} + \sum_b \frac{2\pi}{D}\langle g_{\nu b}^* g_{\nu c}^*\rangle_\nu \left(-i\sum_\mu \frac{g_{\mu a} g_{\mu b}}{E_0 - E_\mu + iI}\right). \tag{11}$$

If one neglects in Eq. (11) the last term as well as the terms with $\mu \neq \nu$ in the first sum over μ and ν, one obtains

$$P_{ac} = \frac{2\pi}{D}\left\langle g_{\mu a} g_{\mu c}^* \left(\sum_b |g_{\mu b}|^2\right)\Gamma_\mu^{-1}\right\rangle_\mu \ . \tag{12}$$

This is the relation used in Ref. 1 to derive the above-mentioned inconsistency with the sum rule (7). The transition from Eq. (11) to Eq. (12) corresponds exactly to the neglect of level-level correlations in Hauser-Feshbach theory,[5] i.e., of the terms denoted by $M_{cc'}$ in Eq. (60) of Ref. 5. We can avoid this approximation by noticing that the first sum on the right-hand side of Eq. (11) is given by Eq. (9). Hence,

$$P_{ac} = \sum_b \langle S_{ab}\rangle \frac{2\pi}{D}\langle g_{\nu b}^* g_{\nu c}^*\rangle_\nu \ . \tag{13}$$

Using the definition (4), and taking the complex conjugate of the resulting equation, we arrive at Eq. (7). The inconsistency between Eq. (12) and Eq. (7) is thus avoided if level-level correlations are not neglected. Moreover, we again see that the sum rule (7) is based on analytic unitarity, this time used in the form of Eq. (9).

5. CONCLUSIONS

We have shown that Moldauer's sum rule is necessary and sufficient for analytic unitarity. This is the reason for its importance and usefulness. We have also shown that the inconsistency pointed out in Ref. 1 is exclusively due to the neglect of level-level correlations. It was shown in Ref. 1 that such neglect, the assumption of a Gaussian distribution, and the combination of Eqs. (12) and (7) determine the statistics of the resonance parameters $g_{\mu a}$, if only terms of lowest order in the p_a are considered.[6] This makes it possible to formulate Hauser-Feshbach theory without a dynamical model for the compound nucleus, and to base it entirely upon the presumed knowledge of $\langle S_{ab}\rangle$. We have now shown that this property is lost if level-level correlations are included, since then Eq. (11) and Eq. (7) become equivalent, and since it is impossible to deduce the statistics of the $g_{\mu a}$ only from Eq. (7). Standard Hauser-Feshbach theory[5] and its extension to the case of direct reactions as formulated in Ref. 1 must thus be viewed as the lowest-order approximation (in the p_a) to the evaluation of $\langle S_{ab}^{\text{fl}} S_{cd}^{\text{fl}*}\rangle$. Only the lowest-order terms are determined by $\langle S_{ab}\rangle$ and general properties of the S matrix. The calculation of terms of higher order requires the knowledge of level-level correlations and, hence, a dynamical model for the compound nucleus. The domain of validity of the usual derivation of the Hauser-Feshbach formula is given by the condition $p_a \ll 1$ for all a. The numbers given in Ref. 3 indicate that this condition does not necessarily imply $\Gamma = \langle \Gamma_\mu \rangle_\mu \ll D$. In the case of many open channels, $p_a \ll 1$ seems to be consistent also with $\Gamma \gtrsim D$.

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[6]It is well known that the neglected terms are of order p_a^2, so that legitimate conclusions can be drawn by considering only linear terms in p_a.