Analytical and numerical calculations for the asymptotic behavior of unitary 9j coefficients

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Previously it was noted in numerical calculations that a certain unitary 9j coefficient

$$U(I,j) = \langle (jj)^{2j}(jj)^{2j}|(jj)^{2j}(jj)^{(2j-2)}\rangle^{I}$$

decreases with increasing j and for fixed small I. The decrease is of the form $Aj^m e^{-\alpha j}$. The exponential decay factor dominates. Analytically we also show, using the Stirling approximation, that $\alpha = 4\ln(2)$ and $m = \frac{3}{2}$.

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I. INTRODUCTION

In previous works [1,2] Zamick and Escuderos addressed the problem of maximum j-pairing. In the course of these studies they found that results were simplified by the fact that a certain coupling matrix element was very small. This was the unitary 9j coefficient

$$U(I,j) = \langle (jj)^{2j}(jj)^{2j}|(jj)^{2j}(jj)^{(2j-2)}\rangle^{I} \tag{1}$$

for small I, e.g., $I = 2$. The work started in the $g_{9/2}$ shell, but as one went to higher shells this $U9j$ became rapidly smaller. Indeed the behavior was parametrized as $Aj^m e^{-\alpha j}$ [2,3]. The consequence of a very weak coupling is that for small total angular momentum I the lowest two states for a maximum J pairing interaction are $\langle (jj)^{2j}(jj)^{2j}|(jj)^{J_p}(jj)^{J_n}\rangle^{I}$ and $\langle (jj)^{2j}(jj)^{(2j-2)}|(jj)^{J_p}(jj)^{J_n}\rangle^{I}$ with J_p and J_n both even [1,2]. In this work we will first conduct numerical studies to much higher angular momenta and with greater precision for the unitary 9j coefficients using *Mathematica*. We will then approach the problem analytically and derive the parameters α and m. We also consider cases where I is large.

II. CALCULATION

A. Asymptotes of small I

As was noted in [1] at first glance $U(2,j)$ seems to fall of exponentially with j. This suggests a form

$$Ae^{-\alpha j}. \tag{2}$$

For this form $\ln(|U(2,j)|) = \ln(A) - \alpha j$. If this were true there would be a linear relationship between $\ln(|U(2,j)|)$ and j. Here we will also consider other values of I as indicated above.

We first plot, in Fig. 1, $\ln(|U(I,j)|)$ vs j for all even I values between $I = 2$ and $I = 32$. The curves indeed approach straight lines indicating that the $U(I,j)$'s drop exponentially with j. This is certainly the dominant trend but there are small deviations indicated by the error analysis.

We try a more elaborate form

$$UA(I,j) = Aj^m e^{-\alpha j}. \tag{3}$$

We consider the ratio

$$RR = \frac{U(I,j+1)^2}{U(I,j)\,U(I,j+2)}. \tag{4}$$

If we assume that $U9j = Aj^m e^{-\alpha j}$, then we have

$$RR = \frac{(A(j+1)^m e^{-\alpha(j+1)})^2}{Aj^m e^{-\alpha j} \times A(j+2)^m e^{-\alpha(j+2)}.} \tag{5}$$

With some algebra this becomes

$$RR = \frac{e^{-2\alpha j}e^{-2\alpha}(j+1)^{2m}}{e^{-2\alpha j}e^{-2\alpha}j^m(j+2)^m}. \tag{6}$$

It is obvious to see the factors which cancel out, then we take the ln of both sides and obtain

$$\ln(RR) = m\ln\left(\frac{(j+1)^2}{j(j+2)}\right). \tag{7}$$

We therefore have the extracted m:

$$m = \frac{\ln(RR)}{\ln\left(\frac{(j+1)^2}{j(j+2)}\right).} \tag{8}$$

It should be noted that in the large j limit $\frac{(j+1)^{2m}}{(j(j+2))^m}$ approaches $1 + \frac{m}{j^2}$. We plot some cases of m vs. j in Figs. 2 to 4. We find that for all even I from $I = 2$ to $I = 12$, m converges to 1.5 in the large j limit.

It is important to note that in order to obtain the asymptotic value of m in Eq. (3) one must go to a sufficiently large value of j. Furthermore the bigger the value of I the higher the one has to go in j. To show the perils of choosing a too small maximum j suppose we choose it to be 500.5, which *a priori* most would consider to be a very large number. The values of m for $I = 2,4,10,20,30$ are, respectively, 1.495, 1.481, 1.391, 1.085, and 0.577. We now see a steady decrease in m as I increases, which could lead to the false conclusion that there is a different asymptotic value of m for each I. However when we choose j large enough, e.g., up to 7000.5 for $I = 32$ we see that the asymptotic value of m is the same for all even I up to $I = 32$, namely $m = 1.5$. It should be noted that convergence is slower as I increases.

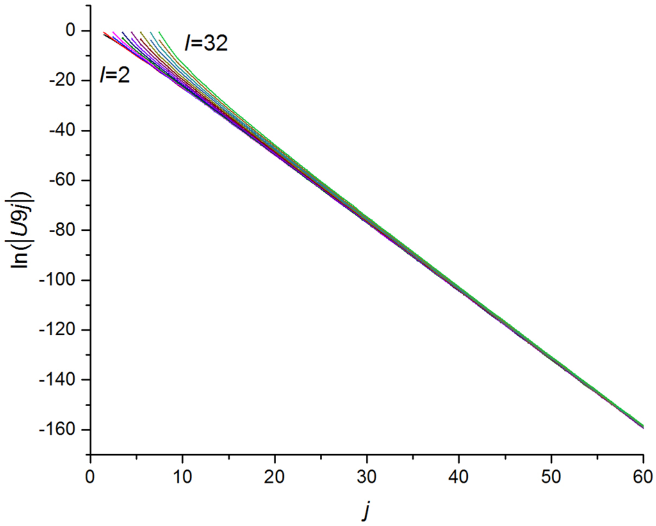

FIG. 1. (Color online) $\ln(|U9j|)$ vs j $I = 2,4,6,\ldots,32$.

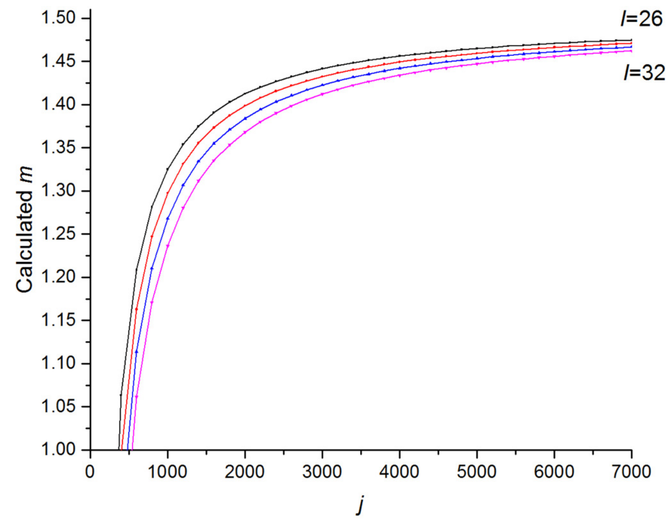

FIG. 3. (Color online) Suspected m vs j $I = 26,28,30,32$.

B. Asymptotes of large *I*

We next consider $U(I,j)$ for the largest values of I. We start with $I = I_{\max} = 4j - 2$ and then also consider $I_{\max} - 2$, $I_{\max} - 4$, etc. We find that $U(I_{\max},j)$ approaches a constant for large j shown in Fig. 5. We assume that the form of the asymptote is

$$U(I_{\max} - 2n, j) = \frac{A}{j^n}. \tag{9}$$

Then we plot $U(I_{\max} - 2n, j) \times j^n$ versus j to determine if this value approaches a constant. The results are shown in Fig. 6. We can conclude then that the asymptote for large I adheres to Eq. (9).

A formula involving many factorials for the case $I = I_{\max}$ is also given by Varshalovich *et al.* in Sec. 10:8:4 Eq. (14) in [4]. We finally remind the reader that our motivation for this work comes from our desire to understand the wave function arising from a "maximum J-pairing" Hamiltonian [1,2].

III. ANALYTICAL RESULTS

A. Asymptotes of small *I*

The numerical results in the previous section for the small I cases lead to the result $m = 1.5$ and the figures show a dominantly exponential decrease with j [3]. We can show some analytical results. We note that there is an explicit formula for the $9j$ symbol associated with the unitary $9j$ coefficient above in the work of Varshalovich *et al.* [4]

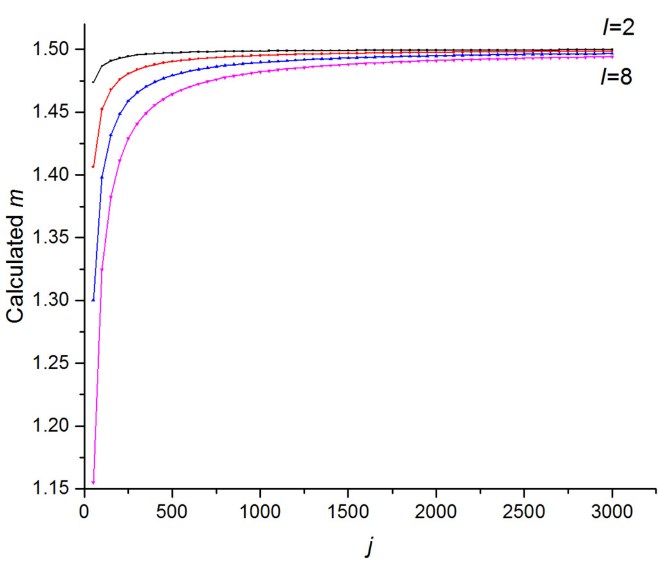

FIG. 2. (Color online) Suspected m vs j $I = 2,4,6,8$.

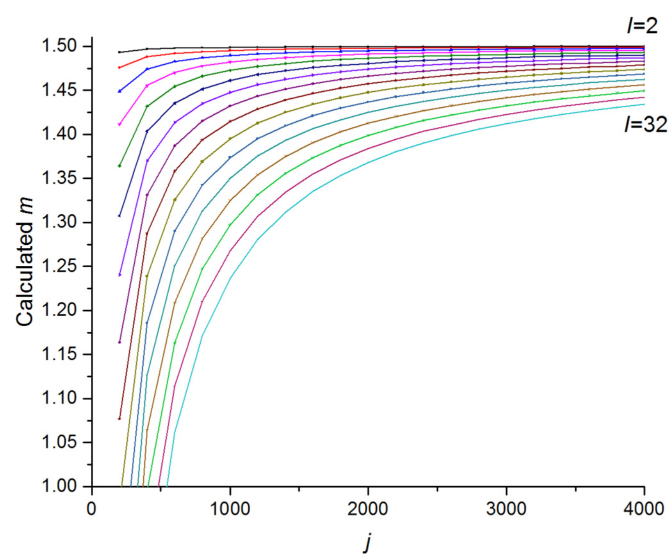

FIG. 4. (Color online) Suspected m vs j $I = 2,4,6,\ldots,32$.

Sec. 10:8:3 Eq. (9) shown here:

$$9j = \left\{ \begin{matrix} a & b & c \\ d & e & f \\ a+d & b+e & j \end{matrix} \right\} = \langle cf\ (a-b)(d-e)|j(a-b+d-e)\rangle$$
$$\times \left[\frac{(2a)!(2b)!(2d)!(2e)!(a+b+d+e+j+1)!(a+d+e+b-j)!}{(2a+2d+1)!(2b+2e+1)!(a+b+c+1)!(a+b-c)!(d+e+f+1)!(d+e-f)!(2j+1)} \right]^{\frac{1}{2}}. \quad (10)$$

We associate $a,b,d,e \to j$; $c,(a+d),(b+e) \to 2j$; $f \to (2j-2)$; and $j \to I$. For some simplification we define a new variable $J = 2j$. We apply that expression to this problem and consider $U9j$ rather than $9j$,

$$U(I,j) = \frac{(J!)^2}{(2J)!} \left[\frac{(2J+I+1)!(2J-I)!(2J+1)(2J-3)}{(2J+1)!(2J-1)!} \right]^{\frac{1}{2}} \times \sqrt{\frac{1}{2(2I+1)}} \times \langle J(J-2)00|I0\rangle. \quad (11)$$

Thus we have related the $U9j$ to a Clebsch-Gordan (CG) coefficient.

For the particular $U9j$ above and for $I = 2$ we obtain the following expression:

$$U(2,j) = \frac{(J!)^2}{(2J)!} \left[\frac{(2J+1)(2J+3)(2J+2)(2J-3)}{(2J-1)} \right]^{\frac{1}{2}} \times \sqrt{\frac{1}{10}} \times \langle J(J-2)00|20\rangle. \quad (12)$$

This special $U9j$ is proportional to a CG coefficient. There is a useful formula in Talmi's book [5] for the associated $3j$ symbol shown here:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(1+(-1)^{j_1+j_2+j_3})(-1)^g \times \sqrt{\frac{(2g-2j_1)!(2g-2j_2)!(2g-2j_3)!}{(2g+1)!}} \times \frac{g!}{(g-j_1)!(g-j_2)!(g-j_3)!}, \quad (13)$$

where $2g = j_1 + j_2 + j_3$ and

$$\mathrm{CG} = \sqrt{(2j_3+1)}(-1)^{j_1-j_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

There is a simpler formula in Talmi's book [5] for this coefficient when $I = 2$:

$$\langle J(J-2)00|20\rangle = -\sqrt{\frac{15J(J-1)^2}{((2J-3)(2J-2)(2J-1)(2J+1))}}. \quad (15)$$

It is easy to see that the CG coefficient falls off at $\frac{1}{\sqrt{J}}$. We now get the combined expression

$$U(2,j) = \frac{(J!)^2}{(2J)!} \sqrt{\frac{3J(J-1)^2(2J+3)(2J+2)}{2(2J-2)(2J-1)^2}}. \quad (16)$$

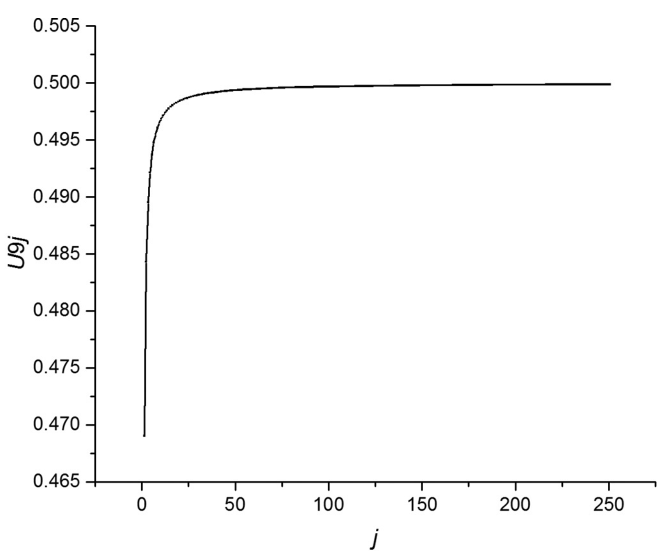

FIG. 5. $U9j$ vs j, $I = I_{\max}(n = 0)$.

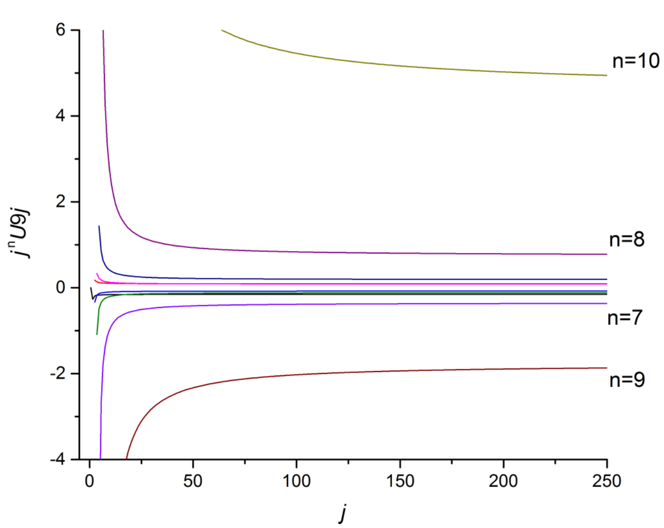

FIG. 6. (Color online) $(j^n U9j)$ vs j, $I = I_{\max} - 2n$, $n = 1,2, \ldots, 10$.

The exponential behavior comes from the factorials via the Stirling approximation

$$\ln(n!) \approx n\ln(n) - n. \tag{17}$$

If we stop there we get

$$\ln\left(\frac{(J!)^2}{(2J)!}\right) \approx 2J\ln(J) - 2J\ln(2J) = -2J\ln 2. \tag{18}$$

However to get the correct asymptotic behavior we must go beyond this and include one more term to obtain the more accurate Stirling approximation

$$\ln(n!) = n\ln(n) - n + \ln(\sqrt{2\pi n}). \tag{19}$$

Using the extended Stirling approximation this becomes

$$\ln\left(\frac{(J!)^2}{(2J!)}\right) \approx -2J\ln(2) + \ln(\sqrt{\pi J}). \tag{20}$$

Recall that we had assigned $J = 2j$ and then taking an inverse logarithm of this yields a contribution

$$\frac{(J!)^2}{(2J!)} \approx e^{-4\ln(2)j}\sqrt{2\pi j}. \tag{21}$$

When we go from j to $j+1$ we get a decrease of about 16 from the exponential factor. This decrease dominates over the increase from the second factor. The second factor and the other terms must contribute to get the j^m part which serves to reduce this ratio a bit.

If the "small" term in the Stirling approximation is neglected a problem arises. The factors under the square root sign clearly go at $j^{3/2}$ in the large j limit. However the CG coefficient decreases with j. This leads to an effective m less than $\frac{3}{2}$. However numerical calculations [3] clearly indicate that $m = \frac{3}{2}$. Hence, although the simplest version of the Stirling approximation gives the right exponential behavior it gives the wrong j^m dependence. By including the "small correction" we take care of this problem.

Analytic expressions of specific $9j$ coefficients have been previously considered for special cases, e.g., for the case of partial dynamical symmetries by Robinson and Zamick [6]. Many relations for $9j$ symbols were found by Zhao and Arima [7] in the context of maximum j-pairing Hamiltonians. Explicit studies of the asymptotic behaviors of $9j$ coefficients have been performed by Anderson *et al.* [8] and by Yu and Littlejohn [9]. What distinguishes the present work from the ones just mentioned is that only here do we consider $9j$s which display an exponential decrease with increasing j. This is called nonclassical behavior by the experts. The large difference in behavior comes from the fact that we are considering coupling matrix elements involving two different J values $2j$ and $2j-2$ whereas in Zhao and Arima [7] for the problem they are addressing they have the same J values. Ironically we have to be in the nonclassical region mathematically to reach the classical limit for the physical problem in question.

B. Asymptotes of large I

We now consider the region near $I = I_{\max} = 4j-2$. It should be pointed out that whereas in the small I case we kept I fixed as we increased j, here as we change j we change I. Thus we are making different comparisons. The figures confirm that for this analysis there is a power-law behavior rather than an exponential one. The $U9j$ goes as $\frac{1}{j^n}$ does, where $n = \frac{(I_{\max}-I)}{2}$. It should be noted that for $I = I_{\max} = 4j-2$ the value of $U9j$ was shown by Talmi [10] to be

$$U9j = \frac{\sqrt{(2j-1)(8j-1)}}{(8j-2)}. \tag{22}$$

Note that this $9j$ approaches 1/2 when j becomes very large. Subsequently an alternate proof was provided by Bayman [11].

For large I we write $I = 4j-2-2n$ and assume n is much smaller than j. We use a more general formula from Talmi's book [5] (top of p. 960) for $\langle 2j(2j-2)00|I0\rangle$.

We can get an expression for all n by using the Stirling approximation for factorials involving large parameters but not for those involving only n. We obtain the following result:

$$U9j = \frac{(-1)^n}{2\sqrt{2}(16)^n}\frac{\sqrt{((2n+2)!(2n)!)}}{(n!)j^n} \tag{23}$$

as j becomes very large. One can verify that for $n=0$ this is indeed $\frac{1}{2}$ and note that for $n=1$ we get $\frac{\sqrt{3/2}}{8j}$. One notes that in this limit (n smaller than j) the CG coefficient goes as $\frac{1}{j^{1/4}}$ (alternatively the $3j$ goes as $\frac{1}{j^{3/4}}$) in the large j limit. In more detail we have

$$\mathrm{CG} = \frac{\sqrt{(2n)!}}{n!(2^n)}\left(\frac{1}{\pi j}\right)^{1/4}(-1)^n. \tag{24}$$

The $\frac{1}{j^{1/4}}$ behavior in the large I limit is in contrast to the behavior in the previous section where I was fixed at a small value while j was increased. In that case the CG coefficient from Eq. (14) was proportional to $\frac{1}{j^{1/2}}$. In this work our motivation for studying the specific $U9j$ coefficients above was to better understand the wave functions of a maximum J-pairing Hamiltonian. What we had previously shown numerically we now have attempted to show analytically. We found the numerical results crucial in guiding us to the analytical ones. We have succeeded in getting analytic expressions for the asymptotic behaviors for small I by using the extended Stirling approximation. We are also able to make statements about the large I problem.

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