Covariant spectator theory of *np* scattering: Deuteron magnetic moment

Franz Gross*

Thomas Jefferson National Accelerator Facility, Newport News, Virginia 23606, USA and College of William and Mary, Williamsburg, Virginia 23185, USA

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The deuteron magnetic moment is calculated using two model wave functions obtained from 2007 high-precision fits to *np* scattering data. Included in the calculation are a new class of isoscalar *np* interaction currents, which are automatically generated by the nuclear force model used in these fits. After normalizing the wave functions, nearly identical predictions are obtained: model WJC-1, with larger relativistic P-state components, gives 0.863(2), while model WJC-2 with very small P-state components gives 0.864(2). These are about 1% larger than the measured value of the moment, 0.857 n.m., giving a new CST prediction for the size of the $\rho\pi\gamma$ exchange, and other purely transverse interaction currents that are largely unconstrained by the nuclear dynamics. The physical significance of these results is discussed, and general formulas for the deuteron form factors, expressed in terms of deuteron wave functions and a new class of interaction current wave functions, are given.

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I. INTRODUCTION, SUMMARY, AND CONCLUSIONS

A. Background

This work is the second in a series of four planned papers (the first, Ref. [1], accompanies this paper) that will present the fourth-generation calculation of the deuteron form factors using what is now called the covariant spectator theory (CST) [2–4].

This new generation of calculations are required by the new fits to the 2007 *np* data base [5] obtained using the CST with a one boson exchange (OBE) kernel. It was found that a high-precision fit (one with $\chi^2/\text{datum} \simeq 1$) was possible only if the $NN\sigma_0$ vertices associated with the exchange of a scalar-isoscalar meson σ_0 included momentum-dependent terms in the form

$$\Lambda^{\sigma_0}(p,p') = g_{\sigma_0}\mathbf{1} - \nu_{\sigma_0}[\Theta(p) + \Theta(p')], \tag{1.1}$$

where ν_{σ_0} is a new parameter determined by fitting the NN scattering data, p and p' are the four-momenta of the outgoing and incoming nucleons, respectively, and the Θ are projection operators

$$\Theta(p) = \frac{m - \not{p}}{2m}, \tag{1.2}$$

which are nonzero for off-shell particles, and hence are a feature of Bethe-Salpeter or CST equations.

Two high-precision models were found with somewhat different properties. Model WJC-1, designed to give the best fit possible, has 27 parameters, $\chi^2/\text{datum} \simeq 1.06$, and a large $\nu_{\sigma_0} = -15.2$. Model WJC-2, designed to give a excellent fit with as few parameters as possible, has only 15 parameters, $\chi^2/\text{datum} \simeq 1.12$, and a smaller $\nu_{\sigma_0} = -2.6$. Both models also predict the correct triton binding energy. The deuteron wave functions predicted by both of these models [6] have small P-state components of relativistic origin, and the normalization of the wave functions includes a term coming from the energy dependence of the kernel, which contributes -5.5% for WJC-1 and -2.3% for WJC-2.

This momentum dependence of the kernel implies the existence of a new class of *np* isoscalar interaction currents that will contribute to the electromagnetic interaction of the deuteron. These currents were fixed in Ref. [1], and this paper completes the derivation started there by decomposing the deuteron current into three independent form factors [7,8] and expressing each of these form factors in terms of integrals over bilinear products of eight invariant functions, or alternatively, the two familiar nonrelativistic S- and D-state wave functions, u and w, the two small P-state components, v_t and v_s, and four additional amplitudes, referred to collectively as χ_ℓ, that appear when both particles are off-shell [9,10]. This paper also discusses the contributions of the interaction currents to the charge and the magnetic moment. Calculation of the quadrupole moment and the dependence of the form factors on the momentum transfer of the scattered electron, Q^2, will be discussed in the remaining two papers, under preparation.

B. Organization of the paper

This paper is long and detailed, so the principal results and conclusions have been extracted and summarized in this section. The interaction current makes significant contributions to the wave function normalization (the charge) and these are reviewed in some detail in Sec. I C. Then, one of the principal new results of this paper, the calculation of the deuteron magnetic moment including the contributions from the interaction current, are presented in Sec. I D. Conclusions are given in Sec. I E.

The remainder of the paper includes four more sections and four Appendixes where many of the details are presented. The two-body current from which all of the results are derived is introduced in Sec. II. The entity that contains the relativistic structure of the deuteron is the *dnp* vertex function with one nucleon on shell. In Sec. II this vertex function is written as

*gross@jlab.org

a sum of products of scalar invariant functions multiplied by covariant Dirac spin operators. This expansion in terms of invariants was first introduced by Blanckenbecler and Cook in 1960 [9], but we use the notation of Ref. [10]. Appendix A shows how to expand these invariant functions in terms of the CST deuteron wave functions u, w, v_t, and v_s (previously reported in the literature), and $\chi_\ell = \{z_0^{--}, z_1^{--}, z_0^{-+}, z_1^{-+}\}$, the negative ρ-spin helicity amplitudes for particle 1. The χ_ℓ are not zero even when both particles are on shell and are needed for a complete calculation of the magnetic moment.

Next, Sec. III describes how the deuteron form factors are extracted from the helicity amplitudes of the deuteron current, and general formulas for the form factors, valid to all Q^2, are assembled. The final results, Eqs. (3.28) and (3.36), give the form factors as a sum of products of the nucleon form factors $F_i(Q^2)$ (with $i = 1,2,3$, with F_3 a new nucleon form factor that contributes to the nucleon current only when both the incoming and outgoing nucleons are off shell) multiplied by body form factors expressed as integrals over traces of bilinear products of invariant functions from which the *dnp* vertex is constructed. The interaction current contributions are conveniently expressed in terms of two new types of wave functions, $\Psi^{(2)}$ and $\widehat{\Psi}$. Explicit formulas for the 18 independent traces that appear in the final results are given in Appendix B. The formulas are manifestly covariant; once the rest frame wave functions are known these formulas reduce the calculation of the deuteron form factors at any Q^2 to quadratures. These formulas will be used to calculate the form factors in the fourth paper of this series, and are one of the principal new results of this paper.

Finally, the last two sections discuss how the charge (Sec. IV) and magnetic moment (Sec. V) are built up from individual contributions from the wave function components, the off-shell nucleon current, and the interaction current. These sections assemble details given in Appendixes C and D. This work is summarized in the following Secs. I C and I D.

C. Charge and normalization

The normalization condition ensures that the charge of the deuteron is one. There are many ways to write this condition; Sec. IV expresses the contributions from the interaction currents in terms of two new wave functions, $\Psi^{(2)}$, a wave function that depends only on the Θ contributions from off-shell particle 2, and $\widehat{\Psi}$, a wave function with *both* particles off shell, which, because of the interaction current contributions, reduces to Ψ when particle 1 is on shell. In this language, the normalization condition (charge) can be expressed as a sum of contributions from the components of Ψ, $\widehat{\Psi}$, and $\Psi^{(2)}$:

$$1 = \int_0^\infty k^2 dk \sum_{\ell=1}^{4} \{1 + a_\ell(k)\} z_\ell^2 + \left\langle \frac{\partial \widetilde{V}}{\partial P_0} \right\rangle, \qquad (1.3)$$

where the notation $z_\ell = z_\ell(k)$ is used generically to denote the wave functions u, w, v_t, or v_s [not to be confused with the helicity amplitudes denoted by $z_\ell^{\rho_1\rho_2}$ and given in Eq. (A26)] with ℓ denoting the angular momentum of the state (so that $z_0 = u$, $z_2 = w$, and $z_1 = v_t$ or v_s). In Sec. IV it is shown how the derivative of the reduced kernel can be expressed in terms

TABLE I. Contributions to the normalization sum (1.3) for model WJC-1. All entries are rounded to three decimal places; all totals are subject to round-off error. Note that the total of columns four and five equals the total in column six, confirming (1.4).

z_ℓ	z_ℓ^2	$a_\ell z_\ell^2$	$z_\ell \widehat{z}_\ell$	$z_\ell z_\ell^{(2)}$	$\langle \frac{\partial \tilde V}{\partial P_0} \rangle$
u	0.974	0.014	−0.035	−0.020	−0.054
w	0.077	0.022	−0.017	−0.002	−0.019
v_t	0.001	−0.003	−0.007	−0.001	−0.007
v_s	0.002	−0.008	0.001	−0.001	0.000
total	1.055	0.025	−0.057	−0.023	−0.080

of products involving the new wave functions

$$\left\langle \frac{\partial \widetilde{V}}{\partial P_0} \right\rangle = \int_0^\infty k^2 dk \sum_{\ell=1}^{4} \left\{ \left\langle z_\ell z_\ell^{(2)} \right\rangle + \langle z_\ell \widehat{z}_\ell' \rangle \right\} \qquad (1.4)$$

and the contributions from the derivative of the strong form factor contribute terms proportional to $a_\ell(k)$, with

$$a_\ell(k) = \begin{cases} -4a(p^2)(E_k - m_d)\delta_k & \ell = 0,2 \\ +4a(p^2)(E_k - m_d)m_d & \ell = 1 \end{cases}, \qquad (1.5)$$

where $a(p^2)$ was defined in Eq. (3.25) with $p^2 = m_d^2 + m^2 - 2m_d E_k$ here, and $\delta_k = 2E_k - m_d$. The budget for these contributions is shown in Tables I and II, where all contributions have been rounded to three decimal places.

Note that, except for the P-state contributions from model WJC-2, all of these contributions are important at the level of 0.001. If the magnetic moment is to be calculated to this accuracy (a goal of this paper), then all of these terms must be included.

D. Magnetic moment

The algebraic expression for the magnetic moment is considerably more complicated than the simple form (1.3) for the charge. While it is possible to calculate the exact result from the formulas given in the appendixes, this will not give much insight into the underlying physics. The goal in this paper is to simplify these formulas, retaining all terms that contribute to 1–2 parts per 1000.

Table III will be used to guide the calculation. It suggests that sufficient accuracy is obtained if the coefficients of all terms but those involving products of the leading wave functions, namely u and w, are retained to leading order in the small parameter $\delta_E = (E_k - m)/E_k$ (a few of the other terms

TABLE II. Contributions to the normalization sum (1.3) for model WJC-2 (see caption to Table I).

z_ℓ	z_ℓ^2	$a_\ell z_\ell^2$	$z_\ell \widehat{z}_\ell$	$z_\ell z_\ell^{(2)}$	$\langle \frac{\partial \tilde V}{\partial P_0} \rangle$
u	0.957	0.007	−0.022	−0.012	−0.034
w	0.065	0.011	−0.010	0.001	−0.009
v_t	0.000	0.000	0.002	0.000	0.002
v_s	0.000	0.000	0.000	0.000	0.000
total	1.023	0.018	−0.030	−0.011	−0.041

TABLE III. Integrated products of wave functions for model WJC-1 with the largest P states. Entries above the diagonal are the products $z_\ell z_{\ell'}$; those along the diagonal and below are products weighted by $(E_k - m)/E_k$.

	u	w	v_t	v_s
u	0.007	0.094	−0.004	–
w	−0.001	0.006	−0.009	−0.010
v_t	–	−0.001	–	0.001
v_s	0.001	−0.001	–	–

are as large as 0.001, but neglecting all of these corrections is not expected to change the results significantly, and all terms of higher order in δ_E are negligible). Guided by these results the formulas for the magnetic moment are simplified.

If the deuteron is treated as a nonrelativistic superposition of S and D states, normalized to unity so that

$$1 = \int_0^\infty k^2\, dk(u^2 + w^2) = P_S + P_D, \tag{1.6}$$

then the well-known result for the magnetic moment is

$$\mu_d = \mu_s + \tfrac{3}{4}(1 - 2\mu_s)P_D = \mu_s + \mu_{\rm NR}, \tag{1.7}$$

where $\mu_s = 0.880$ is the isoscalar nucleon magnetic moment. Inserting the measured deuteron magnetic moment, 0.857 (in nuclear magnetons) gives the famous prediction of 4% for the deuteron D state, a result too low for most modern models.

The CST results for the leading contributions to the magnetic moment (with an estimated accuracy of ± 0.002) were derived in Sec. V and Appendix D. After some simplification, the results can be written [see Eq. (5.2)]

$$\mu_d = \mu_s + \Delta\mu_d, \tag{1.8}$$

where $\Delta\mu_d$ is the sum of eight different types of corrections given in Eqs. (5.3) and (5.6) and listed in Tables IV and V. The physical origin of each of these eight corrections is summarized in Table IV, and their numerical size for each of the models WJC-1 and WJC-2 are summarized in Table V. A running sum of the correction terms is plotted in Fig. 1.

From these results I conclude that the CST is not able to explain the magnetic moment precisely. Within the theoretical errors, the missing contribution is about $\delta\mu_d \simeq -0.006 \pm 0.002$, less than 1% of the magnetic moment and closer to the the experimental value than the nonrelativistic D-state contribution (assuming the $P_d \simeq 5-6\%$ found in most fits). This small difference is a new prediction for the total size of the famous $\rho\pi\gamma$ exchange current that has been extensively studied [11–15] and other purely transverse contributions not constrained by the np dynamics. Predictions for these contributions will be the subject of a future paper.

TABLE IV. Physical origin of the eight different types of corrections that contribute to the magnetic moment.

term	physical origin
$\mu_{\rm NR}$	nonrelativistic D-state contribution
μ_{Rc}	relativistic corrections to S, D terms
$\mu_{h'}$	dependence on the strong form factor, h
μ_{V_2}	interaction currents: off-shell particle 2
μ_{V_1}	interaction currents: on-shell particle 1
$\mu_{\rm int}$	interference of P-states with S- and D-states
μ_P	P-state squared terms
μ_χ	P-state and negative ρ-spin z_ℓ^{--} interference

TABLE V. Contributions to the magnetic moment from the eight different types of corrections discussed in the text. To get the correct experimental value, these corrections must equal −0.023.

	WJC-1		WJC-2	
	u,w only	all	u,w only	all
$\mu_{\rm NR}$	−0.044	−0.044	−0.037	−0.037
μ_{Rc}	0.021	0.021	0.009	0.009
$\mu_{h'}$	−0.010	−0.009	−0.005	−0.005
μ_{V_2}	0.001	0.004	−0.001	−0.009
μ_{V_1}	0.013	0.006	0.008	0.008
$\mu_{\rm int}$	–	0.016	–	0.001
μ_P	–	−0.004	–	0.000
μ_χ	–	−0.007	–	0.000
total	−0.019	−0.017	−0.026	−0.016

E. Conclusions

The calculation of the magnetic moment given in this paper is the first precise consequence of the interaction current derived in Ref. [1]. Using this interaction current, and the deuteron wave functions obtained from the precision CST fits to the np scattering data, model WJC-1 predicts the magnetic moment to be 0.863(2), while model WJC-2 predicts it to be 0.864(2), where the theoretical error is an estimate of the size of the many small terms omitted from the calculation. Taking the value given by the most precise model (WJC-1) and increasing the error to ± 0.003 to allow for the model dependence, my overall prediction is 0.863(3). This result is larger than the experimental value by 0.006(3), implying that the total size of the many missing purely transverse interaction currents unconstrained by the np dynamics (including the $\rho\pi\gamma$ and $\omega\sigma\gamma$ currents) is much smaller than previously estimated. Either these currents are individually quite small, or they tend to cancel when added together. The CST prediction for the magnetic moment, obtained without any adjustable parameters, is within 1% of the experimental value.

The prediction is almost the same for both models, even though the two models have quite different properties. This is illustrated in Fig. 1, which shows the running sum of the eight contributions, added in the order listed in Tables IV and V. For both models the NR correction (1.7) is too small and the relativistic corrections (μ_{Rc}) bring the moment up to equal to, or close to its experimental value. Both of these effects depend on the S and D states only. Then the contributions from the derivative of the strong nucleon form factor, proportional to $a(p^2) = d\log(h)/dp^2$ [see Eq. (3.25)], reduce the moment again, giving an almost identical value near −0.032 for the two models. The two interaction current contributions, V_2

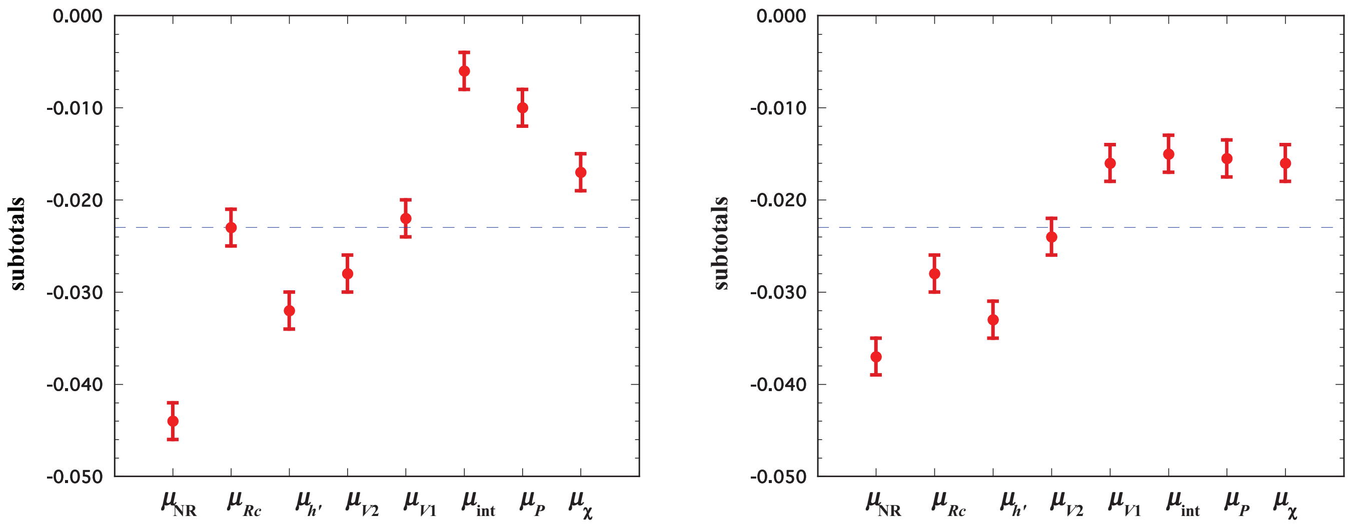

FIG. 1. (Color online) Running sum of the corrections to the magnetic moment, in the order that they are listed in Tables IV and V. The dashed line is $\Delta\mu_d = -0.023$, the correction needed to give the experimental value. The error bars are ± 0.002, an estimate of the size of the terms missing from the approximation of Eq. (1.8). Model WJC-1 (left) and model WJC-2 (right).

(arising from the momentum dependence associated with the Θ attached to the off-shell particle 2) and V_1 (arising from the momentum dependence associated with the Θ attached to particle 1, which only contributes when *both* particles are off shell), both give positive contributions, pushing the total back up to a value equal, or close to the experimental value. These interaction current contributions contain significant contributions from the P states as well as the S and D states. Perhaps the most surprising result comes from the last three terms ($\mu_{\rm int}, \mu_P$, and μ_χ), all of which are zero if the P states v_t and v_s are zero. In model WJC-2 where the P states are very small, these terms add very little, but their contributions are significant for model WJC-1, where they give large canceling effects just sufficient to produce the same total prediction as is obtained for model WJC-2. Note that even the term μ_χ, which is an interference between the P states and the negative ρ-spin contributions from particle 1 [which contribute only to the diagrams (B) of Fig. 2 when both particles are off shell] is important to obtaining agreement between the two models. As shown in Appendix C, these terms cancel in the charge, but make a small but significant contribution to the model WJC-1 prediction for the magnetic moment.

A full comparison of my results with the many other calculations in the literature will be postponed until I have completed my calculation of the quadrupole moment and the form factors. Here I note only that in a recent work based on χEFT [16,17] the deuteron magnetic moment is used to constrain the low-energy constants of χEFT, and hence the magnetic moment itself is not predicted.

We now turn to the derivation of these results, as already outlined in Sec. I B above.

II. WAVE AND VERTEX FUNCTIONS

In the CST, the two-body current is given by the five diagrams shown in Fig. 2 (completely equivalent to the four shown in Fig. 1 of Ref. [1]). These include the interaction current contributions derived in Ref. [1], expressed in terms of the the effective wave functions $\Psi^{(2)}$ and the subtracted vertex functions $\widehat{\Gamma}$ (directly related to $\widehat{\Psi}$) with two particles off shell. Although these diagrams are written for particle 2 off shell, the symmetry of the NN interaction is built into the formalism from the start and they are completely equivalent to an alternative set with particle 1 off shell. At the conclusion of Ref. [1] it was shown that these diagrams can be written as a trace over the product of covariant wave functions (or vertex functions) of the initial and final deuteron, and a current operator describing the interaction of the virtual photon with the off-shell nucleon. In this section the covariant wave and vertex functions will be discussed in detail.

A. General definitions

The covariant wave function of the deuteron is defined in terms of the covariant dnp vertex function, $\mathcal{G}$,

$$\Psi^{\lambda_d}_{\alpha\beta}(k,P) = \left(\Psi^{\lambda_d}_0\right)_{\alpha\beta'}(k,P)\mathcal{C}_{\beta'\beta}$$
$$= S_{\alpha\alpha'}(p)\mathcal{G}^{\lambda_d}_{\alpha'\beta}(k,P), \tag{2.1}$$

where $\mathcal{C}$ is the Dirac charge conjugation matrix, S is the bare nucleon propagator (with the factor of $-i$ removed)

$$S(p) = \frac{1}{m - \not{p}} \tag{2.2}$$

and, for an incoming deuteron of four-momentum P and polarization four-vector ξ, $\mathcal{G}$ is written

$$\mathcal{G}^{\lambda_d}_{\alpha\beta}(k,P) = (\Gamma_\nu \mathcal{C})_{\alpha\beta}(k,P)\xi^\nu_{\lambda_d}(P)$$
$$= \Gamma^{\lambda_d}_{\alpha\beta'}(k,P)\mathcal{C}_{\beta'\beta}, \tag{2.3}$$

with k the four-momentum of particle 1 (with Dirac index β), and $p = P - k$ the four-momentum of particle 2 (with Dirac index α). Care must be taken to distinguish Ψ (which includes the charge conjugation matrix) from Ψ_0 (which does

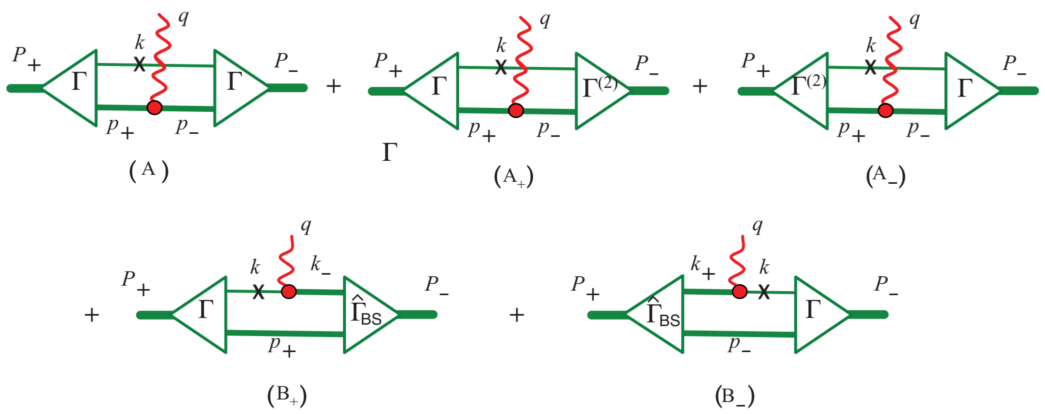

FIG. 2. (Color online) Diagrammatic representation of the two-body current operator in the covariant spectator theory using vertex functions with particle 2 off shell. The interaction current contributions are contained in diagrams ($A_\pm$) and parts of the (B) diagrams, as discussed in the text. Off-shell nucleon lines are thicker than on-shell lines, which are also labeled with an $\times$. Diagrams (A) and ($A_\pm$) describe the interaction of the photon with particle 2, allowing particle 1 to be on shell in both the initial and final state. Diagrams ($B_\pm$) describe the interaction of the photon with particle 1, so that both particles must be off shell in either the initial state (diagram B_+) or in the final state (diagram B_-).

not). These wave (or vertex) functions satisfy the bound-state CST equation

$$S^{-1}_{\alpha\alpha'}(p)\Psi^{\lambda_d}_{\alpha'\beta}(k,P) = -\int_{k'} \overline{V}_{\beta\gamma,\alpha\alpha'}(k,k';P)\Psi_{\alpha'\gamma'}(k',P)\Lambda^T_{\gamma'\gamma}(k), \tag{2.4}$$

where $\overline{V}$ is the symmetrized kernel (introduced in Ref. [5]), the positive energy Dirac projection operator is

$$\Lambda_{\gamma\gamma'}(k) = \frac{(m+\not{k})_{\gamma\gamma'}}{2m} = \sum_{\lambda'} u_\gamma(\mathbf{k},\lambda')\bar{u}_{\gamma'}(\mathbf{k},\lambda'), \tag{2.5}$$

with the Dirac spinors $u = u^+$ discussed in Appendix A, and the volume integral is

$$\int_k = \int \frac{d^3k}{(2\pi)^3}\frac{m}{E_k}. \tag{2.6}$$

Here particle 1, with four-momentum $k = \{E_k, \mathbf{k}\}$, is on shell (so that $E_k = \sqrt{m^2 + \mathbf{k}^2}$).

In the OBE models that are the basis of the work reported here, the strong form factors at the meson-NN vertices are products of strong form factors for each particle entering or leaving the vertex. The strong form factor $h(p)$ [where $h(p)$ is a function of p^2] associated with each external nucleon line can be factored out of the NN scattering kernel, leading to

$$\overline{V}(k,k';P) = h(k)h(p)\widetilde{V}(k,k';P)h(k')h(p'), \tag{2.7}$$

where $\widetilde{V}$ is the reduced kernel, and we recall that, for both primed and unprimed variables, $p = P - k$. If a particle with momentum k is on shell, so that $k^2 = m^2$, the strong form factor is defined so that $h(k) = 1$. Note that the expression (2.7) for the kernel is written allowing for the possibility that any (or all four) of the particles could be off shell.

The next step in the computation of the form factors is to express the wave and vertex functions in terms of scalar invariant functions, so that when the traces (3.28) and (3.33) are computed, the result will be a sum of bilinear products of these scalar functions multiplied by covariant kinematical factors. The result is manifestly covariant, and the effect of boosting the incoming and outgoing states is easily accounted for by correctly shifting the arguments of the invariant functions.

B. Expansion of the wave or vertex functions

When particle 1 is on-shell, the covariant dnp deuteron nucleon vertex function defined in Eq. (2.3) (with the charge conjugation matrix removed) can be expanded into four independent Dirac invariants

$$\Gamma^\mu(k,P) = F\gamma^\mu + \frac{G}{m}k^\mu - 2\Theta(p)\left[H\gamma^\mu + \frac{I}{m}k^\mu\right], \tag{2.8}$$

where k is the four-momentum of the on-shell particle 1, so that $k^2 = m^2$, $p = P - k$ is the four-momentum of the off-shell particle 2, and $\Theta(p)$ is the negative energy projection operator of particle 2 [recall Eq. (1.2)]. The scalar functions $F,G,H,$ and I are all functions of p^2, the only free scalar variable. Note that

$$\overline{\Gamma}^\mu(k,P) = \gamma^0[\Gamma^\mu(k,P)]^\dagger\gamma^0 = F\gamma^\mu + \frac{G}{m}k^\mu - \left[H\gamma^\mu + \frac{I}{m}k^\mu\right]2\Theta(p). \tag{2.9}$$

It is sometimes convenient to work directly with wave function Ψ^μ_0 defined in Eq. (2.1) (with the charge conjugation matrix removed), and the related amplitude $\overline{\Psi}^\mu_0$,

$$\Psi^\mu_0(k,P) \equiv S(p)\Gamma^\mu(k,P) = A\gamma^\mu + \frac{B}{m}k^\mu - 2\Theta(p)\left[C\gamma^\mu + \frac{D}{m}k^\mu\right], \tag{2.10}$$

where $S(p)$ is the undressed propagator of the off-shell particle, and

$$(m^2 - p^2)\,C = m\,F \quad (m^2 - p^2)\,D = m\,G$$
$$m\,A = 2m\,C - H \quad m\,B = 2m\,D - I. \tag{2.11}$$

The $F,G,H,$ and I are related to the deuteron wave functions, as discussed in Appendix A and many previous references [6,10,18,19]. When the spectator is on shell, these invariants depend only on p^2, the mass of the off-shell particle.

C. Bethe-Salpeter vertex functions

The (B) diagrams of Fig. 2 require Bethe-Salpeter (BS) vertex functions with both particles off shell. These can be expanded in terms of invariant functions that depend on the two invariant variables p^2 and $k^2 \neq m^2$. To describe these, the expansion (2.8) is generalized

$$\begin{aligned}\Gamma^\mu_{BS}(k,P) &= F\gamma^\mu + \frac{G}{m}k^\mu - 2\Theta(p)\left[H\gamma^\mu + \frac{I}{m}k^\mu\right]\\ &\quad - \left[K_1\gamma^\mu + \frac{K_2}{m}k^\mu\right]2\Theta(-k)\\ &\quad + 4\Theta(p)\left[K_3\gamma^\mu + \frac{K_4}{m}k^\mu\right]\Theta(-k)\\ &= \Gamma^\mu(k,P) - \Gamma^\mu_{\rm off}(k,P)\,2\Theta(-k),\end{aligned} \tag{2.12}$$

where the invariants in Γ^μ (F,G,H,I) are distinguished from the old only by their arguments (two instead of one). The appearance of the operator on the right of the last terms, $\Theta(-k)$ [instead of $\Theta(k)$, as might have been expected], comes from moving the charge conjugation matrix past the projection operator of particle 1: $\mathcal{C}\,\Theta^T(k) = \Theta(-k)\,\mathcal{C}$. Particle interchange symmetry relates H and I to K_1 and K_2, but we will ignore this constraint for now; it is a numerical feature of the solutions for the matrix elements.

As it turns out all eight invariant functions are present in Γ^μ, even when particle 1 is on shell. (A proof of this can be found in Appendix B of the original longer version (v1) of the present paper in the preprint archive [20].) The $\Gamma^\mu_{\rm off}$ part of the vertex function constructed from the four invariant functions K_i is not zero when $k^2 = m^2$. However, because of the presence of the projection operator $\Theta(-k)$ it does not contribute to diagrams where both $k^2 = m^2$ *and* the vertex function is contracted with an on-shell projection operator (or the on-shell u spinor). Thus it makes no contribution to the (A) diagrams, but a full understanding of the content of the (B) diagrams requires that it be included.

In the rest frame, when both particles are off shell, the covariant variables are related to $\mathbf{k}^2$, the square magnitude of the spectator three-momentum, and k_0, the off-shell energy of particle 1, through the relations

$$\begin{aligned}p^2 &= (P-k)^2 = k^2 + m_d(m_d - 2k_0)\\ k^2 &= m^{*2} \equiv k_0^2 - \mathbf{k}^2.\end{aligned} \tag{2.13}$$

Solving these relations for k_0 and $\mathbf{k}^2$ gives

$$\begin{aligned}k_0 &\to R_0 \equiv \frac{P\cdot k}{m_d} = \frac{m_d^2 + m^{*2} - p^2}{2m_d}\\ \mathbf{k}^2 &\to R^2 \equiv k_0^2 - m^{*2} = \frac{(P\cdot k)^2}{m_d^2} - m^{*2}.\end{aligned} \tag{2.14}$$

These relations provide the unique covariant generalization of the rest frame variables k_0 and $\mathbf{k}^2$ (denoted by R_0 and R^2). Stated more precisely, if the spectator associated with a deuteron with four-momentum P has energy k_0 and a squared three-momentum $\mathbf{k}^2$, then the equivalent rest frame values of these quantities are R_0 and R^2. Note that R_0 and $R = \sqrt{R^2}$ are quite different quantities.

It is instructive to derive these relations by a direct boost from the moving frame to the rest frame. To do this, consider (for definiteness) that the moving deuteron has momentum $\{D_0,\mathbf{0}_\perp,Q/2\}$, with

$$D_0 = \sqrt{m_d^2 + \tfrac{1}{4}Q^2}. \tag{2.15}$$

Then if the spectator has four-momentum $k = \{k_0,\mathbf{k}_\perp,k_z\}$, in the rest frame these values are

$$\begin{aligned}R_0 &= \frac{1}{m_d}\left(D_0k_0 - \frac{1}{2}Qk_z\right) = \frac{P\cdot k}{m_d}\\ R_z &= \frac{1}{m_d}\left(D_0k_z - \frac{1}{2}Qk_0\right)\end{aligned} \tag{2.16}$$

with the transverse momentum, $\mathbf{k}_\perp$, unchanged. The first of the two relations (2.14) emerges immediately, and to obtain the second simply compute the square of the three-momentum in the rest frame

$$\begin{aligned}R^2 &= \mathbf{k}_\perp^2 + R_z^2\\ &= \mathbf{k}^2 + \eta^2\big(k_z^2 + k_0^2\big) - \sqrt{\eta}\,k_zk_0D_0\\ &= \frac{(P\cdot k)^2}{m_d^2} + \mathbf{k}^2 - k_0^2\end{aligned} \tag{2.17}$$

in agreement with (2.14). It is also easy to use (2.16) to confirm that $k^2 = m^{*2}$ is covariant by computing

$$\begin{aligned}R_0^2 - R_z^2 - \mathbf{k}_\perp^2 &= \frac{1}{m_d^2}\left(D_0^2 - \frac{Q^2}{4}\right)\big(k_0^2 - k_z^2\big) - \mathbf{k}_\perp^2\\ &= k_0^2 - \mathbf{k}^2.\end{aligned} \tag{2.18}$$

A word of caution: depending on the context, k is sometimes used to denote either the magnitude of the three-momentum (i.e., R) or the four-momentum (and, when the square of the four-momentum is involved, m^{*2} will sometimes be used instead of k^2). Earlier discussions of deuteron wave functions were restricted to cases when particle 1 was on shell, and were evaluated in the rest frame [6,10,19] or used wave functions boosted from the rest frame [21], where there was no need to make a distinction between R and k.

All of the invariants defined in (2.12) depend on the two variables R and R_0, so that, for example $F = F(R,R_0)$. However, because of the cancellation between the contributions from the (B) diagram and the $\langle V_1\rangle$ interaction currents, discussed in Ref. [1], the effective BS vertex function of interest reduces to the CST function when particle 1 is on shell. The frame-independent way to express this on-shell condition is to introduce E_R, where

$$E_R \equiv \sqrt{m^2 + R^2} \tag{2.19}$$

is the straightforward generalization of E_k. Note that $E_R = E_k$ in the rest frame. Using this notation, the invariant functions satisfy the condition

$$Z(R,E_R) = Z(R), \tag{2.20}$$

where Z is a generic name for any of the eight invariant functions.

III. DEUTERON FORM FACTORS

A. Definitions of the form factors

The most general form of the covariant deuteron electromagnetic current can be expressed in terms of three deuteron form factors

$$\langle d\lambda|J^\mu|d'\,\lambda'\rangle = -2D^\mu\left\{G_1\,\xi^*_\lambda\cdot\xi'_{\lambda'} - G_3\frac{(\xi^*_\lambda\cdot q)(\xi'_{\lambda'}\cdot q)}{2m_d^2}\right\} - G_M[\xi'^{\mu}_{\lambda'}(\xi^*_\lambda\cdot q) - \xi^{*\mu}_\lambda(\xi'_{\lambda'}\cdot q)], \tag{3.1}$$

where the form factors G_1, G_3, and $G_M = G_2$ are all functions of the square of the momentum transfer $q = P_+ - P_-$, with $Q^2 = -q^2$, $D^\mu = \frac{1}{2}(P_+ + P_-)^\mu$, and P_- (P_+) the four-momentum of the incoming (outgoing) deuterons, and $\xi'_{\lambda'}$ (ξ_λ) are the four-vector polarizations of the incoming (outgoing) deuterons with helicities λ' (λ). The polarization vectors satisfy the well-known constraints

$$\begin{aligned} P_+\cdot\xi_\lambda &= P_-\cdot\xi'_{\lambda'} = 0\\ \xi^*_\lambda\cdot\xi_\rho &= -\delta_{\lambda\rho}\\ \xi'^*_{\lambda'}\cdot\xi'_{\rho'} &= -\delta_{\lambda'\rho'}. \end{aligned} \tag{3.2}$$

This notation agrees with that used in Ref. [8], except that now λ denotes the helicity of the outgoing deuteron and λ' the helicity of the incoming deuteron.

The form factors G_1 and G_3 are usually replaced by the charge and quadrupole form factors, defined by

$$\begin{aligned} G_C &= G_1 + \tfrac{2}{3}\eta G_Q\\ G_Q &= G_1 + (1+\eta)G_3 - G_M \end{aligned} \tag{3.3}$$

with $\eta = Q^2/4m_d^2$. At $Q^2 = 0$, the three form factors G_C, G_Q, and G_M give the charge, quadrupole moment, and magnetic moment of the deuteron. Since one unit of the proton charge has been removed from the current, the correct normalizations are

$$\begin{aligned} G_C(0) &= 1\\ G_M(0) &= 2m_d\mu_d = G_2(0)\\ G_Q(0) &= m_d^2 Q_d = G_3(0) + 1 - \mu_d. \end{aligned} \tag{3.4}$$

The form factors can be related to helicity amplitudes. Working in the Breit frame, and choosing the momenta to be

$$\begin{aligned} P^\mu_\pm &= \{D_0,0,0,\pm\tfrac{1}{2}Q\}\\ q^\mu &= \{0,0,0,Q\}, \end{aligned} \tag{3.5}$$

where D_0 was defined in Eq. (2.15), the helicity four-vector polarizations for the deuteron and the photon are

$$\begin{aligned} \xi^\mu_\lambda &= \begin{cases} \{0,\mp 1,-i,0\}/\sqrt{2} & \lambda = \pm 1\\ \{\tfrac{1}{2}Q,0,0,D_0\}/m_d & \lambda = 0 \end{cases}\\ \xi'^\mu_{\lambda'} &= \begin{cases} \{0,\pm 1,-i,0\}/\sqrt{2} & \lambda' = \pm 1\\ \{-\tfrac{1}{2}Q,0,0,D_0\}/m_d & \lambda' = 0 \end{cases}\\ \epsilon^\mu_{\lambda_\gamma} &= \begin{cases} \{0,\mp 1,-i,0\}/\sqrt{2} & \lambda_\gamma = \pm 1\\ \{1,0,0,0\} & \lambda_\gamma = 0, \end{cases} \end{aligned} \tag{3.6}$$

where the polarization vectors for the incoming deuteron (treated as particle 2 in the conventions of Jacob and Wick) have been obtained from those of the outgoing deuteron (particle 1 of Jacob and Wick) by a rotation through π about the $\hat{y}$ axis, multiplied by a phase

$$\xi'_{\lambda'} = (-1)^{1+\lambda}R_y(\pi)\xi_{\lambda'}. \tag{3.7}$$

These definitions agree with Refs. [8] and [22] [except that in Eq. (2.7) of Ref. [22] the $\xi^\mu(\pm 1)$ refer to the spin direction and not the helicity and there is a typo in the expression for $\xi^\mu(0)$].

We will denote the most general helicity amplitude by

$$G^{\lambda_\gamma}_{\lambda\lambda'} \equiv \langle P_+\,\lambda|J_\mu|P_-\,\lambda'\rangle\epsilon^\mu_{\lambda_\gamma}. \tag{3.8}$$

Under rotation by π about the $\hat{z}$ axis, all of the helicity four-vectors (3.6), represented generically by the vector ε, transform as

$$\varepsilon_\lambda = (-1)^\lambda\varepsilon_\lambda, \tag{3.9}$$

giving the condition

$$\lambda_\gamma + \lambda + \lambda' = 0. \tag{3.10}$$

(This relation must be interpreted as arithmetic modulo 2, and can be written in a variety of ways.) In addition, the amplitudes are related to each other by Y-parity conservation (parity followed by rotation π about the $\hat{y}$ axis), which insures that

$$G^{\lambda_\gamma}_{\lambda\lambda'} = G^{-\lambda_\gamma}_{-\lambda-\lambda'}. \tag{3.11}$$

Hence it is sufficient to omit discussion to those nine amplitudes with $\lambda_\gamma = -1$, and of the three amplitudes $G^0_{\lambda-}$ and G^0_{-0}. Of the remaining 14, Eq. (3.10) gives

$$\begin{aligned} &G^+_{++} = G^+_{--} = G^+_{00} = G^+_{-+} = G^+_{+-} = 0\\ &G^0_{+0} = G^0_{0+} = 0, \end{aligned} \tag{3.12}$$

leaving seven possible amplitudes.

A conserved current must have the form (3.1), and direct computation using this gives four further relations

$$\begin{aligned} &G^+_{+0} = G^+_{0+}\\ &G^0_{++} = G^+_{-0} = G^+_{0-} = 0 \end{aligned} \tag{3.13}$$

leaving only the three independent amplitudes G^0_{00}, G^0_{+-} and $G^+_{+0} = G^+_{0+}$. (Note that Eq. (20) of Ref. [8] states incorrectly that G^+_{0-} and G^-_{0+} are nonzero.)

While the sum of all of the individual contributions to the form factors is constructed to give a conserved current, individual terms may not, and for this reason the average of G^+_{+0} and G^-_{0-} (equal to G^+_{0+}), which enjoys a desirable symmetry property discussed below, is used to extract the magnetic contributions from individual terms. The form factors are then extracted from the following combination of helicity amplitudes

$$\begin{aligned} \mathcal{J}_1 &\equiv G^0_{00} = 2D_0\left(G_C + \tfrac{4}{3}\eta\,G_Q\right)\\ \mathcal{J}_2 &\equiv G^0_{+-} = 2D_0\left(G_C - \tfrac{2}{3}\eta\,G_Q\right)\\ \mathcal{J}_3 &\equiv \tfrac{1}{2}(G^+_{+0} + G^-_{0-}) = 2D_0\sqrt{\eta}\,G_M, \end{aligned} \tag{3.14}$$

where $\mathcal{J}_n$ (with $n = 1,2,3$) is a convenient notation for the helicity amplitudes. To calculate the deuteron form factors, it therefore sufficient to calculate the three independent matrix elements (3.14) of the two-body current operator.

The remaining parts of this section assemble the general formulas for the three independent helicity amplitudes, $\mathcal{J}_n$ starting from the results of Eqs. (3.30) and (3.31) of Ref. [1]. From these amplitudes the charge, quadrupole, and magnetic form factors are obtained. Explicit expressions for the charge will be given in Sec. IV and for the magnetic moment in Sec. V. Results for the quadrupole moment will be given in a subsequent paper.

B. Off-shell nucleon current

Following the method of Riska and Gross [23], a conserved two-nucleon current can be constructed [21] using the dressed single-nucleon off-shell current

$$\begin{aligned} j^\mu(p,p') &= h(p)h(p')j_R^\mu(p,p') \\ &= e_0\, f_0(p',p)\mathcal{F}_1^\mu + e_0\, g_0(p',p)\Theta(p')\mathcal{F}_3^\mu\Theta(p) \\ &\quad + e_0\, f_2(p',p)F_2(Q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}, \end{aligned} \tag{3.15}$$

where j_R is the reduced current, f_0,g_0,f_2 are off-shell functions discussed below, $e_0 = \frac{1}{2}$ is the isoscalar charge, the off-shell projection operator Θ was defined in (1.2),

$$\begin{aligned} \mathcal{F}_i^\mu &= [F_i(Q^2) - 1]\widetilde{\gamma}^\mu + \gamma^\mu \\ &= F_i(Q^2)\widetilde{\gamma}^\mu + \frac{\not{q}q^\mu}{q^2}, \end{aligned} \tag{3.16}$$

and the transverse γ matrix is

$$\widetilde{\gamma}^\mu = \gamma^\mu - \frac{\not{q}q^\mu}{q^2} \tag{3.17}$$

with $q = p' - p$. The nucleon form factors are $F_i(Q^2)$, with $Q^2 = -q^2$ [and F_3, subject to the constraint that $F_3(0) = 1$, a new form factor that contributes only when both nucleons are off shell]. The second form of (3.16) displays the interesting fact that the important physics is contained in the transverse part of the current; the longitudinal part that is constrained by the WT identities will not contribute to any observable since it is proportional to q^μ, which vanishes when contracted into any conserved current or any of the three polarization vectors of an off-shell photon.

The off-shell functions f_0 and g_0 are determined from the requirement that the reduced current, j_R^μ, satisfy the Ward-Takahashi (WT) identity

$$q_\mu\, j_R^\mu(p,p') = e_0\big[S_d^{-1}(p') - S_d^{-1}(p)\big], \tag{3.18}$$

where S_d the dressed propagator

$$S_d^{-1}(p) = \frac{m-\not{p}}{h^2(p)} = \frac{S^{-1}(p)}{h^2(p)}, \tag{3.19}$$

where h occurs squared because one comes from the initial and one from the final interactions that connect the propagator.

In all previous references it was assumed that the off-shell function $f_2 = f_0$, but since the $\sigma^{\mu\nu}q_\nu$ term is transverse, the WT identity places no constraint on f_2. Since consistency requires that any variation of f_2 also include the overall factors of hh', so that the relationship (3.15) between the dressed and reduced currents can be maintained, a simple ansatz for possible variations of f_2 is

$$f_2(p.p') = (1-\omega_2)\,hh' + \omega_2\, f_0(p,p'), \tag{3.20}$$

where $\omega_2 = 1$ is the choice previously discussed, and $\omega_2 = 0$ a reasonable alternative. In this paper it was found that the variation in the results for $\omega_2 = 0$ and $\omega_2 = 1$ was less that 0.001, the size of other terms omitted from the calculation. As a result, ω_2 was set to unity (our original assumption) and is no longer considered a parameter. However, for completeness, this dependence is recorded in the formulas given in Sec. V and Appendix D.

Using the shorthand notation $h = h(p)$ and $h' = h(p')$, the simplest solution to (3.18) gives

$$\begin{aligned} f_0(p',p) &= \frac{h'}{h}\frac{(m^2-p^2)}{p'^2-p^2} + \frac{h}{h'}\frac{(m^2-p'^2)}{p^2-p'^2} \\ g_0(p',p) &= \frac{4m^2}{p'^2-p^2}\left(\frac{h}{h'} - \frac{h'}{h}\right). \end{aligned} \tag{3.21}$$

An important simplification of the current occurs if it is contracted into the real (or virtual) photon polarization vectors defined in (3.6), with the property that $q_\mu\epsilon^\mu_{\lambda_\gamma} = 0$. In this case the q^μ terms in (3.15) can be dropped, and setting $f_2 = f_0$ from now on gives

$$\begin{aligned} j^\mu(p',p) \to\; & f_0(p',p)j_N^\mu(p',p) \\ & + e_0 g_0(p',p)F_3(Q^2)\Theta(p')\gamma^\mu\Theta(p), \end{aligned} \tag{3.22}$$

where j_N^μ is the familiar on-shell nucleon current

$$j_N^\mu(p',p) = e_0F_1(Q^2)\gamma^\mu + e_0F_2(Q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}. \tag{3.23}$$

In addition, the following limits are useful

$$\begin{aligned} f_{00} &\equiv \lim_{p'^2\to p^2} f_0(p',p) = 1 + 2a(p^2)(m^2-p^2) \\ g_{00} &\equiv \lim_{p'^2\to p^2} g_0(p',p) = -8m^2\,a(p^2) \end{aligned} \tag{3.24}$$

with

$$a(p^2) = \frac{1}{h}\frac{dh}{dp^2}. \tag{3.25}$$

C. Contributions from the (A) diagrams

The contributions from diagram (A) and (A$_\pm$) were written as a trace in Eq. (3.30) of Ref. [1]. Here the diagrams (A$_\pm$) are those parts of the interaction current that arise from the $\nu\Theta(p)$ and $\nu\Theta(p')$ terms in the sNN and vNN couplings (denoted by $\langle V_2^\mu\rangle$ in Ref. [1]). Using the wave functions and currents introduced above, the corresponding helicity amplitudes, defined in Eq. (3.8), can be written

$$G^{\lambda_\gamma}_{\lambda\lambda'}(q)\big|_{\mathrm{A}+V_2} = -\int_k \mathrm{tr}\Big[\Big\{\overline{\Psi}^{\lambda}_0(k,P_+)\big[f_0(p_+,p_-)j^{\lambda_\gamma}_N(q) + g_0(p_+,p_-)\Theta(p_+)F_3(Q^2)e_0\gamma^{\lambda_\gamma}\Theta(p_-)\big]\Psi^{\lambda'}_0(k,P_-) - \overline{\Psi}^{\lambda}_0(k,P_+)\frac{h_+}{h_-}j^{\lambda_\gamma}_N(q)\Psi^{(2)\lambda'}_0(k,P_-) - \overline{\Psi}^{(2)\lambda}_0(k,P_+)\,j^{\lambda_\gamma}_N(q)\frac{h_-}{h_+}\Psi^{\lambda'}_0(k,P_-)\Big\}\Lambda(-k)\Big], \tag{3.26}$$

where $p_\pm = P_\pm - k$, $j^{\lambda_\gamma}_N = j^\mu_N(\epsilon_{\lambda_\gamma})_\mu$ and $\gamma^{\lambda_\gamma} = \gamma_\mu\epsilon^\mu_{\lambda_\gamma}$ are the vector currents j^μ_N and γ^μ contracted with the photon polarization vector $\epsilon^\mu_{\lambda_\gamma}$. Part of the interaction current contribution is contained in the new wave function $\Psi^{(2)}$ (or $\Psi^{(2)}_0$ when the charge conjugation matrix has been removed), obtained from a truncated kernel proportional to the off-shell couplings depending on $\Theta(p)$ and $\Theta(p')$ (for details see Ref. [1]). Calculation of the three independent helicity amplitudes defined in Eq. (3.14), labeled by $n = \{1,2,3\}$, requires the helicity combinations $n \to \{\lambda_\gamma,\lambda,\lambda'\}$ where $1 \to \{0,0,0\}, 2 \to \{0,+,-\}$ and $3 \to \{+,+,0\} + \{-,0,-\}$. With this correspondence implied in the equations below, six generic traces $\mathcal{A}_{n,i}$, where and $i = \{1,2\}$ and $n = \{1,3\}$, are defined

$$\begin{aligned}\mathcal{A}_{n,i}(\Psi_1\Psi_2) &\equiv -\mathrm{tr}\big[\overline{\Psi}^{\lambda}_1(k,P_+)\,j^{\lambda_\gamma}_i(q)\,\Psi^{\lambda'}_2(k,P_-)\,\Lambda(-k)\big]\\ &= -(-1)^{\lambda_\gamma}\mathrm{tr}\big[\overline{\Psi}^{-\lambda'}_2(k,P_-)\,j^{-\lambda_\gamma}_i(-q)\,\Psi^{-\lambda}_1(k,P_+)\,\Lambda(-k)\big]\\ &= (-1)^{\lambda_\gamma}\mathcal{A}_{n,i}(\Psi_2\Psi_1)\big|_{q\to-q},\end{aligned} \tag{3.27}$$

where the transformations in the second line of (3.27) follow from the identity $\mathrm{tr}[\mathcal{O}] = \mathrm{tr}[\mathcal{O}^\dagger] = \mathrm{tr}[\gamma^0\mathcal{O}^\dagger\gamma^0]$ and the properties $\epsilon^*_{\lambda_\gamma} = (-1)^{\lambda_\gamma}\epsilon_{-\lambda_\gamma}$ and $\xi'^\mu_{-\lambda}(q) = \xi^\mu_\lambda(-q)$. The third line of (3.27) follows immediately from the second line for the $n = 1$ or 2 helicity amplitudes (where $\lambda_\gamma = 0$, and $\lambda' \leftrightarrow -\lambda$). However, the second line interchanges the two terms that contribute to the helicity average for the $n = 3$ combination, transforming $\{+,+,0\} \leftrightarrow -\{-,0.-\}$. Hence choosing the average of the two contributions ensures that the symmetry relation (3.27) holds, even if the individual contribution under study does not, by itself, conserve current. With this notation the trace (3.26), for each independent helicity amplitude, can be written

$$\begin{aligned}\mathcal{J}_n(q)|_{\mathrm{A}+V_2} = {}& e_0F_1(Q^2)\int_k\Big\{f_0(p_+,p_-)\mathcal{A}_{n,1}(\Psi_+\Psi_-) - \frac{h_+}{h_-}\mathcal{A}_{n,1}(\Psi_+\Psi^{(2)}_-) - \epsilon_{n3}\frac{h_-}{h_+}\mathcal{A}_{n,1}(\Psi_+\Psi^{(2)}_-)\big|_{q\to-q}\Big\}\\ &+ e_0F_2(Q^2)\int_k\Big\{f_0(p_+,p_-)\mathcal{A}_{n,2}(\Psi_+\Psi_-) - \frac{h_+}{h_-}\mathcal{A}_{n,2}(\Psi_+\Psi^{(2)}_-) - \epsilon_{n3}\frac{h_-}{h_+}\mathcal{A}_{n,2}(\Psi_+\Psi^{(2)}_-)\big|_{q\to-q}\Big\}\\ &+ e_0F_3(Q^2)\int_k\frac{g_0(p_+,p_-)}{4m^2}\mathcal{A}_{n,1}(\Gamma_+\Gamma_-),\end{aligned} \tag{3.28}$$

where $\epsilon_{n3} = (1 - 2\delta_{n3})$ is the extra phase that appears for the $n = 3$ helicity amplitudes, as derived in Eq. (3.27), and $\Psi_\pm = \Psi_0(k,P_\pm)$ and $\Psi^{(2)}_\pm = \Psi^{(2)}_0(k,P_\pm)$. The last term uses the reduction $\Theta\,\Psi_0 \to \Gamma/(2m)$.

The formulas for the six $\mathcal{A}_{n,i}$, expressed in terms of the invariant functions introduced in Sec. II, are lengthy and are given in Appendix B.

D. Contributions from the (B) diagrams

Diagrams ($\mathrm{B}_\pm$) of Fig. 2 are not identical to the ($\mathrm{B}_\pm$) diagrams shown in Fig. 1 of Ref. [1]. Here the diagrams involve the vertex function $\widehat{\Gamma}_{BS}$, which includes parts of of the interaction current arising from the $\nu\Theta(k)$ and $\nu\Theta(k')$ terms in the sNN and vNN couplings (which can contribute only when k or k' are off shell, and are denoted by $\langle V^\mu_1\rangle$ in Ref. [1]). They were written as a trace in Eq. (3.31) of Ref. [1]. Contracting these results with the photon polarization vector, and using the notation

$$\begin{aligned}E_\pm &= \sqrt{m^2 + \left(\mathbf{k} \pm \tfrac{1}{2}\mathbf{q}\right)^2}\\ \widetilde{k}_\pm &= \left\{k_0,\mathbf{k} \pm \tfrac{1}{2}\mathbf{q}\right\}\end{aligned} \tag{3.29}$$

gives

$$G^{\lambda_\gamma}_{\lambda\lambda'}(q)\big|_{\mathrm{B}+V_1} = \int_k\left[\frac{mE_k}{\mathbf{k}\cdot\mathbf{q}}\right]\mathrm{tr}\bigg\{\frac{1}{k_0}\,\overline{\widehat{\Gamma}}{}^{\lambda}_{BS}(\widetilde{k}_+,P_+)\,S_d(\widetilde{p})\,\widetilde{\Gamma}^{\lambda'}(\widetilde{k}_-,P_-)\Lambda(-\widetilde{k}_-)\,j^{\lambda_\gamma}_N(q)\,\Lambda(-\widetilde{k}_+)\Big|_{k_0=E_-} - \frac{1}{k_0}\,\overline{\widetilde{\Gamma}}{}^{\lambda}(\widetilde{k}_+,P_+)\,S_d(\widetilde{p})\,\widehat{\Gamma}^{\lambda'}_{BS}(\widetilde{k}_-,P_-)\Lambda(-\widetilde{k}_-)\,j^{\lambda_\gamma}_N(q)\,\Lambda(-\widetilde{k}_+)\big|_{k_0=E_+}\bigg\}, \tag{3.30}$$

where $\widetilde{p} = P_\pm - \widetilde{k}_\pm$. When $k_0 = E_+$, the outgoing particle is on shell, with $\widetilde{k}_+ = \widehat{k}_+$ and $\widetilde{k}_- = k_- = \{E_+, \mathbf{k} - \mathbf{q}/2\}$. Similarly, when $k_0 = E_-$, the incoming particle is on shell, with $\widetilde{k}_- = \widehat{k}_-$ and $\widetilde{k}_+ = k_+ = \{E_-, \mathbf{k} + \mathbf{q}/2\}$. The form of the expression (3.30) show clearly how the singularities in the two diagrams at $E_+ = E_-$ cancel, giving a finite result. Part of the interaction current contribution is contained in the new subtracted wave function $\widehat{\Psi}_{BS}(k,P) = S(p)\widehat{\Gamma}_{BS}(k,P)$, obtained through a cancellation of the vertex factors $\Theta(\widetilde{k}_\pm)$ that could be present if particle 1 is off shell (for details see Ref. [1]).

Note that the projection operator $\Lambda(-\widetilde{k}_\pm)$ always accompanies the vertex functions $\widehat{\Gamma}^\lambda_{BS}(\widetilde{k}_\pm, P_\pm)$. Following the discussion in Sec. II C, when $\widetilde{k}_\pm$ is off shell, the product of the subtracted vertex function and projection operator, $\widehat{\Gamma}^\lambda_{BS}(\widetilde{k}_\pm, P_\pm)\Lambda(-\widetilde{k}_\pm)$, breaks into two terms

$$\widehat{\Gamma}^\lambda_{BS}(\widetilde{k}_\pm, P_\pm)\Lambda(-\widetilde{k}_\pm) = \widehat{\Gamma}^\lambda(\widetilde{k}_\pm, P_\pm)\Lambda(-\widetilde{k}_\pm) - \frac{(m^2-\widetilde{k}^2_\pm)}{2m^2}\widehat{\Gamma}^\lambda_{\rm off}(\widetilde{k}_\pm, P_\pm), \quad (3.31)$$

where $\widehat{\Gamma}$ is identical to the on-shell vertex function $\widetilde{\Gamma}$ when $\widetilde{k}_\pm$ is on shell (because the cancellation shown in Ref. [1] ensures that there is no extra $\widetilde{k}_\pm$ dependence).

Introducing the new amplitudes

$$\Upsilon^\lambda(\widetilde{k}, P) = \widehat{\Gamma}^\lambda(\widetilde{k}, P)\Lambda(-\widetilde{k}) \quad (3.32)$$

leads to the following expressions for the independent helicity amplitudes (labeled by the index n as discussed above):

$$\mathcal{J}_n(q)\big|_{{\rm B}+V_1} = \int_k \left[\frac{mE_k}{\mathbf{k}\cdot\mathbf{q}}\right] {\rm tr}\Bigg\{\frac{1}{k_0}\left[\overline{\Upsilon}^\lambda(\widetilde{k}_+, P_+) - \frac{\mathbf{k}\cdot\mathbf{q}}{m^2}\overline{\widehat{\Gamma}}{}^\lambda_{\rm off}(\widetilde{k}_+, P_+)\right] S_d(\widetilde{p})\,\Upsilon^{\lambda'}(\widetilde{k}_-, P_-)\, j_N^{\lambda_\gamma}(q)\Big|_{k_0=E_-}$$
$$- \frac{1}{k_0}\overline{\Upsilon}^\lambda(\widetilde{k}_+, P_+)\, S_d(\widetilde{p})\left[\Upsilon^{\lambda'}(\widetilde{k}_-, P_-) + \frac{\mathbf{k}\cdot\mathbf{q}}{m^2}\widehat{\Gamma}^{\lambda'}_{\rm off}(\widetilde{k}_-, P_-)\right] j_N^{\lambda_\gamma}(q)\Big|_{k_0=E_+}\Bigg\}, \quad (3.33)$$

where the off-shell terms have been reduced using

$$(m^2 - \tilde{k}^2_\pm)\big|_{k_0=E_\mp} = \pm 2\mathbf{k}\cdot\mathbf{q}. \quad (3.34)$$

Equation (3.33) is further reduced by shifting $\mathbf{k} \pm \frac{1}{2}\mathbf{q} \to \mathbf{k}$ in the terms involving $\Gamma_{\rm off}$, and introducing the generic traces

$$\begin{aligned}\mathcal{B}_{n,i}(k_0) &\equiv {\rm tr}\big[\overline{\Upsilon}^\lambda(\tilde{k}_+, P_+)S_d(\tilde{p})\Upsilon^{\lambda'}(\tilde{k}_-, P_-)j_i^{\lambda_\gamma}(q)\big]\\ &= (-)^{\lambda_\gamma}{\rm tr}\big[\overline{\Upsilon}^{-\lambda'}(\tilde{k}_-, P_-)S_d(\tilde{p})\Upsilon^{-\lambda}(\tilde{k}_+, P_+)j_i^{-\lambda_\gamma}(q)\big] = (-)^{\lambda_\gamma}\mathcal{B}_{n,i}(k_0)\big|_{q\to -q}\end{aligned} \quad (3.35a)$$

$$\begin{aligned}\mathcal{C}_{n,i}(\Gamma\,\Gamma_{\rm off}) &= {\rm tr}\big[\overline{\Upsilon}^\lambda(k, P_+)S_d(P_+ - k)\widehat{\Gamma}^{\lambda'}_{\rm off}(k-q, P_-)j_i^{\lambda_\gamma}(q)\big]\\ &= (-)^{\lambda_\gamma}{\rm tr}\big[\overline{\widehat{\Gamma}}{}^{-\lambda'}_{\rm off}(k-q, P_-)S_d(P_+ - k)\Upsilon^{-\lambda}(k, P_+)j_i^{-\lambda_\gamma}(-q)\big] = (-)^{\lambda_\gamma}\mathcal{C}_{n,i}(\Gamma_{\rm off}\,\Gamma)\big|_{q\to -q}\end{aligned} \quad (3.35b)$$

where the labeling of the momenta in (3.35b) is as in Fig. 2, with the four-vector k always on shell. This allows the B $+\langle V_1\rangle$ contributions to the helicity amplitudes to be written

$$\mathcal{J}_n(q)\big|_{{\rm B}+V_1} = e_0F_1(Q^2)\int_k\left\{\left[\frac{mE_k}{\mathbf{k}\cdot\mathbf{q}}\right]\left(\frac{\mathcal{B}_{n,1}(k_0)}{k_0}\Big|_- - \frac{\mathcal{B}_{n,1}(k_0)}{k_0}\Big|_+\right) - \frac{1}{m}\mathcal{C}_{n,1}(\Gamma\,\Gamma_{\rm off}) - \frac{1}{m}\epsilon_{n3}\mathcal{C}_{n,1}(\Gamma\,\Gamma_{\rm off})\Big|_{q\to -q}\right\}$$
$$+ e_0F_2(Q^2)\int_k\left\{\left[\frac{mE_k}{\mathbf{k}\cdot\mathbf{q}}\right]\left(\frac{\mathcal{B}_{n,2}(k_0)}{k_0}\Big|_- - \frac{\mathcal{B}_{n,2}(k_0)}{k_0}\Big|_+\right) - \frac{1}{m}\mathcal{C}_{n,2}(\Gamma\,\Gamma_{\rm off}) - \frac{1}{m}\epsilon_{n3}\mathcal{C}_{n,2}(\Gamma\,\Gamma_{\rm off})\Big|_{q\to -q}\right\}, \quad (3.36)$$

where $|_\pm \to |_{k_0=E_\pm}$.

The formulas for the $\mathcal{B}$ and $\mathcal{C}$ traces, when expressed in terms of the invariant functions introduced in Sec. II, are lengthy and are given in Appendix B.

E. Numerical calculation of the form factors

Computation of the form factors involves not only the wave function Ψ and the vertex function Γ, but also the special wave function $\Psi^{(2)}$ and the subtracted vertex functions $\widehat{\Gamma}$. The calculation of the interaction current contributions has been simplified by introducing the special functions $\Psi^{(2)}$ and $\widehat{\Gamma}$, and their Dirac conjugates. The kernels that produce the bound-state functions $\Psi^{(2)}$ and $\widehat{\Gamma}$ were already been given in a very general form in Ref. [1], but, for convenience, are given in more explicit detail in Appendix C of the original, longer version (v1) of the present paper in the preprint archive [20].

The numerical calculation of the form factors involves the following steps.

(i) Start from the invariant functions $\{F, G, H, I\}$ and $\{A, B, C, D\}$ given in (2.8) and (2.11), or the K_i defined in Eq. (2.12). In the rest frame these are functions of $k = |\mathbf{k}|$ and k_0, and are constructed from the eight helicity amplitudes $z_0^{\pm\pm}$, and $z_1^{\pm\pm}$ as described in Appendix A.

(ii) Replace the rest frame arguments k, and k_0 by the correctly transformed arguments R and R_0 using the general definitions given in Eqs. (2.14). The specific realization of these general definitions depends on the diagram being evaluated and detailed expressions for each diagram are given in Eqs. (B3), (B8), and (B11), (B12).

(iii) Using the invariants with the proper arguments, evaluate the A+V_2 contributions to the helicity amplitudes (3.28) using Eqs. (B1)–(B2). Evaluate the B+V_1 contributions (3.36) using Eqs. (B6) and (B7) and (B9)–(B10). The total result is the sum of these two contributions.

(iv) Extract the individual form factors using the relations (3.14).

These general results do not reduce to simple expressions for the form factors in terms of the the familiar u,w,v_t, and v_s wave functions previously defined in the literature and shown in Eqs. (A29) and (A30). Still, to make connections with the older literature it is useful to express the result for the static moments in terms of leading terms involving integrals over products of u,w,v_t,v_s and corrections. The charge and magnetic moment will be reduced in this way in the following sections.

IV. CHARGE

The charge and normalization have been previously discussed in many references, including Ref. [1], so the purpose here is to see how the same result emerges from the general expressions (3.28) and (3.36). Using the results of Eqs. (C7) and (C8), the contributions from (3.28) are

$$
\begin{aligned}
G_C(0)\big|_{A+V_2} &= e_0\int_0^\infty k^2dk\Big\{ f_{00}\big[u^2+w^2+v_t^2+v_s^2\big] \\
&\quad+\frac{g_{00}}{4m^2}\big[(2E_k-m_d)^2(u^2+w^2)+m_d^2\big(v_t^2+v_s^2\big)\big] \\
&\quad-2\big[uu^{(2)}+ww^{(2)}+v_tv_t^{(2)}+v_sv_s^{(2)}\big]\Big\} \\
&= e_0\int_0^\infty k^2dk\big\{u^2+w^2+v_t^2+v_s^2-4a(p^2)(E_k-m_d) \\
&\quad\times\big[(2E_k-m_d)(u^2+w^2)-m_d\big(v_t^2+v_s^2\big)\big] \\
&\quad-2\big[uu^{(2)}+ww^{(2)}+v_tv_t^{(2)}+v_sv_s^{(2)}\big]\big\}, \qquad (4.1)
\end{aligned}
$$

with f_{00} and g_{00} defined in Eq. (3.24) [with a defined in Eq. (3.25)], and the second line was obtained by using $p = P-k$, which reduces f_{00} in the rest frame to

$$
f_{00} = 1 + a(p^2)\,2m_d(2E_k - m_d). \qquad (4.2)
$$

The special wave functions $z^{(2)}$ are obtained from $\Psi^{(2)}$ in precisely the same way that the z are obtained from Ψ.

Next, using the general results (C14) the contributions to the charge from (3.36) are

$$
\begin{aligned}
G_C(0)\big|_{B+V_1} = e_0\int_0^\infty k^2dk\big\{&u^2+w^2+v_t^2+v_s^2-4a(p^2)(E_k-m_d)\big[(2E_k-m_d)(u^2+w^2)-m_d\big(v_t^2+v_s^2\big)\big] \\
&-2\big(u[\delta_+\hat{u}]_{k_0}+w[\delta_+\hat{w}]_{k_0}\big)+2\big(v_t[\delta_-\hat{v}_t]_{k_0}+v_s[\delta_-\hat{v}_s]_{k_0}\big)\big\}, \qquad (4.3)
\end{aligned}
$$

where the functions $\delta_+\hat{u},\ldots,\delta_-\hat{v}_s$ were defined in (C12), and if $z = h\,\hat{z}$, the derivative is $z_{k_0} = h\,d\hat{z}(k_0)/dk_0|_{k_0=Ek}$.

The charge must be sum of the two contributions (4.1) and (4.3)

$$
1 = G_c(0)\big|_{A+V_2} + G_c(0)\big|_{B+V_1}, \qquad (4.4)
$$

which is also identical to the normalization condition (2.55) of Ref. [1].

The first parts of (4.1) and (4.3) are identical; their sum is the RIA contribution. This contribution arises from diagrams A and B in different ways. The contribution from the A diagram includes the f_0 and g_0 factors in the off-shell current; these factors do not appear in the B diagram, but similar contributions arise from the expansion of the dressed propagator S_d. Of course, the fact that these contributions are identical is not really surprising; it is a consequence of current conservation. The remaining factors originate from the interaction currents generated by the reduced kernel.

The remaining terms from (4.4) must equal the contribution from the energy derivative of the reduced kernel, $\partial\widetilde{V}/\partial P_0$, which appears in the normalization condition discussed in Ref. [1] and elsewhere. This leads to the identity

$$
\begin{aligned}
&-\frac{1}{2m_d}\int_k\int_{k'}\overline{\Psi}^{\lambda}_{\lambda_n\alpha}(k,P)h(p)\,\frac{\partial}{\partial P_0}\widetilde{V}_{\lambda_n\lambda_n',\alpha\alpha'}(k,k';P)\,h(p')\Psi^{\lambda'}_{\alpha'\lambda_n'}(k',P) \\
&\quad= -\int_0^\infty k^2dk\big\{uu^{(2)}+ww^{(2)}+v_tv_t^{(2)}+v_sv_s^{(2)}+u[\delta_+\hat{u}]_{k_0}+w[\delta_+\hat{w}]_{k_0}-v_t[\delta_-\hat{v}_t]_{k_0}-v_s[\delta_-\hat{v}_s]_{k_0}\big\}, \qquad (4.5)
\end{aligned}
$$

where we have set $e_0 = \frac{1}{2}$. This interesting identity, discussed already in Sec. I, shows how the energy derivative of $\tilde{V}$ can be expressed in terms of special wave functions $z^{(2)}$ and $\hat{z}$.

V. MAGNETIC MOMENT

Predictions for the magnetic moment are presented in this section. The new interaction current current contributions, which together account for about 5% of the charge, ensure that many new terms not previously encountered will contribute, and the result for the magnetic moment is the first important test of the CST.

The contributions from diagrams (A) and ($A_\pm$) were given in Eqs. (D6)–(D14) and from the diagrams (B) in Eqs. (D27) and (D29). Adding these together and keeping the leading $z_\ell^{(2)}$ contributions and setting $e_0 = \frac{1}{2}$ gives

$$\mu_d \simeq \mu_s\big(P_S - \tfrac{1}{2}P_D\big) + \tfrac{3}{4}P_D + \widetilde{\Delta}\mu_d, \tag{5.1}$$

where the correction terms are the sum of several contributions of different origin. This form resembles the nonrelativistic result, but is misleading because the sum of the S- and D-state probabilities is not equal to unity in the relativistic theory. Instead, it is more instructive to write the result in the form

$$\mu_d = \mu_s + \Delta\mu_d, \tag{5.2}$$

where, for the nonrelativistic theory, the correction is

$$\Delta\mu_d \to \mu_{\rm NR} = \tfrac{3}{4}P_D(1 - 2\mu_s). \tag{5.3}$$

To obtain a similar form from the CST, we use the relativistic normalization condition. In the approximations used to obtain the leading terms for the magnetic moment, the normalization (or charge) is

$$1 = \int_0^\infty k^2dk\big\{u^2 + w^2 + v_t^2 + v_s^2 + 4a(p^2)m\big[\delta_k(u^2+w^2) - 2m\big(v_t^2+v_s^2\big)\big] - u[\delta_+\hat u]_{k_0} - w[\delta_+\hat w]_{k_0} + v_t[\delta_-\hat v_t]_{k_0} + v_s[\delta_-\hat v_s]_{k_0} - uu^{(2)} - ww^{(2)} - v_tv_t^{(2)} - v_sv_s^{(2)}\big\}. \tag{5.4}$$

Multiplying this by μ_s, and adding and subtracting it from (5.1), gives an expression of the from (5.2) for the magnetic moment, where the correction will be written as a sum of terms

$$\Delta\mu_d = \mu_{\rm NR} + \mu_{Rc} + \mu_{h'} + \mu_{V_2} + \mu_{V_1} + \mu_{\rm int} + \mu_P + \mu_\chi, \tag{5.5}$$

where the individual contributions are

$$\begin{aligned}
\mu_{Rc} &= \int_0^\infty k^2dk\left[\frac{E_k-m}{E_k}\right]\left\{-\mu_s\left(u^2+\frac{1}{2}w^2-\sqrt{2}uw\right)-\frac{1}{4}\left(5u^2-\frac{89}{4}w^2+\frac{79}{2\sqrt{2}}uw\right)\right\}\\
\mu_{h'} &= \int_0^\infty k^2dk\,a(p^2)m\left\{4(1-\mu_s)(1-\omega_2)\delta_ku^2-\mu_s\big[2(2+\omega_2)\,\delta_kw^2-m\big(6\,v_t^2+4v_s^2+4\sqrt{2}v_tv_s\big)\big]\right.\\
&\quad\left.+\frac{\delta_k}{2}[(3+4\omega_2)w^2-\sqrt{2}uw]-\frac{m}{2}\big(9v_t^2+6v_s^2+8\sqrt{2}v_tv_s\big)\right\}\\
\mu_{V_2} &= \int_0^\infty\frac{k^2dk}{2}\left\{(2\mu_s-1)\frac{3}{2}ww^{(2)}+\mu_s\big(3v_tv_t^{(2)}+2v_sv_s^{(2)}\big)+(\mu_s-1)\sqrt{2}\big(v_tv_s^{(2)}+v_sv_t^{(2)}\big)-\frac{1}{2}v_tv_t^{(2)}+v_sv_s^{(2)}-m^{I(2)}\right\}\\
\mu_{V_1} &= \int_0^\infty k^2dk\left\{(2\mu_s-1)\left(\frac{3}{4}w[\delta_+\hat w]_{k_0}-\frac{1}{4}v_t[\delta_-\hat v_t]_{k_0}-\frac{1}{2}v_s[\delta_-\hat v_s]_{k_0}\right)-\frac{\mu_s}{\sqrt{2}}\big(v_t[\delta_-\hat v_s]_{k_0}+v_s[\delta_-\hat v_t]_{k_0}\big)\right\}\\
\mu_{\rm int} &= -\frac{m}{2\sqrt{6}}\int_0^\infty k^2dk\left\{u'(v_t-\sqrt{2}v_s)-w(\sqrt{2}v_t+v_s)'+\frac{1}{k}w(\sqrt{2}\,v_t+v_s)\right\}.\\
\mu_P &= \int_0^\infty k^2dk\left\{-\mu_s\big(v_t^2+v_s^2+\sqrt{2}v_tv_s\big)-\frac{1}{4}v_t^2-\frac{1}{2}v_s^2+\sqrt{2}v_tv_s+\frac{3k}{8\sqrt{2}}(v_tv_s'-v_t'v_s)\right\}\\
\mu_\chi &= -\int_0^\infty k^2dk\left\{\frac{m\,z_0^{--}}{2k}(\sqrt{2}\,v_s+kv_t')+\frac{m\,z_1^{--}}{2k}(\sqrt{2}\,v_t+v_s+kv_s')\right\},
\end{aligned} \tag{5.6}$$

where $\omega_2 = 1$ was defined in Eq. (3.20) and $m^{I(2)}$ in Eq. (D12). Each of these terms has a different physical origin, as discussed in Sec. I D.

Many remaining details are discussed in the Appendixes.

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APPENDIX A: CONNECTIONS BETWEEN THE INVARIANT FUNCTIONS AND COMPONENT WAVE FUNCTIONS

This Appendix shows how to connect the invariant functions defined in Sec. II to the helicity amplitudes that are calculated in the code described in Refs. [5,6]. The four particle 1 positive ρ-spin helicity amplitudes are simple linear combinations of the more familiar component wave functions u, w, v_t, and v_s. The traces given in Appendix B are bilinear products of the invariant functions.

In the rest frame the relativistic wave function (2.1) can be expanded in a set of helicity spinors $u^\rho(\mathbf{k},\lambda)$

$$\Psi^{\lambda_d}_{\alpha\beta}(k,P) = \frac{1}{N_d}\frac{m}{E_k}\sum_{\substack{\lambda_1\rho_1\\ \lambda_2\rho_2}} u^{\rho_2}_{2\alpha}(\mathbf{k},\lambda_2)\, u^{\rho_1 \mathrm{T}}_{1\beta'}(\mathbf{k},\lambda_1)\gamma^0_{\beta'\beta} \times\phi^{\rho_1\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k})$$
$$\Psi^{\lambda_d}_{\alpha\lambda_1}(k,P) = \frac{1}{N_d}\sum_{\lambda_2\rho_2} u^{\rho_2}_{2\alpha}(\mathbf{k},\lambda_2)\phi^{\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k}), \tag{A1}$$

where $\rho=\pm$ is the ρ spin of the particle (if particle 1 is on shell, $\rho_1=+$), $\mathbf{k}$ is the three-momentum of particle 1 in the deuteron rest frame, and $\phi^{\rho_1\rho_2}_{\lambda_1\lambda_2,\lambda_d}$ are normalized helicity amplitudes defined by this expansion. The second form of (A1), obtained from the first using the orthogonality relations (A6) below, will be used only when $\rho_1=+$; reference to ρ_1 is suppressed for simplicity. The transpose symbol is to remind us that, if $\psi_{\alpha\beta}$ is to be viewed as a matrix, then $u_{\beta'}$ must be interpreted as a row vector, but is redundant when the indices are shown explicitly. The normalization constant N_d is

$$N_d = \frac{1}{\sqrt{(2\pi)^3 2m_d}} \tag{A2}$$

and the helicity spinors [cf. Ref. [5], Eqs. (E1) and (E7)] are

$$u^\rho_1(\mathbf{k},\lambda) = N_\rho(k,\lambda)\otimes\chi_\lambda(\theta)$$
$$u^\rho_2(\mathbf{k},\lambda) = N_\rho(k,\lambda)\otimes\chi_{-\lambda}(\theta) \tag{A3}$$

with

$$N_+(k,\lambda) = \begin{pmatrix}\cosh\frac12\zeta\\ 2\lambda\sinh\frac12\zeta\end{pmatrix}$$
$$N_-(k,\lambda) = \begin{pmatrix}-2\lambda\sinh\frac12\zeta\\ \cosh\frac12\zeta\end{pmatrix}, \tag{A4}$$

where $\tanh\zeta = k/E_k$, and, for momenta limited to the $\hat{x}\hat{z}$ plane, so that $\mathbf{k}=\{k\sin\theta,0,k\cos\theta\}$, the two-component helicity spinors are

$$\chi_{1/2}(\theta) = \begin{pmatrix}\cos\frac12\theta\\ \sin\frac12\theta\end{pmatrix}\quad \chi_{1/2}(\theta) = \begin{pmatrix}-\sin\frac12\theta\\ \cos\frac12\theta\end{pmatrix}. \tag{A5}$$

These helicity spinors are real, so that $\bar{u}=u^{\mathrm{T}}\gamma^0$, and they satisfy the orthogonality relations

$$\bar{u}^{\rho'}(\mathbf{k},\lambda')\gamma^0 u^\rho(\mathbf{k},\lambda) = \delta_{\lambda'\lambda}\delta_{\rho'\rho}\frac{E_k}{m}$$
$$\bar{u}^{\rho}(\mathbf{k},\lambda')u^\rho(\mathbf{k},\lambda) = \rho\,\delta_{\lambda'\lambda}, \tag{A6}$$

leading to the inverse relation

$$\phi^{+\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k}) = N_d\frac{m}{E_k}\,\bar{u}^{\rho_2}_2(\mathbf{k},\lambda_2)\gamma^0\Psi^{\lambda_d}(k,P)\bar{u}^{+\mathrm{T}}_1(\mathbf{k},\lambda_1)$$
$$= N_d\frac{m}{E_k}\,\bar{u}^{\rho_2}_{2\alpha}(\mathbf{k},\lambda_2)\gamma^0_{\alpha\alpha'}\Psi^{\lambda_d}_{\alpha'\lambda_1}(k,P). \tag{A7}$$

This is further reduced by writing the wave function in terms of the vertex function, $\mathcal{G}$, and the propagator of particle 2, $S(p)$, and decomposing the rest frame propagator for particle 2 into positive and negative energy parts (or its ρ spin $\pm$ components)

$$S_{\alpha\alpha'}(p) = \frac{m}{E_k}\sum_{\rho,\lambda} G^\rho(k_0,\mathbf{k})u^\rho_2(\mathbf{k},\lambda)\bar{u}^\rho_2(\mathbf{k},\lambda), \tag{A8}$$

where, if particle 1 is also off shell so that $k=\{k_0,\mathbf{k}\}$, the components of the propagator are

$$G^+(k_0,\mathbf{k}) = \frac{1}{E_k+k_0-m_d} \equiv \frac{1}{\delta_+}$$
$$G^-(k_0,\mathbf{k}) = \frac{-1}{m_d+E_k-k_0} \equiv -\frac{1}{\delta_-}, \tag{A9}$$

where the arguments of $\delta_\pm$ will be suppressed. In most cases particle 1 is on shell so that $k_0=E_k$, and (A9) reduce to (cf. Eq. (E14) of Ref. [5])

$$G^+(E_k,\mathbf{k}) = \frac{1}{2E_k-m_d} \equiv \frac{1}{\delta_k}$$
$$G^-(E_k,\mathbf{k}) = -\frac{1}{m_d}. \tag{A10}$$

Using the expansion (A9), the helicity amplitudes (A7) reduce to the previously published form (cf. Eq. (3.10) of Ref. [6], except here ϕ is used in place of ψ and there are other changes in notation)

$$\phi^{+\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k}) = N_d\frac{m}{E_k}G^{\rho_2}\,\mathcal{G}^{+\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k})$$
$$= N_d\frac{m}{E_k}G^{\rho_2}\bar{u}^{\rho_2}_2(\mathbf{k},\lambda_2)\Gamma^{\lambda_d}(k,P)\mathcal{C}\bar{u}^{+\mathrm{T}}_1(\mathbf{k},\lambda_1), \tag{A11}$$

where *no* sum over the repeated index ρ_2 is implied.

In the general case (when $k_0\neq E_k$), the projection operators present in the Γ of (A11) can be simplified by recalling that $p=P-k$, $k=\{k_0,\mathbf{k}\}$, and $\rho=\pm1$, giving

$$2m\bar{u}^{\rho_2}_2(\mathbf{k},\lambda)\Theta(p) = \bar{u}^\rho_2(\mathbf{k},\lambda)[m-\gamma^0(m_d-k_0)-\boldsymbol{\gamma}\cdot\mathbf{k}]$$
$$= -m\,d_2\,\bar{u}^\rho_2(\mathbf{k},\lambda)\gamma^0$$
$$2m\bar{u}^{\rho_1}_1(\mathbf{k},\lambda)\Theta(k) = \bar{u}^\rho_1(\mathbf{k},\lambda)[m-\gamma^0 k_0+\boldsymbol{\gamma}\cdot\mathbf{k}]$$
$$= -m\,d_1\,\bar{u}^\rho_1(\mathbf{k},\lambda)\gamma^0, \tag{A12}$$

where

$$m\,d_1 = k_0-\rho E_k$$
$$m\,d_2 = m_d-k_0-\rho E_k. \tag{A13}$$

Because the helicity spinors are written as a direct product of N_ρ and χ_λ, each operating in its own 2×2 space, it is convenient to similarly decompose the matrix Γ^{λ_d}. To this end note that

$$\gamma_\mu \xi^\mu_{\lambda_d} = \begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot\boldsymbol{\xi}_{\lambda_d} \\ \boldsymbol{\sigma}\cdot\boldsymbol{\xi}_{\lambda_d} & 0 \end{pmatrix} = -i\tau_2 \otimes \boldsymbol{\sigma}\cdot\boldsymbol{\xi}_{\lambda_d}, \tag{A14}$$

where τ_i are the 2×2 operators operating in the N_ρ Dirac space and σ_i operate on the χ_λ spin space, and the three-component deuteron polarization vectors (for an incoming deuteron), $\boldsymbol{\xi}^i_{\lambda_d}$ (with $i=1,2,3$), are related to the four-vectors by

$$\begin{aligned} \xi^\mu_0 &= \{0,0,0,1\} = \{0,\boldsymbol{\xi}^i_0\} \\ \xi^\mu_\pm &= \frac{1}{\sqrt{2}}\{0,\pm 1,-i,0\} = \{0,\boldsymbol{\xi}^i_\pm\}. \end{aligned} \tag{A15}$$

Also note that, in 2×2 form,

$$\mathcal{C} = -\tau_1 \otimes i\sigma_2. \tag{A16}$$

Using this notation, the matrix elements are reduced to the following convenient form

$$\begin{aligned} \phi^{\rho_1\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k}) &= A^{\rho_1\rho_2}_{\lambda_1\lambda_2}(k)\big(\chi^\dagger_{-\lambda_2} i\sigma_2\chi_{\lambda_1}\big)\big(\hat{\mathbf{k}}\cdot\boldsymbol{\xi}_{\lambda_d}\big) \\ &\quad + B^{\rho_1\rho_2}_{\lambda_1\lambda_2}(k)\big(\chi^\dagger_{-\lambda_2}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}_{\lambda_d} i\sigma_2\chi_{\lambda_1}\big) \\ &= -2\lambda_1\delta_{\lambda_1,\lambda_2}\, d^1_{\lambda_d,0}(\theta) A^{\rho_1\rho_2}_{\lambda_1\lambda_2}(k) \\ &\quad + \sqrt{2}^{|\lambda|} d^1_{\lambda_d,\lambda}(\theta)\, B^{\rho_1\rho_2}_{\lambda_1\lambda_2}(k), \end{aligned} \tag{A17}$$

where the identities (C26) from Ref. [6] were used to evaluate the angular matrix elements. The coefficient A contributes only when $\lambda_1=\lambda_2$ and both of the coefficients are independent of the deuteron polarization and the angle θ. Using the definition of Γ when both particles are off shell, Eq. (2.12), and the simplifications (A12), they reduce to

$$\begin{aligned} A^{\rho_1\rho_2}_{\lambda_1\lambda_2}(k) &= N_d\frac{m}{E_k}G^{\rho_2}(k)\overline{N}_{\rho_2}(k,\lambda_2)\frac{k}{m}\{G - K_4\, d_2\, d_1 \\ &\quad + (I d_2 - K_2\, d_1)\tau_3\}(\tau_1)\overline{N}^{\rm T}_{\rho_1}(k,\lambda_1) \\ B^{\rho_1\rho_2}_{\lambda_1\lambda_2}(k) &= N_d\frac{m}{E_k}G^{\rho_2}(k)\overline{N}_{\rho_2}(k,\lambda_2)\{(F + K_3\, d_2\, d_1)i\tau_2 \\ &\quad + (H\, d_2 + K_1\, d_1)\tau_1\}(\tau_1)\overline{N}^{\rm T}_{\rho_1}(k,\lambda_1), \end{aligned} \tag{A18}$$

where $\gamma^0\to\tau_3$, and $\mathcal{C}\gamma^0 = -\gamma^0\mathcal{C}$ was used.

Before evaluating the matrix elements (A18), it is convenient to project the partial wave amplitudes from (A17). Using the definition given in Eq. (3.21) of Ref. [6], these are

$$\begin{aligned} \phi^{\rho_1\rho_2}_{\lambda_1\lambda_2,\lambda_d}(k) &= \sqrt{\frac{3}{4\pi}}\int d\Omega_k\, D^{1*}_{\lambda_d,\lambda}(\phi,\theta,0)\phi^{\rho_1\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k}) \\ &= \sqrt{3\pi}\int_0^1 \sin\theta d\theta\, d^1_{\lambda_d,\lambda}(\theta)\phi^{\rho_1\rho_2}_{\lambda_1\lambda_2,\lambda_d}(\mathbf{k}), \end{aligned} \tag{A19}$$

where $\lambda=\lambda_1-\lambda_2$ and the $d^J_{\lambda',\lambda}(\theta)$ are the rotation matrices, in this case for $J=1$ and $\lambda'=\lambda_d$, where J is the angular momentum and λ_d the helicity of the deuteron. The normalization of the $J=1$ d matrices is independent of helicity

$$\int_0^1 \sin\theta\, d\theta\, \big[d^1_{\lambda_d,\lambda}(\theta)\big]^2 = \frac{2}{3}, \tag{A20}$$

and hence the result for the partial waves is independent of the deuteron helicity. Under parity, the amplitudes transform into each other under the substitution $\lambda_1,\lambda_2\to-\lambda_1,-\lambda_2$. Hence the partial wave amplitudes can conveniently written in terms of a standard helicity with $\lambda_1=\lambda_0=\frac{1}{2}$. Writing a separate formula for cases when $\lambda_1=\lambda_2=\lambda_0$ and $\lambda_1=-\lambda_2=\lambda_0$ gives

$$\begin{aligned} \phi^{\rho_1\rho_2}_{\lambda_0\lambda_0,\lambda_d}(k) &\equiv z^{\rho_1\rho_2}_0(k) = \sqrt{\frac{4\pi}{3}}\big[B^{\rho_1\rho_2}_{\lambda_0,\lambda_0}(k) - A^{\rho_1\rho_2}_{\lambda_0,\lambda_0}(k)\big] \\ \phi^{\rho_1\rho_2}_{\lambda_0,-\lambda_0,\lambda_d}(k) &\equiv z^{\rho_1\rho_2}_1(k) = \sqrt{\frac{8\pi}{3}}\, B^{\rho_1\rho_2}_{\lambda_0,-\lambda_0}(k). \end{aligned} \tag{A21}$$

There are therefore eight independent amplitudes from which the eight invariants that define Γ can be determined.

It is instructive to show explicitly that the parity relation holds. To this end, introduce the matrix elements

$$D^{\rho_1\rho_2}_{i\,\lambda_1\lambda_2}(k) = \overline{N}_{\rho_2}(k,\lambda_2)\bar{\tau}_i\overline{N}^{\rm T}_{\rho_1}(k,\lambda_1), \tag{A22}$$

where $i=0,1,2,3$ with $\bar\tau_i=\tau_i$ for $i=1,3$, $\bar\tau_2=i\tau_2$, and $\bar\tau_0=\mathbf{1}$. The entire helicity dependence of the partial waves is contained in the helicity dependence of the D's, and this can be established from the symmetry property

$$\overline{N}_\rho(k,\lambda) = \rho\overline{N}_\rho(k,-\lambda)\tau_3, \tag{A23}$$

which leads to the relations

$$\begin{aligned} D^{\rho_1\rho_2}_{j\,\lambda_1\lambda_2}(k) &= \rho_1\rho_2 D^{\rho_1\rho_2}_{j\,-\lambda_1,-\lambda_2}(k) \quad j=0,3 \\ D^{\rho_1\rho_2}_{j\,\lambda_1\lambda_2}(k) &= -\rho_1\rho_2 D^{\rho_1\rho_2}_{j\,-\lambda_1,-\lambda_2}(k) \quad j=1,2. \end{aligned} \tag{A24}$$

Examination of the definitions (A18) shows that only D_0 and D_3 contribute to B, and only D_1 and D_2 contribute to A. The extra factor of $2\lambda_1$ multiplying A is just sufficient to show that both of the helicity amplitudes (A21) satisfy the expected relation for a $J=1$ amplitude: $\phi^{\rho_1,\rho_2}=\rho_1\rho_2\phi^{-\rho_1,-\rho_2}$ (cf. Eq. (E22) of Ref. [5]), concluding the proof.

Working out the matrix elements (A18) gives the eight independent helicity amplitudes in terms of the eight invariants that define the two-particle of-shell vertex function (2.12). Using the notation

$$\mathcal{K} = \pi\sqrt{2m_d} \tag{A25}$$

and recalling the definitions of $\delta_\pm$ and $G^\pm$ from Eq. (A9)

$$\begin{aligned} z^{++}_0 &= \frac{G^+m}{\sqrt{6}\mathcal{K}E_k}\bigg\{F + \frac{k^2}{m^2}G - \frac{E_k}{m^2}\delta_+H \\ &\quad - \frac{(E_k-k_0)}{m^2}\bigg[E_kK_1 - \delta_+\bigg(K_3 - \frac{k^2}{m^2}K_4\bigg)\bigg]\bigg\} \\ z^{++}_1 &= \frac{G^+}{\sqrt{3}\mathcal{K}}\bigg\{F - \frac{\delta_+}{E_k}H - \frac{(E_k-k_0)}{E_k}\bigg[K_1 - \frac{E_k\delta_+}{m^2}K_3\bigg]\bigg\} \\ z^{+-}_0 &= -\frac{G^-k}{\sqrt{6}\mathcal{K}E_k}\bigg\{F - G + \frac{E_k\delta_-}{m^2}I \\ &\quad + \frac{(E_k-k_0)}{m^2}[E_kK_2 - \delta_-(K_3+K_4)]\bigg\} \end{aligned}$$

$$z_1^{+-} = \frac{G^- k}{\sqrt{3}\,\mathcal{K}\, m E_k}\{\delta_- H - (E_k - k_0)K_1\}$$
$$z_0^{-+} = -\frac{G^+ k}{\sqrt{6}\,\mathcal{K} E_k}\Big\{F - G + \frac{E_k\delta_+}{m^2} I + \frac{(E_k + k_0)}{m^2}[E_k K_2 - \delta_+(K_3 + K_4)]\Big\}$$
$$z_1^{-+} = \frac{G^+ k}{\sqrt{3}\,\mathcal{K}\, m E_k}\{\delta_+ H - (E_k + k_0)K_1\}$$
$$z_0^{--} = -\frac{G^- m}{\sqrt{6}\,\mathcal{K} E_k}\Big\{F + \frac{k^2}{m^2} G - \frac{E_k}{m^2}\delta_- H - \frac{(E_k + k_0)}{m^2}\Big[E_k K_1 - \delta_-\Big(K_3 - \frac{k^2}{m^2}K_4\Big)\Big]\Big\}$$
$$z_1^{--} = -\frac{G^-}{\sqrt{3}\,\mathcal{K}}\Big\{F - \frac{\delta_-}{E_k} H - \frac{(E_k + k_0)}{E_k}\Big[K_1 - \frac{E_k\delta_-}{m^2}K_3\Big]\Big\}. \tag{A26}$$

Inverting these equations gives the eight invariants in terms of the helicity amplitudes. The results are

$$F = \frac{\sqrt{3}\,\mathcal{K}}{2E_k m_d}\delta_+\delta_-\Big\{(E_k + k_0)\Big[z_1^{++} - \frac{m}{k}z_1^{+-}\Big] - (E_k - k_0)\Big[z_1^{--} + \frac{m}{k}z_1^{-+}\Big]\Big\}$$
$$G = \sqrt{\frac{3}{2}}\frac{m\mathcal{K}}{E_k m_d k^2}\delta_+\delta_-\Big\{(E_k + k_0)\Big[E_k\, z_0^{++} - m\frac{z_1^{++}}{\sqrt{2}} - k\frac{z_1^{+-}}{\sqrt{2}}\Big] - (E_k - k_0)\Big[E_k\, z_0^{--} - m\frac{z_1^{--}}{\sqrt{2}} + k\frac{z_1^{-+}}{\sqrt{2}}\Big]\Big\}$$
$$H = -\frac{\sqrt{3}m\mathcal{K}}{2m_d\, k}\{(E_k + k_0)\delta_-\, z_1^{+-} + (E_k - k_0)\delta_+\, z_1^{-+}\}$$
$$I = \sqrt{\frac{3}{2}}\frac{m^2\mathcal{K}}{E_k m_d k^2} \times\Big\{(E_k + k_0)\Big[\delta_+\Big(m\, z_0^{++} - \frac{E_k}{\sqrt{2}}z_1^{++}\Big) + \delta_-\, k\, z_0^{+-}\Big] - (E_k - k_0)\Big[\delta_-\Big(m\, z_0^{--} - \frac{E_k}{\sqrt{2}}z_1^{--}\Big) - \delta_+\, k\, z_0^{-+}\Big]\Big\}$$
$$K_1 = -\frac{\sqrt{3}m\mathcal{K}}{2m_d\, k}\delta_+\delta_-[z_1^{+-} + z_1^{-+}]$$
$$K_2 = \sqrt{\frac{3}{2}}\frac{m^2\mathcal{K}}{E_k m_d k^2}\delta_-\delta_+\Big(m\, z_0^{++} - E_k\frac{z_1^{++}}{\sqrt{2}} - k z_0^{+-} - m\, z_0^{--} + E_k\frac{z_1^{--}}{\sqrt{2}} - k z_0^{-+}\Big)$$
$$K_3 = -\frac{\sqrt{3}m^2\mathcal{K}}{2E_k m_d\, k}\{\delta_+(k\, z_1^{++} + m\, z_1^{-+}) - \delta_-(k\, z_1^{--} - m\, z_1^{+-})\}$$
$$K_4 = \frac{\sqrt{3}m^3\mathcal{K}}{2m_d\, k^2}\Big\{\delta_+\Big(\sqrt{2}z_0^{++} - \frac{m}{E_k}z_1^{++} + \frac{k}{E_k}z_1^{-+}\Big) - \delta_-\Big(\sqrt{2}z_0^{--} - \frac{m}{E_k}z_1^{--} - \frac{k}{E_k}z_1^{+-}\Big)\Big\}. \tag{A27}$$

When particle 1 is on shell, so that $k_0 = E_k$, the first four amplitudes reduce to

$$F = \sqrt{3}\,\mathcal{K}\,\delta_k\Big[z_1^{++} - \frac{m}{k}z_1^{+-}\Big]$$
$$G = \sqrt{3}\,\mathcal{K}\frac{m\,\delta_k}{k^2}[\sqrt{2}E_k z_0^{++} - m z_1^{++} - k\, z_1^{+-}]$$
$$H = -\sqrt{3}\,\mathcal{K}\, m\, E_k\frac{z_1^{+-}}{k}$$
$$I = \sqrt{6}\,\mathcal{K}\frac{m^2}{k^2}\Big[\frac{m\delta_k}{m_d}\Big(z_0^{++} - \frac{E_k}{\sqrt{2}m}z_1^{++}\Big) + k\, z_0^{+-}\Big] \tag{A28}$$

with $\delta_k = \delta_+(E_k,\mathbf{k}) = 2E_k - m_d$. These are uniquely determined by the the four on-shell helicity amplitudes with $\rho_1 = +$. If these amplitudes are expressed in terms of the u, w, v_t, and v_s amplitudes previously defined in the literature (see Eq. (C31) of Ref. [6]),

$$z_0^{++} = \frac{1}{\sqrt{6}}(u + \sqrt{2}w)$$
$$z_1^{++} = \frac{1}{\sqrt{6}}(\sqrt{2}u - w)$$
$$z_0^{+-} = -\frac{1}{\sqrt{2}}v_s$$
$$z_1^{+-} = -\frac{1}{\sqrt{2}}v_t, \tag{A29}$$

the well-known expansions of F, G, H, and I derived in Ref. [10] are obtained, reproduced here for completeness:

$$F = \mathcal{K}\delta_k\Big[u - \frac{w}{\sqrt{2}} + \sqrt{\frac{3}{2}}\frac{m}{k}v_t\Big]$$
$$G = \mathcal{K}\delta_k\, m\Big[\frac{u}{E_k + m} + (2E_k + m)\frac{w}{\sqrt{2}\,k^2} + \sqrt{\frac{3}{2}}\frac{v_t}{k}\Big]$$
$$H = \mathcal{K}E_k\, m\sqrt{\frac{3}{2}}\frac{v_t}{k}$$
$$I = -\frac{\mathcal{K}\delta_k m^2}{m_d}\Big[\frac{u}{E_k + m} - (E_k + 2m)\frac{w}{\sqrt{2}\,k^2}\Big] - \sqrt{3}\mathcal{K}m^2\frac{v_s}{k}. \tag{A30}$$

The on-shell values of the K_i invariants depend on all eight of the helicity amplitudes. Because of their historical importance, we will continue to express the helicity amplitudes (A29) in terms of the u,w,v_t,v_s wave functions, but will use the original notation for the others, giving

$$K_1 = \sqrt{\frac{3}{2}}\frac{\mathcal{K}\delta_k m}{2k}[v_t - \sqrt{2}\, z_1^{-+}]$$
$$K_2 = -\frac{\mathcal{K}\delta_k m^2}{2E_k}\bigg[\frac{u}{E_k+m} - (E_k+2m)\frac{w}{\sqrt{2}\,k^2} - \sqrt{3}\,\frac{v_s}{k} - \frac{\sqrt{3}}{k^2}(E_k z_1^{--} - \sqrt{2}\, m z_0^{--}) + \sqrt{6}\,\frac{z_0^{-+}}{k}\bigg]$$
$$K_3 = -\frac{\mathcal{K} m^2}{2E_k}\bigg[\frac{\delta_k}{m_d}\left(u - \frac{w}{\sqrt{2}}\right) - \sqrt{\frac{3}{2}}\frac{m}{k}v_t - \sqrt{3}\left(z_1^{--} - \frac{\delta_k\, m}{k\, m_d} z_1^{-+}\right)\bigg]$$
$$K_4 = \frac{\mathcal{K} m^3}{2E_k}\bigg[\frac{\delta_k}{m_d}\left(\frac{u}{E_k+m} + (2E_k+m)\frac{w}{\sqrt{2}\,k^2} + \sqrt{3}\frac{z_1^{-+}}{k}\right) - \sqrt{\frac{3}{2}}\frac{v_t}{k} + \frac{\sqrt{3}}{k^2}(m\, z_1^{--} - \sqrt{2}\, E_k z_0^{--})\bigg]. \tag{A31}$$

The u,w,v_t,v_s wave functions are sometimes transformed into coordinate space (for a full discussion see Ref. [6]). Denoting the typical wave function by z_ℓ (so that $z_0 = u$, $z_2 = w$, and $z_1 = v_t$ or v_s), the momentum and position space wave functions are relatedby the spherical Bessel transforms

$$z_\ell(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty r dr\; j_\ell(kr)\, z_\ell(r)$$
$$\frac{z_\ell(r)}{r} = \sqrt{\frac{2}{\pi}}\int_0^\infty k^2 dk\; j_\ell(kr)\, z_\ell(k), \tag{A32}$$

where j_ℓ is the spherical Bessel function of order ℓ with the convenient recursion relation

$$j_\ell(z) = z^\ell\left(-\frac{1}{z}\frac{d}{dz}\right)^\ell \frac{\sin z}{z}. \tag{A33}$$

The normalization condition for the spherical Bessel functions,

$$\int_0^\infty k^2 dk j_\ell(kr) j_\ell(kr') = \frac{\pi}{2r^2}\delta(r-r') \tag{A34}$$

can be used to transform integrals from momentum space to coordinate space. Another convenient identity is

$$\int_0^\infty dk\,\frac{d}{dk}(k^2\, z_\ell z_{\ell'}) = \int_0^\infty k^2\, dk\left(\frac{2z_\ell z_{\ell'}}{k} + z_\ell z'_{\ell'} + z'_\ell z_{\ell'}\right) = 0. \tag{A35}$$

APPENDIX B: RESULTS FOR THE TRACES

1. Contributions from the (A) diagrams

In this section the traces (3.27) needed for each of the helicity amplitudes defined in Eq. (3.14) are evaluated. Using the compact notation $Z_\pm = Z(R_\pm)$ (where Z is the generic name for $\{A,B,C,D\}$) with $R_\pm$ defined in Eq. (B3) below, the results are

$$\begin{aligned}\mathcal{A}_{n,1}(\Psi_+\Psi_-) = \frac{2}{m}\bigg\{&A_+A_-(a_-z_+ + a_+z_- - a_0z_0) + B_+B_-\frac{a_0\,a_+a_-}{m^2} + D_+D_-\frac{a_+a_-}{m^4}[2b_0(c_+ + c_-) - \zeta_0 a_0]\\
&+ C_+C_-[4m^2(2b_0z_0 - b_+z_- - b_-z_+) + 8a_+a_-(a_0 - b_0) + a_0z_0(\zeta_0 - 8m^2) - 2a_0b_+b_-\\
&- 2b_0z_0(c_+ + c_-) + a_+(2b_0b_- + 4c_-z_- - \zeta_0z_-) + a_-(2b_0b_+ + 4c_+z_+ - \zeta_0z_+)\\
&+ 2b_+c_+z_- + 2b_-c_-z_+] - A_+B_-a_-z_+ - B_+A_-a_+z_-\\
&+ A_+C_-[z_0(2a_0 - b_0) - z_-(2a_+ - b_+)] + C_+A_-[z_0(2a_0 - b_0) - z_+(2a_- - b_-)]\\
&+ A_+D_-\frac{a_-}{m^2}(a_+b_0 - a_0b_+ + z_+c_-) + D_+A_-\frac{a_+}{m^2}(a_-b_0 - a_0b_- + z_-c_+) - (B_+D_- + D_+B_-)\frac{b_0}{m^2}a_+a_-\\
&+ B_+C_-\frac{a_+}{m^2}(2m^2z_- - 2a_0a_- + b_0a_- - z_-c_-) + C_+B_-\frac{a_-}{m^2}(2m^2z_+ - 2a_0a_+ + b_0a_+ - z_+c_+)\\
&+ C_+D_-\frac{a_-}{2m^2}[z_+(\zeta_0 - 4c_-) + 2b_+(2a_0 - b_0)] + D_+C_-\frac{a_+}{2m^2}[z_-(\zeta_0 - 4c_+) + 2b_-(2a_0 - b_0)]\bigg\}\end{aligned} \tag{B1}$$

$$\begin{aligned}\mathcal{A}_{n,2}(\Psi_+\Psi_-) = \frac{1}{m}\bigg\{&A_+A_-(b_+z_- + b_-z_+) - A_+B_-\frac{a_-}{m^2}[a_0b_+ + z_+(c_+ - c_-)] - B_+A_-\frac{a_+}{m^2}[a_0b_- + z_-(c_- - c_+)]\\
&+ D_+D_-a_+a_-\frac{b_0Q^2}{m^4} + C_+C_-\frac{1}{m^2}\big[\big(4m^2 + m_d^2\big)(b_+z_- + b_-c_+) + (b_0 - 2a_0)(2a_+b_- + 2a_-b_+ + Q^2z_0)\\
&- 2(b_0 - 2a_0)b_+b_- + Q^2(a_+z_- + a_-z_+) + 4(a_+z_- - a_-z_+)(c_+ - c_-) - 4(b_+c_+z_- + b_-c_-z_+)\big]\\
&- A_+C_-\frac{1}{2m^2}[4m^2(b_+z_- + b_-z_+) - 2b_0(a_+b_- - a_-b_+) - 2a_0b_+(2a_- - b_-) - Q^2(a_0z_0 - a_+z_- - a_-z_+)\end{aligned}$$

TABLE VI. Vector products that depend on n used in the expansions of $\mathcal{A}_{n,i}$. All are evaluated in the Breit frame using (3.5) and (3.6). The helicity amplitude $\mathcal{A}_{3,i} = \frac{1}{2}(\mathcal{A}_{3_+,i} + \mathcal{A}_{3_-,i})$, as explained in Eq. (3.14). Not shown are $\zeta_0 = 2m_d^2 + Q^2$ and $c_\pm = P_\pm \cdot k = D_0 E_k \mp \frac{1}{2} k_z Q$, which are the same for all helicity combinations.

coefficient	$n = 1\ (J^0_{00})$	$n = 2\ (J^0_{+-})$	$n = 3_+\ (J^+_{+0})$	$n = 3_-\ (J^-_{0-})$
$a_+ = k \cdot \xi^*$	$(E_k Q - 2k_z D_0)/(2m_d)$	$\frac{1}{\sqrt{2}}(k_x - ik_y)$	$\frac{1}{\sqrt{2}}(k_x - ik_y)$	$(E_k Q - 2k_z D_0)/(2m_d)$
$a_- = k \cdot \xi'$	$-(E_k Q + 2k_z D_0)/(2m_d)$	$\frac{1}{\sqrt{2}}(k_x + ik_y)$	$-(E_k Q + 2k_z D_0)/(2m_d)$	$\frac{1}{\sqrt{2}}(k_x + ik_y)$
$a_0 = k \cdot \epsilon$	E_k	E_k	$\frac{1}{\sqrt{2}}(k_x + ik_y)$	$-\frac{1}{\sqrt{2}}(k_x - ik_y)$
$b_+ = P_- \cdot \xi^*$	$D_0 Q/m_d$	0	0	$D_0 Q/m_d$
$b_- = P_+ \cdot \xi'$	$-D_0 Q/m_d$	0	$-D_0 Q/m_d$	0
$b_0 = P_+ \cdot \epsilon = P_- \cdot \epsilon$	D_0	D_0	0	0
$z_+ = \epsilon \cdot \xi^*$	$Q/(2m_d)$	0	-1	0
$z_- = \epsilon \cdot \xi'$	$-Q/(2m_d)$	0	0	1
$z_0 = \xi^* \cdot \xi'$	$-\zeta_0/(2m_d^2)$	-1	0	0

$$
\begin{aligned}
&+2(c_+ - c_-)(b_0 z_0 - 2a_- z_+) - 2b_+ c_+ z_- - 2b_- c_- z_+] + A_+ D_- \frac{a_-}{2m^2}[z_+ Q^2 + 2b_0\, b_+] \\
&-C_+ A_- \frac{1}{2m^2}[4m^2(b_- z_+ + b_+ z_-) - 2b_0(a_+ b_+ - a_+ b_-) - 2a_0 b_-(2a_+ - b_+) - Q^2(a_0 z_0 - a_- z_+ - a_+ z_-) \\
&-2(c_+ - c_-)(b_0 z_0 - 2a_+ z_-) - 2b_- c_- z_+ - 2b_+ c_+ z_-] + D_+ A_- \frac{a_+}{2m^2}[z_- Q^2 + 2b_0\, b_-] \\
&+B_+ C_- \frac{a_+}{2m^2}[z_-(Q^2 - 4(c_+ - c_-)) - 2b_-(b_0 - 2a_0)] + C_+ B_- \frac{a_-}{2m^2}[z_+(Q^2 + 4(c_+ - c_-)) - 2b_+(b_0 - 2a_0)] \\
&+C_+ D_- \frac{a_-}{m^4}\left[a_0\left(a_+ Q^2 - b_+ m_d^2\right) + b_0\left(2b_+(c_+ - m^2) - a_+(Q^2 + 2c_+ - 2c_-) - z_+\left(m^2 Q^2 + m_d^2(c_+ - c_-)\right)\right)\right] \\
&+D_+ C_- \frac{a_+}{m^4}\left[a_0\left(a_- Q^2 - b_- m_d^2\right) + b_0(2b_-(c_- - m^2) - a_-(Q^2 - 2c_+ + 2c_-) - z_-\left(m^2 Q^2 - m_d^2(c_+ - c_-)\right)\right)\right] \\
&-B_+ D_- \frac{a_+ a_-}{2m^4}[a_0 Q^2 - 2b_0(c_+ - c_-)] - D_+ B_- \frac{a_+ a_-}{2m^4}[a_0 Q^2 + 2b_0(c_+ - c_-)]\Bigg\},
\end{aligned}
\tag{B2}
$$

where the vector products needed for this expansion are defined in Table VI. The results for the the traces $\mathcal{A}_{n,3}$ are obtained by the substitutions $A \to F, B \to G, C \to H, D \to I$ in $\mathcal{A}_{n,1}$. These expressions are sums of products of invariant functions and four-vector scalar products and hence are manifestly covariant.

In the terms above, the spectator momentum k is always on shell. In this case the arguments (2.14) of the wave functions for the incoming and outgoing deuterons become

$$
\begin{aligned}
R_\pm^2 &= \frac{(P_\pm \cdot k)^2}{m_d^2} - m^2 = \mathbf{k}^2 \mp k_z\, Q \frac{D_0 E_k}{m_d^2} + \eta\left(E_k^2 + k_z^2\right) \\
E_{R_\pm} &= \sqrt{m^2 + R_\pm^2}.
\end{aligned}
\tag{B3}
$$

Careful examination of the formulas for $\mathcal{A}$ show that they are unchanged under the transformation $+ \leftrightarrow -$. For $n = 1$ helicity amplitudes, the plus and minus coefficients transform into each other as $Q \to -Q$ (as do the arguments of the wave functions), so that the $\mathcal{A}_{1,i}$ satisfy the symmetry property (3.27) by inspection. For the $n = 2$ helicity amplitudes, the $a_\pm$ to not change with Q, but since the $b_\pm$ and $z_\pm$ coefficients are zero in this case, the terms that remain contain either no factors of $a_\pm$ or the product $a_+ a_-$, preserving the symmetry in Q. Finally, the separate terms $n = 3_\pm$ show no special symmetry, but it can be shown that their sum again satisfies the symmetry (3.27) appropriate to the $n = 3$ amplitude.

Although the expressions for $\mathcal{A}$ are given for identical wave functions in initial and final states, this property has not been used in the derivation of the equations and they can easily be extended to the case when $\mathcal{A} \to \mathcal{A}(\Psi_+ \Psi_-^{(2)})$ needed for the calculation of the interaction current terms. Consider the operation of changing the sign of Q in a typical term. Using the fact that the arguments $R_\pm^2 \to R_\mp^2$ when $Q \to -Q$, a typical pair of terms in the expansions for $\mathcal{A}$ transforms to

$$
\begin{aligned}
&Z_+ Y_-^{(2)} C_{ZY}(Q) \pm Y_+ Z_-^{(2)} C_{YZ}(Q) \\
&\quad \to Z_- Y_+^{(2)} C_{ZY}(-Q) \pm Y_- Z_+^{(2)} C_{YZ}(-Q) \\
&\quad = \pm[Z_+^{(2)} Y_- C_{YZ}(-Q) \pm Y_+^{(2)} Z_- C_{ZY}(-Q)].
\end{aligned}
\tag{B4}
$$

Using the symmetry properties just discussed, the coefficients have the property

$$
C_{ZY}(Q) = \pm \epsilon_{n3} C_{YZ}(-Q), \tag{B5}
$$

conforming to the symmetry properties used in (3.28). This simplifies the calculations of the interaction current contributions.

2. Canceling singular contributions from the (B) diagrams

Here the traces (3.35a) needed for the B contributions are evaluated. In these terms k_0 is not fixed until the subtraction shown in Eq. (3.36) is carried out. The results for the traces that depend on k_0 are

$$\begin{aligned}
\mathcal{B}_{n,1}(k_0) = \frac{\zeta}{16m^2}\Big\{ & -2\widetilde{F}_+\widetilde{F}_-[2z_0X_2 + X_3] + \widetilde{G}_+\widetilde{G}_-\frac{1}{m^2}(2\tilde{a}_- - \tilde{b}_-)(2\tilde{a}_+ - \tilde{b}_+)[X_2 - 16m^2\tilde{a}_0] \\
& + 2\widetilde{H}_+\widetilde{H}_-\frac{1}{m^2}X_1[2z_0X_2 + X_3 - 16m^2(2a_0z_0 - b_+z_- - b_-z_+)] - \widetilde{I}_+\widetilde{I}_-\frac{1}{m^4}X_1X_2(2\tilde{a}_- - \tilde{b}_-)(2\tilde{a}_+ - \tilde{b}_+) \\
& + 2\widetilde{F}_+\widetilde{G}_-(2a_- - b_-)[Y_1^+ + 8\tilde{a}_0(\tilde{a}_+ - \tilde{b}_+) + 8\tilde{b}_0\tilde{b}_+ - 4z_+c_q] \\
& + 2\widetilde{G}_+\widetilde{F}_-(2a_+ - b_+)[Y_1^- + 8\tilde{a}_0(\tilde{a}_- - \tilde{b}_-) + 8\tilde{b}_0\tilde{b}_- + 4z_-c_q] \\
& + 16(\widetilde{F}_+\widetilde{H}_- + \widetilde{H}_+\widetilde{F}_-)X_1(2a_0z_0 - b_+z_- - b_-z_+) - 2(\widetilde{G}_+\widetilde{H}_- + \widetilde{I}_+\widetilde{F}_-)X_1\frac{(2\tilde{a}_+ - \tilde{b}_+)}{m^2}[Y_1^- + 4\tilde{b}_0\tilde{b}_-] \\
& - 2(\widetilde{H}_+\widetilde{G}_- + \widetilde{F}_+\widetilde{I}_-)X_1\frac{(2\tilde{a}_- - \tilde{b}_-)}{m^2}[Y_1^+ + 4\tilde{b}_0\tilde{b}_+] \\
& + 8(\widetilde{G}_+\widetilde{I}_- + \widetilde{I}_+\widetilde{G}_-)X_1\frac{\tilde{a}_0}{m^2}(2\tilde{a}_- - \tilde{b}_-)(2\tilde{a}_+ - \tilde{b}_+) + 2\widetilde{I}_+\widetilde{H}_-X_1\frac{(2\tilde{a}_+ - \tilde{b}_+)}{m^2}[Y_1^- - 8\tilde{a}_0(\tilde{a}_- - \tilde{b}_-) - 4c_qz_-] \\
& + 2\widetilde{H}_+\widetilde{I}_-X_1\frac{(2\tilde{a}_- - \tilde{b}_-)}{m^2}[Y_1^+ - 8\tilde{a}_0(\tilde{a}_+ - \tilde{b}_+) + 4c_qz_+]\Big\} \qquad \text{(B6)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{n,2}(k_0) = \frac{\zeta}{16m^2}\Big\{ & 2\widetilde{F}_+\widetilde{F}_-(2X_4 - X_5) + 2\widetilde{G}_+\widetilde{G}_-\frac{Q^2}{m^2}(2\tilde{a}_0 - \tilde{b}_0)(2\tilde{a}_+ - \tilde{b}_+)(2\tilde{a}_- - \tilde{b}_-) + 2\widetilde{H}_+\widetilde{H}_-\frac{1}{m^2}X_1X_5 \\
& + 2\widetilde{I}_+\widetilde{I}_-X_1\frac{b_0\,Q^2}{m^4}(2\tilde{a}_+ - \tilde{b}_+)(2\tilde{a}_- - \tilde{b}_-) - 2(\widetilde{F}_+\widetilde{H}_- + \widetilde{H}_+\widetilde{F}_-)\frac{1}{m^2}X_1X_4 \\
& - \widetilde{F}_+\widetilde{G}_-\frac{1}{m^2}(2\tilde{a}_- - \tilde{b}_-)(Y_+^2 + 4m^2z_+Q^2) - \widetilde{G}_+\widetilde{F}_-\frac{1}{m^2}(2\tilde{a}_+ - \tilde{b}_+)(Y_-^2 + 4m^2z_-Q^2) \\
& + 4(\widetilde{F}_+\widetilde{I}_- + \widetilde{H}_+\widetilde{G}_-)\frac{Q^2}{m^2}z_+(2\tilde{a}_- - \tilde{b}_-)X_1 + 4(\widetilde{I}_+\widetilde{F}_- + \widetilde{G}_+\widetilde{H}_-)\frac{Q^2}{m^2}z_-(2\tilde{a}_+ - \tilde{b}_+)X_1 \\
& - 2(\widetilde{G}_+\widetilde{I}_- + \widetilde{I}_+\widetilde{G}_-)X_1\frac{\tilde{a}_0\,Q^2}{m^4}(2\tilde{a}_- - \tilde{b}_-)(2\tilde{a}_+ - \tilde{b}_+) + \widetilde{H}_+\widetilde{I}_-\frac{1}{m^2}(2\tilde{a}_- - \tilde{b}_-)(Y_+^2 - 4m^2z_+Q^2)X_1 \\
& + \widetilde{I}_+\widetilde{H}_-\frac{1}{m^2}(2\tilde{a}_+ - \tilde{b}_+)(Y_-^2 - 4m^2z_-Q^2)X_1, \qquad \text{(B7)}
\end{aligned}$$

where the new vector products are defined in Tables VII and VIII, and use has been made of the compact notation $\widetilde{Z}_+ = \widetilde{Z}(\tilde{R}_+, R_0^+)$ and $\widetilde{Z}_- = \widetilde{Z}(\tilde{R}_-, R_0^-)$ (where $\widetilde{Z}$ is the generic name for the reduced vertex functions $\{\widetilde{F},\widetilde{G},\widetilde{H},\widetilde{I}\}$) and the vertex function arguments $\tilde{R}_\pm$ and $R_0^\pm$ were defined in (2.14). These arguments depend on both k_0 and Q. Recalling that $\tilde{k}_\pm = \{k_0,\mathbf{k}_\pm\}$, with $\mathbf{k}_\pm = \mathbf{k} \pm \mathbf{q}/2$, the arguments of the incoming and outgoing vertex functions are

$$\begin{aligned}
\tilde{R}_\pm^2 &= \frac{(P_\pm \cdot \tilde{k}_\pm)^2}{m_d^2} - k_0^2 + \mathbf{k}_\pm^2 \\
&= \mathbf{k}_\pm^2 \mp (k_\pm)_z k_0\frac{Q D_0}{m_d^2} + \frac{Q^2}{4m_d^2}\big[k_0^2 + (k_\pm)_z^2\big] \\
R_0^\pm &= \frac{1}{2m_d}[2D_0k_0 \mp (k_\pm)_z Q]. \qquad \text{(B8)}
\end{aligned}$$

Note that $\tilde{R}_\pm^2$ [which is not the same as the $R_\pm^2$ of Eq. (B3)] depends on Qk_0, so that all k_0 dependence vanishes when $Q = 0$, and that in this limit, the arguments reduce to $\mathbf{k}^2$ and k_0. The denominator of ζ contains an additional k_0 dependence through the factor of $m^2 - \widetilde{p}^2 = E_k^2 - (D_0 - k_0)^2$.

The symmetry (3.35a) of the $\mathcal{B}$'s under the transformation $Q \to -Q$ can be confirmed using arguments similar to those used for the $\mathcal{A}$'s.

TABLE VII. Vector products that depend on n used in the expansions of $\mathcal{B}_{n,i}$. All are evaluated in the Breit frame using (3.5) and (3.6). The helicity amplitude $\mathcal{B}_{3,i} = \frac{1}{2}(\mathcal{B}_{3_+,i} + \mathcal{B}_{3_-,i})$, as explained in Eq. (3.14). Not shown are $\zeta = h^2(\tilde{p})/(m^2 - \tilde{p}^2)$, $\zeta_1 = D^2 = D_0^2$, $c_0 = D \cdot \tilde{k} = D_0 k_0$, and $c_q = q \cdot \tilde{k} = -Qk_z$, which are the same for all helicity combinations. Convenient combinations of these vector products are given in Table VIII.

coefficient	$n = 1\ (J^0_{00})$	$n = 2\ (J^0_{+-})$	$n = 3_+\ (J^+_{+0})$	$n = 3_-\ (J^-_{0-})$
$\tilde{a}_+ = \tilde{k} \cdot \xi^*$	$(k_0 Q - 2k_z D_0)/(2m_d)$	$\frac{1}{\sqrt{2}}(k_x - ik_y)$	$\frac{1}{\sqrt{2}}(k_x - ik_y)$	$(k_0 Q - 2k_z D_0)/(2m_d)$
$\tilde{a}_- = \tilde{k} \cdot \xi'$	$-(k_0 Q + 2k_z D_0)/(2m_d)$	$\frac{1}{\sqrt{2}}(k_x + ik_y)$	$-(k_0 Q + 2k_z D_0)/(2m_d)$	$\frac{1}{\sqrt{2}}(k_x + ik_y)$
$\tilde{a}_0 = \tilde{k} \cdot \epsilon$	k_0	k_0	$\frac{1}{\sqrt{2}}(k_x + ik_y)$	$-\frac{1}{\sqrt{2}}(k_x - ik_y)$
$\tilde{b}_+ = -q \cdot \xi^*$	$D_0 Q/m_d$	0	0	$D_0 Q/m_d$
$\tilde{b}_- = q \cdot \xi'$	$-D_0 Q/m_d$	0	$-D_0 Q/m_d$	0
$\tilde{b}_0 = D \cdot \epsilon$	D_0	D_0	0	0
$z_+ = \epsilon \cdot \xi^*$	$Q/(2m_d)$	0	-1	0
$z_- = \epsilon \cdot \xi'$	$-Q/(2m_d)$	0	0	1
$z_0 = \xi^* \cdot \xi'$	$-\zeta_0/(2m_d^2)$	-1	0	0

3. Regular contributions from the (B) diagrams

The results for the $\mathcal{C}_{n,i}$ traces that involve the four invariant functions K_i (contributing to $\Gamma_{\rm off}$ in the initial state) are

$$\begin{aligned}
\mathcal{C}_{n,1}(\Gamma\,\Gamma_{\rm off}) = \frac{\zeta_B}{2m^2}\Big\{&-4\widetilde{F}_+K_1 m^2(\tilde{b}_0\, z_0 + 2a_+z_- - \tilde{b}_- z_+) + 2\widetilde{F}_+K_2(a_- - \tilde{b}_-)[2a_+(2a_0 - \tilde{b}_0) + z_+(2c_0' + c_q)]\\
&+ (\widetilde{F}_+K_3 + \widetilde{H}_+K_1)T_1T_2 - (\widetilde{F}_+K_4 + \widetilde{H}_+K_2)T_1 z_+(a_- - \tilde{b}_-)\\
&+ 2\widetilde{G}_+K_1 a_+[z_-(4m^2 - 2c_0' - c_q) + 2a_-\tilde{b}_0 - 2a_0\,\tilde{b}_-] - 4\widetilde{G}_+K_2 a_+(2a_0 - \tilde{b}_0)(a_- - \tilde{b}_-)\\
&- (\widetilde{G}_+K_3 + \widetilde{I}_+K_1)T_1 a_+ z_- + (\widetilde{G}_+K_4 + \widetilde{I}_+K_2)\frac{a_0}{m^2}T_1 a_+(a_- - \tilde{b}_-)\\
&- \widetilde{H}_+K_3T_1[z_0(2a_0 - \tilde{b}_0) - z_+(2a_- - \tilde{b}_-)] + \widetilde{H}_+K_4\frac{a_- - \tilde{b}_-}{2m^2}T_1[z_+(4m^2 - 2c_0' - c_q) - 2a_+(2a_0 - \tilde{b}_0)]\\
&- \widetilde{I}_+K_3\frac{a_+}{2m^2}T_1[2a_-\tilde{b}_0 - 2a_0\,\tilde{b}_- - z_-(2c_0' + c_q)] - \widetilde{I}_+K_4\frac{\tilde{b}_0}{m^2}T_1 a_+(a_- - \tilde{b}_-)\Big\}
\end{aligned} \tag{B9}$$

$$\begin{aligned}
\mathcal{C}_{n,2}(\Gamma\,\Gamma_{\rm off}) = \frac{\zeta_B}{2m}\Big\{&-2\widetilde{F}_+K_1T_3 + 2\widetilde{F}_+K_2(a_- - \tilde{b}_-)(2\tilde{b}_0\,\tilde{b}_+ - Q^2 z_+) - \widetilde{F}_+K_3T_1(\tilde{b}_+z_- + \tilde{b}_-z_+)\\
&- (\widetilde{F}_+K_4 + \widetilde{H}_+K_2)\frac{a_- - \tilde{b}_-}{m^2}T_1(a_0\,\tilde{b}_+ + c_q z_+) - 2\widetilde{G}_+K_1 a_+[2\tilde{b}_-(2a_0 - \tilde{b}_0) - z_-(Q^2 + 4c_q)]\\
&+ 2\widetilde{G}_+K_2 a_+(a_- - \tilde{b}_-)(a_0\,Q^2 + 2\tilde{b}_0\,c_q) + (\widetilde{G}_+K_3 + \widetilde{I}_+K_1)\frac{a_+}{m^2}T_1(a_0\,\tilde{b}_- - c_q z_-) - \widetilde{H}_+K_1T_1(\tilde{b}_+z_- + \tilde{b}_-z_+)\\
&+ \widetilde{H}_+K_3\frac{1}{2m^2}T_1[4m^2(\tilde{b}_+z_- + \tilde{b}_-z_+) + T_3] + \widetilde{H}_+K_4\frac{a_- - \tilde{b}_-}{2m^2}T_1[2\tilde{b}_+(2a_0 - \tilde{b}_0) + z_+(Q^2 + 4c_q)]\\
&- \widetilde{I}_+K_3\frac{a_+}{2m^2}T_1(z_-Q^2 + 2\tilde{b}_0\,\tilde{b}_-) - \widetilde{I}_+K_4\frac{a_+}{2m^4}T_1(a_- - \tilde{b}_-)(a_0\,Q^2 + 2\tilde{b}_0\,c_q)\Big\},
\end{aligned} \tag{B10}$$

where the vector products that enter into these formulas are defined in Tables VI, VII, and IX, $m^2 - p_+^2 = m^2 - (P_+ - k)^2 = 2D_0E_k - m_d^2 - Qk_z$, $K_i = K_i(\hat{R}_-, \hat{R}_0^-)$, and the final state is on shell, so that $\tilde{Z}$ depends on only one argument $\tilde{Z}_+ = Z(R_+)$.

These terms are finite, so calculations of the static moments require them to order Q^2 only. The arguments of the K_i are

$$\begin{aligned}
\hat{R}_-^2 &= \frac{1}{m_d^2}\Big[D_0E_k + \frac{1}{2}(k_z - Q)\,Q\Big]^2 - (m^2 + 2k_zQ - Q^2)\\
&\to \mathbf{k}^2 - \frac{k_zQ}{m_d}(2m_d - E_k) + \eta\big[(2m_d - E_k)^2 + k_z^2\big]
\end{aligned}$$

$$\begin{aligned}
\hat{R}_0^- &= \frac{1}{m_d}\Big[D_0E_k + \frac{1}{2}(k_z - Q)\,Q\Big]\\
&\to E_k + \frac{k_zQ}{2m_d} + \frac{1}{2}\eta(E_k - 4m_d).
\end{aligned} \tag{B11}$$

The argument of the Z_+ is

$$\begin{aligned}
R_+^2 &= \frac{1}{m_d^2}\Big[D_0E_k - \frac{1}{2}k_zQ\Big]^2 - m^2\\
&\to \mathbf{k}^2 - \frac{k_zQ}{m_d}E_k + \eta\big(E_k^2 + k_z^2\big).
\end{aligned} \tag{B12}$$

TABLE VIII. Combinations of vector products from Table VII that simplify Eqs. (B6) and (B7).

$Y_1^\pm = 4z_\pm(m^2-\tilde k^2) - z_\pm Q^2 + 8\tilde a_0\tilde a_\pm - 4\tilde b_0\tilde b_\pm$

$Y_2^\pm = 4(m^2-\tilde k^2)(\tilde a_0\tilde b_\pm \pm c_q z_\pm) - 4\tilde b_0\tilde b_\pm(m^2+\tilde k^2) + 8\tilde a_0\tilde b_\pm c_0 \mp c_q[8\tilde b_0\tilde a_\pm - \tilde z_\pm(8c_0-Q^2)] + Q^2[\tilde b_0\tilde b_\pm + \tilde a_0(4\tilde a_\pm - 3\tilde b_\pm)]$

$X_1 = m^2 - \tilde k^2 + 2c_0 - \zeta_1$

$X_2 = 4(m^2-\tilde k^2)(\tilde a_0+\tilde b_0) + Q^2(\tilde a_0 - \tilde b_0) + 8\tilde a_0 c_0$

$X_3 = 4(m^2-\tilde k^2)[z_+(2\tilde a_- - 3\tilde b_-) + z_-(2\tilde a_+ - 3\tilde b_+)] - 8(2\tilde a_0 - \tilde b_0)(\tilde a_+\tilde b_- + \tilde a_-\tilde b_+) - Q^2[z_+(2\tilde a_- - \tilde b_-) + z_-(2\tilde a_+ - \tilde b_+)]$
$\quad + 32\tilde a_0\,\tilde a_+\tilde a_- + 8c_q(\tilde a_+ z_- - \tilde a_- z_+) - 8c_0(\tilde b_+ z_- + \tilde b_- z_+)$

$X_4 = (\tilde b_+ z_- + \tilde b_- z_+)[4(m^2+\tilde k^2) - Q^2] - 4\tilde a_0(2\tilde a_+\tilde b_- + 2\tilde a_-\tilde b_+ - z_0\,Q^2) - 8c_q(\tilde a_- z_+ - \tilde a_+ z_-)$

$X_5 = -8(\tilde a_0 - \tilde b_0)(\tilde a_+\tilde b_- + \tilde a_-\tilde b_+) - 8(c_0 - \tilde k^2)(\tilde b_+ z_- + \tilde b_- z_+) + Q^2[4z_0(\tilde a_0 - \tilde b_0) - 2z_-(2\tilde a_+ - \tilde b_+) - 2z_+(2\tilde a_- - \tilde b_-)]$
$\quad - 8c_q(\tilde a_- z_+ - \tilde a_+ z_-)$

APPENDIX C: CHARGE

In this Appendix, the charge is evaluated by taking the $Q^2 = 0$ limit of the contributions from Eqs. (B1), (B6), and (B9). The results from this Appendix were collected in Sec. IV and discussed in Sec. I C. Here, for simplicity, we return to the notation $\mathbf{k}^2 \to k^2$.

1. (A) contributions

At $Q^2 = 0$, $Z_\pm = Z(k)$ and all $\mathcal{A}_{n,2} = 0$. Averaging over θ using $\langle k_z^2\rangle = \langle k_x^2\rangle = \langle k_y^2\rangle = \frac{1}{3}\langle k^2\rangle$ gives

$$\begin{aligned}\mathcal{A}_{1,1} = \mathcal{A}_{2,1} = \frac{2E_k}{m}\bigg\{ & A^2 + \frac{k^2}{3m^2}\left[B^2 + m_R^2 D^2 - \frac{2m_d}{E_k}BD\right] \\ & + C^2\left[4 + m_R^2 + \frac{4k^2}{3m^2} - \frac{4m_d}{E_k}\left(1+\frac{k^2}{3m^2}\right)\right] \\ & + \frac{2m_d k^2}{3m^2E_k}AD - 2AC\left(2 - \frac{m_d}{E_k}\right) \\ & - 2BC\frac{k^2}{3m^2}\left(2-\frac{m_d}{E_k}\right)\bigg\} \\ \mathcal{A}_{1,3} = \mathcal{A}_{2,3} = \frac{E_k}{m}\bigg\{ & F^2 + \frac{k^2}{3m^2}\left[G^2 + m_R^2 I^2 - \frac{2m_d}{E_k}GI\right] \\ & + H^2\left[4 + m_R^2 + \frac{4k^2}{3m^2} - \frac{4m_d}{E_k}\left(1+\frac{k^2}{3m^2}\right)\right] \\ & + \frac{2m_d k^2}{3m^2E_k}FI - 2FH\left(2-\frac{m_d}{E_k}\right) \\ & - 2GH\frac{k^2}{3m^2}\left(2-\frac{m_d}{E_k}\right)\bigg\}. \end{aligned} \tag{C1}$$

TABLE IX. Combinations of vector products used in the expansions of $\mathcal{C}_{n,i}$. The only new terms are $\zeta_B = h^2(p_+)/(m^2 - p_+^2)$ and $c_0' = D\cdot k = D_0E_k$; the $\tilde b$'s are taken from Table VII and the others from Table VI.

$T_1 = Q^2 + 8c_0' + 4c_q - 4\zeta_1$

$T_2 = a_0z_0 + a_+z_- - a_-z_+$

$T_3 = Q^2T_2 - 2a_0\,\tilde b_-(2a_+ + \tilde b_+) + 2\tilde b_0(a_+\tilde b_- + a_-\tilde b_+ + c_qz_0)$
$\quad - 2c_0'(\tilde b_+z_- + \tilde b_-z_+) + c_q(4a_+z_- - \tilde b_+z_- - 3\tilde b_-z_+)$

The contributions from the ($\mathrm{A}_\pm$) diagrams (referred to as the $\langle V_2^\mu\rangle$ part of the exchange current in Ref. [1]) can be easily added. Using the symmetry relation (3.27) at $Q = 0$, the generic XY term in the expansions (C1) can be transformed as follows:

$$\begin{aligned} c_0XY &\to \tfrac{1}{2}c_0(X^fY^i + Y^fX^i) \\ &\to \to c_0(XY^{(2)} + YX^{(2)}), \end{aligned} \tag{C2}$$

where c_0 is independent of Q. The first step uses the symmetry relation to uncover the structure of the generic XY term in the case when the initial and final wave functions are not identical; the symmetry relation guarantees that this replacement is unique. Then, the second step merely applies the result to the special case when the generic final-state functions are X and the generic initial-state functions are $X^{(2)}$. The two terms in (3.28) are identical in this case, giving a factor of 2.

2. (B) contributions

These contributions are obtained from Eq. (3.36) and the traces $\mathcal{B}_{n,i}$ (B6) and the traces $\mathcal{C}_{n,i}$ (B9). The magnetic terms (B7) and (B10) are zero and do not contribute.

A correct calculation of the singular term contribution to the charge requires expansion of the invariants to order Q. Only the k_0 dependence coming from argument $R_0^\pm$ will contribute, and expanding around $k_0 = E_k$ gives

$$\widetilde Z_\pm = \widetilde Z + (E_k - k_0)\widetilde Z_{k_0}, \tag{C3}$$

where $\widetilde Z_{k_0} = d\widetilde Z/(dk_0)$ evaluated at $k_0 = E_k$. Hence

$$\begin{aligned}\lim_{Q\to 0}\frac{mE_k}{k_zQ}&\left[\frac{\mathcal{B}_{1,1}(k_0)}{k_0}\bigg|_- - \frac{\mathcal{B}_{1,1}(k_0)}{k_0}\bigg|_+\right], \\ &= \mathcal{I}_Z + \mathcal{I}_{Z'}, \end{aligned} \tag{C4}$$

where $\mathcal{I}_{Z'}$ includes derivatives of the vertex functions Z, and $\mathcal{I}_Z$ all of the rest [including contributions from the k_0 expansion of the strong form factors $h(\tilde p)$]. The results for $\mathcal{I}_Z$ and $\mathcal{I}_{Z'}$ are

$$\begin{aligned}\mathcal{I}_Z &= \frac{2E_k}{m\,\delta_k^2}\left[F^2+\frac{k^2}{3m^2}G^2+\frac{4k^2}{3m_d^2}(F-G)^2\left(1-\frac{m_d}{E_k}\right)-\frac{\delta_k^2}{m^2}\left(H^2+\frac{k^2}{3m^2}I^2\right)\right]-\frac{8E_k}{m\,\delta_k}a(p^2)(E_k-m_d)\left\{F^2+\frac{k^2}{3m^2}G^2\right.\\ &\quad\left.-\frac{2k^2}{3E_km_d}(F-G)^2-\frac{2\delta_k}{E_k}\left[FH+\frac{k^2}{3m^2}(FI-H^2+GH-GI)\right]-\frac{\delta_km_d}{m^2}\left[\frac{k^2}{3m^2}I^2+\left(1-\frac{2m^2}{m_dE_k}\right)H^2\right]\right\}\\ \mathcal{I}_{Z'} &= -\frac{4E_k}{m\,\delta_k}\left\{FF_{k_0}+\frac{k^2}{3m^2}GG_{k_0}-\frac{2k^2}{3E_km_d}(F-G)(F_{k_0}-G_{k_0})-\frac{\delta_km_d}{m^2}\left[\frac{k^2}{3m^2}II_{k_0}+\left(1-\frac{2m^2}{m_dE_k}\right)HH_{k_0}\right]\right.\\ &\quad\left.-\frac{\delta_k}{E_k}\left[FH_{k_0}+F_{k_0}H+\frac{k^2}{3m^2}(FI_{k_0}+F_{k_0}I-2HH_{k_0}+GH_{k_0}+G_{k_0}H-GI_{k_0}-G_{k_0}I)\right]\right\},\end{aligned}\tag{C5}$$

where the strong form factor $h(\tilde p)$ [where h is evaluated at $\tilde p^2=m^2-m_d(2E_k-m_d)$] has been reabsorbed into the Z's (so that they may be expressed in terms of the u,w,v_t,v_s), $Z_{k_0}=h\,d\tilde Z/(dk_0)$, and the contributions to $\mathcal{I}_Z$ from the derivative of the strong form factor have been isolated in the term proportional to $a(p^2)$.

The contribution from the regular terms is straightforward:

$$\begin{aligned}\mathcal{I}_C = -\frac{2}{m}\mathcal{C}_{1,1} = -\frac{4}{m\,\delta_k}\left\{FK_1-\frac{E_k\delta_k}{m^2}(HK_1+FK_3)+\frac{\delta_k^2}{m^2}HK_3+\frac{k^2}{3m^2}GK_1\right.\\ \left.+\frac{k^2\delta_k}{3m^4m_d}\left[m^2(F-G)K_2-m_d^2I(K_3+K_4)+E_km_d(GK_4+IK_2)-m_d\delta_kHK_4\right]\right\}.\end{aligned}\tag{C6}$$

3. Expressions in terms of the wave functions z_ℓ

Expanding the Z in terms of the wave functions $z_\ell(k)$ (where z_ℓ is the generic name for $\{u,w,v_t,v_s\}$) using (2.11) and (A30), reduces (C1) to the following simple forms

$$\begin{aligned}\mathcal{A}_{n,1} &= 4\pi^2m_d\frac{E_k}{m}\{u^2+w^2+v_t^2+v_s^2\}\\ \mathcal{A}_{n,3} &= 4\pi^2m_d\frac{E_k}{m}\{\delta_k^2(u^2+w^2)+m_d^2(v_t^2+v_s^2)\}.\end{aligned}\tag{C7}$$

Using (C2), the the result for the interaction current contribution is

$$\mathcal{A}_{n,1}^{(2)} = 4\pi^2m_d\frac{E_k}{m}\{2uu^{(2)}+2ww^{(2)}+2v_tv_t^{(2)}+2v_sv_s^{(2)}\},\tag{C8}$$

where $z_\ell^{(2)}$ is the generic name for the wave functions that contribute to $\Psi^{(2)}$. The contribution of these terms to the in normalization condition is discussed in Sec. IV.

For the (B) contributions, the vertex functions Z are expanded in terms of the wave functions $z_\ell(k)$ using (A28) and (A29). This gives

$$\begin{aligned}\mathcal{I}_Z = 4\pi^2m_d\frac{E_k}{m}\left\{u^2+w^2-v_t^2-v_s^2+\sqrt{\frac{2}{3}}\frac{E_k+2m}{m_dk}[uv_t\,m_d+wv_s\,\delta_k]+\frac{2}{\sqrt3}\frac{E_k-m}{m_dk}[wv_t\,m_d-uv_s\,\delta_k]\right.\\ \left.-4a(p^2)(E_k-m_d)\left[\delta_k(u^2+w^2)-m_d(v_t^2+v_s^2)\right]\right\}.\end{aligned}\tag{C9}$$

To obtain the result for $\mathcal{I}_{Z'}$, the derivatives of the invariants must be evaluated using the general results (A27) which give the k_0 dependence of the invariants. These give

$$\begin{aligned}F_{k_0} &= \frac{\partial F}{\partial k_0}\Big|_{k_0=E_k} = \mathcal{K}\left[u-\frac{w}{\sqrt2}+\sqrt{\frac32}\frac{m}{k}v_t\right]\left(1-\frac{\delta_k^2}{2E_km_d}\right)+\mathcal{K}\delta_k\left[u_{k_0}-\frac{1}{\sqrt2}w_{k_0}+\sqrt{\frac32}\frac{m}{k}v_{tk_0}+\frac{\sqrt3}{2E_kk}(kz_1^{--}+mz_1^{-+})\right]\\ G_{k_0} &= \frac{\partial G}{\partial k_0}\Big|_{k_0=E_k} = \mathcal{K}m\left[\frac{u}{E_k+m}+(2E_k+m)\frac{w}{\sqrt2 k^2}+\sqrt{\frac32}\frac{v_t}{k}\right]\left(1-\frac{\delta_k^2}{2E_km_d}\right)\\ &\quad+\mathcal{K}m\,\delta_k\left[\frac{u_{k_0}}{E_k+m}+(2E_k+m)\frac{w_{k_0}}{\sqrt2 k^2}+\sqrt{\frac32}\frac{v_{tk_0}}{k}+\frac{\sqrt3}{2E_kk^2}(\sqrt2E_kz_0^{--}-mz_1^{--}+kz_1^{-+})\right]\end{aligned}$$

$$H_{k_0}=\frac{\partial H}{\partial k_0}\bigg|_{k_0=E_k}=\sqrt{\frac{3}{2}}\frac{\mathcal{K}E_k\,m}{k}\,v_{tk_0}-\sqrt{\frac{3}{2}}\frac{\mathcal{K}\delta_k\,m}{2k\,m_d}(v_t-\sqrt{2}\,z_1^{-+})$$

$$\begin{aligned}I_{k_0}=\frac{\partial I}{\partial k_0}\bigg|_{k_0=E_k}&=-\frac{2\mathcal{K}E_k\,m^2}{m_d^2}\left[\frac{u}{E_k+m}-(E_k+2m)\frac{w}{\sqrt{2}\,k^2}\right]\left(1-\frac{\delta_k^2}{2E_km_d}\right)+\frac{\sqrt{3}\,\mathcal{K}\delta_km^2}{2E_km_dk}v_s\\&\quad-\frac{\mathcal{K}\,m^2}{m_d}\left[\delta_k\left(\frac{u_{k_0}}{E_k+m}-(E_k+2m)\frac{w_{k_0}}{\sqrt{2}\,k^2}\right)+\sqrt{3}\,m_d\frac{v_{sk_0}}{k}\right]\\&\quad+\sqrt{\frac{3}{2}}\frac{\mathcal{K}m^2}{E_km_dk^2}\left[m_dmz_0^{--}-\delta_kkz_0^{-+}-\frac{E_km_d}{\sqrt{2}}z_1^{--}\right],\end{aligned}\tag{C10}$$

where $z_{\ell k_0}=h\,\partial\tilde{z}_\ell(k,k_0)/(\partial k_0)|_{k_0=E_k}$ and $z_\ell=h\,\tilde{z}_\ell$. Note the appearance of the negative ρ-spin helicity amplitudes for particle 1, referred to generically as χ_ℓ. Substituting these expressions and the expansions (A30) into $\mathcal{I}_{Z'}$ gives

$$\begin{aligned}\mathcal{I}_{Z'}=4\pi^2m_d\frac{E_k}{m}\bigg\{&\frac{\delta_k^2}{E_km_d}(u^2+w^2)-\sqrt{\frac{2}{3}}\frac{E_k+2m}{m_dk}[uv_t\,m_d+wv_s\,\delta_k]-\frac{2}{\sqrt{3}}\frac{E_k-m}{m_dk}[wv_t\,m_d-uv_s\,\delta_k]\\&-2(u[\delta_+u]_{k_0}+w[\delta_+w]_{k_0})+\left(2-\frac{\delta_k}{E_k}\right)\left(v_t^2+v_s^2\right)+2(v_t[\delta_-v_t]_{k_0}+v_s[\delta_-v_s]_{k_0})\\&-\sqrt{\frac{2}{3}}\frac{\delta_k}{E_k}[u(z_0^{--}+\sqrt{2}z_1^{--})+w(\sqrt{2}z_0^{--}-z_1^{--})-\sqrt{3}(v_tz_1^{-+}+v_sz_0^{-+})]\\&+\frac{2\delta_k(E_k-m_d)}{\sqrt{3}\,E_km_dk}z_1^{-+}[(E_k+2m)u+\sqrt{2}(E_k-m)w]+\frac{2(E_k-m_d)}{E_kk}v_s\left[\sqrt{2}mz_0^{--}-E_kz_1^{--}\right]\bigg\}.\end{aligned}\tag{C11}$$

where the new functions

$$\begin{aligned}\delta_+\{u,w\}&=(E_k+k_0-m_d)\{u,w\}\\\delta_-\{v_t,v_s\}&=(E_k-k_0+m_d)\{v_t,v_s\}\end{aligned}\tag{C12}$$

have been introduced. Finally, the contribution from $\mathcal{I}_C$ is obtained by substituting the expansions (A30) and (A31), giving

$$\begin{aligned}\mathcal{I}_C=-4\pi^2m_d\frac{E_k}{m}\bigg\{&\frac{\delta_k^2}{E_km_d}(u^2+w^2)-\frac{\delta_k}{E_k}\left(v_t^2+v_s^2\right)-\sqrt{\frac{2}{3}}\frac{\delta_k}{E_k}[u(z_0^{--}+\sqrt{2}z_1^{--})+w(\sqrt{2}z_0^{--}-z_1^{--})-\sqrt{3}(v_tz_1^{-+}+v_sz_0^{-+})]\\&+\frac{2\delta_k(E_k-m_d)}{\sqrt{3}\,E_km_dk}z_1^{-+}[(E_k+2m)u+\sqrt{2}(E_k-m)w]+\frac{2(E_k-m_d)}{E_kk}v_s[\sqrt{2}mz_0^{--}-E_kz_1^{--}]\bigg\}.\end{aligned}\tag{C13}$$

Note that all terms with particle 1 in a negative ρ-spin state cancel in $\mathcal{I}_{Z'}$ and $\mathcal{I}_C$. The charge is independent of the amplitudes $z_j^{-\pm}$. Finally, the sum of all the (B) terms is

$$\begin{aligned}\mathcal{I}_Z+\mathcal{I}_{Z'}+\mathcal{I}_C=4\pi^2m_d\frac{E_k}{m}\big\{&u^2+w^2+v_t^2+v_s^2-4a(p^2)(E_k-m_d)\left[\delta_k(u^2+w^2)-m_d\left(v_t^2+v_s^2\right)\right]\\&-2(u[\delta_+u]_{k_0}+w[\delta_+w]_{k_0})+2(v_t[\delta_-v_t]_{k_0}+v_s[\delta_-v_s]_{k_0})\big\}.\end{aligned}\tag{C14}$$

This result is discussed further in Sec. IV.

APPENDIX D: MAGNETIC MOMENT

1. (A) contributions

The contributions to the magnetic moment, in units of $e/(2m)$, from diagrams (A) and ($\text{A}_\pm$) are obtained from the limit

$$\begin{aligned}\mu_d=\lim_{Q\to0}\frac{m}{m_d}\frac{\mathcal{J}_3}{Q}\bigg|_{\text{A}+V_2}&=e_0\int_k\bigg\{f_{00}[\overline{M}_{1A}+\kappa_s\,\overline{M}_{2A}]\\&\quad+\frac{g_{00}}{4m^2}\overline{M}_{3A}-\overline{M}_1^{(2)}-\kappa_s\,\overline{M}_2^{(2)}\bigg\},\end{aligned}\tag{D1}$$

where $\kappa_s=\kappa_p+\kappa_n$ is the isoscalar anomalous moment of the nucleon and the $\overline{M}_{iA}=\overline{M}_{iA}(k)$ are the limits

$$\overline{M}_{iA}(k)=\frac{m}{m_d}\lim_{Q\to0}\frac{\mathcal{A}_{3,i}(k)}{Q}.\tag{D2}$$

Since the anomalous moment term (B2) is linear in Q, application of the symmetry condition (3.35a) gives

$$\lim_{Q\to0}\frac{I_{XY}^{(2),2}}{Q}\to c_1(XY^{(2)}+YX^{(2)})\tag{D3}$$

and hence $\overline{M}_2^{(2)}$ can be obtained directly from $\overline{M}_{2A}$, just as was done for the charge. To calculate the $\overline{M}_1^{(2)}$ term is more subtle, leading to the substitution

$$\lim_{Q\to 0}\frac{I_{XY}^{(2),1}}{Q} \to c_1(XY^{(2)}+YX^{(2)})+c_1'(X'Y^{(2)} + X^{(2)\prime}Y - XY^{(2)\prime} - X^{(2)}Y'), \tag{D4}$$

where $X^{(2)\prime}=dX^{(2)}/(dk)$, and c_1 and c_1' are additional factors. (More details can be found in Appendix E of the original, longer version (v1) of the present paper in the preprint archive [20].) This result displays the substitution rule, which for all terms (i.e., with or without the derivative) is

$$XY' \to XY^{(2)\prime}+X^{(2)}Y'. \tag{D5}$$

Since the exact result for the magnetic moment does not simplify as it did for the charge, the goal here is to understand the physical content of the leading terms only (those that are expected to be larger than about 0.001 nuclear magnetons). These terms are the products of the wave functions, including some products of one large (u,w) and one small (v_t,v_s) component multiplied by the enhancement m/k, and products of one wave function and one derivative, multiplied by m or k. In addition the leading corrections to order $\delta_E=(E_k-m)/E_k$ to products of u and w are retained. The results, expressed in terms of the wave functions $z_\ell(k)$ (where z_ℓ is the generic name for $\{u,w,v_t,v_s\}$), are

$$\overline{M}_{1A}(k)=2\pi^2\frac{E_k}{m}\left\{u^2+\frac14 w^2-\frac14 v_t^2-\frac12 v_s^2 + \frac{m}{\sqrt6}\left[m_1(k)+\frac12 m_2(k)\right]+\Delta M_1(k)\right\} \tag{D6}$$

$$\overline{M}_{2A}(k)=2\pi^2\frac{E_k}{m}\left\{u^2-\frac12 w^2-\frac12 v_t^2-\sqrt2 v_t v_s+\Delta M_2(k)\right\} \tag{D7}$$

$$\overline{M}_{3A}(k)=2\pi^2\frac{E_k}{m}\left\{m^2\left(\frac32 v_t^2+v_s^2+2\sqrt2 v_t v_s\right)\right\}, \tag{D8}$$

the interference terms are

$$\begin{aligned} m_1(k)&=\frac1k[u(v_t-\sqrt2 v_s)-2w(\sqrt2 v_t+v_s)]\\ m_2(k)&=uv_t'-u'v_t-\sqrt2(uv_s'-u'v_s-wv_t'+w'v_t)\\ &\quad+wv_s'-w'v_s, \end{aligned} \tag{D9}$$

and the standard notation $z_\ell'=dz_\ell/(dk)$ has been used. The leading corrections are

$$\begin{aligned} \Delta M_1(k)&\simeq -\frac{E_k-m}{3E_k}\left[u^2-w^2+\frac{1}{\sqrt2}uw\right]\\ \Delta M_2(k)&\simeq -\frac{E_k-m}{3E_k}\left[u^2+\frac12 w^2-\sqrt2 uw\right]. \end{aligned} \tag{D10}$$

The contributions from the $z^{(2)}$ wave functions can be obtained from $\overline{M}_{1A}$ and $\overline{M}_{2A}$ by the substitution (D5), but first we transform the expression for $\overline{M}_{1A}$. The interference terms can be rearranged and, recalling that the volume integral over k is k^2dk/E_k, integrated by parts, giving

$$\begin{aligned} \overline{M}_{1A}^{\,\rm int}(k)&=2\pi^2\frac{E_k}{m}\frac{m}{2\sqrt6}\left\{\left[u(v_t-\sqrt2 v_s)'-u'(v_t-\sqrt2 v_s)+\frac2k u(v_t-\sqrt2 v_s)\right]\right.\\ &\quad\left.+\left[w(\sqrt2 v_t+v_s)'-w'(\sqrt2 v_t+v_s)-\frac4k w(\sqrt2 v_t+v_s)\right]\right\}\\ &=2\pi^2\frac{E_k}{m}\frac{m}{\sqrt6}\left\{-u'(v_t-\sqrt2 v_s)+w\left[(\sqrt2 v_t+v_s)'-\frac1k(\sqrt2 v_t+v_s)\right]\right\}\equiv 2\pi^2\frac{E_k}{m}\,m^I(k). \end{aligned} \tag{D11}$$

The interference contribution from the $z^{(2)}$ wave functions can therefore be written

$$\begin{aligned} \overline{M}_{1A}^{\,\rm int(2)}(k)&=2\pi^2\frac{E_k}{m}\frac{m}{\sqrt6}\left\{-u^{(2)\prime}(v_t-\sqrt2 v_s)-u'(v_t-\sqrt2 v_s)^{(2)}+w^{(2)}\left[(\sqrt2 v_t+v_s)'-\frac1k(\sqrt2 v_t+v_s)\right]\right.\\ &\quad\left.+w\left[(\sqrt2 v_t+v_s)^{(2)\prime}-\frac1k(\sqrt2 v_t+v_s)^{(2)}\right]\right\}\\ &=2\pi^2\frac{E_k}{m}\frac{m}{\sqrt6}\left\{u^{(2)}(v_t-\sqrt2 v_s)'-u'(v_t-\sqrt2 v_s)^{(2)}+\frac2k u^{(2)}(v_t-\sqrt2 v_s)+w^{(2)}(\sqrt2 v_t+v_s)'\right.\\ &\quad\left.-w'(\sqrt2 v_t+v_s)^{(2)}-\frac1k\left[w^{(2)}(\sqrt2 v_t+v_s)+3w(\sqrt2 v_t+v_s)^{(2)}\right]\right\}\equiv 2\pi^2\frac{E_k}{m}\,m^{I(2)}(k). \end{aligned} \tag{D12}$$

With these definitions, the total contributions from the $z^{(2)}$ wave functions are

$$\overline{M}_1^{(2)}(k)=2\pi^2\frac{E_k}{m}\left\{2uu^{(2)}+\frac12 ww^{(2)}-\frac12 v_t v_t^{(2)}-v_s v_s^{(2)}+m^{I(2)}(k)\right\} \tag{D13}$$

$$\overline{M}_2^{(2)}(k)=2\pi^2\frac{E_k}{m}\left\{2uu^{(2)}-ww^{(2)}-v_t v_t^{(2)}-\sqrt2\left[v_t v_s^{(2)}+v_t^{(2)}v_s\right]\right\}. \tag{D14}$$

There are also small corrections $\Delta M_1^{(2)}$ and $\Delta M_2^{(2)}$ but these can be neglected.

The leading contributions proportional to the derivative of the strong form factor, expressed in terms of $a(p^2)$ defined in Eq. (3.25), are assembled from $\overline{M}_{1A}$ and $\overline{M}_{3A}$ using (3.24). Dropping all terms proportional to δ_k except for the large u^2 and w^2 terms gives

$$\overline{M}_{A1,\text{a term}} \simeq a(p^2)[4m\delta_k\overline{M}_{A1}(k) - 2\overline{M}_{A3}(k)] \simeq 2\pi^2\frac{E_k}{m}a(p^2)\left\{4m\delta_k\left(u^2+\frac{1}{4}w^2\right) - 2m^2\left(\frac{3}{2}v_t^2+v_s^2+2\sqrt{2}v_tv_s\right)\right\}$$
$$\overline{M}_{A2,\text{a term}} \simeq (2\omega_2-1)\,a(p^2)4m\delta_k\overline{M}_{A2}(k) \simeq 2\pi^2\frac{E_k}{m}a(p^2)\,(2\omega_2-1)\,4m\delta_k\left(u^2-\frac{1}{2}w^2\right), \tag{D15}$$

where $\omega_2 = 1$ is the parameter defined in Eq. (3.20).

In view of the rich history and importance of the magnetic moment, it is instructive to rewrite the largest terms in expressions (D6) and (D7) as coordinate space integrals. In momentum space, the leading terms for the deuteron magnetic moment can be rearranged into the following form

$$\mu_d\big|_0 = e_0\int_0^\infty k^2dk\left\{u^2+\frac{1}{4}w^2-\frac{1}{4}v_t^2-\frac{1}{2}v_s^2+\frac{m}{2\sqrt{6}}\left[\left(uv_t'-u'v_t+\frac{2uv_t}{k}\right)+\sqrt{2}\left(u'v_s-uv_s'-\frac{2uv_s}{k}\right)\right.\right.$$
$$\left.\left.+\sqrt{2}\left(wv_t'-w'v_t-\frac{4wv_t}{k}\right)+\left(wv_s'-w'v_s-\frac{4wv_s}{k}\right)\right]\right\}$$
$$+\,e_0\,\kappa_s\int_0^\infty k^2dk\left\{u^2-\frac{1}{2}w^2-\frac{1}{2}v_t^2-\sqrt{2}\,v_tv_s\right\}. \tag{D16}$$

These can be cast into integrals over the wave functions in coordinate space, defined by the transforms (A32). The squared terms and the v_tv_s term are straightforwardly reduced using the normalization condition (A34). The interference terms can be reduced by using the identity (A35) to shift derivatives, giving

$$\int_0^\infty k^2dk\left\{uv'-u'v+\frac{2uv}{k}\right\} = -2\int_0^\infty k^2dk\,u'(k)v(k) = 2\int_0^\infty r\,dr\,u(r)v(r)$$
$$\int_0^\infty k^2dk\left\{wv'-w'v-\frac{4wv}{k}\right\} = 2\int_0^\infty k^2dk\,w(k)\left(v'(k)-\frac{v(k)}{k}\right) = -2\int_0^\infty r\,dr\,w(r)v(r) \tag{D17}$$

(where v can be either v_t or v_s) and the final integrals are are evaluated using the relations

$$u'(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty r^2dr\left(\frac{1}{r}\frac{d}{dk}\right)j_0(kr)u(r) = -\sqrt{\frac{2}{\pi}}\int_0^\infty r^2dr\;j_1(kr)u(r)$$
$$v'(k)-\frac{v(k)}{k} = \sqrt{\frac{2}{\pi}}\int_0^\infty r^2dr\left(\frac{1}{r}\frac{d}{dk}-\frac{1}{kr}\right)j_1(kr)v(r) = -\sqrt{\frac{2}{\pi}}\int_0^\infty r^2dr\;j_2(kr)v(r) \tag{D18}$$

and then using the normalization condition (A34). Writing the final result in terms of the isoscalar magnetic moment, $\mu_s = \kappa_s + 1$, gives

$$\mu_d|_0 = e_0\,\mu_s\int_0^\infty dr\left\{u^2-\frac{1}{2}w^2-\frac{1}{2}v_t^2\right\} - e_0\,\kappa_s\,\sqrt{2}\int_0^\infty dr\;v_tv_s$$
$$\times\,e_0\int_0^\infty dr\left\{\frac{3}{4}w^2+\frac{1}{4}v_t^2-\frac{1}{2}v_s^2+\frac{1}{\sqrt{6}}mr[u(v_t-\sqrt{2}\,v_s)-w(\sqrt{2}v_t+v_s)]\right\}. \tag{D19}$$

In Ref. [22], interaction currents were ignored and the (B) diagrams were assumed to be equal, to the (A) diagram (the RIA approximation); in this case the normalization condition was

$$1 = \int_0^\infty dr\left\{u^2+w^2+v_t^2+v_s^2\right\}. \tag{D20}$$

With this assumption, the results of Eq. (D19) agree with Ref. [22].

2. (B) contributions

The contributions to the magnetic moment (in nuclear magnetons) from the singular terms (involving the $\mathcal{B}_{3,i}$ traces) can be written

$$\mu_d = e_0 \int_k \{\overline{M}_{1B}(k) + \kappa_s \overline{M}_{2B}(k)\}, \tag{D21}$$

where the $\overline{M}_{iB}$ are

$$\overline{M}_{iB} = \frac{m}{m_d} \lim_{Q\to 0} \frac{mE_k}{k_z Q}\left[\frac{\mathcal{B}_{3,i}(k_0)}{Qk_0}\bigg|_- - \frac{\mathcal{B}_{3,i}(k_0)}{Qk_0}\bigg|_+\right]. \tag{D22}$$

The $\overline{M}_{iB}$ are a sum of terms of the form

$$\overline{M}_{iB} \to -\frac{2m^2}{m_d}[d_1 XY + c_1(X_{k_0}Y + Y_{k_0}X)] + d_0\frac{m^2k_z}{m_d}\left[\frac{1}{E_k}(X_{k_0}Y - Y_{k_0}X) - \frac{(m_d - E_k)}{m_d\, k}(X'Y - Y'X)\right], \tag{D23}$$

where d_0,d_1 and c_1 are factors coming from the coefficients of the XY expansion, and $d_0 = x_2 k_z$, so that these terms will not be zero when integrated over k_z. (Details can be found in Appendix E of the original, longer version (v1) of the present paper in the preprint archive [20].) The trace $B_{3,2}$ is already linear in Q and hence for this term the d_0 terms vanish.

Reviewing the above discussion, the actual calculation proceeds in two steps. First, keeping the arguments of the structure functions fixed, expand the traces to first order in Q and $\delta_{k_0} \equiv k_0 - E_k$. Then make the following substitutions (for the d_0,c_1,d_1 terms respectively):

$$\begin{aligned}
k_z\delta_{k_0}(X_+Y_- - Y_+X_-) &\to \frac{m^2k_z^2}{m_dE_k}(X_{k_0}Y - Y_{k_0}X) + \mathcal{D}_2(X'Y - Y'X)\\
Q(X_+Y_- + Y_+X_-) &\to -\frac{2m^2}{m_d}(X_{k_0}Y + Y_{k_0}X)\\
Q\delta_{k_0}(X_+Y_- + Y_+X_-) &\to -\frac{2m^2}{m_d}XY,
\end{aligned} \tag{D24}$$

where

$$\mathcal{D}_2 = -\frac{m^2k_z^2}{m_d^2\, k}(m_d - E_k) \tag{D25}$$

and the factor of k_z that is part of d_0 has been shown explicitly. The $a(p^2)$ contribution is obtained from the special substitution

$$Q(X_+Y_- + Y_+X_-) \to \frac{8m^2a(p^2)}{m_d}(m_d - E_k)XY. \tag{D26}$$

Using these substitutions, and expressing the $\overline{M}$'s directly in terms of the wave functions z_ℓ, gives the following leading-order results

$$\begin{aligned}
\overline{M}_{1B}(k) = 2\pi^2\frac{E_k}{m}\bigg\{& u^2 - \frac{1}{8}w^2 - \frac{3\,uw}{4\sqrt{2}} - \left(2 - \frac{3m^2}{4k^2}\right)v_t^2 - \frac{1}{2}v_s^2 - \frac{1}{4\sqrt{2}}\left(7 - \frac{6m^2}{k^2}\right)v_tv_s + \frac{m}{k}(\sqrt{2}v_t - v_s)(\sqrt{2}z_0^{--} - z_1^{--})\\
&+ \frac{k}{4\sqrt{2}}(u'w - uw') + \frac{m^2}{2\sqrt{2}\,k}(v_tv_s' - v_t'v_s) - 2u[\delta_+\hat{u}]_{k_0} - w[\delta_+\hat{w}]_{k_0} + \sqrt{2}\left(\frac{1}{2}u[\delta_+\hat{w}]_{k_0} - v_t[\delta_-\hat{v}_s]_{k_0} - v_s[\delta_-\hat{v}_t]_{k_0}\right)\\
&+ 2v_t[\delta_-\hat{v}_t]_{k_0} + v_s[\delta_-\hat{v}_s]_{k_0} + 2\,a(p^2)m\left[\delta_k\left(2u^2 + w^2 - \frac{1}{\sqrt{2}}uw\right) - m_d\left(2v_t^2 + v_s^2 - \frac{3}{\sqrt{2}}v_tv_s\right)\right] + \Delta M_{1B}(k)\bigg\}\\
\overline{M}_{2B}(k) = 2\pi^2\frac{E_k}{m}\bigg\{& u^2 - \frac{1}{2}w^2 - \frac{1}{2}v_t^2 - \sqrt{2}v_tv_s + \frac{\sqrt{2}\,m}{k}(\sqrt{2}z_0^{--} - z_1^{--})v_t - 2u[\delta_+\hat{u}]_{k_0} + w[\delta_+\hat{w}]_{k_0} - \sqrt{2}[v_t[\delta_-\hat{v}_s]_{k_0}\\
&+ v_s[\delta_-\hat{v}_t]_{k_0}] + v_t[\delta_-\hat{v}_t]_{k_0} - 4a(p^2)(E_k - m_d)\left[\delta_k\left(u^2 - \frac{1}{2}w^2\right) - m_d\left(\frac{1}{2}v_t^2 - \sqrt{2}v_tv_s\right)\right] + \Delta M_{2B}(k)\bigg\},
\end{aligned} \tag{D27}$$

where $z_{\ell k} = z'_\ell$ and the leading-order correction terms are

$$\begin{aligned}
\Delta M_{1B}(k) &= \frac{E_k - m}{12E_k}\left[u^2 + \frac{131}{4}w^2 + \frac{29}{2\sqrt{2}}uw\right]\\
\Delta M_{2B}(k) &= -\frac{E_k - m}{3E_k}\left[u^2 + \frac{1}{2}w^2 - \sqrt{2}uw\right].
\end{aligned} \tag{D28}$$

Note the unexpected presence of a leading uw contribution to $\overline{M}_{1B}$. This term does not reduce the the expected nonrelativistic limit, but is canceled by a similar contribution from $\overline{M}_{1C}$, which we discuss now.

It is surprising that significant contributions come from the finite terms that depend on the traces $\mathcal{C}_{3,i}$. These give the following additional leading contributions

$$\begin{aligned}\overline{M}_{1C}(k) &= 2\pi^2\frac{E_k}{m}\bigg\{-\frac{3}{8}w^2+\frac{9\,uw}{4\sqrt{2}}+\left(1-\frac{m^2}{4k^2}\right)v_t^2-\frac{1}{4\sqrt{2}}\left(5+\frac{2m^2}{k^2}\right)v_tv_s+\frac{m}{k}(v_tz_0^{--}+v_sz_1^{--})\\&\quad+\frac{k}{4\sqrt{2}}(u'w+3uw')-\frac{1}{2}kww'-\frac{k}{4\sqrt{2}}(v_tv_s'+7v_t'v_s)+\frac{m^2}{2\sqrt{2}\,k}(v_tv_s'+3v_t'v_s)-\left(\frac{1}{2}-\frac{m^2}{k^2}\right)v_tv_t'\\&\quad+\frac{1}{2}m(v_t'z_0^{--}+3v_tz_0^{--\prime}+v_s'z_1^{--}+3v_sz_1^{--\prime})+\frac{1}{2}w[\delta_+\hat{w}]_{k_0}-\frac{1}{\sqrt{2}}(u[\delta_+\hat{w}]_{k_0}+v_s[\delta_-\hat{v}_t]_{k_0})\\&\quad-\frac{1}{2}v_t[\delta_-\hat{v}_t]_{k_0}-a(p^2)m\,m_d\big(v_t^2+\sqrt{2}v_tv_s\big)+\Delta M_{1C}(k)\bigg\}\\\overline{M}_{2C}(k) &= 2\pi^2\frac{E_k}{m}\left\{v_t^2-\frac{\sqrt{2}\,m}{k}(\sqrt{2}z_0^{--}-z_1^{--})v_t\right\}\end{aligned}\tag{D29}$$

with only one correction term

$$\Delta M_{1C}(k)=-\frac{E_k-m}{4E_k}\left[5u^2-\frac{5}{4}w^2+\frac{49}{2\sqrt{2}}uw\right].\tag{D30}$$

Adding the (B) and (C) contributions together, and rearranging some terms, gives

$$\begin{aligned}\overline{M}_{1BC}(k) &= 2\pi^2\frac{E_k}{m}\bigg\{u^2+\frac{1}{4}w^2-\frac{1}{2}\left\{\frac{3}{2}w^2+kww'\right\}+\frac{1}{2\sqrt{2}}\{3uw+ku'w+kuw'\}-\frac{1}{4}v_t^2-\frac{1}{2}\left\{\frac{3}{2}v_t^2+kv_tv_t'\right\}-\frac{1}{2}v_s^2\\&\quad+\frac{m^2}{2k^2}\{v_t^2+2kv_tv_t'\}-\frac{1}{\sqrt{2}}\{3v_tv_s+kv_t'v_s+kv_tv_s'\}+\frac{3k}{4\sqrt{2}}(v_tv_s'-v_t'v_s)+\frac{m^2}{\sqrt{2}\,k^2}\{v_tv_s+kv_tv_s'+kv_t'v_s\}\\&\quad-2u[\delta_+\hat{u}]_{k_0}-\frac{1}{2}w[\delta_+\hat{w}]_{k_0}-\sqrt{2}\big(v_t[\delta_-\hat{v}_s]_{k_0}+v_s[\delta_-\hat{v}_t]_{k_0}\big)+\frac{3}{2}v_t[\delta_-\hat{v}_t]_{k_0}+v_s[\delta_-\hat{v}_s]_{k_0}\\&\quad-m(v_t'z_0^{--}+v_s'z_1^{--})+\frac{3}{2}m\left\{v_t'z_0^{--}+v_tz_0^{--\prime}+\frac{2}{k}v_tz_0^{--}\right\}+\frac{3}{2}m\left\{v_s'z_1^{--}+v_sz_1^{--\prime}+\frac{2}{k}v_sz_1^{--}\right\}\\&\quad-\frac{m}{k}[\sqrt{2}v_sz_0^{--}+(\sqrt{2}\,v_t+v_s)z_1^{--}]+2a(p^2)m\left[\delta_k\left(2u^2+w^2-\frac{1}{\sqrt{2}}uw\right)-m_d\left(\frac{5}{2}v_t^2+v_s^2-\sqrt{2}v_tv_s\right)\right]\\&\quad+\Delta M_{1B}(k)+\Delta M_{1C}(k)\bigg\}\\\overline{M}_{2BC}(k) &= 2\pi^2\frac{E_k}{m}\bigg\{u^2-\frac{1}{2}w^2+\frac{1}{2}v_t^2-\sqrt{2}v_tv_s-2u[\delta_+\hat{u}]_{k_0}+w[\delta_+\hat{w}]_{k_0}-\sqrt{2}[v_t[\delta_-\hat{v}_s]_{k_0}+v_s[\delta_-\hat{v}_t]_{k_0}]\\&\quad+v_t[\delta_-\hat{v}_t]_{k_0}-4a(p^2)(E_k-m_d)\left[\delta_k\left(u^2-\frac{1}{2}w^2\right)-m_d\left(\frac{1}{2}v_t^2-\sqrt{2}v_tv_s\right)\right]+\Delta M_{2B}(k)\bigg\}.\end{aligned}\tag{D31}$$

The expression for $\overline{M}_{1BC}$ has been arranged so that terms in the interior curly braces integrate to zero (recalling that the volume of integration is k^2dk/E_k). Note that the correct leading uw term (which equals 0) and w^2 term are only obtained by the summing the $\mathcal{B}$ and $\mathcal{C}$ traces and retaining the k derivative contributions.

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