Anomalous viscosity of the quark-gluon plasma

Juhee Hong[1] and P. H. Diamond[1,2]

[1]*WCI Center for Fusion Theory, National Fusion Research Institute, Daejeon 305-806, Korea*
[2]*CASS and Department of Physics, University of California San Diego, La Jolla, California 92093, United States*

The shear viscosity of the quark-gluon plasma is predicted to be lower than the collisional viscosity for weak coupling. The estimated ratio of the shear viscosity to entropy density is rather close to the ratio calculated by $\mathcal{N} = 4$ super Yang–Mills theory for strong coupling, which indicates that the quark-gluon plasma might be strongly coupled. However, in the presence of momentum anisotropy, the Weibel instability can arise and drive the turbulent transport. Shear viscosity can be lowered by enhanced collisionality due to turbulence, but the decorrelation time and its relation to underlying dynamics and color-magnetic fields have not been calculated self-consistently. In this paper, we use resonance broadening theory for strong turbulence to calculate the anomalous viscosity of the quark-gluon plasma for nonequilibrium. For saturated Weibel instability, we estimate the scalings of the decorrelation rate and viscosity and compare these with collisional transport. This calculation yields an explicit connection between the underlying momentum-space anisotropy and the viscosity anomaly.

I. INTRODUCTION

At sufficiently high temperature, transport in quark-gluon plasma can be described by weakly coupled theories. Given that typical particles have momentum $\sim T$, there are several important kinetic scales, in terms of weak coupling $g \ll 1$ [1]. First, static color-electric fields are screened at the Debye length $\sim 1/(gT)$. Second, (unlike traditional electromagnetic plasmas) static color-magnetic fields are confined at nonperturbative scales $\sim 1/(g^2 T)$. Finally, dynamics is governed by particle collisions at macroscopic scales $\sim 1/(g^4 T)$ where hydrodynamics can be applied. Transport in quark-gluon plasma has been studied primarily based on macroscopic hydrodynamics. However, there are mesoscopic scales, $1/T \ll \text{(distance)} \ll 1/(g^2 T)$, where collective effects can be important and a magnetohydrodynamic description can be applied, as in electromagnetic plasmas.

Transport coefficients have been calculated using the linearized Boltzmann equation [2,3]. Taking into account two-particle collisions, the ratio of the shear viscosity to entropy density is

$$\frac{\eta_C}{s} \sim \frac{1}{g^4 \ln(1/g)}. \tag{1.1}$$

On the other hand, experimental data can be described by hydrodynamic simulations with an anomalously low viscosity. Comparing elliptic flow data with simulations, the shear viscosity of the quark-gluon plasma is deduced to be (see a review [4])

$$\frac{\eta}{s} \sim \frac{1 \leftrightarrow 5}{4\pi}. \tag{1.2}$$

Much thought has been devoted to the fact that the shear viscosity as deduced from data is much lower than the collisional viscosity Eq. (1.1) when the coupling constant is small enough. Equation (1.2) is rather close to the ratio predicted using $\mathcal{N} = 4$ super Yang–Mills theory for strong coupling, $\eta/s = 1/4\pi$ [5,6]. One way to resolve the problem of why $\eta < \eta_C$ is indicated is to assume that the quark-gluon plasma is strongly coupled. Alternatively, instability effects have been suggested as a means for enhanced collisionality which can reduce η [7,8]. We will discuss this scenario in this work.

When anisotropic momentum distributions occur, the Weibel[1] instability can arise at soft momentum $\sim gT$. The Weibel instability of the quark-gluon plasma has been studied by transport theory, and, equivalently, in hard thermal loop dynamics [10–12]. There have been numerical simulations and analytic studies of thermalization and cascade [13–21]. In electromagnetic plasmas, Weibel-excited random fields coherently scatter particles and so reduce the rate of momentum transport [22,23]. Similarly, turbulent color-magnetic fields might affect transport properties of the quark-gluon plasma. In that case, viscosity is not obtained solely by particle collisions, but instability effects must be also accounted for.

Viscosity measures stress per velocity gradient. Since the stress tensor is $\sim T^4$ and the collision frequency is $\sim g^4 T \ln(1/g)$ for soft momentum transfer, the collisional viscosity is $\eta_C \sim T^3/[g^4 \ln(1/g)]$, as in Eq. (1.1). In the presence of instability-driven fluctuations, we must take a possibly enhanced decorrelation frequency (due to interaction between particles and turbulent fields) into account when computing the transport. Although it depends on which mechanism (collisions or instabilities) is dominant for the relevant kinetic regime, the effective viscosity is roughly determined by

$$\eta \sim \frac{(\text{Stress})}{(\text{Collision Frequency}) + (\text{Decorrelation Frequency})}. \tag{1.3}$$

In high-temperature non-Abelian plasmas, instability arises at momentum $\lesssim gT$. So, we guess that the decorrelation

---

[1]The Weibel instability arises in presence of momentum space anisotropy or temperature gradient [9]. It is cumulative effects of counter streams and develops current filamentation.

frequency is $\lesssim gT$. Since the decorrelation frequency can be higher than the collision frequency, it follows that instability and momentum space scattering might lower the viscosity of the quark-gluon plasma [7,8].

The actual quark-gluon plasma produced in relativistic heavy-ion collisions is a complicated dynamic system. Calculating the viscosity requires us to understand the fluctuation dynamics and transport properties of the plasma in each stage. However, to investigate instability effects on viscosity, we consider a rather simple case in this work. According to numerical simulations, there is no significant difference between Abelian plasmas and non-Abelian plasmas in $1 + 1$ dimensions: instability grows exponentially [14,16,18]. Such Abelianization disappears in $3 + 1$ dimensions, where instability growth is subexponential. To estimate the lower bound of the anomalous viscosity, we assume an Abelian regime in $1 + 1$ dimensions which can be used to determine the maximum intensity of plasma instabilities and transport. In Sec. II, we briefly review the linear instability. We focus on the turbulent Weibel state for soft momentum $k \sim gT$. In Sec. III, we analyze nonlinear particle-wave interaction using resonance-broadening theory for strong turbulence. For the saturated Weibel instability, we obtain the relation between the decorrelation frequency and turbulent color-magnetic fields. Following Refs. [24,25] in electromagnetic plasmas, we calculate the decorrelation frequency and the anomalous viscosity of the quark-gluon plasma for nonequilibrium. Finally, we summarize our results in Sec. IV.

II. LINEAR INSTABILITY

In this section, we briefly review the linear analysis for the Weibel instability. We assume an Abelian regime by linearizing the equations of motion in the gauge field. In the next section, we consider nonlinear particle-wave interaction due to resonance broadening for strong turbulence.

We linearize the distribution of hard particles as[2]

$$f = \langle f \rangle + \delta f, \tag{2.1}$$

where $\langle f \rangle$ is color neutral and anisotropic in momentum $\boldsymbol{p}$, and δf is colored fluctuations. At mesoscopic scales, the kinetic equation of particles is the Vlasov equation

$$v^\mu \partial_\mu \delta f^a + g(\boldsymbol{E}^a + \boldsymbol{v} \times \boldsymbol{B}^a) \cdot \frac{\partial \langle f \rangle}{\partial \boldsymbol{p}} = 0, \tag{2.2}$$

where $v^\mu = p^\mu / E_{\boldsymbol{p}}$. Color-electromagnetic fields obey the non-Abelian Maxwell equation

$$\partial_\nu F^{\mu\nu,a} = J^{\mu,a} = g \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} v^\mu \delta f^a. \tag{2.3}$$

In Fourier space, the linear solution of the Vlasov equation is

$$\delta f^a(\omega, \boldsymbol{k}) = -\frac{g(\boldsymbol{E}^a + \boldsymbol{v} \times \boldsymbol{B}^a) \cdot \frac{\partial \langle f \rangle}{\partial \boldsymbol{p}}}{-i\omega + i\boldsymbol{v} \cdot \boldsymbol{k} + \epsilon}, \tag{2.4}$$

where ϵ is positive and infinitesimal. By plugging the solution to the non-Abelian Maxwell equation, we have

$$ik_\nu F^{\mu\nu,a} = -g^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{v^\mu(\boldsymbol{E}^a + \boldsymbol{v} \times \boldsymbol{B}^a) \cdot \frac{\partial \langle f \rangle}{\partial \boldsymbol{p}}}{-i\omega + i\boldsymbol{v} \cdot \boldsymbol{k} + \epsilon}. \tag{2.5}$$

This can be written as

$$\epsilon^{\mu\nu} A_\nu^a = 0, \tag{2.6}$$

where we defined a tensor

$$\epsilon^{\mu\nu} \equiv (-\omega^2 + k^2) g^{\mu\nu} - k^\mu k^\nu + \Pi^{\mu\nu}, \tag{2.7}$$

with the self-energy

$$\Pi^{\mu\nu} = g^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{\partial \langle f \rangle}{\partial p^i} \left[-v^\mu g^{i\nu} + \frac{v^\mu v^\nu k^i}{-\omega + \boldsymbol{v} \cdot \boldsymbol{k} - i\epsilon} \right]. \tag{2.8}$$

In the temporal gauge $A_0 = 0$, we have $\epsilon^{ij} E_j = 0$, and the linear dispersion relation is

$$\det \epsilon^{ij} = 0. \tag{2.9}$$

Depending on the sign of Im ω, we have exponentially growing or damping solutions $\omega(\boldsymbol{k})$. If there is an exponentially growing solution with Im $\omega > 0$, the quark-gluon plasma has instability that can drive turbulence.

III. NONLINEAR PARTICLE-WAVE INTERACTION

In this section, we consider nonlinear particle-wave interaction due to resonance broadening. Resonance broadening theory is well defined for traditional electromagnetic plasmas (see Appendix A) and amounts to calculating phase space eddy diffusivity and its effects on particle trajectories.[3] We can apply resonance broadening theory to the relativistic quark-gluon plasma in momentum space. For strong turbulence, the linear dispersion relation can be extended to the nonlinear regime with a simple correction in the self-energy. For the Weibel instability at saturation, we calculate the diffusion coefficient (which is related to color-magnetic fields), the particle-wave decorrelation time, and the anomalous viscosity. The momentum-space diffusion coefficient is determined by the saturation condition. This sets an effective root-mean-square turbulence intensity. This approach is made in the spirit of Prandtl's theory of pipe flow turbulence than of the familiar Kolmogorov cascade.

A. Resonance broadening

The distribution function is written as

$$f = \langle f \rangle + f_{\omega,\boldsymbol{k}} + \tilde{f}, \tag{3.1}$$

where $\langle f \rangle$ is the average over space, $f_{\omega,\boldsymbol{k}}$ is the coherent part with respect to color-electromagnetic fields, and $\tilde{f}$ represents

---

[2]For plasmas consisting of gluons, $f = 2N_c f_g$, where f_g is the distribution function of gluons per helicity and color. $\delta f = \delta f^a T^a$, where δf^a and generators T^a are in the adjoint representation.

[3]These enter the linear response which determines the instability.

fluctuations due to noise.[4] Taking the average over space, the mean-field Vlasov equation becomes (see, for example, Ref. [26])

$$\frac{\partial}{\partial t}\langle f\rangle + g\left\langle (E^a + v\times B^a)\cdot\frac{\partial f^a_{\omega,k}}{\partial p}\right\rangle = 0, \qquad (3.2)$$

where we used the fact that f does not diverge at infinity[5] and $\langle E^a\rangle = \langle B^a\rangle = 0$.

Similar to the linear solution Eq. (2.4), the coherent response $f^a_{\omega,k}$ has a peak $\sim 1/(\omega - v\cdot k)$ corresponding to the resonance where particle velocity is equal to the phase velocity of color-electromagnetic waves. In presence of nonlinear interaction between particles and waves, the former are scattered by the ensemble of wave fields. As a result, the peak of the resonance is broadened (see, for example, Ref. [27]). To explain resonance broadening, we consider test particle dynamics in one dimension. In the linear order, the particle trajectory is assumed to be unperturbed, since nonlinear particle-wave interaction scatters the trajectory from the unperturbed one by δx. So, the coherent response is

$$f^a_{\omega,k} = -\int_0^\infty dt\, e^{i(\omega t - kx)+ik\,\delta x} g\left(E^a_{\omega,k}+v\times B^a_{\omega,k}\right)\cdot\frac{\partial\langle f\rangle}{\partial p}. \qquad (3.3)$$

By plugging this response to the quasilinear equation Eq. (3.2), we obtain a diffusion equation [22,23]

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial p}\cdot D(p)\cdot\frac{\partial}{\partial p}\right)\langle f\rangle = 0, \qquad (3.4)$$

where the diffusion tensor is given by the Lorentz force-force correlator with $F^a_{\omega,k} = g(E^a_{\omega,k}+v\times B^a_{\omega,k})$,

$$D(p) = \int_0^\infty dt\, e^{i(\omega t - kx)+ik\,\delta x}\langle F^a_{\omega,k}\,F^a_{\omega,k}\rangle. \qquad (3.5)$$

Since color-electromagnetic fields are turbulent, particles perform a random walk in momentum space. This diffusion scatters particles from their unperturbed trajectories, weakens the response, and eventually saturates the instability.

The scatter of a trajectory can be calculated by taking the average over the probability density function (pdf). We assume that δp has a Gaussian pdf

$$\text{pdf}\,[\delta p] = \frac{1}{\sqrt{\pi Dt}}\,e^{-\frac{(\delta p)^2}{Dt}}. \qquad (3.6)$$

FIG. 1. (Color online) The coherent response $f_{\omega,k}$ has a resonance at $\omega = v\cdot k$. Due to nonlinear particle-wave interaction, the resonance peak of a delta function $\delta(\omega - v\cdot k)$ is broadened with a width proportional to the decorrelation rate $1/t_c$.

Performing the Gaussian integral, we have

$$\langle e^{i(\omega t - kx)+ik\delta x}\rangle_{\text{pdf}} = \int\frac{d\,(\delta p)}{\sqrt{\pi Dt}}e^{-\frac{(\delta p)^2}{Dt}}e^{i(\omega t - kx)+ik\int dt\,(\delta v)}$$

$$\simeq e^{i(\omega - vk)t - \frac{k^2 Dt^3}{4\bar{E}_p^2}}, \qquad (3.7)$$

where we approximated $\int dt\,(\delta v)\simeq t\,(\delta p)/E_p$[6] and replaced E_p by the averaged $\bar{E}_p \equiv (\int d^3p\, E_p\langle f\rangle)/(\int d^3p\,\langle f\rangle)$. From the coefficient of the t^3 term, we define the particle-wave decorrelation time t_c:

$$\left(\frac{1}{t_c}\right)^3 \equiv \frac{k^2 D}{4\bar{E}_p^2}. \qquad (3.8)$$

Here, t_c is the time scale it takes the wave ensemble to scatter a particle by wavelength $\sim 1/k$ from its unperturbed trajectory.

The principal effect of nonlinear particle-wave interaction is to broaden the resonance peak of a delta function to a resonance with a width proportional to the decorrelation rate $1/t_c$. Thus, we can use the Lorentzian approximation for strong turbulence as an approximation (see Fig. 1):

$$\int_0^\infty dt\, e^{i(\omega - v\cdot k)t - t^3/t_c^3}\simeq \frac{i}{\omega - v\cdot k + i/t_c}. \qquad (3.9)$$

In this regard, within resonance broadening theory for strong turbulence, the self-energy Eq. (2.8) acquires a nonlinear correction which amounts to the replacement $\omega\to\omega + i/t_c$.

---

[4] $\tilde{f}$ is ignored in the quasilinear order.

[5] For spatially homogeneous $\langle f\rangle$,

$$\left\langle\frac{\partial f}{\partial x}\right\rangle = \lim_{L\to\infty}\frac{1}{L}\int_{-L/2}^{L/2}dx\,\frac{\partial f}{\partial x}$$

$$= \lim_{L\to\infty}\frac{1}{L}\left[f\left(x = \frac{L}{2}\right) - f\left(x = -\frac{L}{2}\right)\right] = 0.$$

In Sec. III D, the $v\cdot\frac{\partial}{\partial x}$ term will be revived in calculating the viscosity.

---

[6] Since $v = p/E_p$, $\delta v = (\delta p)/E_p - p\,(\delta E_p)/E_p^2$, where we ignore δE_p with a diffusive pdf of Eq. (3.6). Assuming a similar Gaussian pdf of δE_p, we obtain a consistent t^3 factor of resonance broadening.

B. Diffusion coefficient

In the absence of static color-electromagnetic fields, the diffusion tensor due to color-magnetic excitations is[7] [28]

$$D = \sum_{\omega,\boldsymbol{k}} \left(g\boldsymbol{v} \times \delta\boldsymbol{B}_{\omega,\boldsymbol{k}}^a\right) \frac{i}{\omega - \boldsymbol{v}\cdot\boldsymbol{k} + i/t_c} \left(g\boldsymbol{v} \times \delta\boldsymbol{B}_{\omega,\boldsymbol{k}}^a\right), \tag{3.10}$$

where we used the Lorentzian approximation Eq. (3.9). For most unstable modes, the wave vector is along the direction of anisotropy and color-magnetic excitations is perpendicular to the direction[8]

$$\boldsymbol{k} = k\hat{\boldsymbol{z}} \quad \text{and} \quad \delta\boldsymbol{B}_{\omega,\boldsymbol{k}}^a = \delta B_{\omega,\boldsymbol{k}}^a \,\hat{\boldsymbol{y}}. \tag{3.11}$$

Since the Weibel instability is purely growing, we set $\omega = i\gamma$, where γ is the growth rate. Then the diffusion coefficient is

$$D = \sum_{\omega,k} g^2 v_x^2 \left|\delta B_{\omega,k}^a\right|^2 \frac{1}{\gamma + 1/t_c + iv_z k}. \tag{3.12}$$

We now consider how large color-magnetic excitations can grow. When the Weibel instability saturates, color-magnetic excitations stop growing ($\gamma = 0$). So, we have

$$D = \sum_{\omega,k} g^2 v_x^2 \left|\delta B_{\omega,k}^a\right|^2 \frac{1/t_c}{(1/t_c)^2 + (v_z k)^2}, \tag{3.13}$$

where the imaginary part vanished because it is an odd function of k. It can be simplified for "strong turbulence" where the particle-wave decorrelation time t_c is so short compared to the time scale $\sim 1/(\boldsymbol{v}\cdot\boldsymbol{k})$ that the condition $(1/t_c)^2 \gg (v_z k)^2$ is satisfied. Ignoring $(v_z k)^2$ in the denominator, we obtain

$$D \simeq \sum_{\omega,k} g^2 v_T^2 \left|\delta B_{\omega,k}^a\right|^2 \frac{1}{1/t_c}, \tag{3.14}$$

where we replaced v_x^2 by the thermal velocity v_T^2.[9] With the definition of the decorrelation time Eq. (3.8), we determine the relation between the decorrelation time and the intensity of color-magnetic excitations at saturation; namely,

$$\left(\frac{1}{t_c}\right)^4 \simeq \frac{k^2}{4\bar{E}_{\boldsymbol{p}}^2} \sum_{\omega',k'} g^2 v_T^2 \left|\delta B_{\omega',k'}^a\right|^2. \tag{3.15}$$

Here, t_c gives the time scale for scattering of a particle, that is, the trajectory mixing time.

C. Decorrelation time

The particle-wave decorrelation time can be determined from the nonlinear dispersion relation. As discussed below

---

[7]In this work, we consider nonlinear particle-wave interaction for saturated Weibel instability. So, ω and $\boldsymbol{k}$ in the summation satisfy the linear dispersion relation.

[8]Color-electric excitations are $\delta\boldsymbol{E}_{\omega,\boldsymbol{k}}^a = \delta E_{\omega,\boldsymbol{k}}^a \hat{\boldsymbol{x}}$. Since color-electric fields are related to color-magnetic fields by the non-Abelian Maxwell equation, we consider only color-magnetic fields.

[9]The thermal velocity squared $v_T^2 \sim 1$ is a typical velocity squared of particles in the quark-gluon plasma.

Eq. (3.9), the self-energy has a nonlinear correction due to the resonance broadening,

$$\Pi_{\text{NL}}^{ij} = g^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{\partial\langle f\rangle}{\partial p^l} \left[-v^i g^{lj} + \frac{v^i v^j k^l}{-\omega + \boldsymbol{v}\cdot\boldsymbol{k} - i/t_c}\right]. \tag{3.16}$$

Following Ref. [29], given an isotropic distribution $\langle f(\boldsymbol{p}^2)\rangle_{\text{iso}}$, we make an anisotropic distribution by the rescaling of the $\hat{\boldsymbol{z}}$ direction:

$$\langle f\rangle = \left\langle f\left(\boldsymbol{p}^2 + \xi p_z^2\right)\right\rangle_{\text{iso}}. \tag{3.17}$$

Here, $\xi > -1$ is the anisotropy parameter: $-1 < \xi < 0$ corresponds to a stretch and $\xi > 0$ corresponds to a squeeze in the $\hat{\boldsymbol{z}}$ direction. By a change of variables to $\tilde{p} \equiv p\sqrt{1 + \xi v_z^2}$, Eq. (3.16) can be calculated as

$$\Pi_{\text{NL}}^{ij} = m_D^2 \int \frac{d\Omega}{4\pi} \frac{v^i}{\left(1 + \xi v_z^2\right)^2}$$

$$\times \left[v^j + \xi v_z \hat{z}^j + \frac{(\xi + 1)v^j v_z k}{\omega - v_z k + i/t_c}\right], \tag{3.18}$$

where

$$m_D^2 = -\frac{g^2}{2\pi^2} \int_0^\infty dp\, p^2 \frac{d\langle f\rangle_{\text{iso}}}{dp}. \tag{3.19}$$

In the case of Eq. (3.11), the dispersion relation is

$$-\omega^2 + k^2 + \Pi_{\text{NL}}^{xx} = 0. \tag{3.20}$$

For strong turbulence, when the Weibel instability saturates, the self-energy term is

$$\Pi_{\text{NL}}^{xx}$$
$$\simeq \frac{m_D^2}{4}\left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]$$
$$- \frac{m_D^2 t_c^2 k^2}{4}\left[-\frac{3(\xi+1)}{\xi^2} + \frac{(\xi+1)(\xi+3)}{\xi^2}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]. \tag{3.21}$$

From Eq. (3.20), we determine the decorrelation time

$$t_c^2 \simeq \frac{k^2 + \frac{m_D^2}{4}\left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]}{\frac{m_D^2}{4}\left[-\frac{3(\xi+1)}{\xi^2} + \frac{(\xi+1)(\xi+3)}{\xi^2}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]k^2} \tag{3.22}$$

for strong turbulence, where functions of ξ in the square brackets are positive. Since the decorrelation time is taken to be short for strong turbulence, it must satisfy

$$\frac{1}{t_c^2 k^2} \simeq \frac{\frac{m_D^2}{4}\left[-\frac{3(\xi+1)}{\xi^2} + \frac{(\xi+1)(\xi+3)}{\xi^2}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]}{k^2 + \frac{m_D^2}{4}\left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]} \gg v_z^2, \tag{3.23}$$

which gives the validity regime for the anisotropy parameter ξ (see Fig. 2). As anisotropy grows, the decorrelation time decreases until $\xi_k^* > 0$ for the wave vector k. Noting $v_z^2 \ll 1$, ξ around ξ_k^* most likely satisfies the strong-turbulence condition. For low k, this regime corresponds to an extreme squeeze in the momentum $\hat{\boldsymbol{z}}$ direction of an initially isotropic distribution.
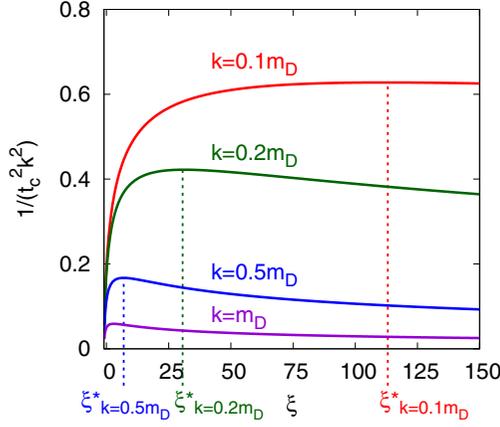
FIG. 2. (Color online) As anisotropy grows, the decorrelation time decreases until $\xi_k^* > 0$ for the wave vector k. For strong turbulence, the anisotropic parameter must be in the regime where $1/(t_c^2 k^2) \gg v_z^2$. Since $v_z^2 \ll 1$, $\xi \approx \xi_k^*$ most likely satisfies the condition. For low k, this regime corresponds to an extreme squeeze in the momentum $\hat{z}$ direction of an initially isotropic distribution.

This might apply to the early stage of relativistic heavy-ion collisions.

At soft momentum $k \sim gT$, the scale of the decorrelation time Eq. (3.22) is

$$t_c \sim \frac{1}{k}. \tag{3.24}$$

By using Eqs. (3.15) and (3.22), we determine the saturation level of color-magnetic excitations:

$$\frac{1}{4\bar{E}_{\boldsymbol{p}}^2} \sum_{\omega,k} g^2 v_T^2 \left| \delta B_{\omega,k}^a \right|^2$$

$$\simeq \left[\frac{\frac{m_D^2}{4} \left[-\frac{3(\xi+1)}{\xi^2} + \frac{(\xi+1)(\xi+3)}{\xi^2} \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}} \right]}{k^2 + \frac{m_D^2}{4} \left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi} \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}} \right]} \right]^2 k^2 \tag{3.25}$$

for strong turbulence. Thus, the scale of the saturated color-magnetic field is[10]

$$\delta B_{\omega,k} \sim \frac{k E_{\boldsymbol{p}}}{g}. \tag{3.26}$$

D. Anomalous viscosity

In this section, we follow the strategy in Ref. [25] to calculate the anomalous viscosity. This is a somewhat artificial way to obtain viscosity, but it allows us to estimate its basic scalings. We assume $\langle f \rangle$ is spatially inhomogeneous. For simplicity, we make v_x depend on x by the replacement

$$v_x \rightarrow v_x - \frac{\tilde{p}^2}{p^2} u(x), \tag{3.27}$$

where $\tilde{p}^2 = p^2(1 + \xi v_z^2)$ and $u(x)$ is the mean flow.[11] Then we take a second moment $(2p_x^2 - p_y^2 - p_z^2)$ of the diffusion equation Eq. (3.4). The corresponding energy-momentum tensor is

$$2T^{xx} - T^{yy} - T^{zz}$$

$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{2p_x^2 - p_y^2 - p_z^2}{E_{\boldsymbol{p}}} \langle f \rangle,$$

$$= \frac{1}{(2\pi)^3} \int d\Omega \frac{2v_x^2 - v_y^2 - v_z^2}{\left(1 + \xi v_z^2\right)^2} \int_0^\infty dp\, p^3 \langle f \rangle_{\text{iso}}. \tag{3.28}$$

From the coefficient of the velocity gradient in the corresponding tensor, we determine the viscosity

$$\eta_A = \frac{2T^{xx} - T^{yy} - T^{zz}}{-4 \frac{\partial u}{\partial x}}. \tag{3.29}$$

In the case of Eq. (3.11), the diffusion equation is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}} \right) \langle f \rangle \simeq \sum_{\omega,k} g^2 v_T^2 \left| \delta B_{\omega,k}^a \right|^2 \frac{1}{1/t_c} \frac{\partial^2 \langle f \rangle}{\partial p_z^2}, \tag{3.30}$$

where we used the diffusion coefficient Eq. (3.14) for strong turbulence. For inhomogeneous $\langle f \rangle$, we revived the $\boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}}$ term

$$\boldsymbol{v} \cdot \frac{\partial \langle f \rangle}{\partial \boldsymbol{x}} \simeq -v_T^2 \tilde{p} \frac{d\langle f \rangle}{d\tilde{p}} \frac{\partial u}{\partial x}, \tag{3.31}$$

where we replaced v_x^2 by v_T^2. Taking a moment $(2p_x^2 - p_y^2 - p_z^2)$, the left-hand side (LHS) in Eq. (3.30) becomes

$$(\text{LHS}) = \frac{\partial}{\partial t}(2T^{xx} - T^{yy} - T^{zz})$$

$$+ \frac{v_T^2}{(2\pi)^3} \int d\Omega \frac{2v_x^2 - v_y^2 - v_z^2}{\left(1 + \xi v_z^2\right)^2}$$

$$\times \left[-\int_0^\infty dp\, p^4 \frac{d\langle f \rangle_{\text{iso}}}{dp} \right] \frac{\partial u}{\partial x}. \tag{3.32}$$

The right-hand side (RHS) is approximated as follows: First, we take a derivative

$$\frac{\partial^2 \langle f \rangle}{\partial p_z^2} = \frac{(\xi + 1)^2 p_z^2}{\tilde{p}^2} \frac{d^2 \langle f \rangle}{d\tilde{p}^2} + \frac{(\xi + 1)}{\tilde{p}} \frac{d\langle f \rangle}{d\tilde{p}}$$

$$- \frac{(\xi + 1)^2 p_z^2}{\tilde{p}^3} \frac{d\langle f \rangle}{d\tilde{p}}, \tag{3.33}$$

where only the second term contributes to the viscosity. Second, we take a moment $(2p_x^2 - p_y^2 - p_z^2)$:

$$(\text{RHS}) = (\xi + 1) \sum_{\omega,k} g^2 v_T^2 \left| \delta B_{\omega,k}^a \right|^2 \frac{1}{1/t_c}$$

$$\times \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{2p_x^2 - p_y^2 - p_z^2}{E_{\boldsymbol{p}}} \frac{1}{\tilde{p}} \frac{d\langle f \rangle}{d\tilde{p}}. \tag{3.34}$$

---

[10]This scale corresponds to when the covariant derivative $[D = \partial - igA \sim i(p - gA)]$ cannot be treated perturbatively, $A \sim \frac{E_{\boldsymbol{p}}}{g}$ [14,18].

[11]We work in the local rest frame, $u(x) = 0$.

Finally, we compare with Eq. (3.28) for $(2T^{xx} - T^{yy} - T^{zz})$ to write

$$\text{(RHS)} = -(\xi + 1) \sum_{\omega,k} g^2 v_T^2 |\delta B_{\omega,k}^a|^2 \frac{1}{1/t_c} \frac{\left[-\int_0^\infty dp \, p^2 \frac{d\langle f \rangle_{\text{iso}}}{dp}\right]}{\left[\int_0^\infty dp \, p^3 \langle f \rangle_{\text{iso}}\right]} (2T^{xx} - T^{yy} - T^{zz}). \quad (3.35)$$

For static state where $\frac{\partial}{\partial t}(2T^{xx} - T^{yy} - T^{zz}) = 0$, we equate Eq. (3.32) to Eq. (3.35) to determine the anomalous viscosity

$$\eta_A \simeq \frac{\frac{1}{2(4\pi)^2}\left[\frac{\xi+3}{\xi(\xi+1)^2} + \frac{(\xi-3)}{\xi(\xi+1)}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]\left[-\int_0^\infty dp \, p^4 \frac{d\langle f \rangle_{\text{iso}}}{dp}\right]\left[\int_0^\infty dp \, p^3 \langle f \rangle_{\text{iso}}\right]}{\sum_{\omega,k} g^2 |\delta B_{\omega,k}^a|^2 \frac{1}{1/t_c}\left[-\int_0^\infty dp \, p^2 \frac{d\langle f \rangle_{\text{iso}}}{dp}\right]}, \quad (3.36)$$

where the function of ξ in the square brackets is positive for $\xi \approx \xi_k^*$ in Fig. 2. As excited color-magnetic-field intensity increases due to instability growth, the anomalous viscosity decreases because the effective collision frequency increases. Since we determined the saturation level of color-magnetic fields Eq. (3.15) and the decorrelation time Eq. (3.22), the anomalous viscosity is given by

$$\eta_A \simeq \frac{v_T^2}{2(8\pi)^2} \left[\frac{\xi+3}{\xi(\xi+1)^2} + \frac{(\xi-3)}{\xi(\xi+1)}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right] \left[\frac{k^2 + \frac{m_D^2}{4}\left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]}{\frac{m_D^2}{4}\left[-\frac{3(\xi+1)}{\xi^2} + \frac{(\xi+1)(\xi+3)}{\xi^2}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]}\right]^{3/2}$$

$$\times \frac{\left[-\int_0^\infty dp \, p^4 \frac{d\langle f \rangle_{\text{iso}}}{dp}\right]\left[\int_0^\infty dp \, p^3 \langle f \rangle_{\text{iso}}\right]}{\left[-\int_0^\infty dp \, p^2 \frac{d\langle f \rangle_{\text{iso}}}{dp}\right]\bar{E}_p^2 |k|} \quad (3.37)$$

for strong turbulence. We expect that this gives the lower bound of the anomalous viscosity in presence of the maximum intensity of the Weibel instability.

Similar to the thermal velocity in nonrelativistic electromagnetic plasmas, we define the "thermal momentum" as

$$p_T^2 \equiv \int \frac{d^3 p}{(2\pi)^3} \frac{p_x^2}{E_p} \langle f \rangle$$

$$= \frac{1}{8\pi^2}\left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]\int_0^\infty dp \, p^3 \langle f \rangle_{\text{iso}}, \quad (3.38)$$

where the function of ξ in the square brackets is positive. Then, by using Eq. (3.24) for soft momentum $k \sim gT$, the scaling trend of the anomalous viscosity is[12]

$$\eta_A \sim \frac{p_T^2}{1/t_c}, \quad (3.39)$$

where t_c at saturation is given by Eq. (3.22). This corresponds to Eq. (A18), in that viscosity is roughly the ratio of the thermal velocity squared to the decorrelation frequency in electromagnetic plasmas. We note that $1/t_c$ sets the effective collision frequency.

IV. SUMMARY AND DISCUSSIONS

In this work, we used resonance-broadening theory for strong turbulence, $(1/t_c)^2 \gg (\boldsymbol{v} \cdot \boldsymbol{k})^2$, to analyze nonlinear particle-wave interaction in the quark-gluon plasma. To determine the maximum intensity of plasma instabilities and transport, we assumed an Abelian regime in $1 + 1$ dimensions.

With the wave vector along the anisotropy axis, the saturation level of color-magnetic excitations is

$$k^2 \sum_{\omega',k'} |\delta B_{\omega',k'}^a|^2 \simeq \frac{4\bar{E}_p^2}{g^2 v_T^2}\left(\frac{1}{t_c}\right)^4, \quad (4.1)$$

where ω' and k' in the summation satisfy the linear dispersion relation, and t_c gives the time scale for scattering of a particle. For saturated Weibel instability, we calculated the particle-wave decorrelation time and the anomalous viscosity

$$t_c^2 \simeq \frac{k^2 + \frac{m_D^2}{4}\left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]}{\frac{m_D^2}{4}\left[-\frac{3(\xi+1)}{\xi^2} + \frac{(\xi+1)(\xi+3)}{\xi^2}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]k^2}, \quad (4.2)$$

$$\eta_A \simeq \frac{v_T^2}{2(8\pi)^2}\left[\frac{\xi+3}{\xi(\xi+1)^2} + \frac{(\xi-3)}{\xi(\xi+1)}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]$$

$$\times \left[\frac{k^2 + \frac{m_D^2}{4}\left[\frac{1}{\xi} + \frac{(\xi-1)}{\xi}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]}{\frac{m_D^2}{4}\left[-\frac{3(\xi+1)}{\xi^2} + \frac{(\xi+1)(\xi+3)}{\xi^2}\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]}\right]^{3/2}$$

$$\times \frac{\left[-\int_0^\infty dp \, p^4 \frac{d\langle f \rangle_{\text{iso}}}{dp}\right]\left[\int_0^\infty dp \, p^3 \langle f \rangle_{\text{iso}}\right]}{\left[-\int_0^\infty dp \, p^2 \frac{d\langle f \rangle_{\text{iso}}}{dp}\right]\bar{E}_p^2 |k|}. \quad (4.3)$$

Here, the anisotropy parameter is $\xi \approx \xi_k^*$ in Fig. 2, which corresponds to an extreme squeeze in the momentum $\hat{z}$ direction of an initially isotropic distribution. We expect that Eq. (4.3) gives the lower bound of the anomalous viscosity in presence of the maximum intensity of the Weibel instability. At soft momentum $k \sim gT$, the typical scales of the color-magnetic fields, the decorrelation time, and the anomalous viscosity are, respectively,

$$\delta B \sim T^2, \quad t_c \sim \frac{1}{gT}, \quad \text{and} \quad \eta_A \sim \frac{T^3}{g}. \quad (4.4)$$

---

[12]The anisotropy parameter ξ is a constant in this work.

We note that the scale of the anomalous viscosity at $k \sim gT$ is much lower than the leading-order collisional viscosity $\eta_C \sim T^3/g^4$.

As discussed in the introduction and Eq. (3.39), the effective viscosity is given by stress per effective collision frequency, so

$$\eta \sim \frac{p_T^2}{1/t_{\text{coll}} + 1/t_c}, \tag{4.5}$$

where $1/t_{\text{coll}}$ is the collision frequency and $1/t_c$ is the decorrelation frequency. Although it depends on the relevant kinetic regime, the scale of the decorrelation frequency $1/t_c$ at $k \sim gT$ is much higher than the collision frequency $1/t_{\text{coll}} \sim g^4 T$. As compared to the collisional viscosity $\eta_C \sim T^3/g^4$, the effective viscosity thus can be lowered to $\eta \sim T^3/g$ due to enhanced collisionality by nonlinear particle-wave interaction. This indicates that instability effects can be dominant in certain stages of quark-gluon plasma transport.

We focused on strong turbulence to consider nonlinear and stochastic particle-wave interaction due to resonance broadening. In addition to particle-wave interactions, there are other nonlinear effects (including wave-wave interactions) which might be important in non-Abelian plasmas. Numerical simulations indicate that gluon self-interactions might control the saturation of the Weibel instability in $3 + 1$ dimensions [16,18]. However, there are limitations to simulations and their interpretation. Therefore, analytic study of nonlinear theory is essential to extract information from the simulations and to understand thermalization of the quark-gluon plasma. We hope to discuss a systematic nonlinear analysis on the quark-gluon plasma instabilities in future presentations.

ACKNOWLEDGMENTS

APPENDIX: ELECTROMAGNETIC PLASMAS

In this appendix, we discuss the Weibel instability in traditional electromagnetic plasmas by using resonance-broadening theory [24,25]. We consider plasmas consisting of electrons, ignoring motions of heavier ions. The analysis parallels that of Sec. III except

(i) The coupling constant g (or plasmon mass $m_D/\sqrt{3}$) is replaced by the electric charge e (or plasma frequency $\omega_p = \sqrt{4\pi ne^2/m}$, where n is the number density and m is the mass of electrons).

(ii) The phase space is velocity $\boldsymbol{v}$ space instead of momentum $\boldsymbol{p}$ space.

Transport can be described by a diffusion equation [22,23]

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{D}(\boldsymbol{v}) \cdot \frac{\partial}{\partial \boldsymbol{v}}\right)\langle f \rangle = 0, \tag{A1}$$

where the diffusion tensor is given by the Lorentz force-force correlator with $\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})/m$. Assuming δv has a Gaussian probability density function

$$\text{pdf}[\delta v] = \frac{1}{\sqrt{\pi D t}}e^{-\frac{(\delta v)^2}{Dt}}, \tag{A2}$$

the pdf average with change of a trajectory is

$$\langle e^{i(\omega t - kx) + ik\delta x}\rangle_{\text{pdf}} = \int \frac{d(\delta v)}{\sqrt{\pi D t}}e^{-\frac{(\delta v)^2}{Dt}}e^{i(\omega t - kx) + ik\int dt(\delta v)}$$

$$\simeq e^{i(\omega - vk)t - \frac{k^2 Dt^3}{4}}. \tag{A3}$$

The particle-wave decorrelation time is defined as

$$\left(\frac{1}{t_c}\right)^3 \equiv \frac{k^2 D}{4}. \tag{A4}$$

By the Lorentzian approximation Eq. (3.9) for strong turbulence, the diffusion coefficient due to magnetic excitations is

$$D = \sum_{\omega, \boldsymbol{k}}\left(\frac{e}{m}\boldsymbol{v} \times \delta\boldsymbol{B}_{\omega, \boldsymbol{k}}\right)\frac{i}{\omega - \boldsymbol{v} \cdot \boldsymbol{k} + i/t_c}\left(\frac{e}{m}\boldsymbol{v} \times \delta\boldsymbol{B}_{\omega, \boldsymbol{k}}\right). \tag{A5}$$

For simplicity, we consider one-dimensional propagation of plasmas; Eq. (3.11). When the Weibel instability saturates, the diffusion coefficient is

$$D \simeq \sum_{\omega, k}\frac{\omega_p^2}{4\pi nm}v_T^2|\delta B_{\omega, k}|^2\frac{1}{1/t_c}, \tag{A6}$$

where we replaced v_x^2 by v_T^2. Using Eq. (A4), the saturation level of magnetic excitations is

$$\left(\frac{1}{t_c}\right)^4 \simeq \frac{k^2}{4}\sum_{\omega', k'}\frac{\omega_p^2}{4\pi nm}v_T^2|\delta B_{\omega', k'}|^2. \tag{A7}$$

In resonance-broadening theory, the nonlinear dispersion relation is given by

$$\omega^2 + \omega_p^2\int d^3\boldsymbol{v}\left[v_x\frac{\partial\langle f\rangle}{\partial v_x} + \frac{kv_x^2}{\omega - v_z k + i/t_c}\frac{\partial\langle f\rangle}{\partial v_z}\right] = k^2. \tag{A8}$$

With an anisotropy parameter $\xi > -1$, electrons obey the Maxwellian distribution

$$\langle f \rangle = \frac{\sqrt{\xi + 1}}{(\sqrt{2\pi}v_T)^3}e^{-\frac{v^2 + \xi v_z^2}{2v_T^2}}, \tag{A9}$$

where v_T is the thermal velocity.[13] At saturation, the particle-wave decorrelation time is

$$t_c^2 \simeq \frac{1}{v_T^2}\left(\frac{1}{k^2} + \frac{1}{\omega_p^2}\right) \tag{A10}$$

for strong turbulence. Since we used the strong-turbulence approximation, $(1/t_c)^2 \gg (v_z k)^2$, it must satisfy

$$\frac{1}{k^2/\omega_p^2 + 1} \gg \frac{v_z^2}{v_T^2}. \tag{A11}$$

---

[13]We normalized the distribution, $\int d^3\boldsymbol{v}\langle f\rangle = 1$. The thermal velocity is the averaged velocity, $\int d^3\boldsymbol{v}\,v_x^2\langle f\rangle = v_T^2$.

Noting $v_z^2 \ll v_T^2$, this condition is valid for $k^2 \lesssim \omega_p^2$. Using Eq. (A7), the saturation level of magnetic excitations is

$$\frac{1}{16\pi nm} \sum_{\omega,k} |\delta B_{\omega,k}|^2 \simeq \frac{\omega_p^2 v_T^2 k^2}{\left(k^2 + \omega_p^2\right)^2} \qquad (A12)$$

for strong turbulence.

To calculate the anomalous viscosity, we assume that the Maxwellian distribution depends on space by the replacement $v_x \to v_x - u(x)$. The diffusion equation is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\langle f \rangle \simeq \sum_{\omega,k} \frac{\omega_p^2}{4\pi nm} v_T^2 |\delta B_{\omega,k}|^2 \frac{1}{1/t_c} \frac{\partial^2 \langle f \rangle}{\partial v_z^2},$$

$$(A13)$$

where we revived the $\boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}}$ term for inhomogeneous $\langle f \rangle$. Taking a second moment $(2v_x^2 - v_y^2 - v_z^2)$ on both sides, we obtain

$$\frac{\partial}{\partial t}\left(2T_{EM}^{xx} - T_{EM}^{yy} - T_{EM}^{zz}\right) + \frac{nmv_T^2\xi}{(\xi+1)}\frac{\partial u}{\partial x}$$

$$\simeq -(\xi+1)\sum_{\omega,k} \frac{\omega_p^2}{4\pi nm}|\delta B_{\omega,k}|^2$$

$$\times \frac{1}{1/t_c}\left(2T_{EM}^{xx} - T_{EM}^{yy} - T_{EM}^{zz}\right), \qquad (A14)$$

where the corresponding stress tensor is

$$2T_{EM}^{xx} - T_{EM}^{yy} - T_{EM}^{zz} = nm \int d^3\boldsymbol{v}\left(2v_x^2 - v_y^2 - v_z^2\right)\langle f \rangle.$$

$$(A15)$$

For static state where $\frac{\partial}{\partial t}(2T_{EM}^{xx} - T_{EM}^{yy} - T_{EM}^{zz}) = 0$, the anomalous viscosity is determined as

$$\eta_A \simeq \frac{\frac{nmv_T^2\xi}{4(\xi+1)^2}}{\sum_{\omega,k} \frac{\omega_p^2}{4\pi nm}|\delta B_{\omega,k}|^2 \frac{1}{1/t_c}}. \qquad (A16)$$

As magnetic field intensity increases, the anomalous viscosity decreases. Since the saturation level of magnetic fields and the decorrelation time are determined by Eqs. (A7) and (A10), we have

$$\eta_A \simeq \frac{nmv_T\xi k^2}{16(\xi+1)^2}\left(\frac{1}{k^2} + \frac{1}{\omega_p^2}\right)^{3/2} \qquad (A17)$$

for strong turbulence. For $k^2 \lesssim \omega_p^2$, the scaling trend of the anomalous viscosity is given by

$$\eta_A \sim \frac{v_T^2}{1/t_c}, \qquad (A18)$$

where we used $v_T k \sim 1/t_c$.

---

[1] P. B. Arnold, Int. J. Mod. Phys. E **16**, 2555 (2007).

[2] P. B. Arnold, G. D. Moore, and L. G. Yaffe, J. High Energy Phys. 11 (2000) 001.

[3] P. B. Arnold, G. D. Moore, and L. G. Yaffe, J. High Energy Phys. 05 (2003) 051.

[4] D. A. Teaney, arXiv:0905.2433.

[5] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).

[6] P. K. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).

[7] M. Asakawa, S. A. Bass, and B. Muller, Phys. Rev. Lett. **96**, 252301 (2006).

[8] M. Asakawa, S. A. Bass, and B. Muller, Prog. Theor. Phys. **116**, 725 (2006).

[9] E. S. Weibel, Phys. Rev. Lett. **2**, 83 (1959).

[10] S. Mrowczynski, Phys. Rev. C **49**, 2191 (1994).

[11] S. Mrowczynski and M. H. Thoma, Phys. Rev. D **62**, 036011 (2000).

[12] J. Randrup and S. Mrowczynski, Phys. Rev. C **68**, 034909 (2003).

[13] P. B. Arnold, J. Lenaghan, and G. D. Moore, J. High Energy Phys. 08 (2003) 002.

[14] P. B. Arnold and J. Lenaghan, Phys. Rev. D **70**, 114007 (2004).

[15] P. B. Arnold, J. Lenaghan, G. D. Moore, and L. G. Yaffe, Phys. Rev. Lett. **94**, 072302 (2005).

[16] A. Rebhan, P. Romatschke, and M. Strickland, J. High Energy Phys. 09 (2005) 041.

[17] A. Rebhan, P. Romatschke, and M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005).

[18] P. B. Arnold, G. D. Moore, and L. G. Yaffe, Phys. Rev. D **72**, 054003 (2005).

[19] P. B. Arnold and G. D. Moore, Phys. Rev. D **73**, 025006 (2006).

[20] P. B. Arnold and G. D. Moore, Phys. Rev. D **73**, 025013 (2006).

[21] A. Dumitru and Y. Nara, Phys. Lett. B **621**, 89 (2005).

[22] T. H. Dupree, Phys. Fluids **9**, 1773 (1966).

[23] T. H. Dupree, Phys. Fluids **11**, 2680 (1968).

[24] T. Abe and K. Niu, J. Phys. Soc. Jpn. **49**, 717 (1980).

[25] T. Abe and K. Niu, J. Phys. Soc. Jpn. **49**, 725 (1980).

[26] D. R. Nicholson, *Introduction to Plasma Theory* (John Wiley and Sons, New York, 1983).

[27] P. H. Diamond, S. I. Itoh, and K. Itoh, *Modern Plasma Physics Vol. 1, Physical Kinetics of Turbulent Plasmas* (Cambridge University Press, Cambridge, 2009).

[28] T. H. Dupree and D. J. Tetreault, Phys. Fluids **21**, 425 (1978).

[29] P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003).