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Weakly interacting massive particle-nucleus elastic scattering response

Nikhil Anand,[1] A. Liam Fitzpatrick,[2] and W. C. Haxton[1,3]

[1]*Department of Physics, University of California, Berkeley, California 94720, USA*

[2]*Stanford Institute for Theoretical Physics, Stanford University, Stanford, California 94305, USA*

[3]*Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

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Background: A model-independent formulation of weakly interacting massive particle (WIMP)-nucleon scattering was recently developed in Galilean-invariant effective field theory.

Purpose: Here we complete the embedding of this effective interaction in the nucleus, constructing the most general elastic nuclear cross section as a factorized product of WIMP and nuclear response functions. This form explicitly defines what can and cannot be learned about the low-energy constants of the effective theory—and consequently about candidate ultraviolet theories of dark matter—from elastic scattering experiments.

Results: We identify those interactions that cannot be reliably treated in a spin-independent/spin-dependent (SI/SD) formulation: For derivative- or velocity-dependent couplings, the SI/SD formulation generally mischaracterizes the relevant nuclear operator and its multipolarity (e.g., scalar or vector) and greatly underestimates experimental sensitivities. This can lead to apparent conflicts between experiments when, in fact, none may exist. The new nuclear responses appearing in the factorized cross section are related to familiar electroweak nuclear operators such as angular momentum $\vec{l}(i)$ and the spin-orbit coupling $\vec{\sigma}(i)\cdot\vec{l}(i)$.

Conclusions: To unambiguously interpret experiments and to extract all of the available information on the particle physics of dark matter, experimentalists will need to (1) do a sufficient number of experiments with nuclear targets having the requisite sensitivities to the various operators and (2) analyze the results in a formalism that does not arbitrarily limit the candidate operators. In an appendix we describe a code that is available to help interested readers implement such an analysis.

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I. INTRODUCTION

Despite the many successes of the Lambda Cold Dark Matter (ΛCDM) cosmological model in predicting the macroscopic behavior of dark matter, attempts at an experimentally significant direct detection of the dark-matter particle have been unsuccessful and its fundamental nature remains uncertain [1,2]. A promising candidate is a weakly interacting massive particle (WIMP) that interacts with standard-model particles through a cross section that is suppressed compared to standard electromagnetic interactions. The challenges associated with observing such a particle notwithstanding, experimental techniques are advancing at a rapid pace, and expectations are high that a definitive measurement of dark-matter interactions is imminent.

In "direct-detection" experiments, an important class of dark-matter searches, the signals are recoil events following WIMP elastic scattering off target nuclei [3–5]. Many models predict rates for such events consistent with the sensitivities some experiments are now reaching. Most models of WIMPs invoke new physics, such as supersymmetry or extra dimensions, associated with electroweak symmetry breaking, where new phenomena can appear at scales that, from a particle-physics perspective, are quite low, e.g., $\gtrsim$100 GeV. However, the momentum transfer in direct detections is still far lower, typically a few hundred MeV or less. Consequently, effective field theory (EFT) provides a general and very efficient way to characterize experiment results: Regardless of the complexity or variety of candidate ultraviolet theories of dark matter, their low-energy consequences can be encoded in a small set of parameters, such as the mass of the WIMP and the effective coupling constants describing the strength of the contact coupling of the WIMP to the nucleon or nucleus. The information that can be extracted from low-energy experiments can be expressed as constraints on the low-energy constants of the EFT.

The WIMP-nucleon scattering is typically treated by modeling the nucleus as a point particle, characterized by a charge and spin, with the charge and spin couplings sometimes allowed to be isospin dependent. This yields the standard spin-independent/spin-dependent (SI/SD) description of the scattering [3,6,7]. Because the momentum transfer in the scattering is large compared to the inverse nuclear size, form factors are generally included. As we discuss here, however, this step is not sufficient: Once momentum transfers reach the point where $\vec{q}\cdot\vec{x}(i)$, where $\vec{x}(i)$ is the nucleon coordinate within the nucleus, is no longer small, not only form factors, but new operators arise. These new operators turn out to be parametrically enhanced for a large class of EFT interactions.

The Galilean-invariant EFT we describe below provides a particularly attractive framework for properly treating dark-matter particle scattering. The procedure yields two effective theories, the first at the level of the WIMP-nucleon scattering amplitude, and the second at the nuclear level, because the embedding of the WIMP-nucleon effective interaction in the nucleus generates the most general form of the elastic nuclear response. Six response functions—not the two conventionally assumed—are produced.

(i) The new responses typically dominate the elastic cross section for candidate interactions involving velocity couplings. The standard SI/SD treatment yields amplitudes for such couplings on the order of the WIMP velocity, $\sim 10^{-3}$. In fact, the amplitude is determined by the velocities of bound nucleons, typically $\sim 10^{-1}$.

(ii) In such cases the standard analysis also incorrectly predicts the dependence of the cross section on WIMP and nuclear target masses and even mischaracterizes the multipolarity of the scattering. That is, an interaction that in the point-nucleus limit appears to be vector, with amplitude proportional to matrix elements of $\vec{\sigma}(i)$, may instead produce a much larger scalar response associated with the composite operator $\vec{\sigma}(i) \cdot \vec{l}(i)$. Thus, a $J = 0$ nuclear target may be highly sensitive to a given interaction, not blind to it.

The nuclear physics treatment presented here follows standard treatments of semileptonic electroweak interactions. The operators that arise in a more complete treatment of WIMP-nucleus scattering also appear in descriptions, for example, of neutrino-nucleus scattering.

The enlarged set of nuclear responses that emerges from a model-independent analysis has important experimental consequences. The EFT analysis shows that elastic scattering can place several new constraints on dark-matter properties, in addition to the two apparent from the conventional SI/SD treatment, provided enough experiments are done. One can successfully turn the nuclear physics "knobs"—the nuclear responses—to determine these constraints by utilizing target nuclei with the requisite ground-state properties. The EFT analysis also shows other ways candidate interactions can be distinguished, e.g., through the nuclear recoil spectrum (which may depend on the v^0, v^2, and v^4 moments of the WIMP velocity distribution) or through the dependence on the mass of the nucleus used in the target.

The basis for our formulation is the description of the WIMP-nucleon interaction in Ref. [8] which, building on the work of Ref. [9], used nonrelativistic EFT to find the most general low-energy form of that interaction. While nuclear calculations were performed in Ref. [8], they were based on a form of the cross section that entangled the unknown particle physics (the WIMP-nucleon couplings) with the nuclear physics. In contrast, here we present a compact and rather elegant form for the WIMP-nucleus elastic cross section as a product of WIMP and nuclear responses. The particle physics is isolated in the former. This expression defines precisely what can and cannot be learned about the EFT's low-energy constants, and consequently the ultraviolet theories that generate those constants, from WIMP-nucleus elastic scattering. It also defines what experimentalists will need to do, in terms of the number of experiments performed and the properties of the nuclear targets they employ, to extract all possible information on the WIMP-matter interaction from elastic scattering.

This paper is organized as follows. In Sec. II we describe the EFT construction of the general WIMP-nucleon Galilean-invariant interaction, including the parametric enhancement of velocity-dependent operators. Relativistic matching to EFT operators is illustrated, using the most general four-fermion interaction. In Sec. III we describe the embedding of this interaction in nuclei. The EFT scattering probability is governed by six nuclear response functions, assuming the nuclear ground state has good parity and CP. We point out the differences between our results, the corresponding EFT cross section in which the finite size of the nucleus is ignored, and the simple SI/SD limit, where only two of the EFT operators are retained. In Sec. IV we present differential and total cross sections and discuss for each of the EFT operators the consequences of taking the allowed limit (thereby reducing the nuclear operators to the SI and SD ones). The concluding Sec. V discusses the implications of our work for experimental searches. We discuss problems that could arise if future search strategies are predicated on treatments of the cross section that exclude plausible operators. Given our ignorance of the WIMP-nucleon interaction, we emphasize the need for a variety experiments using nuclei with the requisite sensitivities. In Appendix A we provide more details on the treatment of velocity-dependent interactions and on the multipole analysis that leads to the general cross section. In Appendix B we describe a *Mathematica* script that we developed to help experimentalists implement the formalism presented here. In addition to its use as an experimental analysis tool, particle and nuclear theorists can use the script to explore the consequences of a specific ultraviolet theory or the implications of new nuclear structure calculations.

II. EFFECTIVE FIELD THEORY CONSTRUCTION OF THE INTERACTION

The idea behind EFT in dark-matter scattering is to follow the usual EFT "recipe," but in a nonrelativistic context, by writing down the relevant operators that obey all of the nonrelativistic symmetries. In the case of elastic scattering of a heavy WIMP off a nucleon, the Lagrangian density will have the contact form

$$\mathcal{L}_{\mathrm{int}}(\vec{x}) = c\Psi_\chi^*(\vec{x})\mathcal{O}_\chi\Psi_\chi(\vec{x})\Psi_N^*(\vec{x})\mathcal{O}_N\Psi_N(\vec{x}), \tag{1}$$

where the $\Psi(\vec{x})$ are nonrelativistic fields and where the WIMP and nucleon operators $\mathcal{O}_\chi$ and $\mathcal{O}_N$ may have vector indices. The properties of $\mathcal{O}_\chi$ and $\mathcal{O}_N$ are then constrained by imposing relevant symmetries. We envision the case where there are a number of candidate interactions $\mathcal{O}_i$ formed from the $\mathcal{O}_\chi$ and $\mathcal{O}_N$. Working to second order in the momenta, one can construct the relevant operators appropriate for use with Pauli spinors, when constructing the Galilean-invariant amplitude,

$$\sum_{i=1}^{\mathcal{N}} \left(c_i^{(n)}\mathcal{O}_i^{(n)} + c_i^{(p)}\mathcal{O}_i^{(p)}\right), \tag{2}$$

where the coupling coefficients c_i may be different for proton and neutrons. The number $\mathcal{N}$ of such operators depends on the generality of the particle-physics description. We find that ten operators arise if we limit our consideration to exchanges involving up to spin-1 exchanges and to operators that are the leading-order nonrelativistic analogs of relativistic operators. Four additional operators arise if more general mediators are allowed.

This interaction can then be embedded in the nucleus. The procedure we follow here—though we discuss generalizations in Appendix B–assumes that the nuclear interaction is the sum of the WIMP interactions with the individual nucleons in the nucleus. The nuclear operators then involve a convolution of the $\mathcal{O}_i$, whose momenta must now be treated as local operators appropriate for bound nucleons, with the plane wave associated with the WIMP scattering, which is an angular and radial operator that can be decomposed with standard spherical harmonic methods. Because momentum transfers are typically comparable to the inverse nuclear size, it is crucial to carry through such a multipole decomposition to identify the nuclear responses associated with the various c_is. The scattering probability is given by the square of the (Galilean) invariant amplitude $\mathcal{M}$, a product of WIMP and nuclear matrix elements, averaged over initial WIMP and nuclear magnetic quantum numbers M_χ and M_N, and summed over final magnetic quantum numbers.

The result can be organized in a way that factorizes the particle and nuclear physics

$$\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}|^2 \equiv \sum_k \sum_{\tau=0,1}\sum_{\tau'=0,1} R_k\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2},\{c_i^\tau c_j^{\tau'}\}\right) W_k^{\tau\tau'}(\vec{q}^{\,2}b^2), \tag{3}$$

where the sum extends over products of WIMP response functions R_k and nuclear response functions W_k. The R_k isolate the particle physics: They depend on specific combinations of bilinears in the low-energy constants of the EFT—the $2\mathcal{N}$ coefficients of Eq. (2)—here labeled by isospin τ (isoscalar, isovector) rather than the n,p of Eq. (2) (see below). The WIMP response functions also depend on the relative WIMP-target velocity $\vec{v}_T^\perp$ and three-momentum transfer $\vec{q}=\vec{p}\,'-\vec{p}=\vec{k}-\vec{k}'$, where $\vec{p}$ ($\vec{p}\,'$) is the incoming (outgoing) WIMP three-momentum and $\vec{k}$ ($\vec{k}'$) the incoming (outgoing) nucleon three-momentum. The nuclear response functions W_k can be varied by experimentalists if they explore a variety of nuclear targets. The W_k are functions of $y\equiv(qb/2)^2$, where b is the nuclear size (explicitly the harmonic oscillator parameter if the nuclear wave functions are expanded in that single-particle basis).

EFT provides an attractive framework for analyzing and comparing direct-detection experiments. It simplifies the analysis of WIMP-matter interactions by exploiting an important small parameter: Typical velocities of the particles comprising the dark-matter halo are $v/c\sim 10^{-3}$ and thus nonrelativistic. Consequently, while there may be a semi-infinite number of candidate ultraviolet theories of WIMP-matter interactions, many of these theories are operationally indistinguishable at low energies. By organizing the EFT in terms of nonrelativistic interactions and degrees of freedom, one can significantly simplify the classification of possible operators [8,9], while not sacrificing generality. In constructing the needed set of independent operators, the equations of motion are employed to remove redundant operators. The operators themselves are expressed in terms of quantities that are more directly related to scattering observables at the relevant energy scale, which makes the relationship between operators and the underlying physics more transparent. Furthermore, it becomes trivial to write operators for arbitrary dark-matter spin, a task that can be rather involved in the relativistic case.

EFT also prevents oversimplification: Because it produces a complete set of effective interactions at low energy, one is guaranteed that the description is general. Provided this interaction is then embedded in the nucleus faithfully, it will then produce the most general nuclear response consistent with the assumed symmetries. Consequently, some very basic questions that do not appear to be answered in the literature can be immediately addressed. How many constraints on dark-matter particle interactions can be obtained from elastic scattering? Conversely, what redundancies exist among the EFT's low-energy constants that cannot be resolved, regardless of the number of elastic-scattering experiments that are done?

A. Constructing the nonrelativistic operators

Because dark-matter–ordinary-matter interactions are more commonly described in relativistic notation, we begin by considering the nonrelativistic reduction of two familiar relativistic interactions. The SI contact interaction between a spin-$\frac{1}{2}$ WIMP and nucleon,

$$\mathcal{L}_{\text{int}}^{\text{SI}}(\vec{x}) = c_1\bar{\Psi}_\chi(\vec{x})\Psi_\chi(\vec{x})\bar{\Psi}_N(\vec{x})\Psi_N(\vec{x}), \tag{4}$$

can be reduced by replacing the spinors within the fields by their low-momentum forms,

$$U(p)=\sqrt{\frac{E+m}{2m}}\begin{pmatrix}\xi\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m_\chi}\xi\end{pmatrix}\sim\begin{pmatrix}\xi\\ \frac{\vec{\sigma}\cdot\vec{p}}{2m}\xi\end{pmatrix}, \tag{5}$$

where we have used Bjorken and Drell γ matrix conventions and spinor normalization (1 instead of the $2m$ used in Ref. [8]). (Consequently, the c's defined here, which carry dimensions of 1/mass^2, differ from those of Ref. [8].) To leading order in p/m_χ and p/m_N, we obtain the nonrelativistic operator

$$c_1 1_\chi 1_N \equiv c_1\mathcal{O}_1. \tag{6}$$

The nonrelativistic analog of the invariant amplitude is obtained by taking the matrix element of this operator between Pauli spinors ξ_χ and ξ_N. In the nonrelativistic reduction of the SD interaction,

$$\mathcal{L}_{\text{int}}^{\text{SD}} = c_4\bar{\chi}\gamma^\mu\gamma^5\chi\bar{N}\gamma_\mu\gamma^5 N, \tag{7}$$

the leading term comes from the spatial components, with $\bar{\chi}\gamma^i\gamma^5\chi\sim\xi_\chi^\dagger\sigma^i\xi_\chi$. As $\sigma^i=2S^i$, we obtain the nonrelativistic operator

$$-4c_4\vec{S}_\chi\cdot\vec{S}_N \equiv -4c_4\mathcal{O}_4. \tag{8}$$

Equations (6) and (8) correspond to the SI and SD operators frequently used in experimental analyses.

One could continue in this manner, constructing all possible relativistic interactions and considering their nonrelativistic reductions. However, this is unnecessary, as the nonrelativistic EFT can be constructed directly from the available operators and momenta, as a systematic expansion. These include 1_χ and 1_N, the three-vectors $\vec{S}_\chi$ and $\vec{S}_N$, and the momenta of the WIMP and nucleon. Of the four momenta involved in the scattering (two incoming and two outgoing), only two combinations are physically relevant owing to inertial frame-independence and momentum conservation. It is convenient to work with the frame-invariant quantities, the momentum transfer $\vec{q}$ and the WIMP-nucleon relative velocity,

$$\vec{v} \equiv \vec{v}_{\chi,\mathrm{in}} - \vec{v}_{N,\mathrm{in}}. \tag{9}$$

It is also useful to construct the related quantity

$$\begin{aligned}\vec{v}^{\perp} &= \vec{v} + \frac{\vec{q}}{2\mu_N} \\ &= \frac{1}{2}(\vec{v}_{\chi,\mathrm{in}} + \vec{v}_{\chi,\mathrm{out}} - \vec{v}_{N,\mathrm{in}} - \vec{v}_{N,\mathrm{out}}) \\ &= \frac{1}{2}\left(\frac{\vec{p}}{m_\chi} + \frac{\vec{p}\,'}{m_\chi} - \frac{\vec{k}}{m_N} - \frac{\vec{k}\,'}{m_N}\right),\end{aligned} \tag{10}$$

which satisfies $\vec{v}^{\perp} \cdot \vec{q} = 0$ as a consequence of energy conservation. Here μ_N is the WIMP-nucleon reduced mass. It was shown in Ref. [8] that operators are guaranteed to be Hermitian if they are built out of the following four three-vectors:

$$i\frac{\vec{q}}{m_N}, \quad \vec{v}^{\perp}, \quad \vec{S}_\chi, \quad \vec{S}_N. \tag{11}$$

Here (in another departure from Ref. [8]) we have introduced m_N as a convenient scale to render $\vec{q}/m_N$ and the constructed $\mathcal{O}_i$ dimensionless: The choice of this scale is not arbitrary, as it leads to an EFT power counting in nuclei that is particularly simple, as we discuss in Secs. II B and IV B. The relevant interactions that we can construct from these three-vectors and that can be associated with interactions involving only spin-0 or spin-1 mediators are

$$\begin{aligned}\mathcal{O}_1 &= 1_\chi 1_N, \\ \mathcal{O}_2 &= (v^{\perp})^2, \\ \mathcal{O}_3 &= i\vec{S}_N \cdot \left(\frac{\vec{q}}{m_N} \times \vec{v}^{\perp}\right), \\ \mathcal{O}_4 &= \vec{S}_\chi \cdot \vec{S}_N, \\ \mathcal{O}_5 &= i\vec{S}_\chi \cdot \left(\frac{\vec{q}}{m_N} \times \vec{v}^{\perp}\right), \\ \mathcal{O}_6 &= \left(\vec{S}_\chi \cdot \frac{\vec{q}}{m_N}\right)\left(\vec{S}_N \cdot \frac{\vec{q}}{m_N}\right), \\ \mathcal{O}_7 &= \vec{S}_N \cdot \vec{v}^{\perp}, \\ \mathcal{O}_8 &= \vec{S}_\chi \cdot \vec{v}^{\perp}, \\ \mathcal{O}_9 &= i\vec{S}_\chi \cdot \left(\vec{S}_N \times \frac{\vec{q}}{m_N}\right), \\ \mathcal{O}_{10} &= i\vec{S}_N \cdot \frac{\vec{q}}{m_N}, \\ \mathcal{O}_{11} &= i\vec{S}_\chi \cdot \frac{\vec{q}}{m_N}.\end{aligned} \tag{12}$$

These 11 operators were discussed in Ref. [8]. We retain 10 of these here, discarding $\mathcal{O}_2$, as this operator cannot be obtained from the leading-order nonrelativistic reduction of a manifestly relativistic operator (see, e.g., Table I of Sec. II C).

We classify these operators as leading order (LO), next-to-leading order (NLO), and next-to-next-to-leading order (N^{2}LO), depending on the total number of momenta and velocities they contain. We see in Sec. IV B that these designations correspond to total cross sections that scale as v_T^0, v_T^2, or v_T^4, where v_T is the WIMP velocity in the laboratory frame.

In addition, one can construct the following operators that do not arise for traditional spin-0 or spin-1 mediators

$$\begin{aligned}\mathcal{O}_{12} &= \vec{S}_\chi \cdot (\vec{S}_N \times \vec{v}^{\perp}), \\ \mathcal{O}_{13} &= i(\vec{S}_\chi \cdot \vec{v}^{\perp})\left(\vec{S}_N \cdot \frac{\vec{q}}{m_N}\right), \\ \mathcal{O}_{14} &= i\left(\vec{S}_\chi \cdot \frac{\vec{q}}{m_N}\right)(\vec{S}_N \cdot \vec{v}^{\perp}), \\ \mathcal{O}_{15} &= -\left(\vec{S}_\chi \cdot \frac{\vec{q}}{m_N}\right)\left[(\vec{S}_N \times \vec{v}^{\perp}) \cdot \frac{\vec{q}}{m_N}\right], \\ \mathcal{O}_{16} &= -\left[(\vec{S}_\chi \times \vec{v}^{\perp}) \cdot \frac{\vec{q}}{m_N}\right]\left(\vec{S}_N \cdot \frac{\vec{q}}{m_N}\right).\end{aligned} \tag{13}$$

It is easy to see that $\mathcal{O}_{16}$ is linearly dependent on $\mathcal{O}_{12}$ and $\mathcal{O}_{15}$,

$$\mathcal{O}_{16} = \mathcal{O}_{15} + \frac{\vec{q}^{\,2}}{m_N^2}\mathcal{O}_{12}, \tag{14}$$

and so should be eliminated. Operator $\mathcal{O}_{15}$ is cubic in velocities and momenta, generating a total cross section of order v^6 (N^{3}LO). It is retained because it arises as the leading-order nonrelativistic limit of certain covariant interactions (see Sec. II C).

Each operator can have distinct couplings to protons and neutrons. Thus, the EFT interaction we employ in this paper takes the form

$$\sum_{\alpha=n,p}\sum_{i=1}^{15} c_i^\alpha \mathcal{O}_i^\alpha, \quad c_2^\alpha \equiv 0. \tag{15}$$

One can factorize the space-spin and proton/neutron components of Eq. (15) by introducing isospin, which is also useful as an approximate symmetry of the nuclear wave functions. Thus, an equivalent form for our interaction is

$$\sum_{i=1}^{15}(c_i^0 1 + c_i^1 \tau_3)\mathcal{O}_i = \sum_{\tau=0,1}\sum_{i=1}^{15} c_i^\tau \mathcal{O}_i t^\tau, \quad c_2^\tau \equiv 0, \tag{16}$$

where $c_i^0 = \frac{1}{2}(c_i^p + c_i^n)$ and $c_i^1 = \frac{1}{2}(c_i^p - c_i^n)$. The isospin states are

$$|p\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |n\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \tag{17}$$

while the isospin operators are

$$t^0 \equiv 1 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad t^1 \equiv \tau_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}. \tag{18}$$

The EFT has a total of 28 parameters, associated with 14 space-spin operators each of which can have distinct couplings to protons and neutrons. If we exclude operators that are not associated with spin-0 or spin-1 mediators, 10 space-spin operators and 20 couplings remain.

TABLE I. Relativistic amplitudes, their nonrelativistic analogs appropriate for evaluation between Paul spinors, the corresponding results as linear combinations of the $\mathcal{O}_i$, and the transformation properties of the interactions [even (E) or odd (O)] under parity and time reversal. Bjorken and Drell spinor and γ matrix conventions are used. The scale $m_{\rm M}$, which appears as an arbitrary normalization below to ensure that kinematic factors are dimensionless, would usually be known from the context of the theory.

j	$\mathcal{L}^j_{\rm int}$	Nonrelativistic reduction	$\sum_i c_i \mathcal{O}_i$	P/T
1	$\bar\chi\chi\bar N N$	$1_\chi 1_N$	$\mathcal{O}_1$	E/E
2	$i\bar\chi\chi\bar N\gamma^5 N$	$i\frac{\vec q}{m_N}\cdot\vec S_N$	$\mathcal{O}_{10}$	O/O
3	$i\bar\chi\gamma^5\chi\bar N N$	$-i\frac{\vec q}{m_\chi}\cdot\vec S_\chi$	$-\frac{m_N}{m_\chi}\mathcal{O}_{11}$	O/O
4	$\bar\chi\gamma^5\chi\bar N\gamma^5 N$	$-\frac{\vec q}{m_\chi}\cdot\vec S_\chi\frac{\vec q}{m_N}\cdot\vec S_N$	$-\frac{m_N}{m_\chi}\mathcal{O}_6$	E/E
5	$\bar\chi\gamma^\mu\chi\bar N\gamma_\mu N$	$1_\chi 1_N$	$\mathcal{O}_1$	E/E
6	$\bar\chi\gamma^\mu\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_{\rm M}}N$	$\frac{\vec q^{\,2}}{2m_N m_{\rm M}}1_\chi 1_N+2\big(\frac{\vec q}{m_\chi}\times\vec S_\chi+i\vec v^\perp\big)\cdot\big(\frac{\vec q}{m_{\rm M}}\times\vec S_N\big)$	$\frac{\vec q^{\,2}}{2m_N m_{\rm M}}\mathcal{O}_1-2\frac{m_N}{m_{\rm M}}\mathcal{O}_3$ $+2\frac{m_N^2}{m_{\rm M}m_\chi}\big(\frac{q^2}{m_N^2}\mathcal{O}_4-\mathcal{O}_6\big)$	E/E
7	$\bar\chi\gamma^\mu\chi\bar N\gamma_\mu\gamma^5 N$	$-2\vec S_N\cdot\vec v^\perp+\frac{2}{m_\chi}i\vec S_\chi\cdot(\vec S_N\times\vec q)$	$-2\mathcal{O}_7+2\frac{m_N}{m_\chi}\mathcal{O}_9$	O/E
8	$i\bar\chi\gamma^\mu\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_M}\gamma^5 N$	$2i\frac{\vec q}{m_{\rm M}}\cdot\vec S_N$	$2\frac{m_N}{m_{\rm M}}\mathcal{O}_{10}$	O/O
9	$\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\chi\bar N\gamma_\mu N$	$-\frac{\vec q^{\,2}}{2m_\chi m_{\rm M}}1_\chi 1_N-2\big(\frac{\vec q}{m_N}\times\vec S_N+i\vec v^\perp\big)\cdot\big(\frac{\vec q}{m_{\rm M}}\times\vec S_\chi\big)$	$-\frac{\vec q^{\,2}}{2m_\chi m_{\rm M}}\mathcal{O}_1+\frac{2m_N}{m_{\rm M}}\mathcal{O}_5$ $-2\frac{m_N}{m_{\rm M}}\big(\frac{\vec q^{\,2}}{m_N^2}\mathcal{O}_4-\mathcal{O}_6\big)$	E/E
10	$\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_{\rm M}}N$	$4\big(\frac{\vec q}{m_{\rm M}}\times\vec S_\chi\big)\cdot\big(\frac{\vec q}{m_{\rm M}}\times\vec S_N\big)$	$4\big(\frac{\vec q^{\,2}}{m_{\rm M}^2}\mathcal{O}_4-\frac{m_N^2}{m_{\rm M}^2}\mathcal{O}_6\big)$	E/E
11	$\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\chi\bar N\gamma^\mu\gamma^5 N$	$4i\big(\frac{\vec q}{m_{\rm M}}\times\vec S_\chi\big)\cdot\vec S_N$	$4\frac{m_N}{m_{\rm M}}\mathcal{O}_9$	O/E
12	$i\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_M}\gamma^5 N$	$-\big[i\frac{\vec q^{\,2}}{m_\chi m_{\rm M}}-4\vec v^\perp\cdot\big(\frac{\vec q}{m_{\rm M}}\times\vec S_\chi\big)\big]\frac{\vec q}{m_{\rm M}}\cdot\vec S_N$	$-\frac{m_N}{m_\chi}\frac{\vec q^{\,2}}{m_{\rm M}^2}\mathcal{O}_{10}-4\frac{\vec q^{\,2}}{m_{\rm M}^2}\mathcal{O}_{12}-4\frac{m_N^2}{m_{\rm M}^2}\mathcal{O}_{15}$	O/O
13	$\bar\chi\gamma^\mu\gamma^5\chi\bar N\gamma_\mu N$	$2\vec v^\perp\cdot\vec S_\chi+2i\vec S_\chi\cdot\big(\vec S_N\times\frac{\vec q}{m_N}\big)$	$2\mathcal{O}_8+2\mathcal{O}_9$	O/E
14	$\bar\chi\gamma^\mu\gamma^5\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_{\rm M}}N$	$4i\vec S_\chi\cdot\big(\frac{\vec q}{m_{\rm M}}\times\vec S_N\big)$	$-4\frac{m_N}{m_{\rm M}}\mathcal{O}_9$	O/E
15	$\bar\chi\gamma^\mu\gamma^5\chi\bar N\gamma^\mu\gamma^5 N$	$-4\vec S_\chi\cdot\vec S_N$	$-4\mathcal{O}_4$	E/E
16	$i\bar\chi\gamma^\mu\gamma^5\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_{\rm M}}\gamma^5 N$	$4i\vec v^\perp\cdot\vec S_\chi\frac{\vec q}{m_{\rm M}}\cdot\vec S_N$	$4\frac{m_N}{m_{\rm M}}\mathcal{O}_{13}$	E/O
17	$i\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\gamma^5\chi\bar N\gamma_\mu N$	$2i\frac{\vec q}{m_{\rm M}}\cdot\vec S_\chi$	$2\frac{m_N}{m_{\rm M}}\mathcal{O}_{11}$	O/O
18	$i\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\gamma^5\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_{\rm M}}N$	$\frac{\vec q}{m_{\rm M}}\cdot\vec S_\chi\big[i\frac{\vec q^{\,2}}{m_N m_{\rm M}}-4\vec v^\perp\cdot\big(\frac{\vec q}{m_{\rm M}}\times\vec S_N\big)\big]$	$\frac{\vec q^{\,2}}{m_{\rm M}^2}\mathcal{O}_{11}+4\frac{m_N^2}{m_{\rm M}^2}\mathcal{O}_{15}$	O/O
19	$i\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\gamma^5\chi\bar N\gamma_\mu\gamma^5 N$	$-4i\frac{\vec q}{m_{\rm M}}\cdot\vec S_\chi\vec v_\perp\cdot\vec S_N$	$-4\frac{m_N}{m_{\rm M}}\mathcal{O}_{14}$	E/O
20	$i\bar\chi i\sigma^{\mu\nu}\frac{q_\nu}{m_{\rm M}}\gamma^5\chi\bar N i\sigma_{\mu\alpha}\frac{q^\alpha}{m_{\rm M}}\gamma^5 N$	$4\frac{\vec q}{m_{\rm M}}\cdot\vec S_\chi\frac{\vec q}{m_{\rm M}}\cdot\vec S_N$	$4\frac{m_N^2}{m_{\rm M}^2}\mathcal{O}_6$	E/E

As WIMP searches are motivated in part by the "WIMP miracle"—WIMPs will naturally freeze-out in the early universe, when their annihilation rate falls behind the expansion rate, to produce a relic density today consistent with the dark-matter density—it is convenient to express the coefficients c_i in weak-scale units. $\mathcal{O}_4$ is related by an isospin rotation to the charge-changing weak axial or Gamow-Teller operator of the standard model,

$$c_4\mathcal{O}_4 t^1 \equiv c_4\mathcal{O}_4\tau_3 \rightarrow \frac{G_F}{\sqrt{2}}\mathcal{O}_4\tau_\pm, \tag{19}$$

where $G_F \sim 1.166\times 10^{-5}$ GeV^{-2} is the Fermi constant and $\tau_\pm$ is the isospin raising or lowering operator. G_F defines a standard-model weak interaction mass scale,

$$m_v \equiv \langle v\rangle = (2G_F)^{-1/2} = 246.2 \text{ GeV}, \tag{20}$$

where $\langle v\rangle$ is the Higgs vacuum expectation value. Consequently, it is natural to characterize experimental constraints on a given c_i in terms of this normalization, that is, in terms of the dimensionless quantity $\tilde c_i$, where $c_i = \tilde c_i/m_v^2$. This normalization is employed in the *Mathematica* script discussed in Appendix B.

B. EFT power counting and $\vec q/m_N$: Parametric enhancement

The EFT formulation leads to an attractive power counting that is helpful in understanding the dependence of laboratory total cross sections on the physically relevant parameters: the WIMP velocity $\vec v_T^\perp$, the ratio of the WIMP-nuclear target reduced mass μ_T to m_N, and the ratio of μ_T to the inverse nuclear size. The scaling behavior we discuss in Sec. IV B takes on a simple form if m_N is used to construct the dimensionless quantity $\vec q/m_N$, a parameter related to the relative velocities of nucleons bound in the nucleus, as explained below. The fact that internucleon velocities are much greater than the WIMP velocity leads to a parametric enhancement of the certain "composite operator" contributions to cross sections.

The introduction of the scale m_N would be arbitrary if we limit ourselves to WIMP-nucleon scattering. Any other choice would simply lead to the same scaling of the total cross section on μ_T/m_N, but with the m_N in the denominator replaced with that new scale. There is a single relative velocity $\vec{v}_T^\perp$ in the WIMP-nucleon system, associated with the Jacobi coordinate, the distance between the WIMP and the nucleon.

However, in a system consisting of a WIMP and a nucleus containing A nucleons, there are A independent Jacobi coordinates, and A associated independent velocities. Any WIMP-nucleon velocity-dependent interaction summed over the nucleons in a nucleus must, of course, involve all of these velocities. One of these can be chosen to be the WIMP-target relative velocity, measured with respect to the center of mass of the nucleus, or $\vec{v}_T^\perp$, the analog of the single WIMP-nucleon velocity. However, in addition to this velocity, there are $A-1$ others associated with the $A-1$ independent Jacobi internucleon coordinates. These velocities are Galilean-invariant intrinsic nuclear operators.

An internal velocity carries negative parity, and thus its nuclear matrix element vanishes owing to the nearly exact parity of the nuclear ground state. However, because the nucleus is composite, the nuclear operators built from $\mathcal{O}_i$ are accompanied by an additional spatial operator $e^{-i\vec{q}\cdot\vec{x}(i)}$. A threshold operator carrying the requisite positive parity can thus be formed by combining $i\vec{q}\cdot\vec{x}(i)$ with $\vec{v}(i)=\vec{p}(i)/m_N$. However, $\vec{p}(i)$ and $\vec{x}(i)$ are conjugate operators: The larger the nuclear size, the smaller is the nucleon momentum scale. Thus, when $\vec{p}(i)$ and $\vec{x}(i)$ are combined to form interactions, one obtains operators such as $\vec{l}(i)$, the orbital angular momentum, that have no associated scale: The single-particle eigenvalues of $l_z(i)$ are integers. (Operators built from such internal nuclear coordinates are called composite operators.) Thus, scattering associated with internal velocities is governed by the parameters multiplying $\vec{p}(i)$ and $\vec{x}(i)$, which form the dimensionless ratio $\vec{q}/m_N$. This dimensionless parameter emerges directly from the physics; it is not put in by hand.

Thus, we see that $\vec{q}/m_N$ is associated with the typical velocity of bound nucleons, $\sim 1/10$. The composite operators constructed from nucleon velocities are enhanced relative to those associated with $\vec{v}_T^\perp$ by the ratio of $\vec{q}/m_N$ to $\vec{v}_T^\perp$, or ~ 100. The standard point-nucleus treatment of WIMP scattering retains only the effects of $\vec{v}_T^\perp$. We find in Sec. IV B that the enhancement associated with $\vec{q}/m_N$ leads to an increased sensitivity to derivative couplings of $\sim 10(\mu_T/m_N)^2$ in the total cross section, relative to point-nucleus treatments.

C. Relativistic matching

The operators $\mathcal{O}_i$ can be viewed as the low-energy equivalents of the relativistic operators governing ultraviolet WIMP-matter interactions. By matching to a specific relativistic theory, one can relate the two sets of operators: This procedure would allow a theorist to convert experimental constraints on the c_i into corresponding constraints on the coefficients d_i of a set of interactions appearing in a given ultraviolet model. In Sec. II A we discussed two simple examples, the SI and SD interactions $\mathcal{L}_{\rm int}^{\rm SI}$ and $\mathcal{L}_{\rm int}^{\rm SD}$. Here we repeat the process for the set of relativistic amplitudes listed in Table I. Unlike the two simple cases discussed in Sec. II, the relativistic amplitudes do not always map onto single operators $\mathcal{O}_i$. Instead, the result is frequently of the form $\sum_i \alpha_i \mathcal{O}_i$, where several of the coefficients α_i are nonzero.

The interactions of Table I describe the interactions of spin-$\frac{1}{2}$ WIMPs with nucleons. (More general interactions could be considered, of course.) Four-momentum definitions follow our three-momentum conventions: the incoming (outgoing) four-momentum of the dark-matter particle χ is p^μ (p'^μ); the incoming (outgoing) four-momentum of the nucleon N is k^μ (k'^μ); and the momentum transfer $q^\mu = p'^\mu - p^\mu = k^\mu - k'^\mu$. We also define $P^\mu = p^\mu + p'^\mu$ and $K^\mu = k^\mu + k'^\mu$. The relative velocity operator of Eq. (10) can be written in term of these variables as

$$\vec{v}^\perp \equiv \frac{1}{2}(\vec{v}_{\chi,\rm in} + \vec{v}_{\chi,\rm out} - \vec{v}_{N,\rm in} - \vec{v}_{N,\rm out}) = \frac{1}{2}\left(\frac{\vec{P}}{m_\chi} - \frac{\vec{K}}{m_N}\right). \tag{21}$$

The relativistic WIMP-nucleon interactions are constructed as bilinear WIMP-nucleon products of the available scalar ($\bar{\chi}\chi$, $\bar{\chi}\gamma^5\chi$) and four-vector ($\bar{\chi}P^\mu\chi$, $\bar{\chi}P^\mu\gamma^5\chi$, $\bar{\chi}i\sigma^{\mu\nu}q_\nu\chi$, and $\bar{\chi}\gamma^\mu\gamma^5\chi$) amplitudes. Thus, there are $2^2+4^2=20$ combinations [8]. The nonrelativistic operators obtained after nonrelativistic reduction are listed in Table I, along with the corresponding expansions in terms of our EFT operators, the $\mathcal{O}_i$. The table also gives transformation properties of the interactions under parity and time reversal. Note that all interactions reduce in leading order to combinations of our 15 $\mathcal{O}_i$, and all of the $\mathcal{O}_i$ appear in the table. Thus, they are the minimal set of nonrelativistic interactions needed to represent the listed set of 20 $\mathcal{L}_{\rm int}^j$.

III. THE NUCLEAR RESPONSE IN EFT

Cross sections or rates for WIMP-nucleon/nucleus scattering can be expressed as simple kinematic integrals over a fundamental particle-nuclear function, the square of the invariant amplitude averaged over initial WIMP and nuclear spins and summed over final spins. The key result of this section is the calculation of this quantity for the EFT interaction.

Because much of the literature employs analyses based on the SI/SD formulation, we begin by considering two limits in which such a result is obtained. One way to obtain a SI/SD result while still using a very general interaction, such as the EFT form developed here, is to treat the nucleus as a point particle. Effectively, one replaces $e^{-i\vec{q}\cdot\vec{x}(i)}\mathcal{O}_i$ with $\mathcal{O}_i$, despite the fact that $\vec{q}\cdot\vec{x}(i)$ is typically ~ 1. Alternatively, one can simply restrict the operators initially to $\mathcal{O}_1$ and $\mathcal{O}_4$, the two LO operators in our EFT list. Then one can proceed to do a full nuclear calculation, including form factors. However, it is not known whether the WIMP interaction has the simple $\mathcal{O}_1/\mathcal{O}_4$ form. We present these two limits so that a comparison with the general cross section result of Sec. III D can be made.

A. The EFT nucleon calculation

One could, in principle, detect WIMPs through their elastic scattering off free protons and (hypothetically) neutrons. Such a target can be treated as a point because the inverse nucleon size is large compared to typical momentum transfers in WIMP scattering. In this case the EFT Galilean-invariant amplitude corresponding to Eq. (16) for a proton target becomes

$$\mathcal{M} = \langle \vec{p}\,' S_\chi m_\chi ; \vec{k}' S_N = \tfrac{1}{2} m_N T_N = \tfrac{1}{2} m_T = \tfrac{1}{2} \big| \mathcal{H} \big| \vec{p} S_\chi m_\chi ; \vec{k} S_N = \tfrac{1}{2} m_N T_N = \tfrac{1}{2} m_T = \tfrac{1}{2} \rangle, \tag{22}$$

where we have introduced the proton's isospin quantum numbers for consistency with the isospin form of our Hamiltonian, Eq. (16). An elementary calculation then yields the square of the invariant amplitude, averaged over initial spins and summed over final spins, for WIMP scattering off a proton,

$$\begin{aligned}\frac{1}{2j_\chi+1}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2_{\text{proton}} &= \left[c_1^{p\,2} + \frac{j_\chi(j_\chi+1)}{3}\left(\frac{\vec{q}^{\,2}}{m_N^2}\vec{v}_T^{\perp 2}c_5^{p\,2} + \vec{v}_T^{\perp 2}c_8^{p\,2} + \frac{\vec{q}^{\,2}}{m_N^2}c_{11}^{p\,2}\right)\right]|M_{F;p}|^2 \\ &+ \frac{1}{12}\Bigg\{\left(\frac{\vec{q}^{\,2}}{m_N^2}\vec{v}_T^{\perp 2}c_3^{p\,2} + \vec{v}_T^{\perp 2}c_7^{p\,2} + \frac{\vec{q}^{\,2}}{m_N^2}c_{10}^{p\,2}\right) + \frac{j_\chi(j_\chi+1)}{3}\Bigg[3c_4^{p\,2} + 2\frac{\vec{q}^{\,2}}{m_N^2}\left(c_4^p c_6^p + c_9^{p\,2}\right) + \frac{\vec{q}^{\,4}}{m_N^4}c_6^{p\,2} + \\ &+ 2\vec{v}_T^{\perp 2}c_{12}^{p\,2} + \frac{\vec{q}^{\,2}}{m_N^2}\vec{v}_T^{\perp 2}\left(c_{13}^{p\,2} + c_{14}^{p\,2} - 2c_{12}^p c_{15}^p\right) + \frac{\vec{q}^{\,4}}{m_N^4}\vec{v}_T^{\perp 2}c_{15}^{p\,2}\Bigg]\Bigg\}|M_{GT;p}|^2. \end{aligned}\tag{23}$$

The SI (or Fermi) and SD (or Gamow-Teller) operators evaluated between nonrelativistic Pauli spinors are

$$\begin{aligned}|M_{F;p}|^2 &\equiv \tfrac{1}{2}|\langle 1/2||1||1/2\rangle|^2 = 1,\\ |M_{GT;p}|^2 &\equiv \tfrac{1}{2}|\langle 1/2||\sigma||1/2\rangle|^2 = 3,\end{aligned}\tag{24}$$

where $||$ denotes a matrix element reduced in spin and the subscript p is an explicit reminder that this is a proton matrix element.

B. The EFT point-nucleus limit

The corresponding result for a point nucleus of spin j_N,

$$\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}|^2_{\text{ptnucleus}}, \tag{25}$$

is obtained by making two substitutions in Eq. (23). First, the proton Fermi and Gamow-Teller matrix elements are replaced with their nuclear analogs,

$$\begin{aligned}|M_{F;p}|^2 &\to \left|M^N_{F;p}(0)\right|^2 \equiv Z^2 \\ &= \left[\frac{1}{2j_N+1}|\langle j_N||\sum_{i=1}^{A}\frac{1+\tau_3(i)}{2}||j_N\rangle|^2\right],\\ |M_{GT;p}|^2 &\to \left|M^N_{GT;p}(0)\right|^2 \\ &\equiv \left[\frac{1}{2j_N+1}|\langle j_N||\sum_{i=1}^{A}\frac{1+\tau_3(i)}{2}\sigma(i)||j_N\rangle|^2\right],\end{aligned}\tag{26}$$

where we have assumed that the WIMP coupling is only to protons—enforced by the introduction of the isospin operators—to produce a result analogous to Eq. (23). Second, the velocity $\vec{v}_T^\perp$ that in the nucleon case represented the WIMP-nucleon relative velocity now becomes the analogous parameter measured with respect to the nuclear center of mass. There are no intrinsic nuclear velocities because the nucleus is a point.

On integrating over phase space, one obtains a cross section that depends on the two particle-physics quantities within the square brackets of Eq. (23), with the associated kinematic factors evaluated by averaging over the WIMP velocity distribution. Thus, this limit yields a SI/SD cross section—the nuclear operators are just the charge and the spin—though the WIMP response functions multiplying the squares of the two operators are considerably more complicated than in the standard SI/SD analysis, containing the coefficients of all of the EFT operators. In the point-nucleus limit, one can thus place two constraints on the EFT coefficients by doing an SI experiment ($J=0$ nuclear target) to isolate the Fermi response and an SD experiment ($J>0$) to probe the Gamow-Teller response. If one extends the analysis to include isospin, two additional experiments would be needed: on a $J=0$ target with a distinct N/Z ratio and on an $J>0$ odd-neutron target.

C. The SI/SD nuclear cross section

The SI/SD result most often seen in the literature properly accounts for the momentum transfer in the scattering, but simplifies the WIMP-nucleon operator by assuming it is formed from a linear combination of $\mathcal{O}_1$ and $\mathcal{O}_4$. Other possible operators are neglected.

The WIMP-nucleus interaction is then written as the sum over these WIMP interactions with the bound nucleons, taking into account the finite spatial extent of the nuclear charge and spin-current densities,

$$\begin{aligned}1_\chi\rho_N(\vec{x}) &= 1_\chi\sum_{i=1}^{A}\left[c_1^0 + c_1^1\tau_3(i)\right]e^{-i\vec{q}\cdot\vec{x}_i}\\ &\to c_1^p 1_\chi\sum_{i=1}^{A}\frac{1+\tau_3(i)}{2}e^{-i\vec{q}\cdot\vec{x}_i},\\ \vec{S}_\chi\cdot\vec{j}_N(\vec{x}) &= \vec{S}_\chi\cdot\sum_{i=1}^{A}\left[c_4^0 + c_4^1\tau_3(i)\right]\frac{\vec{\sigma}(i)}{2}e^{-i\vec{q}\cdot\vec{x}_i}\\ &\to c_4^p\vec{S}_\chi\cdot\sum_{i=1}^{A}\frac{1+\tau_3(i)}{2}\frac{\vec{\sigma}(i)}{2}e^{-i\vec{q}\cdot\vec{x}_i},\end{aligned}\tag{27}$$

where on the right we have again simplified the result by restricting the couplings to protons, to allow comparisons with Eqs. (23) and (26).

The spin averaged/summed transition probability can be easily evaluated by the spherical harmonic methods outlined in Appendix A, yielding

$$\begin{aligned}
&\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}|^2\\
&= c_1^{p\,2}\left[\frac{4\pi}{2j_N+1}\sum_{J=0,2,\ldots}^{\infty}|\langle j_N||\sum_{i=1}^{A}M_J(qx_i)\frac{1+\tau_3(i)}{2}||j_N\rangle|^2\right]\\
&\quad + c_4^{p\,2}\frac{j_\chi(j_\chi+1)}{12}\left\{\frac{4\pi}{2j_N+1}\sum_{J=1,3,\ldots}^{\infty}\left[|\langle j_N||\sum_{i=1}^{A}\Sigma''_J(qx_i)\frac{1+\tau_3(i)}{2}||j_N\rangle|^2+|\langle j_N||\sum_{i=1}^{A}\Sigma'_J(qx_i)\frac{1+\tau_3(i)}{2}||j_N\rangle|^2\right]\right\}\\
&\equiv c_1^{p\,2}\left|M^N_{F;p}(0)\right|^2 F_F^{p\,2}(q^2) + c_4^{p\,2}\frac{j_\chi(j_\chi+1)}{12}\left|M^N_{GT;p}(0)\right|^2 F_{GT}^{p\,2}(q^2).
\end{aligned} \tag{28}$$

Here $M_J(qx_i)$ is the charge multipole operator and $\Sigma''_J(qx_i)$ and $\Sigma'_J(qx_i)$ are the longitudinal and transverse spin multipole operators of rank J, which are standard in treatments of electroweak nuclear interactions and are defined below. The assumption of nuclear wave functions of good parity and CP restricts the sums to even and odd J, respectively.

The form factors $F_F^p(q^2)$ and $F_{GT}^p(q^2)$ are defined so that $F_F^p(0) = F_{GT}^p(0) = 1$ and can be computed from a nuclear model

$$\begin{aligned}
F_F^{p\,2}(q^2) &= \frac{\sum_{J=0,2,\ldots}^{\infty}|\langle j_N||\sum_{i=1}^{A}M_J(qx_i)\frac{1+\tau_3(i)}{2}||j_N\rangle|^2}{\frac{1}{4\pi}|\langle j_N||\sum_{i=1}^{A}\frac{1+\tau_3(i)}{2}||j_N\rangle|^2},\\
F_{GT}^{p\,2}(q^2) &= \frac{\sum_{J=1,3,\ldots}^{\infty}\left[|\langle j_N||\sum_{i=1}^{A}\Sigma''_J(qx_i)\frac{1+\tau_3(i)}{2}||j_N\rangle|^2+|\langle j_N||\sum_{i=1}^{A}\Sigma'_J(qx_i)\frac{1+\tau_3(i)}{2}||j_N\rangle|^2\right]}{\frac{1}{4\pi}|\langle j_N||\sum_{i=1}^{A}\frac{1+\tau_3(i)}{2}\sigma(i)||j_N\rangle|^2}.
\end{aligned} \tag{29}$$

The spin form factor has the above form because of the identity

$$\vec{S}_\chi\cdot\vec{S}_N \equiv (\vec{S}_\chi\cdot\hat{q})(\vec{S}_N\cdot\hat{q}) + (\vec{S}_\chi\times\hat{q})\cdot(\vec{S}_N\times\hat{q}), \tag{30}$$

where $\hat{q}$ is the unit vector along the momentum transfer to the nucleus. Thus, the use of $\mathcal{O}_4$ implies equal couplings to the longitudinal and transverse spin operators Σ''_J and Σ'_J, which cannot interfere if one sums over spins. In a more general treatment of the WIMP-nucleon interaction, these operators would be independent. For example, in the EFT expansion $\mathcal{O}_4 = \vec{S}_\chi\cdot\vec{S}_N$ and $\mathcal{O}_6 = (\vec{S}_\chi\cdot\vec{q})(\vec{S}_N\cdot\vec{q})$ have distinct coefficients.

Often in the literature $F_F^p(q^2)$ and $F_{GT}^p(q^2)$ are not calculated microscopically, but are represented by simple phenomenological forms.

The operators M_J, Σ''_J, and Σ'_J are, respectively, the vector charge, axial longitudinal, and axial transverse electric multipole operators familiar from electroweak nuclear physics. The latter two operators are also frequently designated as L_J^5 and $T_J^{\mathrm{el}\,5}$ in the literature to emphasize their multipole and axial character.

While we have simplified the above expressions by assuming all couplings are to protons, to allow a comparison with our free-proton result, the expressions for arbitrary isospin are also simple:

$$\begin{aligned}
\frac{1}{2j_\chi+1}\frac{1}{2j+1}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{4\pi}{2j_N+1}\Bigg(\sum_{J=0,2,\ldots}^{\infty}|\langle j_N||\sum_{i=1}^{A}M_J(qx_i)\left[c_1^0+c_1^1\tau_3(i)\right]||j_N\rangle|^2\\
&\quad+\frac{j_\chi(j_\chi+1)}{12}\sum_{J=1,3,\ldots}^{\infty}\Bigg\{|\langle j_N||\sum_{i=1}^{A}\Sigma''_J(qx_i)\left[c_4^0+c_4^1\tau_3(i)\right]||j_N\rangle|^2\\
&\quad+|\langle j_N||\sum_{i=1}^{A}\Sigma''_J(qx_i)\left[c_4^0+c_4^1\tau_3(i)\right]||j_N\rangle|^2\Bigg\}\Bigg).
\end{aligned} \tag{31}$$

D. General EFT form of the WIMP-nucleus response

The general form of the WIMP-nucleus interaction consistent with the assumption of nuclear ground states with good P and CP can be derived by building an EFT at the nuclear level, or by embedding the EFT WIMP-nucleon interaction into the nucleus, without making assumptions of the sort just discussed. We follow the second strategy here, as it allows us to connect the nuclear responses back to the single-nucleon interaction and consequently to the ultraviolet theories which map onto that single-nucleon interaction, on nonrelativistic reduction.

We relegate most of the details to Appendix A, giving just the essentials here. First, the basic model assumption is that the nuclear interaction is the sum of the interactions of the

WIMP with the individual nucleons in the nucleus. Thus, the mapping from the nucleon-level effective operators to nuclear operators is made by the following generalization of Eq. (16),

$$\sum_{\tau=0,1}\sum_{i=1}^{15} c_i^\tau \mathcal{O}_i t^\tau \rightarrow \sum_{\tau=0,1}\sum_{i=1}^{15} c_i^\tau \sum_{j=1}^{A} \mathcal{O}_i(j) t^\tau(j). \tag{32}$$

Now the nuclear operators appearing in this expression are built from $i\vec{q}/m_N$, a c number, $\vec{S}_N$, which acts on intrinsic nuclear coordinates, and the relative velocity operator $\vec{v}^\perp$, which now represents a set of A internal WIMP-nucleus system velocities, $A-1$ of which involve the relative coordinates of bound nucleons (the Jacobi velocities), and one of which is the velocity of the dark matter (DM) particle relative to the nuclear center of mass,

$$\vec{v}^\perp \rightarrow \left\{\tfrac{1}{2}[\vec{v}_{\chi,\text{in}} + \vec{v}_{\chi,\text{out}} - \vec{v}_{N,\text{in}}(i) - \vec{v}_{N,\text{out}}(i)], i = 1,\ldots,A\right\}$$
$$\equiv \vec{v}_T^\perp - \{\vec{v}_{N,\text{in}}(i) + \vec{v}_{N,\text{out}}(i), i = 1,\ldots,A-1\}. \tag{33}$$

The DM particle/nuclear center of mass relative velocity is a c number,

$$\vec{v}_T^\perp = \tfrac{1}{2}[\vec{v}_{\chi,\text{in}} + \vec{v}_{\chi,\text{out}} - \vec{v}_{T,\text{in}}(i) - \vec{v}_{T,\text{out}}(i)], \tag{34}$$

while the internal nuclear Jacobi velocities $\vec{v}_N$ are operators acting on intrinsic nuclear coordinates. [That is, for a single-nucleon (A=1) target, $\vec{v}_T^\perp \equiv \vec{v}^\perp$, while for all nuclear targets, there are $A-1$ additional velocity degrees of freedom associated with the Jacobi internucleon velocities.] This separation is discussed in more detail in Appendix A.

In analogy with Eq. (27), one then obtains the WIMP-nucleus interaction

$$\sum_{\tau=0,1}\Bigg\{ l_0^\tau \sum_{i=1}^{A} e^{-i\vec{q}\cdot\vec{x}_i} + l_0^{A\tau}\sum_{i=1}^{A}\frac{1}{2M}\left[-\frac{1}{i}\overleftarrow{\nabla}_i\cdot\vec{\sigma}(i)e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\vec{\sigma}(i)\cdot\frac{1}{i}\overrightarrow{\nabla}_i\right]$$
$$+ \vec{l}_5^\tau\cdot\sum_{i=1}^{A}\vec{\sigma}(i)e^{-i\vec{q}\cdot\vec{x}_i} + \vec{l}_M^\tau\cdot\sum_{i=1}^{A}\frac{1}{2M}\left(-\frac{1}{i}\overleftarrow{\nabla}_i e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\frac{1}{i}\overrightarrow{\nabla}_i\right)$$
$$+ \vec{l}_E^\tau\cdot\sum_{i=1}^{A}\frac{1}{2M}[\overleftarrow{\nabla}_i\times\vec{\sigma}(i)e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\vec{\sigma}(i)\times\overrightarrow{\nabla}_i]\Bigg\}_{\text{int}} t^\tau(i), \tag{35}$$

where the subscript "int" instructs one to take the intrinsic part of the nuclear operators (that is, the part dependent on the internal Jacobi velocities). Comparing to Eq. (27), one sees that three new velocity-dependent densities appear: the nuclear axial charge operator, familiar as the β decay operator that mediates $0^+ \leftrightarrow 0^-$ decays; the convection current, familiar from electromagnetism; and a spin-velocity current that is less commonly discussed, but does arise as a higher-order correction in weak interactions. The associated WIMP tensors contain the EFT input:

$$\begin{aligned}
l_0^\tau &= c_1^\tau + i\left(\frac{\vec{q}}{m_N}\times\vec{v}_T^\perp\right)\cdot\vec{S}_\chi c_5^\tau + \vec{v}_T^\perp\cdot\vec{S}_\chi c_8^\tau + i\frac{\vec{q}}{m_N}\cdot\vec{S}_\chi c_{11}^\tau,\\
l_0^{A\tau} &= -\frac{1}{2}\left[c_7^\tau + i\frac{\vec{q}}{m_N}\cdot\vec{S}_\chi c_{14}^\tau\right],\\
\vec{l}_5 &= \frac{1}{2}\Bigg[i\frac{\vec{q}}{m_N}\times\vec{v}_T^\perp c_3^\tau + \vec{S}_\chi c_4^\tau + \frac{\vec{q}}{m_N}\frac{\vec{q}}{m_N}\cdot\vec{S}_\chi c_6^\tau + \vec{v}_T^\perp c_7^\tau + i\frac{\vec{q}}{m_N}\times\vec{S}_\chi c_9^\tau + i\frac{\vec{q}}{m_N}c_{10}^\tau\\
&\quad + \vec{v}_T^\perp\times\vec{S}_\chi c_{12}^\tau + i\frac{\vec{q}}{m_N}\vec{v}_T^\perp\cdot\vec{S}_\chi c_{13}^\tau + i\vec{v}_T^\perp\frac{\vec{q}}{m_N}\cdot\vec{S}_\chi c_{14}^\tau + \frac{\vec{q}}{m_N}\times\vec{v}_T^\perp\frac{\vec{q}}{m_N}\cdot\vec{S}_\chi c_{15}^\tau\Bigg],\\
\vec{l}_M &= i\frac{\vec{q}}{m_N}\times\vec{S}_\chi c_5^\tau - \vec{S}_\chi c_8^\tau,\\
\vec{l}_E &= \frac{1}{2}\left[\frac{\vec{q}}{m_N}c_3^\tau + i\vec{S}_\chi c_{12}^\tau - \frac{\vec{q}}{m_N}\times\vec{S}_\chi c_{13}^\tau - i\frac{\vec{q}}{m_N}\frac{\vec{q}}{m_N}\cdot\vec{S}_\chi c_{15}^\tau\right].
\end{aligned} \tag{36}$$

In Appendix A the products of plane waves and scalar/vector operators appearing in Eq. (35) are expanded in spherical and vector spherical harmonics, and the resulting amplitude is squared, averaged over initial spins and summed over final spins. One obtains

$$\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}|^2_{\text{nucleus/EFT}} = \frac{4\pi}{2j_N+1}\sum_{\tau=0,1}\sum_{\tau'=0,1}\Bigg\{\sum_{J=0,2,\ldots}^{\infty}\Bigg[R_M^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)\langle j_N||M_{J;\tau}(q)||j_N\rangle\langle j_N||M_{J;\tau'}(q)||j_N\rangle$$
$$+\frac{\vec{q}^{\,2}}{m_N^2}R_{\Phi''}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)\langle j_N||\Phi''_{J;\tau}(q)||j_N\rangle\langle j_N||\Phi''_{J;\tau'}(q)||j_N\rangle$$

$$
\begin{aligned}
&+ \frac{\vec{q}^2}{m_N^2} R_{\Phi'' M}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^2}{m_N^2}\right) \langle j_N || \Phi''_{J;\tau}(q) || j_N\rangle \langle j_N || M_{J;\tau'}(q) || j_N\rangle \Big] \\
&+ \sum_{J=2,4,\ldots}^{\infty} \left[\frac{\vec{q}^2}{m_N^2} R_{\tilde{\Phi}'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^2}{m_N^2}\right) \langle j_N || \tilde{\Phi}'_{J;\tau}(q) || j_N\rangle \langle j_N || \tilde{\Phi}'_{J;\tau'}(q) || j_N\rangle \right] \\
&+ \sum_{J=1,3,\ldots}^{\infty} \Bigg[R_{\Sigma''}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^2}{m_N^2}\right) \langle j_N || \Sigma''_{J;\tau}(q) || j_N\rangle \langle j_N || \Sigma''_{J;\tau'}(q) || j_N\rangle \\
&+ R_{\Sigma'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^2}{m_N^2}\right) \langle j_N || \Sigma'_{J;\tau}(q) || j_N\rangle \langle j_N || \Sigma'_{J;\tau'}(q) || j_N\rangle \\
&+ \frac{\vec{q}^2}{m_N^2} R_{\Delta}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^2}{m_N^2}\right) \langle j_N || \Delta_{J;\tau}(q) || j_N\rangle \langle j_N || \Delta_{J;\tau'}(q) || j_N\rangle \\
&+ \frac{\vec{q}^2}{m_N^2} R_{\Delta\Sigma'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^2}{m_N^2}\right) \langle j_N || \Delta_{J;\tau}(q) || j_N\rangle \langle j_N || \Sigma'_{J;\tau'}(q) || j_N\rangle \Bigg] \Bigg\}. \qquad (37)
\end{aligned}
$$

Note that five of the eight terms above are accompanied by a factor of $\vec{q}^{\,2}/m_N^2$. This is the parameter identified in Sec. II B that governs the enhancement of the composite operators with respect to the point operators for those $\mathcal{O}_i$ where composite operators contribute. Thus, one can read off those response functions that are generated by composite operators from this factor. The DM particle response functions are determined by the c_i^τ's,

$$
\begin{aligned}
R_M^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= c_1^\tau c_1^{\tau'} + \frac{j_\chi(j_\chi+1)}{3}\left[\frac{\vec{q}^{\,2}}{m_N^2}\vec{v}_T^{\perp 2} c_5^\tau c_5^{\tau'} + \vec{v}_T^{\perp 2} c_8^\tau c_8^{\tau'} + \frac{\vec{q}^{\,2}}{m_N^2} c_{11}^\tau c_{11}^{\tau'}\right], \\
R_{\Phi''}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= \frac{\vec{q}^{\,2}}{4m_N^2} c_3^\tau c_3^{\tau'} + \frac{j_\chi(j_\chi+1)}{12}\left(c_{12}^\tau - \frac{\vec{q}^{\,2}}{m_N^2} c_{15}^\tau\right)\left(c_{12}^{\tau'} - \frac{\vec{q}^{\,2}}{m_N^2} c_{15}^{\tau'}\right), \\
R_{\Phi'' M}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= c_3^\tau c_1^{\tau'} + \frac{j_\chi(j_\chi+1)}{3}\left(c_{12}^\tau - \frac{\vec{q}^{\,2}}{m_N^2} c_{15}^\tau\right) c_{11}^{\tau'}, \\
R_{\tilde{\Phi}'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= \frac{j_\chi(j_\chi+1)}{12}\left[c_{12}^\tau c_{12}^{\tau'} + \frac{\vec{q}^{\,2}}{m_N^2} c_{13}^\tau c_{13}^{\tau'}\right], \\
R_{\Sigma''}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= \frac{\vec{q}^{\,2}}{4m_N^2} c_{10}^\tau c_{10}^{\tau'} + \frac{j_\chi(j_\chi+1)}{12}\left[c_4^\tau c_4^{\tau'} \frac{\vec{q}^{\,2}}{m_N^2}\left(c_4^\tau c_6^{\tau'} + c_6^\tau c_4^{\tau'}\right) + \frac{\vec{q}^{\,4}}{m_N^4} c_6^\tau c_6^{\tau'} + \vec{v}_T^{\perp 2} c_{12}^\tau c_{12}^{\tau'} + \frac{\vec{q}^{\,2}}{m_N^2}\vec{v}_T^{\perp 2} c_{13}^\tau c_{13}^{\tau'}\right], \qquad (38) \\
R_{\Sigma'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= \frac{1}{8}\left[\frac{\vec{q}^{\,2}}{m_N^2}\vec{v}_T^{\perp 2} c_3^\tau c_3^{\tau'} + \vec{v}_T^{\perp 2} c_7^\tau c_7^{\tau'}\right] + \frac{j_\chi(j_\chi+1)}{12}\left[c_4^\tau c_4^{\tau'} + \frac{\vec{q}^{\,2}}{m_N^2} c_9^\tau c_9^{\tau'} + \frac{\vec{v}_T^{\perp 2}}{2}\left(c_{12}^\tau - \frac{\vec{q}^{\,2}}{m_N^2} c_{15}^\tau\right)\left(c_{12}^{\tau'} - \frac{\vec{q}^{\,2}}{m_N^2} c_{15}^{\tau'}\right)\right. \\
&\quad \left. + \frac{\vec{q}^{\,2}}{2m_N^2}\vec{v}_T^{\perp 2} c_{14}^\tau c_{14}^{\tau'}\right], \\
R_{\Delta}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= \frac{j_\chi(j_\chi+1)}{3}\left[\frac{\vec{q}^{\,2}}{m_N^2} c_5^\tau c_5^{\tau'} + c_8^\tau c_8^{\tau'}\right], \\
R_{\Delta\Sigma'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2}\right) &= \frac{j_\chi(j_\chi+1)}{3}\left[c_5^\tau c_4^{\tau'} - c_8^\tau c_9^{\tau'}\right].
\end{aligned}
$$

The six nuclear operators appearing in Eq. (37), familiar from standard-model electroweak interaction theory, are constructed from the Bessel spherical harmonics and vector spherical harmonics, $M_{JM}(q\vec{x}) \equiv j_J(qx)Y_{JM}(\Omega_x)$ and $\vec{M}_{JL}^M \equiv j_L(qx)\vec{Y}_{JLM}(\Omega_x)$,

$$
\begin{aligned}
M_{JM;\tau}(q) &\equiv \sum_{i=1}^{A} M_{JM}(q\vec{x}_i)t^\tau(i), \\
\Delta_{JM;\tau}(q) &\equiv \sum_{i=1}^{A} \vec{M}_{JJ}^M(q\vec{x}_i)\cdot\frac{1}{q}\vec{\nabla}_i t^\tau(i),
\end{aligned}
$$

$$\begin{aligned}
\Sigma'_{JM;\tau}(q) &\equiv -i\sum_{i=1}^{A}\left\{\frac{1}{q}\vec{\nabla}_i\times\vec{M}_{JJ}^{M}(q\vec{x}_i)\right\}\cdot\vec{\sigma}(i)t^{\tau}(i)\\
&=\sum_{i=1}^{A}\left\{-\sqrt{\frac{J}{2J+1}}\vec{M}_{JJ+1}^{M}(q\vec{x}_i)+\sqrt{\frac{J+1}{2J+1}}\vec{M}_{JJ-1}^{M}(q\vec{x}_i)\right\}\cdot\vec{\sigma}(i)t^{\tau}(i),\\
\Sigma''_{JM;\tau}(q) &\equiv \sum_{i=1}^{A}\left\{\frac{1}{q}\vec{\nabla}_i M_{JM}(q\vec{x}_i)\right\}\cdot\vec{\sigma}(i)t^{\tau}(i)\\
&=\sum_{i=1}^{A}\left\{\sqrt{\frac{J+1}{2J+1}}\vec{M}_{JJ+1}^{M}(q\vec{x}_i)+\sqrt{\frac{J}{2J+1}}\vec{M}_{JJ-1}^{M}(q\vec{x}_i)\right\}\cdot\vec{\sigma}(i)t^{\tau}(i),\\
\tilde{\Phi}'_{JM;\tau}(q) &\equiv \sum_{i=1}^{A}\left\{\left[\frac{1}{q}\vec{\nabla}_i\times\vec{M}_{JJ}^{M}(q\vec{x}_i)\right]\cdot\left[\vec{\sigma}(i)\times\frac{1}{q}\vec{\nabla}_i\right]+\frac{1}{2}\vec{M}_{JJ}^{M}(q\vec{x}_i)\cdot\vec{\sigma}(i)\right\}t^{\tau}(i),\\
\Phi''_{JM;\tau}(q) &\equiv i\sum_{i=1}^{A}\left[\frac{1}{q}\vec{\nabla}_i M_{JM}(q\vec{x}_i)\right]\cdot\left[\vec{\sigma}(i)\times\frac{1}{q}\vec{\nabla}_i\right]t^{\tau}(i).
\end{aligned}\tag{39}$$

Equations (37), (38), and (39) comprise the general expression for the WIMP-nucleon spin-averaged transition probability. M, Δ, Σ', Σ'', $\tilde{\Phi}'$, and Φ'' transform as vector charge, vector transverse magnetic, axial transverse electric, axial longitudinal, vector transverse electric, and vector longitudinal operators, respectively. These are the allowed responses under the assumption that the nuclear ground state is an approximate eigenstate of P and CP, and thus we have derived the most general form of the cross section.

Slater determinants are often constructed in a harmonic oscillator basis because they allow projection of spurious center-of-mass motion. In that case, Eq. (37) gives the cross section as a sum of products of WIMP $R_k^{\tau\tau'}(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2})$ and nuclear $W_k^{\tau\tau'}(y)$ response functions, where $y=(qb/2)^2$ with b the harmonic oscillator size parameter. That is, the evolution of the nuclear responses with q is determined by the single dimensionless parameter y. Equation (37) can then be written compactly as

$$\begin{aligned}
&\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}|^2_{\text{nucleus}-\text{HO/EFT}}\\
&=\frac{4\pi}{2j_N+1}\sum_{\tau=0,1}\sum_{\tau'=0,1}\left\{\left[R_M^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_M^{\tau\tau'}(y)+R_{\Sigma''}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_{\Sigma''}^{\tau\tau'}(y)+R_{\Sigma'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_{\Sigma'}^{\tau\tau'}(y)\right]\right.\\
&\quad+\frac{\vec{q}^{\,2}}{m_N^2}\left[R_{\Phi''}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_{\Phi''}^{\tau\tau'}(y)+R_{\Phi''M}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_{\Phi''M}^{\tau\tau'}(y)+R_{\tilde{\Phi}'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_{\tilde{\Phi}'}^{\tau\tau'}(y)\right.\\
&\quad\left.\left.+R_{\Delta}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_{\Delta}^{\tau\tau'}(y)+R_{\Delta\Sigma'}^{\tau\tau'}\left(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2}\right)W_{\Delta\Sigma'}^{\tau\tau'}(y)\right]\right\},
\end{aligned}\tag{40}$$

where

$$\begin{aligned}
W_O^{\tau\tau'}(y) &\equiv \sum_{J=0,2,\ldots}^{\infty}\langle j_N||O_{J;\tau}(q)||j_N\rangle\langle j_N||O_{J;\tau'}(q)||j_N\rangle \text{ for } O=M,\Phi'',\\
W_O^{\tau\tau'}(y) &\equiv \sum_{J=1,3,\ldots}^{\infty}\langle j_N||O_{J;\tau}(q)||j_N\rangle\langle j_N||O_{J;\tau'}(q)||j_N\rangle \text{ for } O=\Sigma'',\Sigma',\Delta,\\
W_{\tilde{\Phi}'}^{\tau\tau'}(y) &= \sum_{J=2,4,\ldots}^{\infty}\langle j_N||\tilde{\Phi}'_{J;\tau}(q)||j_N\rangle\langle j_N||\tilde{\Phi}'_{J;\tau'}(q)||j_N\rangle,\\
W_{\Phi''M}^{\tau\tau'}(y) &= \sum_{J=0,2,\ldots}^{\infty}\langle j_N||\Phi''_{J;\tau}(q)||j_N\rangle\langle j_N||M_{J;\tau'}(q)||j_N\rangle,\\
W_{\Delta\Sigma'}^{\tau\tau'}(y) &= \sum_{J=1,3,\ldots}^{\infty}\langle j_N||\Delta_{J;\tau}(q)||j_N\rangle\langle j_N||\Sigma'_{J;\tau'}(q)||j_N\rangle.
\end{aligned}\tag{41}$$

Equations (40), (38), and (41) are the key formulas evaluated by the *Mathematica* script described in Appendix B. Parity and CP restrict the sums over multipolarities J to only even or only odd terms, depending on the transformation properties of the operators, as described in Appendix A.

The physics of these six nuclear response functions is more easily seen by examining the long-wavelength forms of the corresponding operators. The operators that are nonvanishing as $q \to 0$ are

$$\begin{aligned}
\sqrt{4\pi}\, M_{00;\tau}(0) &= \sum_{i=1}^{A} t^\tau(i),\\
\sqrt{4\pi}\, \Delta_{1M;\tau}(0) &= -\frac{1}{\sqrt{6}}\sum_{i=1}^{A} l_{1M}(i) t^\tau(i),\\
\sqrt{4\pi}\, \Sigma'_{1M;\tau}(0) &= \sqrt{\frac{2}{3}}\sum_{i=1}^{A} \sigma_{1M}(i) t^\tau(i),\\
\sqrt{4\pi}\, \Sigma''_{1M;\tau}(0) &= \frac{1}{\sqrt{3}}\sum_{i=1}^{A} \sigma_{1M}(i) t^\tau(i),\\
\sqrt{4\pi}\, \tilde{\Phi}'_{2M;\tau}(0) &= -\frac{1}{\sqrt{5}}\sum_{i=1}^{A} \left\{x(i)\otimes\left[\vec{\sigma}(i)\times\frac{1}{i}\vec{\nabla}(i)\right]_1\right\}_2 t^\tau(i),\\
\sqrt{4\pi}\, \Phi''_{JM;\tau}(0) &= \begin{cases} \frac{1}{3\sum_{i=1}^{A}\vec{\sigma}(i)\cdot\vec{l}(i)t^\tau(i)} & J=0,\\ -\frac{1}{\sqrt{5}}\sum_{i=1}^{A}\left\{x(i)\otimes\left[\vec{\sigma}(i)\times\frac{1}{i}\vec{\nabla}(i)\right]_1\right\}_2 t^\tau(i) & J=2,\end{cases}
\end{aligned} \tag{42}$$

where the operator Φ'' has scalar and tensor components that survive. Two combinations of operators are, of course, related to the SI/SD forms,

$$\begin{aligned}
\left|M^N_{F;\tau}(0)\right|^2 &\equiv \frac{4\pi}{2j_N+1}|\langle j_N||M_{0;\tau}(0)||j_N\rangle|^2,\\
\left|M^N_{GT;\tau}(0)\right|^2 &\equiv \frac{4\pi}{2j_N+1}[|\langle j_N||\Sigma''_{1;\tau}(0)||j_N\rangle|^2\\
&\quad +|\langle j_N||\Sigma'_{1;\tau}(q)||j_N\rangle|^2].
\end{aligned} \tag{43}$$

In the next section we describe in more detail some of the differences between this form and the point-nucleus and allow forms, where the only the simple Fermi and Gamow-Teller operators arise. However, one can make some initial observations here.

(i) The most general form of the WIMP-nucleus elastic scattering probability has six, not two, response functions. They are associated with the squares of the matrix elements of the six operators given in Eqs. (39). There are also two interference terms ($\Phi'' \leftrightarrow M$ and $\Delta \leftrightarrow \Sigma'$). Total cross sections thus depend on eight bilinear combinations of WIMP couplings $R^{\tau\tau'}$, not just the two combinations found in the point-nucleus limit.

(ii) The spin response familiar from the standard allowed treatment of WIMP-nucleus interactions splits into separate longitudinal and transverse components, as various candidate effective interactions do not couple to all spin projections symmetrically. The associated operators, Σ'' and Σ', are proportional in the long-wavelength limit, but are distinct at finite $\vec{q}^{\,2}$ because their associated form factors differ.

(iii) Three new response functions are generated from couplings to the intrinsic velocities of nucleons and consequently reflect the composite nature of the nucleus. Reflecting their finite-nuclear-size origin, the three responses appear in Eq. (37) with an explicit factor of $\vec{q}^{\,2}/m_N^2$.

(iv) Two scalar responses appear in Eq. (37), generated by the standard Fermi operator $1(i)$ and the new spin-orbit operator $\vec{\sigma}\cdot\vec{l}(i)$. Thus, both are "spin-independent" responses, responses associated with operators that transform as scalars under rotations.

(v) There are three vector responses, two associated with the (in general, independent) longitudinal and transverse projections of spin and the third with the orbital angular momentum operator $\vec{l}(i)$. These three operators transform under rotations as $\vec{j}_N$, and all thus require a nuclear ground-state spin of $j_N \geqslant 1/2$. It was shown in Ref. [8] that among the various nuclear targets now in use for dark-matter studies, the relative strength of spin and orbital transition probabilities can differ by two orders magnitude or more.

(vi) One response function, generated by $\tilde{\Phi}'$, is tensor, and thus only contributes if $j_N \geqslant 1$. This response function is somewhat exotic, coming from interactions $\mathcal{O}_{12}$, $\mathcal{O}_{13}$, and $\mathcal{O}_{15}$ that we have noted do not arise for traditional spin-0 or spin-1 exchanges.

(vii) The EFT result of Eq. (37) and the SI/SD result of Eq. (28) coincide if one takes $\vec{q}^{\,2} \to 0$ and also $\vec{v}_T^{\perp 2} \to 0$, a limit that zeros out all contributions from low-energy constants other than c_1 and c_4. However, away from this limit they differ. This illustrates the inconsistency of the standard SI/SD formulation with form factors: One selectively includes powers of

$\vec{q}\cdot\vec{x}(i)$ to modify the Fermi $1(i)$ and $\vec{\sigma}(i)$ operators through form factors, while not using those same factors to create new operators.

E. Nuclear response function evaluation: The one-body density matrix

We have expressed the dark-matter particle scattering cross sections in terms of the singly reduced (in angular momentum) nuclear matrix elements of one-body operators of definite angular momentum. These nuclear matrix elements can be conveniently expressed in terms of the one-body density matrix: The density matrix extracts from complicated many-body nuclear wave functions containing all possible correlations, just that information necessary to evaluate one-body operators. Once the density matrix is obtained from a nuclear many-body calculation, all many-body matrix elements then reduce to simple sums over single-particle matrix elements.

In the treatment so far we have labeled the nuclear ground state by its angular momentum j_N, an exact quantum number. Here we add to that label the isospin quantum numbers T, M_T: Isospin T is an approximate but not exact quantum label, as isospin is broken by the electromagnetic interactions among nucleons. However, we employ that label here because most shell-model calculations are isospin conserving, and thus most density matrices derived from such calculations employ T as a quantum label. We stress, however, that everything discussed below can be trivially repeated without the assumption of T as a nuclear-state label: The density matrix would then be defined without this assumption.

With the inclusion of the isospin labels, the singly reduced many-body matrix elements can be written

$$\langle j_N; T M_T || \sum_{i=1}^{A} \hat{O}_{J;\tau}(q\vec{x}_i) || j_N; T M_T\rangle = (-1)^{T-M_T} \begin{pmatrix} T & \tau & T \\ -M_T & 0 & M_T \end{pmatrix} \left\langle j_N; T \vdots\vdots \sum_{i=1}^{A} \hat{O}_{J;\tau}(q\vec{x}_i) \vdots\vdots j_N; T \right\rangle,$$

$$\left\langle j_N; T \vdots\vdots \sum_{i=1}^{A} \hat{O}_{J;\tau}(q\vec{x}_i) \vdots\vdots j_N; T \right\rangle = \sum_{|\alpha|,|\beta|} \Psi^{J;\tau}_{|\alpha|,|\beta|} \langle|\alpha| \vdots\vdots O_{J;\tau}(q\vec{x}) \vdots\vdots |\beta|\rangle. \tag{44}$$

Here $\Psi^{J;\tau}_{|\alpha|,|\beta|}$ is the one-body density matrix for the diagonal ground-state-to-ground-state transition, $|\alpha|$ represents the nonmagnetic quantum numbers in the chosen single-particle basis [e.g., for a single-particle harmonic oscillator state $|\alpha\rangle = |n_\alpha(l_\alpha s_\alpha = 1/2) j_\alpha m_{j_\alpha}; t_\alpha = 1/2 m_{t_\alpha}\rangle \equiv ||\alpha|; m_{j_\alpha} m_{t_\alpha}\rangle$, with n_α the nodal quantum number], $\vdots\vdots$ denotes a doubly reduced matrix element (in spin and isospin), and the sums over $|\alpha|$ and $|\beta|$ extend over complete sets of single-particle quantum numbers. The density matrix can be written in second quantization as

$$\Psi^{J;\tau}_{|\alpha|,|\beta|} \equiv \frac{1}{[J][\tau]} \langle j_N; T \vdots\vdots [c^\dagger_{|\alpha|} \otimes \tilde{c}_{|\beta|}]_{J;\tau} \vdots\vdots j_N; T\rangle, \tag{45}$$

where $c^\dagger_\alpha$ is the single-particle creation operator, $\tilde{c}_\beta = (-1)^{j_\beta - m_{j_\beta} + 1/2 - m_{t_\beta}} c_{|\beta|; -m_{j_\beta}, -m_{t_\beta}}$, and $[J] \equiv \sqrt{2J+1}$. The phases yield a destruction operator $\tilde{c}_\beta$ that transforms as a spherical tensor in single-particle angular momentum and isospin.

Equation (44) is an exact expression for $\langle j_N; T M_T || \hat{O}_J; \tau || j_N; T M_T\rangle$. When one invokes a nuclear model to calculate a dark-matter response function, effectively one is employing some physics-motivated prescription for intelligently truncating the infinite sums over $|\alpha|, |\beta|$ in Eq. (44) to some finite subset, hopefully capturing most of the relevant low-momentum physics.

The isospin matrix element in Eq. (44) is easily performed, yielding

$$\begin{aligned} &\langle|\alpha| \vdots\vdots \hat{O}_J; \tau(q\vec{x}) \vdots\vdots |\beta|\rangle \\ &\quad = \sqrt{2}[\tau] \langle n_\alpha(l_\alpha 1/2) j_\alpha || O_J || n_\beta(l_\beta 1/2) j_\beta\rangle, \end{aligned} \tag{46}$$

where O_J is the space-spin part of the operator. If the single-particle basis is that of a harmonic oscillator, the reduced matrix element for $O_J = \{M_J, \Sigma'_J, \Sigma''_J, \Delta_J, \tilde{\Phi}'_J, \Phi''_J\}$ can be evaluated algebraically,

$$\begin{aligned} &\langle n_\alpha(l_\alpha 1/2) j_\alpha || O_J(q\vec{x}) || n_\beta(l_\beta 1/2) j_\beta\rangle \\ &\quad = \frac{1}{\sqrt{4\pi}} y^{(J-K)/2} e^{-y} p(y), \end{aligned} \tag{47}$$

where $K = 2$ for the normal parity [$\pi = (-1)^J$] operators M_J, $\tilde{\Phi}'_J$, and $\tilde{\Phi}''_J$ and $K = 1$ for the abnormal parity [$\pi = (-1)^{J+1}$] operators Δ, Σ', and Σ''. $y = (qb/2)^2$, where b is the oscillator parameter, and $p(y)$ is a finite polynomial in y. Thus, the nuclear response functions W of Eq. (41) are simple functions of y.

The Mathematica script of Appendix B evaluates nuclear matrix elements from input one-body density matrices.

IV. CROSS SECTIONS AND COMPARISONS WITH THE SD/SI FORM

As the new scalar and vector operators related to $\vec{\sigma}\cdot\vec{\ell}(i)$ and $\vec{\ell}(i)$ have selection rules and coherence properties that are quite different from those of the point-nucleus operators $1(i)$ and $\vec{\sigma}(i)$, the retention of only the point-nucleus operators can lead to numerical errors in cross-section estimates. In this section we quantify this claim by identifying those operators where the contributions of the composite operators are enhanced, isolating the enhancement factor mentioned in Sec. II B. We also use the rate formulas presented below to show that neglect of the composite operators leads, in these cases, to cross sections that lack the proper functional dependence on parameters such as the WIMP and target masses.

A. Cross sections and rates

The cross section for WIMP scattering off a nucleus in the laboratory frame is obtained by folding the transition probability with the corresponding Lorentz-invariant phase space,

$$d\sigma = \frac{1}{v}\frac{m_\chi}{E_\chi^i}\left[\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}|^2\right] \times \frac{m_\chi}{E_\chi^f}\frac{d^3p'}{(2\pi)^3}\frac{m_T}{E_T^f}\frac{d^3k'}{(2\pi)^3}(2\pi)^4\delta^4(p+k-p'-k'), \tag{48}$$

where p,p' and k,k' are the initial and final dark-matter particle and nuclear momenta. $\mathcal{M}$, in most other applications the Lorentz-invariant amplitude, is in our construction the Galilean-invariant amplitude, owing to the nonrelativistic nature of the scattering. As this expression is in the laboratory frame, v is the initial WIMP velocity; the target is at rest.

$\mathcal{M}$ is a function of v and one other variable. If we define a scattering angle by the direction of nuclear recoil relative to the initial WIMP velocity, $\hat{v}\cdot\hat{k}' = -\hat{v}\cdot\hat{q} = \cos\theta$, then that second variable can be taken to be $\vec{q}^{\,2}$, or equivalently the energy of the recoiling nucleus $E_R = \vec{q}^{\,2}/2m_T$, or equivalently, using the laboratory-frame energy conservation condition

$$\frac{\vec{p}^{\,2}}{2m_\chi} - \frac{(\vec{p}-\vec{k}')^2}{2m_\chi} - \frac{\vec{k}'^2}{2m_T} = 0 \Rightarrow \frac{\vec{k}'^2}{2\mu_T} = \vec{v}\cdot\vec{k}' \Rightarrow \frac{\vec{q}'^2}{4\mu_T^2 v^2} = \cos^2\theta = \frac{1}{2}(1+\cos 2\theta), \tag{49}$$

the angular variable $\cos 2\theta$. Note that as $\vec{v}\cdot\vec{k}' \geqslant 0$, $0 \leqslant \theta \leqslant \pi/2$, and thus $0 \leqslant 2\theta \leqslant \pi$. We can integrate Eq. (48) to obtain the differential cross sections

$$\frac{d\sigma(v,E_R)}{dE_R} = 2m_T\frac{d\sigma(v,\vec{q}^{\,2})}{d\vec{q}^{\,2}} = \frac{2m_T}{4\pi v^2}\left[\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}^{Nuc}|^2\right], \tag{50}$$

$$\frac{d\sigma(v,\theta)}{d\cos 2\theta} = 2\mu_T^2 v^2\frac{d\sigma(v,\vec{q}^{\,2})}{d\vec{q}^{\,2}} = \frac{\mu_T^2}{2\pi}\left[\frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}^{Nuc}|^2\right]. \tag{51}$$

The differential scattering rate per detector and per target nucleus averaging over the galactic WIMP velocity distribution can then be calculated,

$$\frac{dR_D}{dE_R} = N_T\frac{dR_T}{dE_R} = N_T\int\frac{d\sigma(v,E_R)}{dE_R}v\,dn_\chi = N_T n_\chi\int_{v>v_{\min}}\frac{d\sigma(v,E_R)}{dE_R}v f_E(\vec{v})d^3v \equiv N_T n_\chi\left\langle v\frac{d\sigma(v,E_R)}{dE_R}\right\rangle_{v>v_{\min}}, \tag{52}$$

where N_T is the number of target nuclei in the detector, n_χ is the local number density of dark-matter particles, and $f_E(\vec{v})$ the normalized velocity distribution of the dark-matter particles. Thus, $n_\chi = \rho_\chi/m_\chi$, where ρ_χ is the dark-matter density. The integral over velocities begins with the minimum velocity required to produce a recoil energy E_R,

$$v_{\min} = v_{\min}(E_R) = \frac{q}{2\mu_T} = \frac{1}{\mu_T}\sqrt{\frac{m_T E_R}{2}}. \tag{53}$$

Similarly,

$$\frac{dR_D}{d\cos 2\theta} = N_T n_\chi\int\frac{d\sigma(v,E_R)}{d\cos 2\theta}v f_E(\vec{v})d^3v \equiv N_T n_\chi\left\langle v\frac{d\sigma(v,E_R)}{d\cos 2\theta}\right\rangle. \tag{54}$$

Here there is no restriction on the recoil energy and thus no requirement for a minimum velocity.

In the same way, one can calculate the total cross section,

$$\sigma(v) = \int_0^{4v^2\mu_T^2}\frac{d\sigma(v,\vec{q}^{\,2})}{d\vec{q}^{\,2}}d\vec{q}^{\,2}. \tag{55}$$

The total scattering rate per detector R_D and per target nucleus R_T become

$$R_D = N_T R_T = N_T n_\chi\int\sigma(v)v f_E(\vec{v})d^3v \equiv N_T n_\chi\langle v\sigma(v)\rangle. \tag{56}$$

B. Parametric dependence of total cross sections

An inspection of Eq. (37) shows that if all operators are evaluated in the long-wavelength limit (that is, ignoring form factors), the equation reduces to the point-nucleus result given in Eq. (26), if in addition operators other than M, Σ'', and Σ' are eliminated. Thus, by working in the long-wavelength limit, keeping all operators in leading order, one has a simple test of the relevance of the new operators, those other than the Fermi and Gamow-Teller ones. A suitable observable for this comparison is $\sigma(v)$, as the integration over $\vec{q}^{\,2}$ in Eq. (37) is easily done using the laboratory-frame relation $\vec{v}_T^{\perp 2} = \vec{v}^{\,2} + \vec{q}^{\,2}/4\mu_T^2$. One finds for each of the EFT interactions [and, for simplicity, considering couplings only to protons, so that the results match Eq. (26)]

$$\sigma_{c_1^p}(v) = c_1^{p2}\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}\langle M_{0;p}(0)\rangle^2\right],$$

$$\sigma_{c_3^p}(v) = c_3^{p2}v^4\frac{\mu_T^2}{\pi}\left\{\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^2\frac{1}{12}\left[\langle\Sigma'_{1;p}(0)\rangle^2 + 16\left(\frac{\mu_T}{m_N}\right)^2(\langle\Phi''_{0;p}(0)\rangle^2 + \langle\Phi''_{2;p}(0)\rangle^2)\right]\right\},$$

$$
\begin{aligned}
\sigma_{c_4^p}(v) &= c_4^{p2}\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}S_\chi(S_\chi+1)\frac{1}{12}(\langle\Sigma'_{1;p}\rangle^2+\langle\Sigma''_{1;p}\rangle^2)\right],\\
\sigma_{c_5^p}(v) &= c_5^{p2}v^4\frac{\mu_T^2}{\pi}\left\{\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^2 S_\chi(S_\chi+1)\frac{2}{9}\left[\langle M_{0;p}\rangle^2+8\left(\frac{\mu_T}{m_N}\right)^2\langle\Delta_{1;p}\rangle^2\right]\right\},\\
\sigma_{c_6^p}(v) &= c_6^{p2}v^4\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^4 S_\chi(S_\chi+1)\frac{4}{9}\langle\Sigma''_{1;p}\rangle^2\right],\\
\sigma_{c_7^p}(v) &= c_7^{p2}v^2\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}\frac{1}{16}\langle\Sigma'_{1;p}\rangle^2\right],\\
\sigma_{c_8^p}(v) &= c_8^{p2}v^2\frac{\mu_T^2}{\pi}\left\{\frac{4\pi}{2J_i+1}S_\chi(S_\chi+1)\frac{1}{6}\left[\langle M_{0;p}\rangle^2+4\left(\frac{\mu_T}{m_N}\right)^2\langle\Delta_{1;p}\rangle^2\right]\right\},\\
\sigma_{c_9^p}(v) &= c_9^{p2}v^2\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^2 S_\chi(S_\chi+1)\frac{1}{6}\langle\Sigma'_{1;p}\rangle^2\right],\\
\sigma_{c_{10}^p}(v) &= c_{10}^{p2}v^2\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^2 \frac{1}{2}\langle\Sigma''_{1;p}\rangle^2\right],\\
\sigma_{c_{11}^p}(v) &= c_{11}^{p2}v^2\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^2 S_\chi(S_\chi+1)\frac{2}{3}\langle M_{0;p}\rangle^2\right],\\
\sigma_{c_{12}^p}(v) &= c_{12}^{p2}v^2\frac{\mu_T^2}{\pi}\left\{\frac{4\pi}{2J_i+1}S_\chi(S_\chi+1)\frac{1}{24}\left[\langle\Sigma''_{1;p}\rangle^2+\frac{1}{2}\langle\Sigma'_{1;p}\rangle^2+4\left(\frac{\mu_T}{m_N}\right)^2(\langle\tilde{\Phi}'_{2;p}\rangle^2+\langle\Phi''_{0;p}\rangle^2+\langle\Phi''_{2;p}\rangle^2)\right]\right\},\\
\sigma_{c_{13}^p}(v) &= c_{13}^{p2}v^4\frac{\mu_T^2}{\pi}\left\{\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^2 S_\chi(S_\chi+1)\frac{1}{18}\left[\langle\Sigma''_{1;p}\rangle^2+8\left(\frac{\mu_T}{m_N}\right)^2\langle\tilde{\Phi}'_{2;p}\rangle^2\right]\right\},\\
\sigma_{c_{14}^p}(v) &= c_{14}^{p2}v^4\frac{\mu_T^2}{\pi}\left[\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^2 S_\chi(S_\chi+1)\frac{1}{36}\langle\Sigma'_{1;p}\rangle^2\right],\\
\sigma_{c_{15}^p}(v) &= c_{15}^{p2}v^6\frac{\mu_T^2}{\pi}\left\{\frac{4\pi}{2J_i+1}\left(\frac{\mu_T}{m_N}\right)^4 S_\chi(S_\chi+1)\frac{1}{18}\left[\langle\Sigma'_{1;p}\rangle^2+24\left(\frac{\mu_T}{m_N}\right)^2((\langle\Phi''_{0;p}\rangle^2+\Phi''_{2;p}\rangle^2)\right]\right\},
\end{aligned}
\tag{57}
$$

where we have used $\langle\hat{O}_{J;p}\rangle$ as shorthand for the matrix element $\langle j_N||\hat{O}_{J;p}||j_N\rangle$.

The pattern one sees in the above results reflects an underlying EFT power counting. Suppose we designate our WIMP-nucleon operators as $\mathcal{O}_i(\alpha_i,\beta_i)$, where $\alpha_i \in \{0,1\}$ and β_i denote the number of powers of $\vec{v}^\perp$ and $\vec{q}/m_N$, respectively, appearing in the operator,

$$\mathcal{O}_i(\alpha_i,\beta_i) \leftrightarrow [\vec{v}^\perp]^{\alpha_i}\left[\frac{\vec{q}}{m_N}\right]^{\beta_i}. \tag{58}$$

The total cross section has the form

$$
\begin{aligned}
\sigma_i(v) \sim{}& c_i^2\mu_T^2(v^2)^{\alpha_i+\beta_i}\left(\frac{\mu_T^2}{m_N^2}\right)^{\beta_i}\\
&\times\left[a_T^i\langle\hat{O}_i^T\rangle^2+a_N^i\delta_{\alpha_i 1}\langle\hat{O}_i^N\rangle^2\left(\frac{\mu_T^2}{m_N^2}\right)^{\alpha_i}\right],
\end{aligned}
\tag{59}
$$

where $\hat{O}_i^T$ and $\hat{O}_I^N$ represent one of the dimensionless points $(M_0,\Sigma'_1,\Sigma''_1)$ or composite $(\Delta_1,\tilde{\Phi}_2,\Phi''_{0,2})$ operators, respectively, and a_T^i and a_N^i represent simple numerical factors, e.g.,

$$a_T^{15}=\frac{S_\chi(S_\chi+1)}{18},\quad a_N^{15}=\frac{2S_\chi(S_\chi+1)}{3}, \tag{60}$$

where typically $a_N^i/a_T^i \sim 10$.

We see that total cross sections and thus total rates depend on the dimensionless parameters v and μ_T/m_N, but that the parametric dependence on μ_T/m_N depends on the operator type, point, or composite. The cross section for the composite operators have the simple behavior

$$\sigma_i(v)|_N \sim \left[v^2\frac{\mu_T^2}{m_N^2}\right]^{\alpha_i+\beta_i}, \tag{61}$$

where the value of $\alpha_i+\beta_i=0,1,2,3$ is equivalent to our EFT designation LO, NLO, NNLO, N^3LO. This reflects the fact that there are $\alpha_i+\beta_i$ powers of $\vec{q}/m_N$ in the composite operator, with one factor ($\alpha_i=1$) coming from $i\vec{q}\cdot\vec{x}(i)$ in combination with $\vec{v}_N(i)$. The cross section contributions of the point-nucleus operator scale as

$$\sigma_i(v)|_T \sim (v^2)^{\alpha_i} \left[v^2 \frac{\mu_T^2}{m_N^2} \right]^{\beta_i}. \tag{62}$$

There are β_i powers of $\vec{q}/m_N$, while the accompanying velocity is not an operator, but the c number $v_T^\perp$.

Both terms are generally present (see the exception below) if there is a velocity coupling. Consequently, the neglect of composite operators for interactions with derivative couplings not only leads to a cross section that is much too small [by a factor $\sim(a_N^i/a_T^i)(\mu_T^2/m_N^2)$], but produces a cross section with the wrong parametric dependence on m_T and m_χ, potentially distorting comparisons among experiments that are using different nuclear targets, as well as sensitivity plots as a function of m_χ.

If this calculation is extended to the full operators rather than just their long-wavelength forms, the two terms comprising Eq. (59) are modified by factors $F_T^2(\gamma)$ and $F_N^2(\gamma)$, where $\gamma = (b\mu_T v)^2$. Thus, three dimensionless parameters, v, γ, and μ_T/m_N, describe the total cross section's dependence on the WIMP velocity, the nuclear size, and the WIMP-to-nucleus mass ratio, respectively.

C. Consequences for operators: EFT vs SI/SD comparisons

The above results should be helpful to those wanting to understand the limitations of standard treatments that retain only the SI/SD responses. The consequences are operator specific.

(1) Operators $\mathcal{O}_1$ and $\mathcal{O}_4$ are the simple-minded SI and SD operators. Their coupling is to total spin and total charge (in the general case, some combination of N and Z, depending on chosen operator isospin). These operators are point operators, and thus the standard treatment is valid in all respects.

(2) The coupling of operator $\mathcal{O}_{11}$ to the nucleus is 1_i, the vector charge operator. As the nuclear physics is identical to that of $\mathcal{O}_1$, a standard SI analysis would correctly model the nuclear physics of this operator. However, the dependence of rates on the WIMP velocity distribution differ for $\mathcal{O}_1$ and $\mathcal{O}_{11}$, and this difference would normally not be addressed in comparisons among experiments if only interaction $\mathcal{O}_1$ is retained [see point (7) below].

(3) The operators $\mathcal{O}_6$ and $\mathcal{O}_{10}$ couple to the nucleus through longitudinal spin, $\vec{q}\cdot\vec{\sigma}(i)$, while $\mathcal{O}_9$ couples through transverse spin, $\vec{q}\times\vec{\sigma}(i)$. For these operators, the standard analysis based on a spin-dependent coupling would yield the right threshold ($\vec{q}\to 0$) coupling to the nucleus, but misrepresent the form factors (as Σ' and Σ'' are described by distinct form factors). The predicted dependence of rates on the galactic WIMP velocity distribution also differs from the standard $\mathcal{O}_4$ interaction [see point (7) below].

(4) The operators $\mathcal{O}_3$, $\mathcal{O}_5$, $\mathcal{O}_8$, $\mathcal{O}_{12}$, $\mathcal{O}_{13}$, and $\mathcal{O}_{15}$ involve velocity-dependent couplings to the nucleus. The standard SI/SD analysis grossly misrepresents the physics of these operators, leading to errors that can exceed several orders of magnitude. They couple dominantly through the new composite operators Δ, $\tilde{\Phi}'$, and Φ'': The contributions of these operators to the cross section are parametrically enhanced relative to those of the standard operators by the factor $(4-24)\times(\mu_T/m_N)^2 \sim 10A^2$. The resulting large errors can be partially mitigated in the case of $\mathcal{O}_5$ and $\mathcal{O}_8$ because the new operators compete with M_0, which can be coherent if isospin couplings are dialed to make the operator primarily isoscalar. However, even in this favorable case, the error can be an order of magnitude.

(5) In all of the cases above, the standard treatment would distort the multipolarity of the coupling. Operators $\mathcal{O}_3$, $\mathcal{O}_{12}$, $\mathcal{O}_{13}$, and $\mathcal{O}_{15}$ would appear in the standard treatment as spin-dependent interactions, coupling through Σ_1' and Σ_1'', and thus could be probed only if the target has $j_N \geqslant 1/2$. In fact, $\mathcal{O}_3$, $\mathcal{O}_{12}$, and $\mathcal{O}_{15}$ have dominant scalar couplings through Φ_0'', which we have noted is proportional to $\vec{\sigma}(i)\cdot\vec{l}(i)$, an operator that is not only scalar, but is quasicoherent, as discussed in Ref. [8]. The dominant contribution from $\mathcal{O}_{13}$ is through the tensor operator $\tilde{\Phi}_2'$, which requires $j_N \geqslant 1$, a possibility totally outside the standard description.

(6) Two operators remain that at first appear puzzling: $\mathcal{O}_7$ and $\mathcal{O}_{14}$ have velocity-dependent couplings to the nucleus, but unlike the operators discussed in point (5), they have standard spin-dependent couplings, and no contribution from the new composite operators. This result is a consequence of the good P and CP of nuclear wave functions. These operators couple to the nucleus through the axial charge, $\vec{S}_N\cdot\vec{v}^\perp$. When one combines $\vec{S}_N\cdot\vec{v}^\perp$ with $e^{-i\vec{q}\cdot\vec{x}_i}$ to produce multipole operators in the standard way, the matrix elements of the even multipoles vanish by parity, while those of the odd multipoles vanish by CP (or, equivalently, time-reversal invariance). Consequently, all contributions of intrinsic velocities to $\mathcal{O}_7$ and $\mathcal{O}_{14}$ vanish. Thus, the only contribution to the axial charge operator that survives is the single degree of freedom corresponding to the nuclear center-of-mass velocity. As this velocity is a c number, the associated nuclear coupling is a conventional spin operator, Σ_1'.

(7) By adopting an interaction having the form $\mathcal{O}_1$ or $\mathcal{O}_4$, one builds in the assumption that detector rates depend on the v^0 moment of the galactic velocity distribution. This assumption is generally in error for operators other than $\mathcal{O}_1$ and $\mathcal{O}_4$, even if the operator is one of those described in points (2) and (3) above, with nuclear physics quite similar to $\mathcal{O}_1$ and $\mathcal{O}_4$. The rates for LO, NLO, NNLO, ..., operators depend on the v^0, v^2, v^4, ..., moments, respectively, of the WIMP velocity distribution. Consequently, the distribution of events as a function of recoil energy E_R could be used to discriminate among classes of candidate interactions.

V. SUMMARY AND DISCUSSION

This paper was written with three goals in mind. The first was to formulate the nuclear physics of dark-matter scattering in a way that is both fully general, so that no unjustified assumptions are made about the nature of the WIMP-nucleon interaction, and transparent physically, so that one sees by inspection what particle-physics quantities can be tested in elastic scattering experiments. This was accomplished by employing a general WIMP-nucleon interaction developed in EFT and applying standard techniques of multipole analysis in semileptonic weak interactions to express the cross section in a factorized form as products of WIMP and nuclear response functions. In the usual context of electron or neutrino scattering, this kind of approach allows one to immediately see how to exploit the lepton—the electron or neutrino—to probe the less-well-understood nucleus. For example, in elastic electron scattering one can study distribution of charge in the nucleus (SI) or the distribution of the magnetization current (SD), by controlling lepton kinematics, leading to the standard Rosenbluth separation of the charge and magnetic elastic form factors. The reverse is the case in dark-matter studies: Here the WIMP properties are the unknowns and the nuclear target becomes the probe. By exploring nuclear targets with different ground-state properties, one can constrain the character of the low-energy WIMP-nucleon interaction. In this paper we sought to define what, in principle, could be learned about dark matter by varying the nuclear probe in this way.

The application of such standard techniques to WIMP-nucleus scattering leads to a factorized cross section that involves three new operators not found in standard SI/SD analysis. There are also differences in the operators treated in common, e.g., distinct form factors for the the transverse and longitudinal spin components. Large differences between an analysis that properly treats the nuclear size, which cannot be neglected because $|\vec{q} \cdot \vec{r}(i)| \sim 1$, and the standard SI/SD analysis are found for approximately half of the EFT operators, the subset with velocity dependence. The SI/SD analysis requires such interactions to be accompanied by at least one factor of $\vec{v}_T^{\perp 2} \sim 10^{-6}$. However, any velocity-dependent interaction necessarily couples to the WIMP velocity and to the internuclear velocities $\vec{v}(i)$ with equal strengths. These intrinsic velocities combine with $\vec{q} \cdot \vec{r}(i)$ to form even-parity operators such as

$$\frac{q}{m_N}\vec{\ell}(i), \quad \frac{q}{m_N}\vec{\sigma}(i) \cdot \vec{\ell}(i).$$

Consequently, any calculation that properly treats internal nuclear degrees of freedom leads to new operators and to associated rates that are suppressed only by $q^2/m_N^2 \sim 10^{-2}$. The standard SI/SD model of dark-matter particle interactions fails dramatically for interactions with derivative couplings: In these cases the unjustified assumption of a point nucleus—effectively insisting that $\vec{q} \cdot \vec{r}(i) \sim 0$ when in fact it is $\sim$1—leads to erroneous results.

Our formulation has implications for WIMP search strategies, specifically for the number and variety of direct-detection experiments that may be required to understand dark matter, once nonzero rates are found. Historically, we know that characterizing unknown interactions is challenging: The form of the low-energy weak interaction—which of five candidate interactions contribute—was debated into the mid-1950s though evidence of its V and A nature came as early as 1936 [10].

While the standard SI/SD description is a sensible way to characterize detector sensitivities now, greater care may be needed once dark-matter events are seen in several detectors. Even among operators closely related to $\mathcal{O}_1$ or $\mathcal{O}_4$—e.g., those carrying additional kinematic factors (e.g., $\mathcal{O}_6$, $\mathcal{O}_9$, $\mathcal{O}_{10}$, $\mathcal{O}_{11}$) or distinct form factors because the spin coupling is purely longitudinal or purely transverse (e.g., $\mathcal{O}_6$, $\mathcal{O}_9$, $\mathcal{O}_{10}$)—differences will appear between targets, reflecting, for example, the impact of the nuclear mass on the typical momentum transfer. Other candidate operators have no relationship to the SI/SD ones: This is generally the case for velocity-dependent interactions. As was shown in Ref. [8], it is not unusual to find changes in the relative sensitivities of detectors of an order of magnitude or more, as operators are varied. For example, the relative sensitivity of a Xe detector to ones using NaI or Ge changes by factors of $\sim$14 and $\sim$7, respectively, if the the operator $\mathcal{O}_4 = \vec{S}_\chi \cdot \vec{S}_N$ is changed to $\mathcal{O}_8 = \vec{S}_\chi \cdot \vec{v}^\perp$ (taking the coupling to be to neutrons). The corresponding nuclear operators are $\vec{\sigma}(i)$ and $\vec{\ell}(i)$, and both NaI and Ge are relatively more sensitive to the latter than the former. Thus, it is possible to get contradictory answers from comparisons among detectors, if an analysis is done assuming $\mathcal{O}_4$, while the underlying interaction is actually $\mathcal{O}_8$.

However, there is another motivation for doing more elastic scattering experiments than just avoiding confusion: There is more to be learned from such experiments than is apparent in SI/SD analyses. To distinguish a SI interaction from a SD one, one simply needs results from two targets, one with $J = 0$ and the second with $J > 0$. If isospin is included, perhaps four are need, two scalar targets with distinct isospin and two spin-sensitive targets, one with an unpaired proton and the second with an unpaired neutron. However, in fact, Eq. (37) states that experimentalists can derive eight distinct constraints from elastic scattering from the rates they measure, provided they systematically vary the “nuclear physics knobs” by exploring targets with the requisite sensitives to the operators described here. This includes constraints on the velocity-dependent interactions. The previous discussion of Ge and Xe sensitivities provides a nice example of the importance of avoiding the kinds of assumptions that are implicit in the SI/SD approach. If comparable experiments in Ge and Xe were to yield events in the SD channel for Ge but none in Xe, confusion would ensue: Xe has the larger SD cross section (barring fine tuning of the isospin dependence). However, in an analysis that uses the general form of the cross section, the conclusion would be both clear and important: WIMPs must couple dominantly through $\vec{v}^\perp$—that is, through the operator $\vec{\ell}(i)$—rather than $\vec{S}_N(i)$. This conclusion could then be tested in other targets with enhanced sensitivity to $\vec{\ell}(i)$, e.g., NaI. In a similar way, the scalar operators $1_N(i)$ and $\sigma(i) \cdot \vec{\ell}(i)$ can be distinguished, if nuclear targets with the requisite properties are used.

As the experimental community is about to begin a process of “downselecting” to fewer experiments and targets, it is important to keep such possibilities in mind. It is not easy to predict at this time how many experiments will eventually be needed, to fully characterize dark-matter elastic scattering.

A further goal of this paper was to simplify the interface between the particle physics and nuclear physics of dark matter. As the theory ranges from the construction of ultraviolet theories of dark matter to the many-body physics governing ground-state properties of heavy nuclei, the integration required to interpret experiment can be challenging. Our approach has been to divide this problem into three components, with the center block, the WIMP-nucleon EFT that leads to the operators $\mathcal{O}_i$, accessible to both communities. The nonrelativistic EFT framework is sufficiently general that almost every candidate ultraviolet theory will match on the $\mathcal{O}_i$: This becomes the particle theorist's job, one that can be done without any knowledge of the nuclear physics. Similarly, the nuclear physics has been framed in terms of the one-body density matrix, thereby factoring the nuclear physics from the particle physics of dark matter. Generating the density matrices for the targets experimentalists have chosen becomes the nuclear theorist's job, one that can be done without any consideration of specific operators, other than their rank in angular momentum and isospin.

The division of the problem into ultraviolet theory, nonrelativistic EFT, and one-body density matrices may help the particle and nuclear communities work together more productively on dark-matter studies. This division was exploited in the construction of the response function *Mathematica* script described in Appendix B. Given the density matrix (density matrices for several of common nuclear targets are built into the script), many-body matrix elements of the nuclear operators in Eq. (37) are reduced to simple sums over single-particle matrix elements, which the script then evaluates analytically. A nuclear theorist can explore the consequences of a new many-body calculation simply by inserting a new density matrix into the code. Similarly, the script contains the table of matching coefficients given in Table I: A particle theorist can explore the consequences of a new field theory of dark matter by specifying the relativistic form of the dark-matter particle interaction with nucleons. One of the attractive features of the nonrelativistic EFT formulation is as a bridge connecting the particle-physics modeling and nuclear-structure aspects of dark matter.

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APPENDIX A: SOME DETAILS OF THE RESPONSE FUNCTION DERIVATION

The algebraic techniques that lead to Eq. (37) are commonly used in treatments of semileptonic weak interactions. We briefly outline the steps, after first taking note of certain simplifications that are made in the many-body theory to obtain the relatively tractable form of Eq. (37).

1. Treatment of the velocity operator

We take as our WIMP-nucleus interaction the sum over the one-body interactions of the WIMP with the individual nucleons in the nucleus. While this is the usual starting point for treatments of electroweak nuclear reactions, it is an assumption. The nucleon is a composite object held together by the exchange of various mesons, which clearly can have their own interactions with the WIMP. There has been some work on the possible size of two-body corrections to WIMP-nucleus interactions [11,12]. Our feeling at this point is that the uncertainty of the WIMP interaction with nucleons, as embodied in our 14 coefficients c_i, is currently so great that the one-body approximation is appropriate. This sentiment would change were dark-matter interactions discovered, making a detailed understanding WIMP-matter interactions important.

Given the assumption of a one-body interaction, we noted that the Galilean invariance then leads to the replacement

$$\begin{aligned}\vec{v}^{\perp} &\rightarrow \{\vec{v}_\chi - \vec{v}_N(i), i = 1,\ldots,A\}\\ &\equiv \vec{v}_T^{\perp} - \{\vec{v}_N(i), i = 1,\ldots,A-1\},\end{aligned} \tag{A1}$$

where $\vec{v}_\chi$ and $\vec{v}_N$ are the symmetrized velocities $(\vec{v}_{\chi,\mathrm{in}} + \vec{v}_{\chi,\mathrm{out}})/2$ and $(\vec{v}_{N,\mathrm{in}} + \vec{v}_{N,\mathrm{out}})/2$, respectively, and where $\{\vec{v}_N(i)\}$ represents the set of $A-1$ independent symmetrized internucleon Jacobi velocities. The DM particle velocity relative to the nuclear center of mass is a c number,

$$\begin{aligned}\vec{v}_T^{\perp} &= \vec{v}_\chi - \vec{v}_T\\ \vec{v}_T &\equiv \frac{1}{2A}\sum_{i=1}^{A}[\vec{v}_{N,\mathrm{in}}(i) + \vec{v}_{N,\mathrm{out}}(i)],\end{aligned}$$

while the internal nuclear Jacobi velocities $\vec{v}_N(i)$ are operators acting on intrinsic nuclear coordinates. It may be helpful to illustrate this division more explicitly, using one of our interactions, the axial charge operator $\mathcal{O}_7$. We take the simplest example of two nucleons in a nucleus. Then

$$\begin{aligned}\vec{v}^{\perp}\cdot\vec{S}_N &\rightarrow \sum_{i=1}^{2}\frac{1}{2}[\vec{v}_{\chi,\mathrm{in}} + \vec{v}_{\chi,\mathrm{out}} - \vec{v}_{N,\mathrm{in}}(i) - \vec{v}_{N,\mathrm{out}}(i)]\cdot\vec{S}_N(i)\\ &= \frac{1}{2}\left[\vec{v}_{\chi,\mathrm{in}} + \vec{v}_{\chi,\mathrm{out}} - \frac{\vec{v}_{\mathrm{N,in}}(1) + \vec{v}_{\mathrm{N,in}}(2)}{2}\right.\\ &\quad \left. - \frac{\vec{v}_{N,\mathrm{out}}(1) + \vec{v}_{N,\mathrm{out}}(2)}{2}\right]\cdot\sum_{i=1}^{2}\vec{S}_N(i)\\ &\quad - \frac{1}{2}\left[\frac{\vec{v}_{N,\mathrm{in}}(1) - \vec{v}_{N,\mathrm{in}}(2)}{2} + \frac{\vec{v}_{N,\mathrm{out}}(1) - \vec{v}_{N,\mathrm{out}}(2)}{2}\right]\\ &\quad \cdot[\vec{S}_N(1) - \vec{S}_N(2)]\\ &\equiv \vec{v}_T^{\perp}\cdot\sum_{i=1}^{2}\vec{S}_N(i) - \vec{v}_N\cdot[\vec{S}_N(1) - \vec{S}_N(2)]\end{aligned} \tag{A2}$$

yields one term proportional to $\vec{v}_T^\perp$,

$$\vec{v}_T^\perp \equiv \frac{1}{2}(\vec{v}_{\chi,\text{in}} + \vec{v}_{\chi,\text{out}} - \vec{v}_{T,\text{in}} - \vec{v}_{T,\text{out}}), \quad \text{where}$$
$$\vec{v}_{T,\text{in}} \equiv \frac{1}{2}\sum_{i=1}^{2} \vec{v}_{N,\text{in}}(i) \quad \text{and} \tag{A3}$$
$$\vec{v}_{T,\text{out}} \equiv \frac{1}{2}\sum_{i=1}^{2} \vec{v}_{N,\text{out}}(i),$$

and a second term that depends only on the relative internucleon velocity,

$$\vec{v}_N \equiv \frac{1}{2}\left[\frac{\vec{v}_{N,\text{in}}(1) - \vec{v}_{N,\text{in}}(2)}{2} + \frac{\vec{v}_{N,\text{out}}(1) - \vec{v}_{N,\text{out}}(2)}{2}\right], \tag{A4}$$

and is thus separately Galilean invariant. This decomposition can be repeated for A nucleons

$$\sum_{i=1}^{A} \frac{1}{2}[\vec{v}_{\chi,\text{in}} + \vec{v}_{\chi,\text{out}} - \vec{v}_{N,\text{in}}(i) - \vec{v}_{N,\text{out}}(i)] \cdot \vec{S}_N(i)$$
$$= \vec{v}_T^\perp \cdot \sum_{i=1}^{A} \vec{S}_N(i) - \left\{\sum_{i=1}^{A} \frac{1}{2}[\vec{v}_{N,\text{in}}(i) + \vec{v}_{N,\text{out}}(i)] \cdot \vec{S}_N(i)\right\}_{\text{int}},$$

where $\vec{v}_T$ is now the target velocity obtained by averaging over A nucleon velocities. The intrinsic operator on the right can be written in a form that makes the dependence on relative nucleon velocities manifest,

$$\frac{1}{2A}\sum_{i>j=1}^{A} [\vec{S}_N(i) - \vec{S}_N(j)] \cdot \{[\vec{v}_{N,\text{in}}(i) + \vec{v}_{N,\text{out}}(i)] - [\vec{v}_{N,\text{in}}(j) + \vec{v}_{N,\text{out}}(j)]\}, \tag{A5}$$

or, alternatively and trivially, it can be written as the difference of two terms,

$$\sum_{i=1}^{A} \frac{1}{2}[\vec{v}_{N,\text{in}}(i) + \vec{v}_{N,\text{out}}(i)] \cdot \vec{S}_N(i) - \frac{1}{2}[\vec{v}_{T,\text{in}} + \vec{v}_{T,\text{out}}] \cdot \sum_{i=1}^{A} \vec{S}_N(i). \tag{A6}$$

We make two technical observations.

(i) The assumption that the WIMP-nuclear interaction is the sum over the individual WIMP-nucleon interactions leads to two interactions that are separately Galilean invariant, one constructed from $\vec{v}_T^\perp$ and one constructed from the internal relative nucleon velocities. However, these two interactions then have a common coefficient, c_7. In contrast, if one were to construct an effective theory at the nuclear level, operators that are separately invariant would be assigned independent strengths. It would be interesting to explore whether the work of [11,12] on more complicated WIMP-nucleus couplings can be viewed as adding corrections to the one-body formulation that, in fact, make the two operators independent.

(ii) While the nuclear matrix elements in the formulas we derive in the text are intrinsic ones, in fact, almost all calculations of the structure of complex nuclei are performed in overcomplete bases in which the coordinates of all A nucleons appear. If the underlying single-particle basis is the harmonic oscillator and if set of included Slater determinants is appropriately chosen, certain separability properties of the harmonic oscillator allow one to remove the extra degrees of freedom by numerical means, forcing the center of mass into the $1s$ state. Yet still the basis is expressed in terms of nucleon coordinates. Largely for this reason, the intrinsic operator is evaluated using Eq. (A6) with the *further* assumption that the second, more complicated, term in Eq. (A6) can be ignored. This clearly greatly simplifies the calculation, allowing one to evaluate the nuclear matrix element from the one-body density matrix. This kind of approximation—or more correctly, simplification—is used almost universally in nuclear physics, as there is no practical alternative. In schematic models it can be shown that the errors induced are typically $o(1/A)$ and associated with a center-of-mass form factor.

2. Multipole decomposition

In the text leading up to Eq. (35), we formed a WIMP-nucleus interaction by assuming the one-body form, as discussed above, interpreting nucleon momenta as operators acting on the wave functions of the bound nucleon. We stressed that the resulting interaction has precisely the same form as that conventionally used in SI/SD (or $\mathcal{O}_1/\mathcal{O}_4$) analyses, except that a complete set of EFT operators have been included. Equation (35), repeated here,

$$\sum_{\tau=0,1}\left\{l_0^\tau \sum_{i=1}^{A} e^{-i\vec{q}\cdot\vec{x}_i} + l_0^{A\tau}\sum_{i=1}^{A}\frac{1}{2M}\left[-\frac{1}{i}\overleftarrow{\nabla}_i\cdot\vec{\sigma}(i)e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\vec{\sigma}(i)\cdot\frac{1}{i}\overrightarrow{\nabla}_i\right] + \vec{l}_5^\tau\cdot\sum_{i=1}^{A}\vec{\sigma}(i)e^{-i\vec{q}\cdot\vec{x}_i} + \vec{l}_M^\tau\cdot\sum_{i=1}^{A}\frac{1}{2M}\left(-\frac{1}{i}\overleftarrow{\nabla}_i e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\frac{1}{i}\overrightarrow{\nabla}_i\right) + \vec{l}_E^\tau\cdot\sum_{i=1}^{A}\frac{1}{2M}[\overleftarrow{\nabla}_i\times\vec{\sigma}(i)e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\vec{\sigma}(i)\times\overrightarrow{\nabla}_i]\right\}_{\text{int}} t^\tau(i),$$

where the WIMP tensors appearing above are defined in Eq. (36) and contain all of the EFT input in the form of the c_i's, is the starting point for our multipole analysis. The invariant amplitude is the matrix element of this interaction,

$$\mathcal{M}_{\text{nucleus/EFT}} = \sum_{\tau=0,1} \langle j_\chi, M_\chi; j_N M_N| \left\{ l_0^\tau \sum_{i=1}^A e^{-i\vec{q}\cdot\vec{x}_i} + l_0^{A\tau} \sum_{i=1}^A \frac{1}{2M}\left[-\frac{1}{i}\overleftarrow{\nabla}_i \cdot \vec{\sigma}(i) e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\vec{\sigma}(i)\cdot \frac{1}{i}\overrightarrow{\nabla}_i \right] \right.$$
$$+ \vec{l}_5^\tau \cdot \sum_{i=1}^A \vec{\sigma}(i) e^{-i\vec{q}\cdot\vec{x}_i} + \vec{l}_M^\tau \cdot \sum_{i=1}^A \frac{1}{2M}\left(-\frac{1}{i}\overleftarrow{\nabla}_i e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\frac{1}{i}\overrightarrow{\nabla}_i \right)$$
$$\left. + \vec{l}_E^\tau \cdot \sum_{i=1}^A \frac{1}{2M}[\overleftarrow{\nabla}_i \times \vec{\sigma}(i) e^{-i\vec{q}\cdot\vec{x}_i} + e^{-i\vec{q}\cdot\vec{x}_i}\vec{\sigma}(i)\times\overrightarrow{\nabla}_i] \right\}_{\text{int}} t^\tau(i) |j_\chi, M_\chi; j_N M_N\rangle, \quad \text{(A7)}$$

where the subscript "int" instructs one to take the intrinsic part of the operator (that is, the part depending on the internal Jacobi coordinates).

The Hamiltonian can be expressed in terms of nuclear operators carrying good angular momentum and parity and transforming simply under time reversal by carrying out a standard multipole decomposition. For the scalar nuclear terms in Eq. (A7) this involves the expansion of the plane wave in terms of the Bessel spherical harmonics,

$$M_{JM}(q\vec{x}_i) \equiv j_J(qx_i) Y_{JM}(\Omega_{x_i}), \quad \text{(A8)}$$

while for the vector nuclear quantities of the form $\vec{A}e^{i\vec{q}\cdot\vec{x}_i} = \sum_\lambda (-1)^\lambda A_{-\lambda}\hat{e}_\lambda e^{-i\vec{q}\cdot\vec{x}_i}$ one uses Bessel vector spherical harmonics,

$$\vec{M}_{JLM}(q\vec{x}_i) \equiv j_L(qx_i)\vec{Y}_{JLM}(\Omega_{x_i}), \quad \vec{Y}_{JLM}(\Omega_{x_i}) \equiv \sum_{m\lambda} Y_{LM}(\Omega_{x_i})\vec{e}_\lambda \langle Lm1\lambda|(L1)JM\rangle, \quad \text{(A9)}$$

where $\vec{e}_\lambda$ denotes a spherical unit vector and $A_\lambda = \hat{e}_\lambda \cdot \vec{A}$, to project out longitudinal, transverse electric, and transverse magnetic components. After some algebra, $\mathcal{M}_{\text{nucleus/EFT}}$ can be written

$$\sum_{\tau=0,1} \langle j_\chi, M_{\chi f}; j_N M_{Nf}| \left(\sum_{J=0}^\infty \sqrt{4\pi(2J+1)}(-i)^J \left[l_0^\tau M_{J0;\tau}(q) - i l_0^{A\tau}\frac{q}{m_N}\tilde{\Omega}_{J0;\tau}(q)\right] \right.$$
$$+ \sum_{J=1}^\infty \sqrt{2\pi(2J+1)}(-i)^J \sum_{\lambda=\pm1}(-1)^\lambda \left\{ l_{5\lambda}^\tau [\lambda\Sigma_{J-\lambda;\tau}(q) + i\Sigma'_{J-\lambda;\tau}(q)] \right.$$
$$\left. - i\frac{q}{m_N} l_{M\lambda}^\tau [\lambda\Delta_{J-\lambda;\tau}(q) + i\Delta'_{J-\lambda;\tau}(q)] - i\frac{q}{m_N} l_{E\lambda}^\tau [\lambda\tilde{\Phi}_{J-\lambda;\tau}(q) + i\tilde{\Phi}'_{J-\lambda;\tau}(q)] \right\}$$
$$\left. + \sum_{J=0}^\infty \sqrt{4\pi(2J+1)}(-i)^J \left[i l_{50}^\tau \Sigma''_{J0;\tau}(q) + \frac{q}{m_N} l_{M0}^\tau \tilde{\Delta}''_{J0;\tau}(q) + \frac{q}{m_N} l_{E0}^\tau \Phi''_{J0;\tau}(q)\right]\right) |j_\chi, M_{\chi i}; j_N M_{Ni}\rangle, \quad \text{(A10)}$$

where we have defined the operators as

$$O_{JM;\tau}(q) \equiv \sum_{i=1}^A O_{JM}(q\vec{x}_i) t^\tau(i). \quad \text{(A11)}$$

The 11 operators appearing above correspond to the charge multipoles of the vector charge (accompanying l_0) and axial-vector charge (l_0^A) operators, and the longitudinal, transverse electric, and transverse magnetic projections of the axial-vector spin current (accompanying $\vec{l}_5$), vector convection current (accompanying $\vec{l}_M$), and vector spin-velocity current (accompanying $\vec{l}_E$) operators. As transverse multipoles must carry at least one unit of angular momentum, the multipole sums in those cases begin with $J = 1$.

In elastic transitions the contributing multipoles are severely restricted by the known approximate good parity and CP of nuclear ground states, as detailed in Table II. Five of the operators (those not defined in the body of this paper) are eliminated entirely; in other cases, only the even or odd multipoles can satisfy the combined parity and CP requirements. Thus, we obtain the simpler expression

$$\mathcal{M}_{\text{nucleus/EFT}}^{\text{elastic}} = \sum_{\tau=0,1} \langle j_\chi, M_{\chi f}; j_N M_{Nf}| \left\{ \sum_{J=0,2,\ldots}^\infty \sqrt{4\pi(2J+1)}(-i)^J \left[l_0^\tau M_{J0;\tau}(q) + \frac{q}{m_N} l_{E0}^\tau \Phi''_{J0;\tau}(q)\right] \right.$$
$$+ \sum_{J=1,3,\ldots}^\infty \sqrt{2\pi(2J+1)}(-i)^J \sum_{\lambda=\pm1}(-1)^\lambda \left[i l_{5\lambda}^\tau \Sigma'_{J-\lambda;\tau}(q) - i\frac{q}{m_N} l_{M\lambda}^\tau \lambda\Delta_{J-\lambda;\tau}(q)\right]$$

TABLE II. The parity-time reversal transformation properties for the 11 operators arising in DM particle scattering off nuclei. The nearly exact parity and CP of nuclear ground states restricts the contributing multipoles in elastic scattering to those that transform under parity and CP as even-even (E-E): These are the even multipoles of the vector charge operator M_{JM} and of the longitudinal and transverse electric projections of the spin-velocity current Φ''_{JM} and $\tilde{\Phi}'_{JM}$ and the odd multipoles of the longitudinal and transverse electric projections of the spin current Σ''_{JM} and Σ'_{JM} and of the transverse magnetic projection of the convection current Δ_{JM}.

Projection	Charge/current	Operator	P,CPproperties Even J	P,CPproperties Odd J
Charge	Vector charge	M_{JM}	E-E	O-O
Charge	Axial-vector charge	$\tilde{\Omega}_{JM}$	O-E	E-O
Longitudinal	Spin current	Σ''_{JM}	O-O	E-E
Transverse magnetic	Spin current	Σ_{JM}	E-O	O-E
Transverse electric	Spin current	Σ'_{JM}	O-O	E-E
Longitudinal	Convection current	$\tilde{\Delta}''_{JM}$	E-O	O-E
Transverse magnetic	Convection current	Δ_{JM}	O-O	E-E
Transverse electric	Convection current	Δ'_{JM}	E-O	O-E
Longitudinal	Spin-velocity current	Φ''_{JM}	E-E	O-O
Transverse magnetic	Spin-velocity current	$\tilde{\Phi}_{JM}$	O-E	E-O
Transverse electric	Spin-velocity current	$\tilde{\Phi}'_{JM}$	E-E	O-O

$$+\sum_{J=2,4,\ldots}^{\infty}\sqrt{2\pi(2J+1)}(-i)^J\sum_{\lambda=\pm1}(-1)^{\lambda}\left[\frac{q}{m_N}l^{\tau}_{E\lambda}\tilde{\Phi}'_{J-\lambda;\tau}(q)\right]$$
$$\left.+\sum_{J=1,3,\ldots}^{\infty}\sqrt{4\pi(2J+1)}(-i)^J\left[il^{\tau}_{50}\Sigma''_{J0;\tau}(q)\right]\right\}|j_\chi,M_{\chi i};j_N M_{Ni}\rangle. \quad \text{(A12)}$$

This expression involves only the six multipole operators of Eq. (39).

The Wigner-Eckart theorem can be used to reduce the nuclear matrix elements. Then after forming $|\mathcal{M}|^2$, averaging over initial nuclear spins, summing over final nuclear spins, and using the orthogonality condition imposed by the two three-j symbols obtained in the reduction, one obtains

$$\frac{1}{2j_N+1}\sum_{M_{Ni},M_{Nf}}\left|\langle j_\chi M_{\chi f};j_N M_{Nf}|\mathcal{M}^{\text{elastic}}_{\text{nucleus/EFT}}|j_\chi M_{\chi i};j_N M_{Ni}\rangle\right|^2$$
$$=\frac{4\pi}{2J_i+1}\sum_{\tau=0,1}\sum_{\tau'=0,1}\left(\sum_{J=0,2,\ldots}^{\infty}\left\{\langle l^{\tau}_0\rangle\langle l^{\tau'}_0\rangle^*\langle j_N||M_{J;\tau}(q)||j_N\rangle\langle j_N||M_{J;\tau'}(q)||j_N\rangle\right.\right.$$
$$+\frac{\vec{q}}{m_N}\cdot\langle\vec{l}^{\tau}_E\rangle\frac{\vec{q}}{m_N}\cdot\langle\vec{l}^{\tau'}_E\rangle^*\langle j_N||\Phi''_{J;\tau}(q)||j_N\rangle\langle j_N||\Phi''_{J;\tau'}(q)||j_N\rangle$$
$$\left.+\frac{2\vec{q}}{m_N}\cdot\text{Re}\left[\langle\vec{l}^{\tau}_E\rangle\langle l^{\tau'}_0\rangle^*\right]\langle j_N||\Phi''_{J;\tau}(q)||j_N\rangle\langle j_N||M_{J;\tau'}(q)||j_N\rangle\right\}$$
$$+\sum_{J=2,4,\ldots}^{\infty}\frac{1}{2}\left(\frac{q^2}{m_N^2}\langle\vec{l}^{\tau}_E\rangle\cdot\langle\vec{l}^{\tau'}_E\rangle^*-\frac{\vec{q}}{m_N}\cdot\langle\vec{l}^{\tau}_E\rangle\frac{\vec{q}}{m_N}\cdot\langle\vec{l}^{\tau'}_E\rangle^*\right)\langle j_N||\tilde{\Phi}'_{J;\tau}(q)||j_N\rangle\langle j_N||\tilde{\Phi}'_{J;\tau'}(q)||j_N\rangle$$
$$+\sum_{J=1,3,\ldots}^{\infty}\left\{\hat{q}\cdot\langle\vec{l}^{\tau}_5\rangle\hat{q}\cdot\langle\vec{l}^{\tau'}_5\rangle^*\langle j_N||\Sigma''_{J;\tau}(q)||j_N\rangle\langle j_N||\Sigma''_{J;\tau'}(q)||j_N\rangle\right.$$
$$+\frac{1}{2}\left(\langle\vec{l}^{\tau}_5\rangle\cdot\langle\vec{l}^{\tau'}_5\rangle^*-\hat{q}\cdot\langle\vec{l}^{\tau}_5\rangle\hat{q}\cdot\langle\vec{l}^{\tau'}_5\rangle^*\right)\langle j_N||\Sigma'_{J;\tau}(q)||j_N\rangle\langle j_N||\Sigma'_{J;\tau'}(q)||j_N\rangle$$
$$+\frac{1}{2}\left(\frac{q^2}{m_N^2}\langle\vec{l}^{\tau}_M\rangle\cdot\langle\vec{l}^{\tau'}_M\rangle^*-\frac{\vec{q}}{m_N}\cdot\langle\vec{l}^{\tau}_M\rangle\frac{\vec{q}}{m_N}\cdot\langle\vec{l}^{\tau'}_M\rangle^*\right)\langle j_N||\Delta_{J;\tau}(q)||j_N\rangle\langle j_N||\Delta_{J;\tau'}(q)||j_N\rangle$$
$$\left.\left.+\frac{\vec{q}}{m_N}\cdot\text{Re}\left[i\langle\vec{l}^{\tau}_M\rangle\times\langle\vec{l}^{\tau'}_5\rangle^*\right]\langle j_N||\Delta_{J;\tau}(q)||j_N\rangle\langle j_N||\Sigma'_{J;\tau'}(q)||j_N\rangle\right\}\right), \quad \text{(A13)}$$

where we have used the shorthand for the WIMP matrix elements

$$\langle l \rangle \equiv \langle j_\chi M_{\chi f} | l | j_\chi M_{\chi i} \rangle. \tag{A14}$$

Note that while our original multipole decomposition was done with a z axis aligned along $\vec{q}$, this result is now frame independent as it is expressed entirely in terms of scalar products.

Finally, we average over initial WIMP spins and sum over final spins, as in the nuclear case. The WIMP tensors involve combinations of 1 and $\vec{S}_\chi$. As we sum over all magnetic quantum numbers, the only surviving terms in the bilinear products of the WIMP tensors must transform as spin scalars, and thus as 1 or as $\vec{S}_\chi^2$. The constant term yields 1. All cross terms linear in $\vec{S}_\chi$ must vanish. The spin terms must be proportional to $j_\chi(j_\chi+1)$. The associated coefficients are easily calculated for the various products,

$$\frac{1}{2j_\chi+1}\sum_{m_{\chi_i} m_{\chi_f}} \langle j_\chi m_{\chi_i} | \left\{ \begin{array}{c} \vec{S}_\chi | j_\chi m_{\chi_f}\rangle \cdot \langle j_\chi m_{\chi_f} | \vec{S}_\chi \\ \vec{A}\cdot\vec{S}_\chi | j_\chi m_{\chi_f}\rangle \cdot \langle j_\chi m_{\chi_f} | \vec{B}\cdot\vec{S}_\chi \\ \vec{A}\times\vec{S}_\chi | j_\chi m_{\chi_f}\rangle \cdot \langle j_\chi m_{\chi_f} | \vec{B}\times\vec{S}_\chi \\ \vec{A}\times\vec{S}_\chi | j_\chi m_{\chi_f}\rangle \cdot \langle j_\chi m_{\chi_f} | \vec{S}_\chi \end{array} \right\} | j_\chi m_{\chi_i}\rangle = \left\{ \begin{array}{c} 1 \\ \vec{A}\cdot\vec{B}/3 \\ 2\vec{A}\cdot\vec{B}/3 \\ 0 \end{array} \right\} j_\chi(j_\chi+1). \tag{A15}$$

The results are further simplified because the resulting scalars $\vec{A}\cdot\vec{B}$ often involve longitudinal and transverse quantities or $\vec{q}\cdot\vec{v}_T^\perp$, which vanish.

Executing the associated algebra yields the final result given in Eqs. (37) and (38). The transition probability is expressed as a product of WIMP and nuclear responses functions, where the former isolates the particle physics in functions that are bilinear in the EFT coefficients, the c_i's.

3. Generalizing the exchange

Our EFT approach has focused on interactions between the WIMP and nucleus mediated by a heavy exchange, so that the interaction is pointlike. However, nothing in the treatment of the WIMP or nuclear vertices depends on this assumption. We believe the adaptation of this code for cases in which the exchange is mediated by a photon or other light particle would be very simple. This would, of course, require one to add the needed momentum-dependent propagator to the code. Once that line is added, however, we see no reason that subsequent integrations over phase space would present any difficulties: Indeed, the operator formalism we employ here is the common formalism for both electron scattering and semileptonic weak interactions. The exchange in the former is a photon, while the latter is treated as a four-fermion interaction analogous to the WIMP case.

APPENDIX B: THE *Mathematica* SCRIPT: DOCUMENTATION

The formalism presented in this paper, with its factorization cross sections into products of WIMP and nuclear responses, is the basis for the *Mathematica* script available as Supplemental Material [13]. The script was constructed so that experimental groups would be able to conduct model independent analyses of their experiments using the EFT framework. We have integrated the particle and nuclear physics in ways that should make the code useful to nuclear structure and particle theorists as well, as described in previous sections.

In this section, which also serves as a `readme` file for the program, we discuss the usage of the program itself.

1. Initialization

Our *Mathematica* package, along with all of the associated documentation, can be found at `http://www.ocf.berkeley.edu/nanand/software/dmformfactor/`. To initialize the package, either put `dmformfactor.m` in your directory for *Mathematica* packages and run

```
<<`dmformfactor
```

or initialize the package file itself from its source directory. For example,

```
<<``/Users/me/myfiles/dmformfactor.m''
```

2. Summary of functions

To compute the WIMP response functions $R_i^{\tau\tau'}(\vec{v}_T^{\perp 2}, \frac{\vec{q}^{\,2}}{m_N^2})$, the user must first call functions setting the dark-matter mass and spin as well as the coefficients of the effective Lagrangian. To compute the nuclear response functions $W_i[(qb/2)^2]$, the user must specify the Z and A of the isotope. The density matrices and the oscillator parameter b needed in the calculation of the W_i are set internally in the script, though there are options to override the internal values. The nuclear ground-state spin and isospin (the script assumes exact isospin, consistent with an input density matrix that is doubly reduced; see text) are also set internally, once Z and A are input.

(i) `SetJChi` and `SetMChi`: These set the dark-matter spin and mass, respectively. Simply call
`SetJChi[j]`
and
`SetMChi[m]`
to set the dark-matter spin to `j` and the dark-matter mass to `m`. The unit `GeV` is recognized by the script; for example, calling `SetMChi[10 GeV]` sets the dark-matter mass to 10 GeV.

(ii) `SetIsotope[Z,A,bFM, filename]`
This sets the nuclear-physics input, including the charge `Z` and atomic number `A` of the isotope, the file

for the density matrices that the user wants to use, and the oscillator parameter b[fm] (that is, b in femtometers). If the users elects to use the default density matrices (which are available for ^{19}F, ^{23}Na, ^{70}Ge, ^{72}Ge, ^{73}Ge, ^{74}Ge, ^{76}Ge, ^{127}I, ^{128}Xe, ^{129}Xe, ^{130}Xe, ^{131}Xe, ^{132}Xe, ^{134}Xe, and ^{136}Xe), then simply take `filename` to be "default" (note that one must still specify the correct `Z` and `A` for the isotope of interest). Otherwise, users must provide their own density matrix file, to be read in by the program. Similarly, entering "default" for b will employ the approximate formula $b[\text{fm}] = \sqrt{41.467/(45A^{-1/3} - 25A^{-2/3})}$. To use another value of b[fm], enter a numerical value. The nuclear mass is set to Am_N.

(iii) `SetCoeffsNonrel[i,value,isospin]`
This sets the coefficients c_i of the EFT operators $\mathcal{O}_i$. The script allows the user to set values for $\{c_1,c_3,c_4,\ldots,c_{15}\}$; note that c_2 is excluded, for reasons discussed in the text. We have chosen a normalization such that the coefficients c_i all have dimensions (Energy)$^{-2}$;[1] to compensate for this, the dimensionless user input for `value` is multiplied by m_V^{-2}, with $m_V \equiv 246.2$ GeV.
The coefficients carry an isospin index α that can be specified in one of two ways, as a coupling to protons and neutrons, $\{c_i^p,c_i^n\}$, in which case the associated operator is

$$\left[c_i^p\frac{1+\tau_3}{2} + c_i^n\frac{1-\tau_3}{2}\right]\mathcal{O}_i, \qquad \text{(B1)}$$

or as a coupling to isospin, $\{c_i^0,c_i^1\}$, where the associated operator is

$$\left[c_i^0 + c_i^1\tau_3\right]\mathcal{O}_i. \qquad \text{(B2)}$$

For the former, the input should be "n" for neutrons and "p" for protons. For example,
`SetCoeffsNonrel[4,12.3, ‘‘p’’]`
whereas for the latter it should be 0 for isoscalar and 1 for isovector. All coefficients are set to 0 by default when the package is initialized. `SetCoeffsNonrel` will change only the coefficient specified and will leave all other coefficients unchanged. So, for example, if one initializes the package and calls `SetCoeffsNonrel[4,12.3, 0]`, then c_4^p and c_4^n will both be 6.15, with all other coefficients vanishing. If one then calls `SetCoeffsNonrel[4,3.3,‘‘p’’]`, then c_4^p will be set to 3.3, but c_4^n will not change and will still be 6.15. Thus, by making two calls, an arbitrary combination of $\{c_4^p,c_4^n\}$ or equivalently $\{c_4^0,c_4^1\}$ can be set.

(iv) `SetCoeffsRel[i,value,isospin]`
These functions are similar to `SetCoeffsNonrel`, except that they set the coefficients d_j of the 20 covariant interactions $\mathcal{L}_{\text{int}}^j$ defined in Table I. The coefficients d_j are dimensionless, by inserting appropriate powers of the user-defined scale m_M, set by the user function `SetMM`. This scale is set by default to be $m_M = m_V \equiv 246.2$ GeV. We adopt a convention where the spinors in $\mathcal{L}_{\text{int}}^j$ are defined as normalized to unity: With this convention a nonrelativistic reduction of the $\mathcal{L}_{\text{int}}^j$ in the second column of Table I would give the results in the fourth column. [As noted in the paper, we use a spinor normalization of $2m$ in our derivations, but extract the factor of $4m_\chi m_N$ to maintain the definition above.]
`SetCoeffsNonrel` and `SetCoeffsRel` cannot be used together. By default, the package assumes you will use `SetCoeffsNonrel`. The first time the user calls `SetCoeffsRel`, the package will first reset all coefficients back to zero before calling `SetCoeffsRel`, after which point it will act normally. A subsequent call to `SetCoeffsNonrel` will similarly first reset all coefficients back to zero and then revert to nonrelativistic mode.
Because the relativistic operators implicitly assume spin-$\frac{1}{2}$ WIMPs, any call to `SetCoeffsRel` automatically sets $j_\chi = 1/2$.

(v) `SetMM[mM]`
Set the fiducial scale m_M for the relativistic coefficients d_i.

(vi) `ZeroCoeffs[]`
Calling `ZeroCoeffs[]` simply resets all operators coefficients to zero.

(vii) `ResponseNuclear[y,i,tau,tau2]`
This function prints out any of the eight nuclear response functions $W_i^{\tau\tau_2}(y)$. This involves a folding of the single-particle matrix elements with the density matrices. The results are printed as analytic functions in the dimensionless variable $y = (qb/2)^2$. The i run from 1 to 8, according to (1) W_M, (2) $W_{\Sigma''}$, (3) $W_{\Sigma'}$, (4) $W_{\Phi''}$, (5) $W_{\tilde{\Phi}'}$, (6) W_Δ, (7) $W_{M\Phi''}$, and (8) $W_{\Sigma'\Delta}$.

(viii) `TransitionProbability[v,q(,IfRel)]`
This is the main user function. It first prints out the Lagrangian that is being used.
Second, it folds the $W_i^{\tau\tau'}(y)$ and $R_i^{\tau\tau'}(\vec{v}_T^{\perp 2},\frac{\vec{q}^{\,2}}{m_N^2})$ to form

$$P_{\text{tot}} = \frac{1}{2j_\chi+1}\frac{1}{2j_N+1}\sum_{\text{spins}}|\mathcal{M}|^2_{\text{nucleus}-\text{HO/EFT}}. \qquad \text{(B3)}$$

It then evaluates the transition probability for the numerical values of b and m_N. As b is in fm, the substitution is $y = [qb/(2\hbar c)]^2 \sim [qb/2(0.197\ \text{GeV fm})]^2$. As m_N is input in GeV, this evaluates Eq. (40) as a function `TransitionProbability[vsq,q]`, where `q` is in GeV. This function can be printed out or plotted numerically.
The conventional relativistic normalization of the amplitude differs from the nonrelativistic normalization by a factor of $1/(4m_\chi m_T)$. Because the

[1] Note that this convention for the c_i's differs from that in Ref. [8].

conventional relativistic normalization is commonly used and produces a dimensionless value for $|\mathcal{M}|^2$, we also provide an optional argument `IfRel`, which if set to `True` will output (B3) with the relativistic normalization convention [that is, it will multiply by $(4m_\chi m_T)^2$ to produce a dimensionless transition probability]. By default, it is set to `False`.

(ix) `DiffCrossSection[ERkeV,v]`
From the transition probability $P_{\rm tot}$, one can immediately obtain the differential cross section per recoil energy:

$$\frac{d\sigma}{dE_R} = \frac{m_T}{2\pi v^2} P_{\rm tot}. \tag{B4}$$

The function `DiffCrossSection[ERkeV,v]` takes as arguments the recoil energy in units of keV and the velocity of the incoming DM particle in the laboratory frame. It first prints out the Lagrangian being used and then outputs the differential cross-section $\frac{d\sigma}{dE_R}$.

(x) `ApproxTotalCrossSection[v]`
From the differential cross section $\frac{d\sigma}{dE_R}$, one can also obtain the total cross section as a function of v by integrating over recoil energies. In general, this depends on energy thresholds and, written in closed form, is a complicated analytic function owing to the exponential damping factor e^{-2y} in the response functions, so for precise values it is simplest to do the energy integration numerically. However, for approximate results we can consider the limit of small nuclear harmonic oscillator parameter b, in which case the exponential factor e^{-2y} can be neglected. For fixed v, the integration over E_R from zero up to the kinematic threshold $E_{R,\rm max} = 2\frac{\mu_T^2 v^2}{m_T}$ can be performed analytically. The function `ApproxTotalCrossSection[v]` takes as argument the velocity v of the incoming DM particle in the laboratory frame and, after printing out the Lagrangian being used, outputs this approximate total cross section $\sigma(v)$.

(xi) `EventRate[`N_T,ρ_χ,q,v_e,v_0(,$v_{\rm esc}$)`]`
One can determine the total detector event rate (per unit time per unit detector mass per unit recoil energy) in terms of the transition probability $P_{\rm tot}$. One simply multiplies $P_{\rm tot}$ by the appropriate prefactor and integrates over the halo velocity distribution, as follows:

$$\frac{dR_D}{dE_R} = N_T \frac{\rho_\chi m_T}{2\pi m_\chi} \left\langle \frac{1}{v} P_{\rm tot}(v^2,q^2) \right\rangle. \tag{B5}$$

Here, $\langle \cdots \rangle$ indicates averaging over the halo velocity distribution. N_T is the number of target nuclei per detector mass, ρ_χ is the local dark-matter density, m_χ is the dark-matter mass, and m_N is the nucleon mass. In general, the halo average integral should include a lower bound on the magnitude of the velocity at $v_{\rm min}$, which is $v_{\rm min} = \frac{q}{2\mu_T}$ for elastic scattering:

$$\langle h(q,\vec{v}) \rangle \equiv \int_{v_{\rm min}(q)}^{\infty} v^2 dv \int d^2\Omega f_v(\vec{v} + \vec{v}_e) h(q,\vec{v}). \tag{B6}$$

The vector $\vec{v}_e$ is Earth's velocity in the galactic rest frame. While there has been much work recently on understanding theoretical constraints on the halo distribution from N-body simulations and from general considerations of dynamics, little is known by direct observation and there are still large uncertainties. A very simple approximation that suffices for general considerations is to take a Maxwell-Boltzmann distribution,

$$f_v(\vec{v}) = \frac{1}{\pi^{3/2} v_0^3} e^{-v^2/v_0^2}, \tag{B7}$$

where v_0 is roughly 220 km/s, about the rms velocity of the visible matter distribution (though N-body simulations suggest that the dark-matter distribution may be shallower, and a larger v_0 may be more appropriate). The function `EventRate[`q,b,v_e,v_0`]` evaluates the event rate $\frac{dR_D}{dE_R}$ assuming this Maxwell-Boltzmann distribution as default. A cutoff Maxwell-Boltzmann distribution is also implemented as an option, in which case

$$f_v(\vec{v}) \propto (e^{-v^2/v_0^2} - e^{v_{\rm esc}^2/v_0^2})\Theta\big(v_{\rm esc}^2 - \vec{v}^2\big), \tag{B8}$$

where $v_{\rm esc}$ is the escape velocity, and the subtraction above is included to make the distribution shut down smoothly. In this case, $v_{\rm esc}$ should be included as an optional argument to EventRate; if it is not included, it is set to a default value of $12v_0$ (which is essentially $v_{\rm esc} = \infty$).

(xii) `SetHALO[halo]`
This sets the halo distribution used. The variable *halo* can be set either to "MB", in which case the Maxwell-Boltzmann distribution is used, or "MBcutoff", in which case the cut-off Maxwell-Boltzmann distribution is used. It is set to "MB" by default.

(xiii) `SetHelm[UseHelm]`
Calling `SetHelm[True]` sets the structure function for the density operator M_J to be given by the Helm form factor, rather than by the structure function obtained from the density matrix. `SetHelm[False]` implements the structure function based on the density matrix, which is the default setting.

3. Examples

A full example for the transition probability would look like the following:

```
<<''/Users/me/mypackages/dmformfactor.m'';
SetJChi[1/2]
SetMChi[50 GeV]
F19filename=''default'';
bFM=''default'';
SetIsotope[9, 19, bFM, F19filename]
```

```
SetCoeffsNonrel[3, 3.1, ‘‘p’’]
TransitionProbability[v,qGeV]
TransitionProbability[v,qGeV,True]
```

To additionally calculate the event rate $\frac{dR_D}{dE_R}$ in a Maxwell-Boltzmann halo velocity distribution, one can call

```
mNucleon=0.938 GeV;
NT=1/(19 mNucleon);
Centimeter=(10^13 Femtometer);
rhoDM=0.3 GeV/Centimeter^3;
ve=232 KilometerPerSecond;
v0=220 KilometerPerSecond;
EventRate[NT,rhoDM,qGeV,ve,v0]
```

For a cutoff Maxwell-Boltzmann halo, an escape velocity must also be specified:

```
mNucleon=0.938 GeV;
NT=1/(19 mNucleon);
Centimeter=(10^13 Femtometer);
rhoDM=0.3 GeV/Centimeter^3;
ve=232 KilometerPerSecond;
v0=220 KilometerPerSecond;
vesc=550 KilometerPerSecond;
SetHalo[‘‘MBcutoff’’];
EventRate[NT,rhoDM,qGeV,ve,v0,vesc]
```

Finally, to get a quick estimate of the experimental bound from the 225 live day run of XENON100, one can use the standard SI isoscalar interaction for a generic isotope of xenon, taking ^{131}Xe, for instance. Taking into account efficiencies, the total effective exposure is approximately 2500 kg days. A relativistic operator coefficient of $2f_p/\text{GeV}^2$ with $f_p = 4\times10^{-9}$ predicts only a couple of events and so should be close to the upper limit of their allowed cross section:

```
mNucleon=0.938 GeV;
NT=1/(131 mNucleon);
Centimeter=(10^13 Femtometer);
rhoDM=0.3 GeV/Centimeter^3;
SetMChi[150 GeV]
ve=232 KilometerPerSecond;
v0=220 KilometerPerSecond;
vesc=550 KilometerPerSecond;
SetHALO[‘‘MBcutoff’’];
Xe131filename=‘‘default’’;
bFM=‘‘default’’;
SetIsotope[54, 131, bFM, Xe131filename]
SetCoeffsRel[1,2fp,0]
myrate[qGeV_]=(2500 KilogramDay)
EventRate[NT,rhoDM,qGeV,ve,v0,vesc];
fp=2.4*10^(-4);
NIntegrate[myrate[qGeV] GeV*(qGeV GeV/(131
mNucleon)),qGeV,0,10]
```

The final line of output should be 2.06 for the value of the integral, which gives the predicted number of events. The factor $\frac{q}{131m_N} = \frac{q}{m_T}$ inside the integral is from the change of variables from dE_R to dq, because $E_R = q^2/2m_T$. In this example, the WIMP is sufficiently heavy that the exact low-energy threshold changes the prediction by less than a factor of two, so to get a rough estimate we have just integrated down to zero energy. Finally, we can look at which nucleon scattering cross section corresponds to $f_p = 2.4\times10^{-4}$,

$$\sigma_p = \frac{(4m_N m_T f_p/m_V^2)^2}{16\pi(m_N+m_T)^2} = 1.7\times10^{-45}\ \text{cm}^2, \qquad \text{(B9)}$$

which agrees to within a factor of a few with the published upper bound on σ_p from the XENON100 collaboration [14]. A more accurate calculation of the bound would include, among other corrections, the exact energy thresholds in the momentum transfer integral, an average over the year as Earth's velocity changes, a sum over different isotopes according to their natural abundance, and a more precise treatment of energy-dependent efficiencies.

4. Density matrix syntax

If one calls `SetIsotope[Z,A, filename]` with a custom density matrix, the input density matrix file must contain the reduced density matrix elements $\Psi^{J,T}(|\alpha|,|\beta|)$ to be used. The in and out states $|\alpha|$ and $|\beta|$ should be specified by their principle quantum number N and their total angular momentum j. See Ref. [15] for more details. The format of the file for each projection onto operators of spin J and isospin J should be as follows:

$$\begin{array}{llllll}
\texttt{ONE-BODY DENSITY MATRIX} & & & \cdots & 2\text{J0} = 2J, \quad \cdots \quad 2T \\
\cdots N_{\text{in}}^1 & 2j_{\text{in}}^1 & N_{\text{out}}^1 & 2j_{\text{out}}^1 & \Psi^{J,T}\left(\{N_{\text{in}}^1, j_{\text{in}}^1\};\{N_{\text{out}}^1, j_{\text{out}}^1\}\right) \\
 & \vdots & & \vdots & \vdots \\
\cdots N_{\text{in}}^n & 2j_{\text{in}}^n & N_{\text{out}}^n & 2j_{\text{out}}^n & \Psi^{J,T}\left(\{N_{\text{in}}^n, j_{\text{in}}^n\};\{N_{\text{out}}^n, j_{\text{out}}^n\}\right)
\end{array}$$

Dots "$\cdots$" indicate places where the code will simply ignore what appears there; the routines reading in the input are searching for regular expressions that match the above syntax. Consequently, additional lines in the file that are not of the above form will also be ignored. This is probably clearest to follow by seeing an explicit example. For instance, the density matrix for ^{19}F is shown below. The density matrices for ^{19}F, ^{23}Na, ^{70}Ge, ^{72}Ge,^{73}Ge,^{74}Ge,^{76}Ge, ^{127}I, ^{128}Xe,^{129}Xe,^{130}Xe,^{131}Xe,^{132}Xe,^{134}Xe, and ^{136}Xe are already built into the program and no external file is needed.

```
INITIAL STATE CHARGE CONJ SYM = 0 TIME
REVERSAL SYM = 0
FINAL STATE CHARGE CONJ SYM = 0 TIME
REVERSAL SYM = 0
-23.88003 -23.88003
```

```
  ONE-BODY DENSITY MATRIX FOR 2JF = 1 2TF =
1 2JI = 1 2TI = 1 2JO = 0 TO =, 0
  NBRA 2*JBRA NKET 2*JKET VALUE
  0 1 0 1 4.00000000
  1 1 1 1 4.00000000
  1 3 1 3 5.65685425
  2 1 2 1 1.22525930
  2 3 2 3 0.20366116
  2 5 2 5 0.85835832
  ONE-BODY DENSITY MATRIX FOR 2JF = 1 2TF =
1 2JI = 1 2TI = 1 2JO = 0 TO =, 2
  NBRA 2*JBRA NKET 2*JKET VALUE
  2 1 2 1 0.36984837
  2 3 2 3 0.04794379
  2 5 2 5 0.32467225
  ONE-BODY DENSITY MATRIX FOR 2JF = 1 2TF =
1 2JI = 1 2TI = 1 2JO = 2 TO =, 0
  NBRA 2*JBRA NKET 2*JKET VALUE
  2 1 2 1 0.44514263
  2 3 2 1 -0.01197751
  2 1 2 3 0.01197751
  2 3 2 3 -0.05428837
  2 5 2 3 -0.12172578
  2 3 2 5 0.12172578
  2 5 2 5 0.12280637
  ONE-BODY DENSITY MATRIX FOR 2JF = 1 2TF =
1 2JI = 1 2TI = 1 2JO = 2 TO =, 2
  NBRA 2*JBRA NKET 2*JKET VALUE
  2 1 2 1 -0.40780345
  2 3 2 1 -0.01278520
  2 1 2 3 0.01278520
  2 3 2 3 0.01209672
  2 5 2 3 0.10547489
  2 3 2 5 -0.10547489
  2 5 2 5 -0.24110544
```


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