Relativistic viscous hydrodynamics for heavy-ion collisions: A comparison between the Chapman-Enskog and Grad methods

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Derivations of relativistic second-order dissipative hydrodynamic equations have relied almost exclusively on the use of Grad's 14-moment approximation to write $f(x,p)$, the nonequilibrium distribution function in the phase space. Here we consider an alternative Chapman-Enskog-like method, which, unlike Grad's, involves a small expansion parameter. We derive an expression for $f(x,p)$ to second order in this parameter. We show analytically that while Grad's method leads to the violation of the experimentally observed $1/\sqrt{m_T}$ scaling of the longitudinal femtoscopic radii, the alternative method does not exhibit such an unphysical behavior. We compare numerical results for hadron transverse-momentum spectra and femtoscopic radii obtained in these two methods, within the one-dimensional scaling expansion scenario. Moreover, we demonstrate a rapid convergence of the Chapman-Enskog-like expansion up to second order. This leads to an expression for $\delta f(x,p)$ which provides a better alternative to Grad's approximation for hydrodynamic modeling of relativistic heavy-ion collisions.

I. INTRODUCTION

The standard model of relativistic heavy-ion collisions relies on relativistic hydrodynamics to simulate the intermediate-stage evolution of the high-energy-density fireball formed in these collisions [1]. Recent simulations generally make use of some version of the Müller-Israel-Stewart second-order theory of causal dissipative hydrodynamics [2,3]. Hydrodynamics has achieved remarkable success in explaining, for example, the observed mass ordering of the elliptic flow [4–6], higher harmonics of the azimuthal anisotropic flow [7,8], and the ridge and shoulder structure in long-range rapidity correlations [9]. The recently measured correlators between event planes of different harmonics [10] too can be understood qualitatively within event-by-event hydrodynamics [11]. Notwithstanding these successes, the basic formulation of the dissipative hydrodynamic equations continues to be an area of considerable activity, largely because of the ambiguities arising due to the variety of ways in which these equations can be derived [12–18].

For a system that is out of equilibrium, the existence of thermodynamic gradients results in thermodynamic forces, which give rise to various transport phenomena. To quantify these nonequilibrium effects, it is convenient to first specify the nonequilibrium phase-space distribution function $f(x,p)$ and then calculate the various transport coefficients. In the context of hydrodynamics, two most commonly used methods to determine the form of the distribution function close to local thermodynamic equilibrium are (1) Grad's 14-moment approximation [19] and (2) the Chapman-Enskog method [20]. Although both the methods involve expanding $f(x,p)$ around the equilibrium distribution function $f_0(x,p)$, there are important differences.

In the relativistic version of Grad's 14-moment approximation, the small deviation from equilibrium is usually approximated by means of a Taylor-like series expansion in momenta truncated at quadratic order [2,17]. Further, the 14 coefficients in this expansion are assumed to be linear in dissipative fluxes. However, it is not apparent why a power series in momenta

should be convergent and whether one is justified in making such an ansatz, without a small expansion parameter.

The Chapman-Enskog method, on the other hand, aims at obtaining a perturbative solution of the Boltzmann transport equation using the Knudsen number (ratio of mean free path to a typical macroscopic length) as a small expansion parameter. This is equivalent to making a gradient expansion about the local equilibrium distribution function [21]. This method of obtaining the form of the nonequilibrium distribution function is consistent [16] with dissipative hydrodynamics, which is also formulated as a gradient expansion.

The above two methods have been compared and shortcomings of Grad's approximation have been pointed out in the literature [22–24]. In spite of these shortcomings, the derivations of relativistic second-order dissipative hydrodynamic equations, as well as particle-production prescriptions, rely almost exclusively on Grad's approximation. The Chapman-Enskog method, on the other hand, has seldom been employed in the hydrodynamic modeling of the relativistic heavy-ion collisions. The focus of the present work is to explore the applicability of the latter method.

In this paper, the Boltzmann equation in the relaxation-time approximation is solved iteratively, which results in a Chapman-Enskog-like expansion of the nonequilibrium distribution function. Truncating the expansion at the second order, we derive an explicit expression for the viscous correction to the equilibrium distribution function. We compare the hadronic spectra and longitudinal Hanbury-Brown-Twiss (HBT) radii obtained using the form of the viscous correction derived here and Grad's 14-moment approximation, within a one-dimensional scaling expansion. We find that at large transverse momenta, the present method yields smaller hadron multiplicities. We also show analytically that while Grad's approximation leads to the violation of the experimentally observed $1/\sqrt{m_T}$ scaling of HBT radii [25–29], the viscous correction obtained here does not exhibit such unphysical behavior. Finally, we demonstrate the rapid convergence of the Chapman-Enskog-like expansion up to second order.

II. RELATIVISTIC VISCOUS HYDRODYNAMICS

Within the framework of relativistic hydrodynamics, the variables that characterize the macroscopic state of a system are the energy-momentum tensor, $T^{\mu\nu}$, particle four-current, N^μ, and entropy four-current, S^μ. The local conservation of net charge ($\partial_\mu N^\mu = 0$) and energy-momentum ($\partial_\mu T^{\mu\nu} = 0$) lead to the equations of motion of a relativistic fluid, whereas the second law of thermodynamics requires $\partial_\mu S^\mu \geqslant 0$. For a system with no net conserved charges, hydrodynamic evolution is governed only by the conservation equations for energy and momentum.

The energy-momentum tensor of a macroscopic system can be expressed in terms of a single-particle phase-space distribution function and can be tensor decomposed into hydrodynamic degrees of freedom [21]. Here we restrict ourselves to a system of massless particles (ultrarelativistic limit) for which the bulk viscosity vanishes, leading to

$$T^{\mu\nu} = \int dp \; p^\mu p^\nu \, f(x,p) = \epsilon u^\mu u^\nu - P \Delta^{\mu\nu} + \pi^{\mu\nu}. \quad (1)$$

Here $dp \equiv g d\mathbf{p}/[(2\pi)^3 |\mathbf{p}|]$, where g is the degeneracy factor, p^μ is the particle four-momentum, and $f(x,p)$ is the phase-space distribution function. In the tensor decomposition, ϵ, P, and $\pi^{\mu\nu}$ are energy density, thermodynamic pressure, and shear stress tensor, respectively. The projection operator $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is orthogonal to the hydrodynamic four-velocity u^μ defined in the Landau frame: $T^{\mu\nu} u_\nu = \epsilon u^\mu$. The metric tensor is Minkowskian, $g^{\mu\nu} \equiv \mathrm{diag}(+, -, -, -)$.

The evolution equations for ϵ and u^μ,

$$\dot{\epsilon} + (\epsilon + P)\theta - \pi^{\mu\nu}\nabla_{(\mu} u_{\nu)} = 0,$$
$$(\epsilon + P)\dot{u}^\alpha - \nabla^\alpha P + \Delta^\alpha_\nu \partial_\mu \pi^{\mu\nu} = 0, \quad (2)$$

are obtained from the conservation of the energy-momentum tensor. We use the standard notation $\dot{A} \equiv u^\mu \partial_\mu A$ for co-moving derivative, $\theta \equiv \partial_\mu u^\mu$ for expansion scalar, $A^{(\alpha} B^{\beta)} \equiv (A^\alpha B^\beta + A^\beta B^\alpha)/2$ for symmetrization, and $\nabla^\alpha \equiv \Delta^{\mu\alpha}\partial_\mu$ for spacelike derivatives. In the ultrarelativistic limit, the equation of state relating energy density and pressure is $\epsilon = 3P \propto \beta^{-4}$. The inverse temperature, $\beta \equiv 1/T$, is determined by the Landau matching condition $\epsilon = \epsilon_0$, where ϵ_0 is the equilibrium energy density. In this limit, the derivatives of β,

$$\dot{\beta} = \frac{\beta}{3}\theta - \frac{\beta}{12P}\pi^{\rho\gamma}\sigma_{\rho\gamma}, \quad (3)$$

$$\nabla^\alpha \beta = -\beta \dot{u}^\alpha - \frac{\beta}{4P}\Delta^\alpha_\rho \partial_\gamma \pi^{\rho\gamma}, \quad (4)$$

can be obtained from Eq. (2), where $\sigma^{\rho\gamma} \equiv \nabla^{(\rho} u^{\gamma)} - (\theta/3)\Delta^{\rho\gamma}$ is the velocity stress tensor [30]. The above identities are used later in the derivations of viscous corrections to the distribution function and shear evolution equation.

For a system close to local thermodynamic equilibrium, the phase-space distribution function can be written as $f = f_0 + \delta f$, where the deviation from equilibrium is assumed to be small ($\delta f \ll f$). Here f_0 represents the equilibrium distribution function of massless Boltzmann particles at vanishing chemical potential, $f_0 = \exp(-\beta u \cdot p)$, where $u \cdot p \equiv u_\mu p^\mu$. From Eq. (1), the shear stress tensor, $\pi^{\mu\nu}$, can be expressed in

terms of the nonequilibrium part of the distribution function, δf, as [17]

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \; p^\alpha p^\beta \, \delta f, \quad (5)$$

where $\Delta^{\mu\nu}_{\alpha\beta} \equiv \Delta^\mu_{(\alpha}\Delta^\nu_{\beta)} - (1/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}$ is a traceless symmetric projection operator orthogonal to u^μ. To make further progress, the form of δf has to be determined. In the following, we adopt a Chapman-Enskog-like expansion for the distribution function, to obtain δf order-by-order in gradients, by solving the Boltzmann equation iteratively in the relaxation-time approximation.

III. CHAPMAN-ENSKOG EXPANSION

Determination of the nonequilibrium phase-space distribution function is one of the central problems in statistical mechanics. This can be achieved by solving a kinetic equation such as the Boltzmann equation. The relativistic Boltzmann equation with the relaxation-time approximation for the collision term is given by [31],

$$p^\mu \partial_\mu f = C[f] = -(u \cdot p)\frac{\delta f}{\tau_R}, \quad (6)$$

where τ_R is the relaxation time. We recall that the zeroth and first moments of the collision term, $C[f]$, should vanish to ensure the conservation of particle current and energy-momentum tensor [21]. This requires that τ_R is independent of momenta, and u^μ is defined in the Landau frame [31]. Therefore, within the relaxation-time approximation, Landau frame is mandatory and not a choice. Momentum-dependent τ_R was considered in Ref. [32] where the authors also studied the consequences of different momentum dependencies of δf for the heavy-ion observables.

Exact solutions of the Boltzmann equation are possible only in rare circumstances. The most common technique of generating an approximate solution to the Boltzmann equation is the Chapman-Enskog expansion, where the distribution function is expanded about its equilibrium value in powers of space-time gradients [20]

$$f = f_0 + \delta f, \quad \delta f = \delta f^{(1)} + \delta f^{(2)} + \cdots, \quad (7)$$

where $\delta f^{(n)}$ is nth-order in derivatives. The Boltzmann equation can be solved iteratively by rewriting Eq. (6) in the form $f = f_0 - (\tau_R/u \cdot p) \, p^\mu \partial_\mu f$ [16,33,34]. We obtain

$$f_1 = f_0 - \frac{\tau_R}{u \cdot p} p^\mu \partial_\mu f_0, \quad f_2 = f_0 - \frac{\tau_R}{u \cdot p} p^\mu \partial_\mu f_1, \; \cdots \quad (8)$$

where $f_n = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots + \delta f^{(n)}$. To first- and second-orders in derivatives, we have

$$\delta f^{(1)} = -\frac{\tau_R}{u \cdot p} p^\mu \partial_\mu f_0, \quad (9)$$

$$\delta f^{(2)} = \frac{\tau_R}{u \cdot p} p^\mu p^\nu \partial_\mu \left(\frac{\tau_R}{u \cdot p} \partial_\nu f_0 \right). \quad (10)$$

In the next section, the above expressions for δf along with Eq. (5) are used in the derivation of the evolution equation for the shear stress tensor.

IV. VISCOUS EVOLUTION EQUATION

In order to complete the set of hydrodynamic equations, Eq. (2), we need to derive an expression for the shear stress tensor, $\pi^{\mu\nu}$. The first-order expression for $\pi^{\mu\nu}$ can be obtained from Eq. (5) using $\delta f = \delta f^{(1)}$ from Eq. (9),

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \, p^\alpha p^\beta \left(-\frac{\tau_R}{u \cdot p} \, p^\gamma \partial_\gamma \, f_0 \right). \quad (11)$$

Using Eqs. (3) and (4) and keeping only those terms which are first-order in gradients, the integral in the above equation reduces to

$$\pi^{\mu\nu} = 2\tau_R \beta_\pi \sigma^{\mu\nu}, \quad (12)$$

where $\beta_\pi = 4P/5$ [16].

The second-order evolution equation for shear stress tensor can also be obtained in a similar way by using $\delta f = \delta f^{(1)} + \delta f^{(2)}$ from Eqs. (9) and (10) in Eq. (5). Performing the integrations and using Eqs. (3), (4) and (12), we get [16,30]

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta, \quad (13)$$

where $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ is the vorticity tensor, and we have used Eq. (12). It is clear from the form of the above equation that the relaxation time τ_R can be identified with the shear relaxation time τ_π. By comparing the first-order evolution Eq. (12) with the relativistic Navier-Stokes equation $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$, we obtain $\tau_\pi = \eta/\beta_\pi$, where η is the coefficient of shear viscosity.

V. CORRECTIONS TO THE DISTRIBUTION FUNCTION

In this section, we derive the expression for the nonequilibrium part of the distribution function, δf, up to second order in gradients of u^μ. For this purpose, we employ Eqs. (9) and (10), which were obtained using a Chapman-Enskog-like expansion. We then recall the derivation of the standard Grad's 14-moment approximation for δf, and compare these two expressions.

Using Eqs. (3) and (4) for the derivatives of β, and Eq. (13) for $\sigma^{\mu\nu}$, in Eqs. (9) and (10), we arrive at the form of the second-order viscous correction to the distribution function:

$$\delta f = \frac{f_0\beta}{2\beta_\pi(u \cdot p)} p^\alpha p^\beta \pi_{\alpha\beta} - \frac{f_0\beta}{\beta_\pi} \left[\frac{\tau_\pi}{u \cdot p} p^\alpha p^\beta \pi_\alpha^\gamma \, \omega_{\beta\gamma} \right.$$

$$- \frac{5}{14\beta_\pi(u \cdot p)} p^\alpha p^\beta \pi_\alpha^\gamma \, \pi_{\beta\gamma} + \frac{\tau_\pi}{3(u \cdot p)} p^\alpha p^\beta \pi_{\alpha\beta}\theta$$

$$- \frac{6\tau_\pi}{5} p^\alpha \dot{u}^\beta \pi_{\alpha\beta} + \frac{(u \cdot p)}{70\beta_\pi} \pi^{\alpha\beta}\pi_{\alpha\beta} + \frac{\tau_\pi}{5} p^\alpha (\nabla^\beta \pi_{\alpha\beta})$$

$$- \frac{3\tau_\pi}{(u \cdot p)^2} p^\alpha p^\beta p^\gamma \pi_{\alpha\beta}\dot{u}_\gamma + \frac{\tau_\pi}{2(u \cdot p)^2} p^\alpha p^\beta p^\gamma (\nabla_\gamma \pi_{\alpha\beta})$$

$$\left. - \frac{\beta + (u \cdot p)^{-1}}{4(u \cdot p)^2\beta_\pi} (p^\alpha p^\beta \pi_{\alpha\beta})^2 \right] + \mathcal{O}(\delta^3), \quad (14)$$

$$\equiv \delta f_1 + \delta f_2 + \mathcal{O}(\delta^3). \quad (15)$$

The first term on the right-hand side of Eq. (14) corresponds to the first-order correction, δf_1, whereas the terms within square brackets are of second order, δf_2 (see Appendix A). Note that $\delta f_1 \neq \delta f^{(1)}$ and $\delta f_2 \neq \delta f^{(2)}$, due to the nonlinear nature of Eqs. (3), (4), and (13). It is straightforward to show that the form of δf in Eq. (14) is consistent with the definition of the shear stress tensor, Eq. (5), and satisfies the matching condition $\epsilon = \epsilon_0$ and the Landau frame definition $u_\nu T^{\mu\nu} = \epsilon u^\mu$ [21], i.e.,

$$\int dp \, (u \cdot p)^2 \, \delta f = 0, \quad \int dp \, \Delta_{\mu\alpha} u_\beta \, p^\alpha p^\beta \, \delta f = 0, \quad (16)$$

order-by-order in gradients (see Appendix A).

On the other hand, Grad's 14-moment approximation for δf can be obtained from a Taylor-like expansion in the powers of momenta [2,17]

$$\delta f_G = f_0[\varepsilon(x) + \varepsilon_\alpha(x)p^\alpha + \varepsilon_{\alpha\beta}(x)p^\alpha p^\beta], \quad (17)$$

where ε's are the momentum-independent coefficients in the expansion, which, however, may depend on thermodynamic and dissipative quantities. For a system of massless particles with no net conserved charges, i.e., in the absence of bulk viscosity and charge diffusion current, the above equation reduces to

$$\delta f_G = \frac{f_0\beta^2}{10\beta_\pi} p^\alpha p^\beta \pi_{\alpha\beta}, \quad (18)$$

where the coefficient is obtained using Eq. (5). We observe that unlike Eq. (14) for the Chapman-Enskog case, Eq. (18) for Grad's is linear in shear stress tensor. However, it is important to note that both the forms of δf, i.e., δf_1 and δf_G, lead to identical evolution equations for the shear stress tensor, Eq. (13), with the same coefficients [13,30].

VI. BJORKEN SCENARIO

In order to model the hydrodynamical evolution of the matter formed in the heavy-ion collision experiments, we use the Bjorken prescription [35] for one-dimensional expansion. We consider the evolution of a system of massless particles ($\epsilon = 3P$) at vanishing net baryon number density. In terms of the Milne coordinates $(\tau, r, \varphi, \eta_s)$, where $\tau = \sqrt{t^2 - z^2}$, $r = \sqrt{x^2 + y^2}$, $\varphi = \tan^{-1}(y/x)$, and $\eta_s = \tanh^{-1}(z/t)$, and with $u^\mu = (1,0,0,0)$, evolution equations for ϵ and $\Phi \equiv -\tau^2\pi^{\eta_s\eta_s}$ become

$$\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}(\epsilon + P - \Phi), \quad (19)$$

$$\frac{d\Phi}{d\tau} = -\frac{\Phi}{\tau_\pi} + \beta_\pi \frac{4}{3\tau} - \lambda \frac{\Phi}{\tau}. \quad (20)$$

The transport coefficients appearing in the above equation reduce to [16]

$$\tau_\pi = \frac{\eta}{\beta_\pi}, \quad \beta_\pi = \frac{4P}{5}, \quad \lambda = \frac{38}{21}. \quad (21)$$

In $(\tau, r, \varphi, \eta_s)$ coordinates, the components of particle four-momenta are given by

$$p^\tau = m_T \cosh(y - \eta_s), \quad p^r = p_T \cos(\varphi_p - \varphi),$$

$$p^\varphi = p_T \sin(\varphi_p - \varphi)/r, \quad p^{\eta_s} = m_T \sinh(y - \eta_s)/\tau, \quad (22)$$

where $m_T^2 = p_T^2 + m^2$, p_T is the transverse momentum, y is the particle rapidity, and φ_p is the azimuthal angle in the momentum space. We note that for the Bjorken expansion, $\theta = 1/\tau$, $\dot{u}^\mu = 0$, $\omega^{\mu\nu} = 0$ and $p_\mu d\Sigma^\mu = m_T \cosh(y - \eta_s)\tau d\eta_s r dr d\varphi$. In this scenario, the nonvanishing factors appearing in Eq. (14) reduce to $u \cdot p = m_T \cosh(y - \eta_s)$, $\pi_{\alpha\beta}\pi^{\alpha\beta} = 3\Phi^2/2$, and

$$p^\alpha p^\beta \pi_{\alpha\beta} = \frac{\Phi}{2} p_T^2 - \Phi m_T^2 \sinh^2(y - \eta_s),$$

$$p^\alpha p^\beta \pi_\alpha^{\;\gamma} \pi_{\gamma\beta} = -\frac{\Phi^2}{4} p_T^2 - \Phi^2 m_T^2 \sinh^2(y - \eta_s),$$

$$(23)$$

$$p^\alpha p^\beta p^\gamma \nabla_\alpha \pi_{\beta\gamma} = 2\frac{\Phi}{\tau} m_T^3 \sinh^2(y - \eta_s)\cosh(y - \eta_s),$$

$$p^\alpha \nabla^\beta \pi_{\alpha\beta} = -\frac{\Phi}{\tau} m_T \cosh(y - \eta_s).$$

Within the framework of the relativistic hydrodynamics, observables pertaining to heavy-ion collisions are influenced by viscosity in two ways: first through the viscous hydrodynamic evolution of the system and second through corrections to the particle production rate via the nonequilibrium distribution function [36]. Hydrodynamic evolution and the nonequilibrium corrections to the distribution function were considered in the previous sections; in the following sections, we focus on two observables, namely transverse-momentum spectra and HBT radii of hadrons.

VII. HADRONIC SPECTRA

The hadron spectra can be obtained using the Cooper-Frye freezeout prescription [37]

$$\frac{dN}{d^2 p_T dy} = \frac{g}{(2\pi)^3} \int p_\mu d\Sigma^\mu f(x,p), \quad (24)$$

where p^μ is the particle four-momentum, $d\Sigma^\mu$ represents the element of the three-dimensional freeze-out hypersurface, and $f(x,p)$ represents the phase-space distribution function at freeze-out.

For the ideal freeze-out case ($f = f_0$), we get

$$\frac{dN^{(0)}}{d^2 p_T dy} = \frac{g}{4\pi^3} m_T \tau A_\perp K_1, \quad (25)$$

where $A_\perp$ denotes the transverse area of the overlap zone of colliding nuclei and $K_n \equiv K_n(z_m)$ are the modified Bessel functions of the second kind with argument $z_m \equiv m_T/T$. In Eq. (25) and hereafter, the hydrodynamical quantities such as T, τ, Φ, P, etc., correspond to their values at freeze-out. The expression for hadron production up to first order ($f = f_0 + \delta f_1$) is obtained as

$$\frac{dN^{(1)}}{d^2 p_T dy} = \left[1 + \frac{\Phi}{4\beta_\pi z_m}\left\{z_p^2 \frac{K_0}{K_1} - 2z_m\right\}\right]\frac{dN^{(0)}}{d^2 p_T dy}, \quad (26)$$

where $z_p \equiv p_T/T$. Here we have used the recurrence relation $K_{n+1}(z) = 2nK_n(z)/z + K_{n-1}(z)$. The derivation of the hadron spectra up to second order, $dN^{(2)}/d^2 p_T dy$ (by setting $f = f_0 + \delta f_1 + \delta f_2$), is presented in the Appendix B.

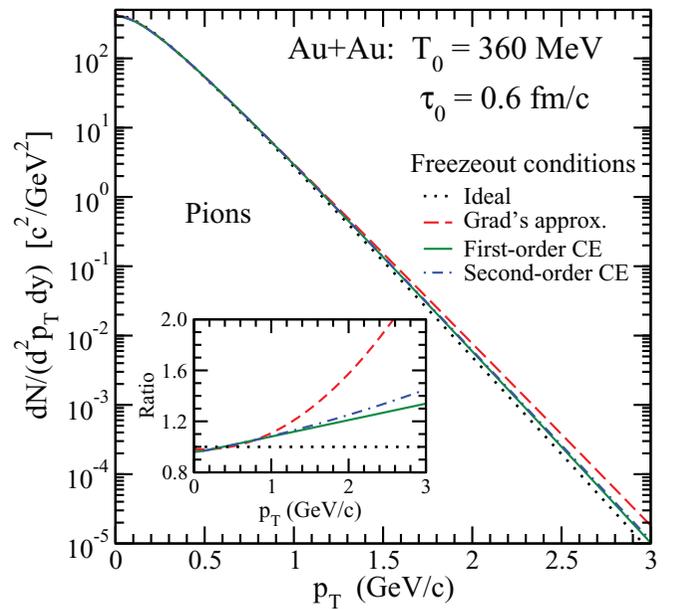For comparison, we also present the result for hadron production obtained using Grad's 14-moment approximation

FIG. 1. (Color online) Pion spectra as a function of the transverse momentum p_T, obtained with the second-order hydrodynamic evolution, followed by freeze-out in various scenarios: ideal, Grad's 14-moment approximation, and first- and second-order Chapman-Enskog. Inset: Pion yields in the above four cases scaled by the corresponding values in the ideal case.

($f = f_0 + \delta f_G$) [36,38]

$$\frac{dN^{(G)}}{d^2 p_T dy} = \left[1 + \frac{\Phi}{20\beta_\pi}\left\{z_p^2 - 2z_m \frac{K_2}{K_1}\right\}\right]\frac{dN^{(0)}}{d^2 p_T dy}. \quad (27)$$

We solve the evolution equations (19) and (20) with initial temperature $T_0 = 360$ MeV, time $\tau_0 = 0.6$ fm/c, and isotropic pressure configuration $\Phi_0 = 0$, corresponding to central ($b = 0$) Au-Au collisions at the Relativistic Heavy-Ion Collider. The system is evolved with shear viscosity to entropy density ratio $\eta/s = 1/4\pi$ corresponding to the Kovtun-Son-Starinets (KSS) lower bound [39], until the freeze-out temperature $T = 150$ MeV is reached. In order to study the effects of the various forms of δf via the freeze-out prescription, Eq. (24), we evolve the system using the second-order viscous hydrodynamic equations (19) and (20) in all the cases.

In Fig. 1, we present the pion transverse-momentum spectra for the four freeze-out conditions discussed above, namely ideal, first- and second-order Chapman-Enskog, and Grad's 14-moment approximation. We observe that nonideal freeze-out conditions tend to increase the high-p_T particle production. While the Chapman-Enskog corrections are small, Grad's 14-moment approximation results in rather large corrections to the ideal case. This is clearly evident in the inset where we show the pion yields in the four cases scaled by the values in the ideal case. These features can be easily understood from Eqs. (26) and (27): The first-order Chapman-Enskog correction is essentially linear in p_T whereas that due to Grad is quadratic. The second-order Chapman-Enskog correction is small, indicating rapid convergence of the expansion up to second order.

VIII. HBT RADII

HBT interferometry provides a powerful tool to unravel the space-time structure of the particle-emitting sources in heavy-ion collisions, because of its ability to measure source sizes, lifetimes, and particle emission durations [40]. The source function, $S(x, K)$, for on-shell particle emission is defined such that it satisfies

$$\frac{dN}{d^2 K_T dy} \equiv \int d^4 x \, S(x, K). \tag{28}$$

By comparing the above equation with Eq. (24), we see that the source function is restricted to the freeze-out hypersurface and is given by

$$S(x, K) = \frac{g}{(2\pi)^3} \int p_\mu d\Sigma^\mu(x') f(x', p) \delta^4(x - x'). \tag{29}$$

At relatively small momenta, certain space-time variances of the source function can be obtained, to a good approximation, from the correlation between particle pairs [41]. Space-time averages with respect to the source function are defined as

$$\langle \alpha \rangle_K \equiv \frac{\int d^4 x \, S(x, K) \alpha}{\int d^4 x \, S(x, K)} = \frac{\int K_\mu d\Sigma^\mu f(x, K) \alpha}{\int K_\mu d\Sigma^\mu f(x, K)}, \tag{30}$$

where K_μ is the pair four-momentum.

The longitudinal HBT radius, R_L, is calculated in terms of the transverse momentum, K_T, of the identical-particle pair [41]:

$$R_L^2(K_T) = \frac{\int K_\mu d\Sigma^\mu f(x, K) z^2}{\int K_\mu d\Sigma^\mu f(x, K)}. \tag{31}$$

In the central-rapidity region, the pair four-momentum is given by $K^\mu = (K^\tau, K^r, K^\varphi, K^{\eta_s}) = (m_T, K_T, 0, 0)$. The integration measure is given by $K_\mu d\Sigma^\mu = m_T \cosh(\eta_s)\tau d\eta_s r dr d\varphi$ with $m_T = \sqrt{K_T^2 + m_p^2}$, m_p being the particle mass. Using the relation $z = \tau \sinh(\eta_s)$, we get

$$R_L^2(K_T) = \tau^2 \left[\frac{\int K_\mu d\Sigma^\mu f(x, K)\cosh^2(\eta_s)}{\int K_\mu d\Sigma^\mu f(x, K)} - 1 \right],$$

$$\equiv \tau^2 \left[\frac{N[f]}{D[f]} - 1 \right]. \tag{32}$$

Note that the integral, $D[f]$, in the denominator in the above equation is the same as that occurring in the Cooper-Frye prescription for particle production, Eq. (24), and was already calculated in the previous section. We next calculate the integral, $N[f]$, in the numerator.

In the ideal case, $f = f_0$, we have

$$N[f_0] = \frac{2A_\perp \tau z_m}{4\beta} (K_3 + 3K_1). \tag{33}$$

This leads to the well-known result of Hermann and Bertsch [42]

$$\left(R_L^2\right)^{(0)} = \frac{\tau^2}{z_m} \frac{K_2}{K_1}, \tag{34}$$

which for large values of z_m results in the Makhlin-Sinyukov formula $\left(R_L^2\right)^{(0)} = \tau^2 T/m_T$ [43,44]. Thus in the ideal case, $(R_L)^{(0)}$ exhibits the so-called $1/\sqrt{m_T}$ scaling.

The first-order calculation requires $N[\delta f_1]$, which is given by

$$N[\delta f_1] = \frac{2A_\perp \tau \Phi}{16\beta \beta_\pi} \left[(2z_p^2 + z_m^2)K_0 + 2z_p^2 K_2 - z_m^2 K_4 \right]. \tag{35}$$

The second-order calculation requires $N[\delta f_2]$, which is given in the Appendix B. For comparison we also calculate R_L in Grad's 14-moment approximation. This requires $N[\delta f_G]$, which we obtain as

$$N[\delta f_G] = \frac{2A_\perp \tau \Phi z_m}{160\beta \beta_\pi} \left[(2z_p^2 - 6z_m^2)K_1 \right.$$
$$\left. + (2z_p^2 - z_m^2)K_3 - z_m^2 K_5 \right]. \tag{36}$$

In the following, we show that the viscous correction to R_L due to Grad's 14-moment approximation violates the experimentally observed $1/\sqrt{m_T}$ scaling [25–29], whereas it is preserved in the Chapman-Enskog case. To this end, we calculate the first-order viscous correction to R_L in both the cases. Expanding the R_L in Eq. (31) to first order in δf and using the relation $z = \tau \sinh(\eta_s)$ we obtain the ideal contribution

$$\left(R_L^2\right)^{(0)} = \frac{\int K^\mu d\Sigma_\mu f_0 \, \tau^2 \sinh^2(\eta_s)}{\int K^\mu d\Sigma_\mu f_0}, \tag{37}$$

and the first viscous correction in the two cases

$$\left(\delta R_L^2\right)^{(1,G)} = -\left(R_L^2\right)^{(0)} \left(\frac{dN^{(1,G)}}{d^2 K_T} - \frac{dN^{(0)}}{d^2 K_T} \right)\bigg/ \frac{dN^{(0)}}{d^2 K_T}$$
$$+ \frac{\int K^\mu d\Sigma_\mu \, \tau^2 \sinh^2(\eta_s) \, \delta f_{1,G}}{\int K^\mu d\Sigma_\mu \, f_0}. \tag{38}$$

The ideal radius $(R_L^2)^{(0)}$ was obtained in Eq. (34). Viscous corrections due to the Chapman-Enskog method and Grad's 14-moment approximation can be obtained similarly. By substituting the viscous correction, δf_1, from Eq. (14) into Eq. (38), using the results for the particle spectra, Eqs. (25) and (26), and the ideal radius, Eq. (34), and performing the η_s integrals, we obtain

$$\frac{\left(\delta R_L^2\right)^{(1)}}{\left(R_L^2\right)^{(0)}} = -\frac{\Phi}{16\beta_\pi} \left[16 + \frac{4z_p^2}{z_m} \left(\frac{K_0}{K_1} - \frac{K_1}{K_2} \right) \right]. \tag{39}$$

Similarly, for Grad's approximation, Eq. (18), we obtain

$$\frac{\left(\delta R_L^2\right)^{(G)}}{\left(R_L^2\right)^{(0)}} = -\frac{\Phi}{20\beta_\pi} \left[20 - 2z_m \left(\frac{K_0}{K_1} - \frac{K_1}{K_2} \right) + 4z_m \frac{K_1}{K_2} \right]. \tag{40}$$

Using the asymptotic expansion of modified Bessel functions of the second kind [45],

$$K_n(z_m) = \left(\frac{\pi}{2z_m} \right)^{\frac{1}{2}} e^{-z_m} \left[1 + \frac{4n^2 - 1}{8z_m} + \cdots \right], \tag{41}$$

for large z_m, we have

$$\frac{K_0}{K_1} - \frac{K_1}{K_2} = \frac{1}{z_m} + \mathcal{O}\left(\frac{1}{z_m^2} \right). \tag{42}$$
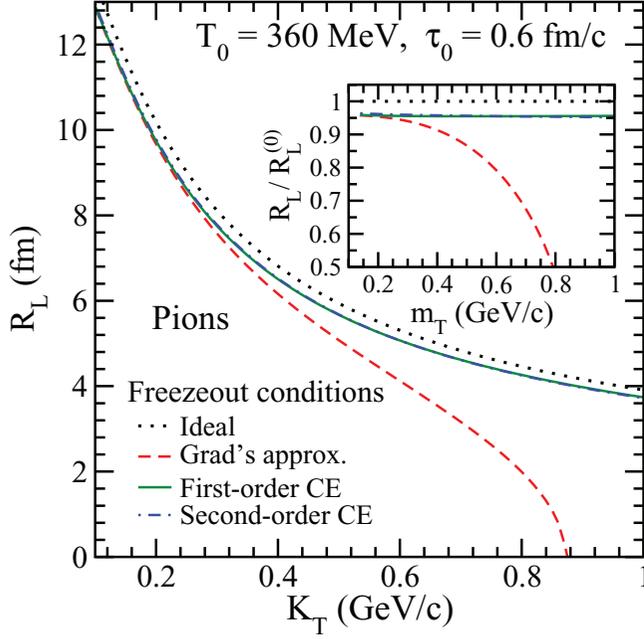
FIG. 2. (Color online) Longitudinal HBT radius as a function of the transverse momentum K_T of the pion pair, obtained with the second-order hydrodynamic evolution, followed by freeze-out in various scenarios: ideal, Grad's 14-moment approximation, and first- and second-order Chapman-Enskog. Inset: HBT radius in the above cases scaled by the corresponding values in the ideal case.

Hence, for large values of z_m, we find

$$\left(\delta R_L^2\right)^{(1)} = -\frac{5\tau^2 T \Phi}{4\beta_\pi m_T}, \tag{43}$$

$$\left(\delta R_L^2\right)^{(G)} = -\frac{\tau^2 T \Phi}{5\beta_\pi m_T}\left(3 + \frac{m_T}{T}\right). \tag{44}$$

It is clear from the above two equations that the viscous correction to R_L in the Chapman-Enskog case preserves the $1/\sqrt{m_T}$ scaling, whereas in Grad's 14-moment approximation it grows as m_T/T relative to the ideal result, and thus violates the scaling [36].

Results for the longitudinal HBT radius, R_L, for identical-pion pairs in central Au-Au collisions, for the four cases discussed above, are displayed in Fig. 2. We note that while there is no noticeable difference between first- and second-order Chapman-Enskog results compared to the ideal case, they predict a slightly smaller value for R_L. On the other hand, R_L corresponding to Grad's approximation exhibits a qualitatively different behavior and even becomes imaginary for $K_T \gtrsim 0.9$ GeV/c, which is clearly unphysical. More importantly, the ratio $R_L/R_L^{(0)}$ shown in the inset of Fig. 2 illustrates that the $1/\sqrt{m_T}$ scaling, which is violated in Grad's approximation, survives in the Chapman-Enskog case.

IX. SUMMARY AND CONCLUSIONS

We derived the form of the viscous correction to the equilibrium distribution function, up to second order in gradients, by employing a Chapman-Enskog-like iterative solution of the Boltzmann equation in the relaxation-time approximation. This approach is in accordance with the formulation of hydrodynamics, which is also a gradient expansion. We used this form of the viscous correction to calculate the hadronic transverse-momentum spectra and longitudinal Hanbury-Brown-Twiss radii and compared them with those obtained in Grad's 14-moment approximation within the one-dimensional scaling expansion. These results demonstrate the rapid convergence of the Chapman-Enskog expansion up to second order, and thus it is sufficient to retain only the first-order correction in the freeze-out prescription. We found that the Chapman-Enskog method results in softer hadron spectra compared with Grad's approximation. We further showed that the experimentally observed $1/\sqrt{m_T}$ scaling of HBT radii, which is also seen in the ideal freeze-out calculation, is maintained in the Chapman-Enskog method. In contrast, the Grad's 14-moment approximation leads to the violation of this scaling as well as an imaginary value for R_L at large momenta. For initial conditions typical of heavy-ion collisions at the Large Hadron Collider ($T_0 = 500$ MeV and $\tau_0 = 0.4$ fm/c), we have found that the above conclusions remain unchanged.

We conclude by recalling the well-known form of the viscous correction due to Grad's 14-moment approximation,

$$\delta f_G = \frac{f_0 \tilde{f}_0}{2(\epsilon + P)T^2} \, p^\alpha p^\beta \pi_{\alpha\beta}, \tag{45}$$

and the alternate form due to Chapman-Enskog method proposed here,

$$\delta f_{CE} = \frac{5 f_0 \tilde{f}_0}{8 P T (u \cdot p)} \, p^\alpha p^\beta \pi_{\alpha\beta}, \tag{46}$$

where $\tilde{f}_0 \equiv 1 - r f_0$, with $r = 1, -1, 0$ for Fermi, Bose, and Boltzmann gases, respectively. In view of the arguments presented in this paper, we advocate that the form of δf_{CE} proposed here should be a better alternative for hydrodynamic modeling of relativistic heavy-ion collisions.

APPENDIX A: CONSTRAINTS ON THE VISCOUS CORRECTION TO THE DISTRIBUTION FUNCTION

In this appendix, we show that the form of the viscous correction to the distribution function, δf, given in Eq. (14) satisfies the matching condition $\epsilon = \epsilon_0$ and the Landau frame definition $u_\nu T^{\mu\nu} = \epsilon u^\mu$, at each order in gradients [21]. We also show that δf is consistent with the definition of the shear stress tensor, Eq. (5).

The first- and second-order viscous corrections to the distribution function can be written separately using Eq. (14). The first-order correction is given by

$$\delta f_1 = \frac{f_0 \beta}{2\beta_\pi (u \cdot p)} \, p^\alpha p^\beta \pi_{\alpha\beta}, \tag{A1}$$

whereas the second-order correction is

$$\delta f_2 = -\frac{f_0 \beta}{\beta_\pi}\left[\frac{\tau_\pi}{u \cdot p} \, p^\alpha p^\beta \pi_\alpha^\gamma \, \omega_{\beta\gamma} - \frac{5}{14\beta_\pi (u \cdot p)} \, p^\alpha p^\beta \pi_\alpha^\gamma \, \pi_{\beta\gamma} \right.$$
$$\left. + \frac{\tau_\pi}{3(u \cdot p)} \, p^\alpha p^\beta \pi_{\alpha\beta} \theta - \frac{6\tau_\pi}{5} \, p^\alpha \dot{u}^\beta \pi_{\alpha\beta} + \frac{(u \cdot p)}{70\beta_\pi} \pi^{\alpha\beta} \pi_{\alpha\beta} \right.$$

$$+ \frac{\tau_\pi}{5} p^\alpha (\nabla^\beta \pi_{\alpha\beta}) - \frac{3\tau_\pi}{(u \cdot p)^2} p^\alpha p^\beta p^\gamma \pi_{\alpha\beta} \dot{u}_\gamma + \frac{\tau_\pi}{2(u \cdot p)^2}$$

$$\times p^\alpha p^\beta p^\gamma (\nabla_\gamma \pi_{\alpha\beta}) - \frac{\beta + (u \cdot p)^{-1}}{4(u \cdot p)^2 \beta_\pi} (p^\alpha p^\beta \pi_{\alpha\beta})^2 \Bigg]. \quad \text{(A2)}$$

In the following, we show that the δf_i given in Eqs. (A1) and (A2) satisfies the conditions

$$L_1[\delta f_i] \equiv \int dp\, (u \cdot p)^2\, \delta f_i = 0, \quad \text{(A3)}$$

corresponding to $\epsilon = \epsilon_0$, and

$$L_2[\delta f_i] \equiv \int dp\, \Delta_{\mu\alpha} u_\beta\, p^\alpha p^\beta\, \delta f_i = 0, \quad \text{(A4)}$$

corresponding to $u_\nu T^{\mu\nu} = \epsilon u^\mu$.

At first order, we obtain

$$L_1[\delta f_1] = \frac{\beta}{2\beta_\pi} \pi_{\alpha\beta} u_\gamma I_{(0)}^{\alpha\beta\gamma}, \quad L_2[\delta f_1] = \frac{\beta}{2\beta_\pi} \pi_{\alpha\beta} \Delta_{\mu\gamma} I_{(0)}^{\alpha\beta\gamma}, \quad \text{(A5)}$$

where we define the integral

$$I_{(r)}^{\mu_1 \mu_2 \cdots \mu_n} \equiv \int \frac{dp}{(u \cdot p)^r} p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} f_0. \quad \text{(A6)}$$

The above momentum integral can be decomposed into hydrodynamic tensor degrees of freedom as

$$I_{(r)}^{\mu_1 \mu_2 \cdots \mu_n} = I_{n0}^{(r)} u^{\mu_1} u^{\mu_2} \cdots u^{\mu_n} + I_{n1}^{(r)}$$
$$\times \left(\Delta^{\mu_1 \mu_2} u^{\mu_3} \cdots u^{\mu_n} + \text{perms} \right) + \cdots, \quad \text{(A7)}$$

where we readily identify $I_{20}^{(0)} = \epsilon$ and $I_{21}^{(0)} = -P$. Using the above tensor decomposition for $I_{(0)}^{\alpha\beta\gamma}$ in Eq. (A5), we obtain

$$L_1[\delta f_1] = 0, \quad L_2[\delta f_1] = 0. \quad \text{(A8)}$$

Similarly, for second-order corrections given in Eq. (A2), we obtain

$$L_1[\delta f_2] = 0 + \frac{5\beta}{14\beta_\pi^2} \pi_{\alpha\beta} \pi^{\alpha\beta} I_{31}^{(0)} + 0 + 0 - \frac{\beta}{70\beta_\pi^2} \pi_{\alpha\beta} \pi^{\alpha\beta} I_{30}^{(0)}$$
$$- \frac{\beta \tau_\pi}{5\beta_\pi} (\nabla^\alpha \pi_{\alpha\beta}) I_{30}^{(0)} u^\beta + 0 - \frac{\beta \tau_\pi}{\beta_\pi} (\nabla_\gamma \pi_{\alpha\beta}) I_{31}^{(0)}$$
$$\times u^{(\alpha} \Delta^{\beta)\gamma} + \frac{\beta}{2\beta_\pi^2} \pi_{\alpha\beta} \pi^{\alpha\beta} \left(\beta I_{42}^{(0)} + I_{42}^{(1)} \right). \quad \text{(A9)}$$

Using the identities

$$I_{nq}^{(r)} = -\frac{1}{2q+1} I_{n-1,q-1}^{(r-1)}, \quad \text{(A10)}$$

$$I_{nq}^{(0)} = \frac{1}{\beta} \left[-I_{n-1,q-1}^{(0)} + (n-2q) I_{n-1,q}^{(0)} \right], \quad \text{(A11)}$$

and Eq. (12), we obtain

$$L_1[\delta f_2] = -\frac{25}{14\beta_\pi} \pi_{\alpha\beta} \pi^{\alpha\beta} - \frac{3}{14\beta_\pi} \pi_{\alpha\beta} \pi^{\alpha\beta} + \frac{12}{8\beta_\pi} \pi_{\alpha\beta} \pi^{\alpha\beta}$$
$$- \frac{5}{2\beta_\pi} \pi_{\alpha\beta} \pi^{\alpha\beta} + \frac{3}{\beta_\pi} \pi_{\alpha\beta} \pi^{\alpha\beta}$$
$$= 0. \quad \text{(A12)}$$

A similar calculation leads to

$$L_2[\delta f_2] = 0 + 0 + 0 + \frac{6\beta\tau_\pi}{5\beta_\pi} I_{31}^{(0)} \Delta_\mu^\alpha \dot{u}^\beta \pi_{\alpha\beta} + 0$$
$$- \frac{\beta\tau_\pi}{5\beta_\pi} I_{31}^{(0)} \Delta_\mu^\alpha (\nabla^\beta \pi_{\alpha\beta}) - \frac{6\beta\tau_\pi}{5\beta_\pi} I_{31}^{(0)} \Delta_\mu^\alpha \dot{u}^\beta \pi_{\alpha\beta}$$
$$- \frac{\beta\tau_\pi}{\beta_\pi} I_{42}^{(1)} \Delta_\mu^\alpha (\nabla^\beta \pi_{\alpha\beta}) + 0$$
$$= 0. \quad \text{(A13)}$$

To obtain the second equality, we have used Eq. (A10) to replace $I_{42}^{(1)} = -I_{31}^{(0)}/5$.

Next we show that the form of the viscous correction to the distribution function, $\delta f = \delta f_1 + \delta f_2$ given in Eqs. (A1) and (A2), is consistent with the definition of the shear stress tensor given in Eq. (5). In other words, we show that $\pi^{\mu\nu} = L_3[\delta f_1] + L_3[\delta f_2]$, where

$$L_3[\delta f_i] \equiv \Delta_{\alpha\beta}^{\mu\nu} \int dp\, p^\alpha p^\beta\, \delta f_i. \quad \text{(A14)}$$

At first order, we get

$$L_3[\delta f_1] = \frac{\beta}{2\beta_\pi} \Delta_{\alpha\beta}^{\mu\nu} \pi_{\gamma\delta} I_{(1)}^{\alpha\beta\gamma\delta}. \quad \text{(A15)}$$

Using the tensor decomposition for $I_{(1)}^{\alpha\beta\gamma\delta}$ in the above equation, we obtain

$$L_3[\delta f_1] = \frac{\beta}{\beta_\pi} I_{42}^{(1)} \pi^{\mu\nu} = \pi^{\mu\nu}. \quad \text{(A16)}$$

Here we have used $I_{42}^{(1)} = \beta_\pi/\beta$, obtained by employing the recursion relations, Eqs. (A10) and (A11).

Similarly, for the second-order correction δf_2 given in Eq. (A2), we obtain

$$L_3[\delta f_2] = -2\tau_\pi \pi_\gamma^{\langle\mu} \omega^{\nu\rangle\gamma} + \frac{5}{7\beta_\pi} \pi_\gamma^{\langle\mu} \pi^{\nu\rangle\gamma} - \frac{2}{3} \tau_\pi \pi^{\mu\nu} \theta + 0$$
$$+ 0 + 0 + 0 + \left(\frac{1}{\beta_\pi} \pi_\gamma^{\langle\mu} \pi^{\nu\rangle\gamma} + 2\tau_\pi \pi_\gamma^{\langle\mu} \omega^{\nu\rangle\gamma} \right.$$
$$\left. + \frac{2}{3} \tau_\pi \pi^{\mu\nu} \theta \right) - \frac{12}{7\beta_\pi} \pi_\gamma^{\langle\mu} \pi^{\nu\rangle\gamma}$$
$$= 0. \quad \text{(A17)}$$

Hence $L_3[\delta f] = L_3[\delta f_1] + L_3[\delta f_2] = \pi^{\mu\nu}$. This result was expected because no second-order term (e.g., $\pi\pi$, $\pi\omega$, etc.) or their linear combinations, when substituted in Eq. (5), can result in a first-order term (π) which we have on the left-hand side of Eq. (5). In fact, each higher-order correction (δf_n) when substituted in Eq. (5) will vanish. The fact that δf given in Eq. (14) satisfies the constraints, as demonstrated in this appendix, shows that our method of obtaining the viscous corrections to the distribution function is quite robust.

APPENDIX B: SECOND-ORDER VISCOUS CORRECTIONS TO HADRON SPECTRA AND HBT RADII

Within the one-dimensional scaling expansion, $\dot{u} = 0 = \omega^{\mu\nu}$, which reduces the number of terms in Eq. (A2). The

nonvanishing terms can be simplified using Eq. (23) as

$$
\begin{aligned}
\delta f_2 = \frac{f_0 \beta}{\beta_\pi} \Bigg[&-\frac{5\Phi^2 m_T \{ p_T^2/(4m_T^2) + \sinh^2(y - \eta_s) \}}{14\beta_\pi \cosh(y - \eta_s)} \\
&-\frac{\tau_\pi \Phi m_T \{ p_T^2/(2m_T^2) - \sinh^2(y - \eta_s) \}}{3\tau \cosh(y - \eta_s)} \\
&-\frac{3\Phi^2 m_T \cosh(y - \eta_s)}{140\beta_\pi} + \frac{\tau_\pi \Phi m_T \cosh(y - \eta_s)}{5\tau} \\
&-\frac{\tau_\pi \Phi m_T \sinh^2(y - \eta_s)}{\tau \cosh(y - \eta_s)} + \frac{\Phi^2 \beta}{4\beta_\pi \cosh^2(y - \eta_s)} \\
&\times \left\{ 1 + \frac{(\beta m_T)^{-1}}{\cosh(y - \eta_s)} \right\} \left\{ \frac{p_T^2}{2m_T^2} - \sinh^2(y - \eta_s) \right\}^2 \Bigg].
\end{aligned}
$$
(B1)

The contribution to the hadronic spectra resulting from these second-order terms is calculated using Eq. (24) as

$$
\begin{aligned}
\frac{\delta dN^{(2)}}{d^2 p_T dy} &\equiv \frac{g}{(2\pi)^3} \int m_T \cosh(y - \eta_s) \tau d\eta_s r dr d\varphi \, \delta f_2 \\
&= \frac{g \tau A_\perp}{4\pi^3 \beta \beta_\pi} \Bigg[\frac{-5\Phi^2}{56\beta_\pi} (z_p^2 K_0 + 4z_m K_1) \\
&- \frac{\Phi \tau_\pi}{6\tau} (z_p^2 K_0 - 2z_m K_1) - \frac{3\Phi^2 z_m^2}{280\beta_\pi} (K_0 + K_2) \\
&+ \frac{\Phi \tau_\pi z_m^2}{10\tau} (K_0 + K_2) - \frac{\Phi \tau_\pi z_m}{\tau} K_1 + \frac{\Phi^2 z_m^2}{4\beta_\pi} \\
&\times \left\{ z_m X^2 \mathcal{I}_1 - 2z_m X K_1 + \frac{z_m}{4} (K_3 + 3K_1) \right. \\
&\left. + X^2 \mathcal{I}_2 - 2X K_0 + \frac{1}{2} (K_0 + K_2) \right\} \Bigg],
\end{aligned}
$$
(B2)

where $X \equiv z_p^2/(2z_m^2) + 1$, $K_n(z_m)$ are the modified Bessel functions of the second kind

$$
K_n(z) \equiv \int_0^\infty dt \, e^{-z\cosh(t)} \cosh(nt),
$$
(B3)

and $\mathcal{I}_n$ are the integrals defined as

$$
\mathcal{I}_n(z) \equiv \int_0^\infty dt \, e^{-z\cosh(t)} \operatorname{sech}^n(t),
$$
(B4)

with the following properties:

$$
\frac{d^n \mathcal{I}_n(z)}{dz^n} = (-1)^n K_0(z), \quad \mathcal{I}_0(z) = K_0(z).
$$
(B5)

The expression for hadron spectra up to second order, by setting $f = f_0 + \delta f_1 + \delta f_2$ in the freeze-out prescription, Eq. (24), becomes

$$
\frac{dN^{(2)}}{d^2 p_T dy} = \frac{dN^{(1)}}{d^2 p_T dy} + \frac{\delta dN^{(2)}}{d^2 p_T dy}.
$$
(B6)

Similarly, within the Bjorken model, one can calculate the longitudinal HBT radii by including the second-order viscous corrections in Eq. (32) using Eq. (B1). To this end, we calculate $N[\delta f_2]$ by setting $f = f_0 + \delta f_1 + \delta f_2$ in Eq. (32) and performing the integrations

$$
\begin{aligned}
N[\delta f_2] &= \int m_T \cosh^3(y - \eta_s) \tau d\eta_s r dr d\varphi \, \delta f_2 \\
&= \frac{2A_\perp \tau}{\beta \beta_\pi} \Bigg[\frac{-5\Phi^2}{112\beta_\pi} \Big\{ (z_p^2 - z_m^2) K_0 + z_p^2 K_2 \\
&+ z_m^2 K_4 \Big\} - \frac{\Phi \tau_\pi}{24\tau} \Big\{ (2z_p^2 + z_m^2) K_0 + 2z_p^2 K_2 \\
&- z_m^2 K_4 \Big\} - \frac{3\Phi^2 z_m^2}{1120\beta_\pi} (3K_0 + 4K_2 + K_4) \\
&+ \frac{\Phi \tau_\pi z_m^2}{40\tau} (3K_0 + 4K_2 + K_4) - \frac{\Phi \tau_\pi z_m^2}{8\tau} \big(K_4 \\
&- K_0 \big) + \frac{\Phi^2 z_m^2}{4\beta_\pi} \left\{ \left(X^2 - X + \frac{3}{8} \right) K_0 + \left(z_m X^2 \right. \right. \\
&\left. - \frac{3}{2} z_m X + \frac{5}{8} z_m \right) K_1 + \left(\frac{1}{2} - X \right) K_2 + \left(\frac{5}{16} z_m \right. \\
&\left. \left. - \frac{1}{2} z_m X \right) K_3 + \frac{1}{8} K_4 + \frac{1}{16} z_m K_5 \right\} \Bigg].
\end{aligned}
$$
(B7)

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