Model-independent aspects of the reaction $\bar{K} + N \to K + \Xi$

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Various model-independent aspects of the $\bar{K} N \to K \Xi$ reaction are investigated, starting from the determination of the most general structure of the reaction amplitude for Ξ baryons with $J^P = \frac{1}{2}^{\pm}$ and $\frac{3}{2}^{\pm}$ and the observables that allow a complete determination of these amplitudes. Polarization observables are constructed in terms of spin-density matrix elements. Reflection symmetry about the reaction plane is exploited, in particular, to determine the parity of the produced Ξ in a model-independent way. In addition, extending the work of Biagi *et al.* [Z. Phys. C **34**, 175 (1987)], a way of determining simultaneously the spin and parity of the ground state of Ξ baryons as well as those of the excited Ξ states is presented.

I. INTRODUCTION

Multistrangeness baryons have played an important role in the development of our understanding of strong interactions. For example, the prediction and discovery of the Ω baryon, with strangeness $S = -3$, have been a spectacular confirmation of how well the SU(3) flavor symmetry works in strong interactions. Nevertheless, more than a half-century later, our knowledge of multistrangeness baryons is still very limited as can be seen from the fact that the spin of $\Omega^-(1672)$ was only recently confirmed [1,2].

In this work, we concentrate on Ξ baryons with strangeness $S = -2$. So far, studies of Ξ physics have been very scarce. The present situation can be summarized as follows. (i) The SU(3) flavor symmetry allows as many Ξ states as there are N^* and Δ^* resonances combined ($\sim$44). However, until now, only 11 Ξ resonances have been discovered [2]. (ii) Being $S = -2$ baryons, if there are no strange particles in the initial state, Ξs are produced only indirectly and have relatively low production rates. In fact, the yield is only of the order of nanobarns in the photoproduction reaction [3], whereas the yield is of the order of microbarns [4] in the hadronic, $\bar{K}$-induced reaction (where the Ξ is produced directly because of the presence of an $S = -1$ $\bar{K}$ meson in the initial state).

With the advent of new particle accelerators capable of reaching higher energies and advances in technologies and experimental techniques, we are now in a better position than ever to study multistrangeness baryons. Indeed, the CLAS Collaboration at the Thomas Jefferson National Accelerator Facility (JLab) plans to initiate a Ξ spectroscopy program using the upgraded 12-GeV machine. The Collaboration is also expected to measure exclusive Ω photoproduction for the

first time [5]. Some data for the Ξ ground state are already available [3], obtained from the 6-GeV machine. In addition, J-PARC proposes to study Ξ baryons via the $\bar{K} N \to K \Xi$ and $\pi N \to K K \Xi$ reactions as well as Ω production [6,7], and at the Facility for Antiproton and Ion Research (FAIR) of GSI, the reaction $\bar{p} p \to \bar{\Xi} \Xi$ will be studied [8]. For a more complete compilation of baryon spectra, Ξ baryons should be studied as an integral part of any baryon spectroscopy program.

Theoretical studies of Ξ baryons are hampered mainly by the scarcity of experimental data. The existing theoretical models cannot be well constrained, and as a consequence, there is strong model dependence in predictions of the Ξ spectrum. In particular, one of the current open issues in the Ξ spectrum concerns the low mass of $\Xi(1690)$ and $\Xi(1620)$, i.e., the nature of the third-lowest Ξ state [9]. Here, different approaches, such as the nonrelativistic and relativistic quark models [10–12], one-boson-exchange model [13], large-N_c model [14–18], QCD sum rules [19,20], and Skyrme model [9], yield contradictory predictions for the nature of these resonances. The planned new experimental studies as mentioned above are expected to play a key role in addressing such open problems. Quite recently, lattice QCD calculations of the baryon spectra, including those of Ξ and Ω baryons, have been reported [21,22].

To extract relevant information on Ξ resonances from the experimental data, a reliable reaction model is required. To date, for photoproduction reactions, there exists only the work of some of the present authors [23,24] analyzing the available CLAS data [3]. In $\bar{K}$-induced reactions, recent calculations are reported by Sharov *et al.* [25] and by Shyam *et al.* [26]. Thus, further theoretical studies on this subject are timely for suggesting directions to experimental studies by providing predictions on Ξ baryon production processes and tools to analyze the forthcoming data. One feature of these production processes is that the t-channel processes of intermediate meson production are suppressed since the exchanged intermediate meson should be exotic, having two units of strangeness. As a consequence, the production of a Ξ state is dominated

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by intermediate $S = -1$ hyperons. Therefore, by analyzing the production mechanisms of the Ξ, one also hopes to gain some insight into the spectrum and couplings of the $S = -1$ hyperons.

Our ongoing efforts to better understand the production process of Ξ baryons focus on two lines of inquiry. On the one hand, to build understanding of the dynamics of reactions involving strange particles, we are engaged in model-dependent analyses within an effective Lagrangian approach along the lines employed in the photoprocesses reported by some of us in Refs. [23] and [24]. Further results of such model-dependent analyses will be reported elsewhere. On the other hand, in the present work we report on the study of *model-independent* aspects of production processes of Ξ baryons, exploiting, in particular, some basic symmetries of the reactions in question to determine the spin and parity quantum numbers of the Ξ resonances.

The present paper is organized as follows. In Sec. II, following the method in Ref. [27], the most general structure of the reaction amplitude for $\bar{K}N \to K\Xi$ is derived for a Ξ of spin 1/2 or of spin 3/2, and a set of observables is identified that determines the reaction amplitude completely. Here, the reflection symmetry about the reaction plane is exploited to determine the parity of the Ξ resonances in a model-independent manner. Furthermore, the coefficients that multiply each spin structure in the reaction amplitude are expressed in terms of partial-wave matrix elements, which readily allows for a partial-wave analysis when sufficient data become available. Since the determination of the basic quantum numbers of the produced Ξ involve its spin observables, the spin-density-matrix (SDM) approach is used, in Sec. III, to discuss all relevant observables in terms of SDM elements. The SDM formalism is also very convenient when dealing with high-spin Ξ resonance productions. In fact, the SDM approach has often been used in the description of higher-spin particle production processes such as the (spin-1) vector meson productions [28,29]. Then, following Refs. [30] and [31], the SDM elements are extracted from the moments associated with the decay processes of the produced Ξ in conjunction with the self-analyzing properties of the ground state Ξ and Λ in weak decays. These moments, in turn, can be extracted from the measured angular distribution of the decay product. Section IV contains a summary, and some technical details of the derivations are given in Appendixes A–D.

II. STRUCTURE OF THE $\bar{K}N \to K\Xi$ REACTION AMPLITUDE

In this section, we derive the most general structure of the amplitude for the reaction of

$$\bar{K}(q) + N(p) \to K(q') + \Xi(p'), \qquad (1)$$

following the method used in Ref. [27]. In the present work, we consider the production of Ξ of spin 1/2 and 3/2 with both positive and negative parities. The method is quite general and, in principle, can be applied to extract the spin structure of any reaction amplitude. In the above equation, the arguments denote the four-momenta of the respective particles.

The reaction in Eq. (1) is described in its center-of-momentum (CM) frame, where $\boldsymbol{q} = -\boldsymbol{p}$ and $\boldsymbol{q}' = -\boldsymbol{p}'$. For further convenience, we define the three mutually orthogonal unit vectors $\hat{\boldsymbol{n}}_i (i = 1,2,3)$ in terms of the independent momenta available in the reaction; i.e.,

$$\hat{\boldsymbol{n}}_1 \equiv \frac{(\boldsymbol{p} \times \boldsymbol{p}') \times \boldsymbol{p}}{|(\boldsymbol{p} \times \boldsymbol{p}') \times \boldsymbol{p}|}, \qquad (2a)$$

$$\hat{\boldsymbol{n}}_2 \equiv \frac{\boldsymbol{p} \times \boldsymbol{p}'}{|\boldsymbol{p} \times \boldsymbol{p}'|}, \qquad (2b)$$

$$\hat{\boldsymbol{n}}_3 \equiv \frac{\boldsymbol{p}}{|\boldsymbol{p}|}, \qquad (2c)$$

where $\boldsymbol{p}$ and $\boldsymbol{p}'$ denote the three-momenta of the nucleon and Ξ, respectively. Note that $\boldsymbol{p}$ and $\boldsymbol{p}'$ define the reaction plane, such that $\hat{\boldsymbol{n}}_2$ is perpendicular to the reaction plane. The coordinate-system setup is shown in Fig. 1. Throughout this paper, the hat notation for vectors is used to indicate unit vectors; i.e., $\hat{\boldsymbol{a}} \equiv \boldsymbol{a}/|\boldsymbol{a}|$ for an arbitrary vector $\boldsymbol{a}$. The quantization axis is chosen to be along $\hat{\boldsymbol{n}}_3$. We also use the alternative Cartesian notation $i = x,y,z$ for the indices of the unit vectors $\hat{\boldsymbol{n}}_i$.

A. Production of Ξ with $J^P = \frac{1}{2}^{\pm}$

First, we consider spin-parity $J^P = \frac{1}{2}^{\pm}$ for the Ξ produced in reaction (1). Following the method in Ref. [27], the most general spin structure of the reaction amplitude, consistent with basic symmetries, is

$$\hat{M} = M_0' + M_2' \boldsymbol{\sigma} \cdot (\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') \quad \text{for } J^P = \frac{1}{2}^+, \qquad (3a)$$

$$\hat{M} = M_1' \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}' + M_3' \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}} \quad \text{for } J^P = \frac{1}{2}^-, \qquad (3b)$$

where $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ stands for the vector built up of the usual Pauli spin operators.[1] Equations (3a) and (3b) are direct consequences of the amplitude's reflection symmetry about the reaction plane [32,33], which is further exploited in our analysis presented below. Note that the coefficients M_1' and M_2' do not contain an S wave in the final state.

For further convenience, we rewrite Eq. (3) as

$$\hat{M} = M_0 + M_2 \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_2 \qquad \text{for } J^P = \frac{1}{2}^+, \qquad (4a)$$

$$\hat{M} = M_1 \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_1 + M_3 \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_3 \quad \text{for } J^P = \frac{1}{2}^-, \qquad (4b)$$

using $\hat{\boldsymbol{p}}' = \cos\theta \, \hat{\boldsymbol{n}}_3 + \sin\theta \, \hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_3 = \hat{\boldsymbol{p}}$. The respective coefficients in Eqs. (3) and (4) are related by

$$M_0' = M_0, \qquad M_2' = \frac{1}{\sin\theta} M_2, \qquad (5a)$$

$$M_1' = \frac{1}{\sin\theta} M_1, \quad M_3' = M_3 - \frac{\cos\theta}{\sin\theta} M_1. \qquad (5b)$$

Following Ref. [27], one may also express these coefficients in terms of partial-wave matrix elements. The corresponding

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[1]Note that the spin structure for the positive-parity Ξ in Eq. (3a) is identical to the familiar structure of the πN elastic scattering amplitude. However, obviously, the isospin structure is different.
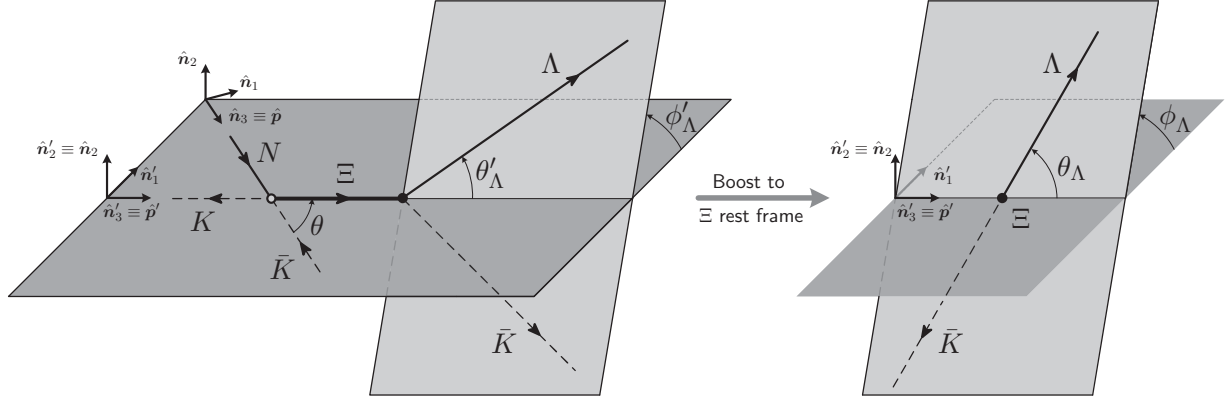
FIG. 1. Coordinate systems used in describing the Ξ production reaction and its subsequent decay process. Left: The production reaction $\bar{K}N \to K\Xi$ is shown in its center-of-momentum (CM) frame. The corresponding reaction plane (dark gray) contains the nucleon and Ξ momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$, respectively. The basis vectors $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ are defined in Eq. (2), with $\hat{\boldsymbol{n}}_3$ aligned with the nucleon momentum $\boldsymbol{p}$ and $\hat{\boldsymbol{n}}_2$ perpendicular to the reaction plane; θ indicates the Ξ emission angle. The (primed) frame $\{\hat{\boldsymbol{n}}_1', \hat{\boldsymbol{n}}_2', \hat{\boldsymbol{n}}_3'\}$ is obtained from $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ by rotating the latter about the $\hat{\boldsymbol{n}}_2$ axis by θ, which aligns $\hat{\boldsymbol{n}}_3'$ with $\boldsymbol{p}'$ and leaves $\hat{\boldsymbol{n}}_2' \equiv \hat{\boldsymbol{n}}_2$. The (light-gray) plane tilted by the angle ϕ_Λ' about the $\hat{\boldsymbol{n}}_3' \equiv \hat{\boldsymbol{p}}'$ axis is spanned by the momenta of the decay products Λ and $\bar{K}$. The polar and azimuthal angles of the decay product Λ in the rotated (primed) CM frame are indicated by θ_Λ' and ϕ_Λ', respectively. Right: In this boosted frame, the decay process of the produced Ξ at rest is described in the $\{\hat{\boldsymbol{n}}_1', \hat{\boldsymbol{n}}_2', \hat{\boldsymbol{n}}_3'\}$ coordinate system. The polar and azimuthal angles of the decay product Λ are indicated here by θ_Λ and ϕ_Λ, respectively. For the latter angle, one has $\phi_\Lambda \equiv \phi_\Lambda'$ since the boost happens along the corresponding tilt axis.

results are given in Appendix A, which show, in particular, that the coefficients M_1 and M_2 vanish identically for Ξ scattering angles $\theta = 0$ and π, as can be seen in Eq. (A3). The partial-wave expansions will become particularly relevant once sufficient experimental data become available to permit their full-fledged partial-wave analysis. The isospin structure of the amplitudes in Eq. (4) [or in Eq. (3)] is contained in the coefficients M_i as given explicitly by Eq. (A3) in Appendix A.

Once the spin structure of the reaction amplitude is determined, all the observables can be readily expressed in terms of the amplitudes M_i multiplying each spin structure. For the reaction under consideration, apart from the cross section $(d\sigma/d\Omega)$, a complete set of observables includes the target asymmetry (T), recoil Ξ polarization (P), and spin-transfer coefficient (K). For arbitrary spin orientations along directions $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$, their coordinate-independent expressions are

$$\frac{d\sigma}{d\Omega} \equiv \frac{1}{2}\,\mathrm{Tr}[\hat{M}\hat{M}^\dagger], \tag{6a}$$

$$\frac{d\sigma}{d\Omega}T_a \equiv \frac{1}{2}\,\mathrm{Tr}[\hat{M}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}}\,\hat{M}^\dagger], \tag{6b}$$

$$\frac{d\sigma}{d\Omega}P_a \equiv \frac{1}{2}\,\mathrm{Tr}[\hat{M}\hat{M}^\dagger\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}}], \tag{6c}$$

$$\frac{d\sigma}{d\Omega}K_{ba} \equiv \frac{1}{2}\,\mathrm{Tr}[\hat{M}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{b}}\,\hat{M}^\dagger\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}}]. \tag{6d}$$

For Cartesian directions $\hat{\boldsymbol{n}}_i$ enumerated by $i = 1,2,3 (=x,y,z)$, in particular, one obtains

$$\frac{d\sigma}{d\Omega}T_i = \frac{1}{2}\,\mathrm{Tr}[\hat{M}\sigma_i\hat{M}^\dagger], \tag{7a}$$

$$\frac{d\sigma}{d\Omega}P_i = \frac{1}{2}\,\mathrm{Tr}[\hat{M}\hat{M}^\dagger\sigma_i], \tag{7b}$$

$$\frac{d\sigma}{d\Omega}K_{ij} = \frac{1}{2}\,\mathrm{Tr}[\hat{M}\sigma_i\hat{M}^\dagger\sigma_j]. \tag{7c}$$

Of course, the T, P, and K observables for arbitrary directions in Eq. (6) can be expressed as linear combinations of the specific Cartesian expressions given in Eq. (7).

Due to symmetries of the reaction, eight observables vanish identically, i.e., $T_i = P_i = K_{iy} = K_{yi} = 0$ for $i = x,z$, and of the remaining eight, only four are independent for a given parity, and they completely determine the amplitudes M_i in Eq. (4). Indeed, for a positive-parity Ξ, we have

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}K_{yy} = |M_0|^2 + |M_2|^2, \tag{8a}$$

$$\frac{d\sigma}{d\Omega}K_{xx} = \frac{d\sigma}{d\Omega}K_{zz} = |M_0|^2 - |M_2|^2, \tag{8b}$$

$$\frac{d\sigma}{d\Omega}T_y = \frac{d\sigma}{d\Omega}P_y = 2\,\mathrm{Re}[M_2 M_0^*], \tag{8c}$$

$$\frac{d\sigma}{d\Omega}K_{xz} = -\frac{d\sigma}{d\Omega}K_{zx} = 2\,\mathrm{Im}[M_2 M_0^*], \tag{8d}$$

and for a negative-parity Ξ, we obtain

$$\frac{d\sigma}{d\Omega} = -\frac{d\sigma}{d\Omega}K_{yy} = |M_1|^2 + |M_3|^2, \tag{9a}$$

$$\frac{d\sigma}{d\Omega}K_{xx} = -\frac{d\sigma}{d\Omega}K_{zz} = |M_1|^2 - |M_3|^2, \tag{9b}$$

$$\frac{d\sigma}{d\Omega}T_y = -\frac{d\sigma}{d\Omega}P_y = 2\,\mathrm{Im}[M_3 M_1^*], \tag{9c}$$

$$\frac{d\sigma}{d\Omega}K_{xz} = \frac{d\sigma}{d\Omega}K_{zx} = 2\,\mathrm{Re}[M_3 M_1^*]. \tag{9d}$$

The respective first two relations in the two equation sets determine the magnitudes of the amplitudes M_0, M_2 and M_1, M_3, respectively, whereas the respective last two relations determine their phase differences. Therefore, apart from an irrelevant overall phase, the observables in Eqs. (8) and (9) determine the amplitudes M_i, $i = 0,\ldots,3$, unambiguously.

These results reveal that it is experimentally demanding to determine the reaction amplitude completely, for it requires measuring both the single- and the double-polarization observables.

Comparing Eqs. (8a) and (9a), one obtains

$$K_{yy} = \pi_\Xi, \tag{10}$$

where π_Ξ stands for the parity of the produced Ξ. This result is actually a direct consequence of the reflection symmetry, as exploited in Bohr's theorem [33,34] and applied in Ref. [35]. It, therefore, provides a model-independent way of determining the parity of the Ξ resonance. Alternative expressions extracted from Eqs. (8) and (9) are [35]

$$T_y = \pi_\Xi P_y, \tag{11}$$

which involves only single polarization observables, and

$$K_{xx} = \pi_\Xi K_{zz} \quad \text{and} \quad K_{xz} = -\pi_\Xi K_{zx}. \tag{12}$$

These results are all consequences of the reflection symmetry about the reaction plane.

In Sec. III, we perform the analysis in terms of the SDM elements, which are equivalent to the observables discussed here. The SDM elements are convenient quantities when dealing with spin observables, especially when higher-spin particles are produced in the reaction. They can be extracted from the information on the subsequent decay processes of the produced particles, in conjunction with the self-analyzing property of the decaying particles via a weak decay, without the explicit measurement of the spin polarizations of these produced particles.

B. Production of Ξ with $J^P = \frac{3}{2}^{\pm}$

We now turn to the spin-parity $J^P = \frac{3}{2}^{\pm}$. Again, following Ref. [27], the most general spin structure of the reaction amplitude is given by

$$
\begin{aligned}
\hat{M} = {}& F_1' \, \boldsymbol{T}^\dagger \cdot (\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') + F_2' \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}}' \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}' \\
& + F_3' \, [\boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}} \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}' + \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}}' \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}] \\
& + F_4' \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}} \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}
\end{aligned} \tag{13a}
$$

for $J^P = \frac{3}{2}^{+}$ and by

$$
\begin{aligned}
\hat{M} = {}& G_1' \, [\boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}} \, \boldsymbol{\sigma} \cdot (\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') + \boldsymbol{T}^\dagger \cdot (\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}] \\
& + G_2' \, [\boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}}' \, \boldsymbol{\sigma} \cdot (\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') + \boldsymbol{T}^\dagger \cdot (\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}') \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}'] \\
& + G_3' \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}}' + G_4' \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{p}}
\end{aligned} \tag{13b}
$$

for $J^P = \frac{3}{2}^{-}$. Here, $\boldsymbol{T}^\dagger$ stands for the (spin-1/2 $\to$ spin 3/2) transition operator. Its explicit representation may be found elsewhere [36]. In contrast to the spin-1/2 case, each parity of the spin-3/2 case has four independent amplitudes, F_i' and G_i' ($i = 1, \ldots, 4$), respectively, and one needs at least eight independent observables to determine them completely apart from an irrelevant overall phase. From the above equations, it is obvious that only F_4' and G_4' contain an S wave in the final

state. Also, F_2' and G_2' contain only D and higher waves in the final state.

The amplitudes in Eq. (13) can be also rewritten as

$$
\begin{aligned}
\hat{M} = {}& F_1 \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_2 + F_2 \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_1 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_1 \\
& + F_3 \, [\boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_3 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_1 + \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_1 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_3] \\
& + F_4 \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_3 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_3
\end{aligned} \tag{14a}
$$

for $J^P = \frac{3}{2}^{+}$ and

$$
\begin{aligned}
\hat{M} = {}& G_1 \, [\boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_3 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_2 + \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_2 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_3] \\
& + G_2 \, [\boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_1 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_2 + \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_2 \, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_1] \\
& + G_3 \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_1 + G_4 \, \boldsymbol{T}^\dagger \cdot \hat{\boldsymbol{n}}_3
\end{aligned} \tag{14b}
$$

for $J^P = \frac{3}{2}^{-}$. The coefficients F_i and G_i are expressed in terms of the partial-wave matrix elements as given in Appendix A. They are also related to the corresponding coefficients F_i' and G_i' in Eq. (13) by

$$F_1' = \frac{1}{\sin\theta} F_1, \tag{15a}$$

$$F_2' = \frac{1}{\sin^2\theta} F_2, \tag{15b}$$

$$F_3' = \frac{1}{\sin\theta} F_3 - \frac{\cos\theta}{\sin^2\theta} F_2, \tag{15c}$$

$$F_4' = F_4 + \frac{\cos^2\theta}{\sin^2\theta} F_2 - 2\frac{\cos\theta}{\sin\theta} F_3 \tag{15d}$$

and

$$G_1' = \frac{1}{\sin\theta} G_1 - \frac{\cos\theta}{\sin^2\theta} G_2, \tag{15e}$$

$$G_2' = \frac{1}{\sin^2\theta} G_2, \tag{15f}$$

$$G_3' = \frac{1}{\sin\theta} G_3, \tag{15g}$$

$$G_4' = G_4 - \frac{\cos\theta}{\sin\theta} G_3. \tag{15h}$$

The polarization observables for this case are discussed in the next section in terms of the SDM elements.

III. SPIN-DENSITY-MATRIX APPROACH

As mentioned before, when dealing with higher-spin Ξ (i.e., spins higher than 1/2), in particular, it is more convenient to continue the analysis of the $\bar{K}N \to K\Xi$ reaction in terms of SDM elements. An analysis similar (but not identical) to the present one based on the SDM formalism was performed in Ref. [37] for a general two-body reaction with an unpolarized initial state. Also, the reaction $\bar{K}N \to \omega\Lambda$ was analyzed within the SDM approach in Ref. [38].

In Sec. II, we have exploited the mirror (or reflection) symmetry about the reaction plane in our analysis, in particular, for the parity determination of the Ξ resonances. In fact, as long as the production process conserves the total parity, the reaction amplitude should have this symmetry [32,33]. This mirror operation is equivalent to doing a parity transformation

followed by a subsequent rotation by $180°$ about the $\hat{n}_2$ axis: $\hat{\mathcal{P}}_y = \hat{R}_y(180°)\hat{\mathcal{P}}$. The resulting symmetry, in terms of the spin matrix element, is

$$\langle S_f\, m_f|\hat{M}|S_i\, m_i\rangle$$
$$= \langle S_f\, m_f|\hat{\mathcal{P}}_y^\dagger \mathcal{P}_y \hat{M} \mathcal{P}_y^\dagger \mathcal{P}_y|S_i\, m_i\rangle$$
$$= \pi_f \pi_i\, (-1)^{(S_f - m_f) - (S_i - m_i)}\, \langle S_f - m_f|\hat{M}|S_i - m_i\rangle \quad (16)$$

and holds as long as the quantization axis is in the production plane. Here, $\pi_{i(f)}$ is the intrinsic parity of the initial (final) state.

Based on this symmetry, the $J^P = \frac{1}{2}^{\pm}$ Ξ production amplitude, $\hat{M}$ given by Eq. (4), is completely described by two complex helicity amplitudes, $\mathcal{H}_1$ and $\mathcal{H}_2$, given by the spin matrix elements,

$$\mathcal{H}_1 \equiv \langle \lambda_\Xi = \tfrac{1}{2}|\, \hat{M}\, |\lambda_N = \tfrac{1}{2}\rangle$$
$$= \pi_\Xi \langle \lambda_\Xi = -\tfrac{1}{2}|\, \hat{M}\, |\lambda_N = -\tfrac{1}{2}\rangle, \quad (17a)$$

$$\mathcal{H}_2 \equiv \langle \lambda_\Xi = \tfrac{1}{2}|\, \hat{M}\, |\lambda_N = -\tfrac{1}{2}\rangle$$
$$= -\pi_\Xi \langle \lambda_\Xi = -\tfrac{1}{2}|\, \hat{M}\, |\lambda_N = \tfrac{1}{2}\rangle, \quad (17b)$$

where λ_N and λ_Ξ denote the helicity of the initial nucleon and final Ξ, respectively. Here, reference to the spin quantum numbers $S_\Xi = S_N = 1/2$ has been suppressed. The helicity amplitudes are related to the coefficient amplitudes in Eq. (4) by

$$\mathcal{H}_1 = M_0 \cos\frac{\theta}{2} + i M_2 \sin\frac{\theta}{2}, \quad (18a)$$

$$\mathcal{H}_2 = -i M_2 \cos\frac{\theta}{2} + M_0 \sin\frac{\theta}{2} \quad (18b)$$

for a positive parity Ξ and by

$$\mathcal{H}_1 = M_3 \cos\frac{\theta}{2} + M_1 \sin\frac{\theta}{2}, \quad (19a)$$

$$\mathcal{H}_2 = M_1 \cos\frac{\theta}{2} - M_3 \sin\frac{\theta}{2} \quad (19b)$$

for a negative parity Ξ. Here, θ is the scattering angle; i.e., $\cos\theta \equiv \hat{p} \cdot \hat{p}'$.

Likewise, the production amplitude $\hat{M}$ for a Ξ with $J^P = \frac{3}{2}^{\pm}$ determined by Eq. (14) is completely described by four complex amplitudes given as

$$\mathcal{H}_1 \equiv \langle \lambda_\Xi = \tfrac{3}{2}|\hat{M}|\lambda_N = \tfrac{1}{2}\rangle$$
$$= \pi_\Xi \langle \lambda_\Xi = -\tfrac{3}{2}|\hat{M}|\lambda_N = -\tfrac{1}{2}\rangle, \quad (20a)$$

$$\mathcal{H}_2 \equiv \langle \lambda_\Xi = \tfrac{3}{2}|\hat{M}|\lambda_N = -\tfrac{1}{2}\rangle$$
$$= -\pi_\Xi \langle \lambda_\Xi = -\tfrac{3}{2}|\hat{M}|\lambda_N = \tfrac{1}{2}\rangle, \quad (20b)$$

$$\mathcal{H}_3 \equiv \langle \lambda_\Xi = \tfrac{1}{2}|\hat{M}|\lambda_N = \tfrac{1}{2}\rangle$$
$$= -\pi_\Xi \langle \lambda_\Xi = -\tfrac{1}{2}|\hat{M}|\lambda_N = -\tfrac{1}{2}\rangle, \quad (20c)$$

$$\mathcal{H}_4 \equiv \langle \lambda_\Xi = \tfrac{1}{2}|\hat{M}|\lambda_N = -\tfrac{1}{2}\rangle$$
$$= \pi_\Xi \langle \lambda_\Xi = -\tfrac{1}{2}|\hat{M}|\lambda_N = \tfrac{1}{2}\rangle. \quad (20d)$$

These helicity amplitudes are related to the coefficient functions in Eq. (14) by

$$\mathcal{H}_1 = \frac{1}{\sqrt{2}}\left[i\cos\frac{\theta}{2}F_1 - \cos\theta \sin\frac{\theta}{2}F_2 \right.$$
$$\left. - \cos\frac{3\theta}{2}F_3 + \sin\theta \cos\frac{\theta}{2}F_4 \right], \quad (21a)$$

$$\mathcal{H}_2 = \frac{1}{\sqrt{2}}\left[i\sin\frac{\theta}{2}F_1 - \cos\theta \cos\frac{\theta}{2}F_2 \right.$$
$$\left. + \sin\frac{3\theta}{2}F_3 - \sin\theta \sin\frac{\theta}{2}F_4 \right], \quad (21b)$$

$$\mathcal{H}_3 = \frac{1}{\sqrt{6}}\left[-i\sin\frac{\theta}{2}F_1 + (2 - 3\cos\theta)\cos\frac{\theta}{2}F_2 \right.$$
$$\left. + 3\sin\frac{3\theta}{2}F_3 - (1 - 3\cos\theta)\cos\frac{\theta}{2}F_4 \right], \quad (21c)$$

$$\mathcal{H}_4 = \frac{1}{\sqrt{6}}\left[i\cos\frac{\theta}{2}F_1 + (2 + 3\cos\theta)\sin\frac{\theta}{2}F_2 \right.$$
$$\left. + 3\cos\frac{3\theta}{2}F_3 - (1 + 3\cos\theta)\sin\frac{\theta}{2}F_4 \right] \quad (21d)$$

for a positive-parity Ξ and by

$$\mathcal{H}_1 = \frac{1}{\sqrt{2}}\left[i(2 - \cos\theta)\cos\frac{\theta}{2}G_1 + 2i\sin^3\frac{\theta}{2}G_2 \right.$$
$$\left. - \cos\theta \cos\frac{\theta}{2}G_3 + \sin\theta \cos\frac{\theta}{2}G_4 \right], \quad (22a)$$

$$\mathcal{H}_2 = \frac{1}{\sqrt{2}}\left[-i(2 + \cos\theta)\sin\frac{\theta}{2}G_1 + 2i\cos^3\frac{\theta}{2}G_2 \right.$$
$$\left. - \cos\theta \sin\frac{\theta}{2}G_3 + \sin\theta \sin\frac{\theta}{2}G_4 \right], \quad (22b)$$

$$\mathcal{H}_3 = \frac{1}{\sqrt{6}}\left[3i\cos\theta \sin\frac{\theta}{2}G_1 + 3i\sin\theta \sin\frac{\theta}{2}G_2 \right.$$
$$\left. + (2 + 3\cos\theta)\sin\frac{\theta}{2}G_3 + (3\cos\theta - 1)\cos\frac{\theta}{2}G_4 \right], \quad (22c)$$

$$\mathcal{H}_4 = \frac{1}{\sqrt{6}}\left[-3i\cos\theta \cos\frac{\theta}{2}G_1 - 3i\sin\theta \cos\frac{\theta}{2}G_2 \right.$$
$$\left. + (2 - 3\cos\theta)\cos\frac{\theta}{2}G_3 + (3\cos\theta + 1)\sin\frac{\theta}{2}G_4 \right] \quad (22d)$$

for a negative-parity Ξ.

The SDM elements are defined by

$$\rho_{\lambda\lambda'}^{\Xi,i} \equiv \langle \lambda|\hat{\rho}^{N,i}|\lambda'\rangle = \tfrac{1}{2}\langle \lambda|\hat{M}\sigma_i \hat{M}^\dagger|\lambda'\rangle \quad (23)$$

for $i = 0, \ldots, 3$, where λ and λ' stand for the helicity of the produced Ξ baryon and $\sigma_0 = \mathbb{1}$ is the 2×2–unit matrix. For completeness, a relevant part of the SDM formalism for the present work is presented in Appendix B. The SDM elements

have the properties

$$\rho_{\lambda,\lambda'}^{\Xi,i} = (-1)^{i+\lambda-\lambda'} \rho_{-\lambda,-\lambda'}^{\Xi,i}, \tag{24a}$$

$$\rho_{\lambda,\lambda'}^{\Xi,i} = \rho_{\lambda',\lambda}^{\Xi,i*} \tag{24b}$$

due to the Hermiticity of $\hat{\rho}$ [cf. Eq. (23)] and the symmetry of the spin matrix elements in Eq. (16).

We now relate the SDM elements, $\rho_{\lambda,\lambda'}^{\Xi,i}$, to the helicity amplitudes $\mathcal{H}_j$ given by Eqs. (17) and (20), which determine the reaction amplitudes. The purpose is to find a set of SDM elements that fixes these helicity amplitudes completely.

A. Ξ of $J^P = \frac{1}{2}^\pm$

Starting with $J = \frac{1}{2}$, there are 16 possible SDM elements $\rho_{\lambda\lambda'}^{\Xi,i}$. However, only four of them are independent for a given parity and they determine the amplitudes $\mathcal{H}_1$ and $\mathcal{H}_2$ apart from an irrelevant overall phase. Inserting Eq. (17) into Eq. (23), a set of four independent SDM elements can be determined as

$$2\rho_{\frac{1}{2},\frac{1}{2}}^0 = 2i\pi_\Xi \rho_{\frac{1}{2},-\frac{1}{2}}^2 = |\mathcal{H}_1|^2 + |\mathcal{H}_2|^2, \tag{25a}$$

$$2\rho_{\frac{1}{2},\frac{1}{2}}^3 = 2\pi_\Xi \rho_{\frac{1}{2},-\frac{1}{2}}^1 = |\mathcal{H}_1|^2 - |\mathcal{H}_2|^2, \tag{25b}$$

$$\rho_{\frac{1}{2},\frac{1}{2}}^2 = i\pi_\Xi \rho_{\frac{1}{2},-\frac{1}{2}}^0 = \text{Im}[\mathcal{H}_1\mathcal{H}_2^*], \tag{25c}$$

$$\rho_{\frac{1}{2},\frac{1}{2}}^1 = -\pi_\Xi \rho_{\frac{1}{2},-\frac{1}{2}}^3 = \text{Re}[\mathcal{H}_1\mathcal{H}_2^*], \tag{25d}$$

where the superindex Ξ in $\rho_{\lambda\lambda'}^{\Xi,i}$ was dropped for simplicity. A complete list of SDM elements $\rho_{\lambda,\lambda'}^i$ in terms of helicity amplitudes $\mathcal{H}_i$ is given in Appendix C.

The SDM elements are directly related to the observables defined by Eq. (6). For example, from Eqs. (24) and (B10d), we have

$$\frac{d\sigma}{d\Omega} = 2\rho_{\frac{1}{2},\frac{1}{2}}^0, \qquad \frac{d\sigma}{d\Omega}K_{yy'} = 2i\rho_{\frac{1}{2},-\frac{1}{2}}^2, \tag{26a}$$

$$\frac{d\sigma}{d\Omega}T_y = 2\rho_{\frac{1}{2},\frac{1}{2}}^2, \qquad \frac{d\sigma}{d\Omega}P_{y'} = 2i\rho_{\frac{1}{2},-\frac{1}{2}}^0, \tag{26b}$$

$$\frac{d\sigma}{d\Omega}K_{xx'} = 2\rho_{\frac{1}{2},-\frac{1}{2}}^1, \qquad \frac{d\sigma}{d\Omega}K_{zz'} = 2\rho_{\frac{1}{2},\frac{1}{2}}^3, \tag{26c}$$

$$\frac{d\sigma}{d\Omega}K_{xz'} = 2\rho_{\frac{1}{2},\frac{1}{2}}^1, \qquad \frac{d\sigma}{d\Omega}K_{zx'} = 2\rho_{\frac{1}{2},-\frac{1}{2}}^3, \tag{26d}$$

where the primed Cartesian components correspond to the rotated frame (see Fig. 1; note that $y' \equiv y$).

From Eqs. (10) and (26), we see, in particular, that

$$K_{yy'} = \frac{i\rho_{\frac{1}{2},-\frac{1}{2}}^2}{\rho_{\frac{1}{2},\frac{1}{2}}^0} = \pi_\Xi. \tag{27}$$

More generally, in terms of the SDM elements, one obtains

$$(-1)^{\frac{1}{2}-\lambda'}\frac{i\rho_{\lambda,-\lambda'}^2}{\rho_{\lambda,\lambda'}^0} = (-1)^{\frac{1}{2}-\lambda'}\frac{\rho_{\lambda,-\lambda'}^1}{\rho_{\lambda,\lambda'}^3} = \pi_\Xi. \tag{28}$$

This result reveals that one needs to measure two SDM elements to determine the parity of the Ξ baryon: either $\rho_{\lambda,\lambda'}^0$, with an unpolarized target nucleon, and $\rho_{\lambda,-\lambda'}^2$, with a polarized target nucleon along the direction $\hat{\boldsymbol{n}}_2 \equiv \hat{\boldsymbol{n}}_2'$ perpendicular to the

reaction plane, or $\rho_{\lambda,\lambda'}^1$, with a transversally polarized target along $\hat{\boldsymbol{n}}_1$, and $\rho_{\lambda,-\lambda'}^3$, with a longitudinally polarized target along $\hat{\boldsymbol{n}}_3 \equiv \hat{\boldsymbol{p}}$. Note that $\rho_{\lambda,\lambda'}^0$ is directly related to the cross section $d\sigma/d\Omega$ when $\lambda = \lambda'$.

B. Ξ of $J^P = \frac{3}{2}^\pm$

For $J = 3/2$, analogously to the $J = 1/2$ case, inserting Eq. (20) into Eq. (23), the SDM elements are related to the four amplitudes $\mathcal{H}_i$ ($i = 1, \ldots, 4$) by

$$2\rho_{\frac{3}{2},\frac{3}{2}}^0 = |\mathcal{H}_1|^2 + |\mathcal{H}_2|^2, \tag{29a}$$

$$2\rho_{\frac{3}{2},-\frac{3}{2}}^1 = \pi_\Xi(|\mathcal{H}_1|^2 - |\mathcal{H}_2|^2), \tag{29b}$$

$$-i\rho_{\frac{3}{2},-\frac{3}{2}}^0 = \pi_\Xi \text{Im}[\mathcal{H}_2\mathcal{H}_1^*], \tag{29c}$$

$$\rho_{\frac{3}{2},\frac{3}{2}}^1 = \text{Re}[\mathcal{H}_2\mathcal{H}_1^*], \tag{29d}$$

$$2\rho_{\frac{1}{2},\frac{1}{2}}^0 = |\mathcal{H}_3|^2 + |\mathcal{H}_4|^2, \tag{30a}$$

$$2\rho_{\frac{1}{2},-\frac{1}{2}}^1 = \pi_\Xi(|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2), \tag{30b}$$

$$-i\rho_{\frac{1}{2},-\frac{1}{2}}^0 = \pi_\Xi \text{Im}[\mathcal{H}_3\mathcal{H}_4^*], \tag{30c}$$

$$\rho_{\frac{1}{2},\frac{1}{2}}^1 = \text{Re}[\mathcal{H}_3\mathcal{H}_4^*], \tag{30d}$$

$$2\rho_{\frac{3}{2},-\frac{1}{2}}^1 = \pi_\Xi(\mathcal{H}_2\mathcal{H}_4^* - \mathcal{H}_1\mathcal{H}_3^*), \tag{31a}$$

$$2\rho_{\frac{3}{2},\frac{1}{2}}^0 = \mathcal{H}_2\mathcal{H}_4^* + \mathcal{H}_1\mathcal{H}_3^*, \tag{31b}$$

$$2\rho_{\frac{3}{2},-\frac{1}{2}}^0 = \pi_\Xi(\mathcal{H}_1\mathcal{H}_4^* - \mathcal{H}_2\mathcal{H}_3^*), \tag{32a}$$

$$2\rho_{\frac{3}{2},\frac{1}{2}}^1 = \mathcal{H}_1\mathcal{H}_4^* + \mathcal{H}_2\mathcal{H}_3^*. \tag{32b}$$

A complete list of SDM elements $\rho_{\lambda,\lambda'}^{\Xi,i}$ in terms of the helicity amplitudes $\mathcal{H}_i$ is given in Appendix C.

As mentioned in the previous section, a set of eight independent SDM elements (e.g., $\rho_{\frac{1}{2},\frac{1}{2}}^0$, $\rho_{\frac{3}{2},\frac{3}{2}}^0$, $\text{Re}[\rho_{\frac{3}{2},\frac{1}{2}}^0]$, $\text{Im}[\rho_{\frac{3}{2},\frac{1}{2}}^0]$, $\text{Re}[\rho_{\frac{3}{2},-\frac{1}{2}}^0]$, $\rho_{\frac{1}{2},-\frac{1}{2}}^1$, $\rho_{\frac{3}{2},-\frac{3}{2}}^1$, $\text{Re}[\rho_{\frac{3}{2},\frac{1}{2}}^1]$) determines all four helicity amplitudes $\mathcal{H}_i$, ($i = 1, \ldots, 4$), apart from an irrelevant overall phase. The analysis here is analogous to the one carried out in Ref. [39] for pion photoproduction.

Equations (24) and (B13) can be used to calculate K_{yy} in terms of SDM elements involving Ξ of spin 3/2. This leads to

$$K_{yy} = \frac{i\left(\rho_{\frac{3}{2},-\frac{3}{2}}^2 - \rho_{\frac{1}{2},-\frac{1}{2}}^2\right)}{\rho_{\frac{3}{2},\frac{3}{2}}^0 + \rho_{\frac{1}{2},\frac{1}{2}}^0} = \pi_\Xi \tag{33}$$

or, more generally,

$$(-1)^{\frac{3}{2}-\lambda'}\frac{i\rho_{\lambda,-\lambda'}^2}{\rho_{\lambda,\lambda'}^0} = (-1)^{\frac{3}{2}-\lambda'}\frac{\rho_{\lambda,-\lambda'}^1}{\rho_{\lambda,\lambda'}^3} = \pi_\Xi. \tag{34}$$

Equations (27), (28), (33), and (34) can be extended to an arbitrary spin J of the Ξ baryon; i.e.,

$$K_{yy} = \frac{i\sum_\lambda(-1)^{J-\lambda}\rho_{\lambda,-\lambda}^2}{\sum_\lambda \rho_{\lambda,\lambda}^0} = \pi_\Xi \tag{35}$$

and

$$(-1)^{J-\lambda'} \frac{i\rho^2_{\lambda,-\lambda'}}{\rho^0_{\lambda,\lambda'}} = (-1)^{J-\lambda'} \frac{\rho^1_{\lambda,-\lambda'}}{\rho^3_{\lambda,\lambda'}} = \pi_\Xi. \qquad (36)$$

Note here that in the last expression two SDM elements are sufficient to determine the parity, whereas one needs a whole sum of SDM elements to achieve the same in terms of K_{yy}.

C. Extracting SDM elements from experiment

To extract SDM elements from experiment, following Chung [30] and Biagi *et al.* [31], one may relate them to moments, H^i, and weak-decay-asymmetry parameters. Their definitions and further details are given in Appendix D. Here, we present only some pertinent results.

For a spin-J Ξ undergoing a single weak-decay process, one obtains the SDM element

$$\rho^{\Xi,i}_{\lambda_\Xi,\lambda'_\Xi} = \sum_L \frac{2L+1}{2J+1} \langle J\,\lambda'_\Xi\, L\, M | J\, \lambda_\Xi \rangle\, t^{J,i}_{LM}, \qquad (37)$$

where $M = \lambda_\Xi - \lambda'_\Xi$. The coefficients $t^{J,i}_{LM}$ can be determined from the ratio of the moments,

$$\left(\frac{d\sigma}{d\Omega}\right) \frac{H^i(L,M)}{H^0(0,0)} = \zeta_L\, t^{J,i}_{LM} \left\langle J\, \tfrac{1}{2}\, L\, 0 \middle| J\, \tfrac{1}{2} \right\rangle, \qquad (38)$$

where $\zeta_L = 1$ for even L and $\zeta_L = \alpha_\Xi$ for odd L, with α_Ξ denoting the Ξ decay-asymmetry parameter.

Note that since all moments vanish identically for $L > 2J$, Eq. (38) offers a way of determining the spin of the Ξ undergoing a single (weak) decay by measuring the moments as a function of L. In other words, the nonvanishing $H^i(L,M)$ with the largest L value for some i and M determines J as $J = L/2$. Experimentally, of course, this may be challenging since it is not *a priori* clear how small the measured values of the next higher moment $H^i(L+1,M)$ would need to be in order to be compatible with 0. And, moreover, one would need to confirm that the smallness of this moment is not accidental.

Similar results are obtained for excited Ξ resonances, Ξ^*, undergoing a double-decay process, as discussed in Appendix D. In this case, we have

$$\left(\frac{d\sigma}{d\Omega}\right) \frac{H^i(0,0,L,M)}{H^0(0,0,0,0)} = t^{J,i}_{LM} \left\langle J\, \tfrac{1}{2}\, L\, 0 \middle| J\, \tfrac{1}{2} \right\rangle \qquad (39a)$$

for even L and

$$\left(\frac{d\sigma}{d\Omega}\right) \frac{H^i(1,0,L,M)}{H^0(0,0,0,0)} = \frac{\alpha_\Lambda}{3}\, t^{J,i}_{LM} \left\langle J\, \tfrac{1}{2}\, L\, 0 \middle| J\, \tfrac{1}{2} \right\rangle \qquad (39b)$$

for odd L. Here, α_Λ denotes the Λ decay-asymmetry parameter for the decay chain $\Xi^* \to \Lambda + \bar{K}$ followed by $\Lambda \to N + \pi$. In the case of $\Xi^* \to \Xi + \pi$ followed by $\Xi \to \Lambda + \pi$ instead, α_Λ needs to be replaced by α_Ξ.

For Ξ resonances decaying along the double-decay chain specified in Eq. (D3), the result of the corresponding moments given by Eq. (D22) leads to [31]

$$\frac{H^0(1,\pm 1,L,M)}{H^0(1,0,L,M)} = \pi_\Xi (-1)^{J+\frac{1}{2}} \frac{2J+1}{\sqrt{2L(L+1)}} \qquad (40)$$

for an unpolarized target and for odd values of $L(\leqslant 2J)$. This offers a way of determining the spin and parity of the excited Ξ resonance simultaneously.

IV. SUMMARY

A model-independent analysis of the $\bar{K} + N \to K + \Xi$ reaction has been performed. Following the method in Ref. [27], we derived the most general spin structure of the reaction amplitude, consistent with basic symmetries, for Ξ baryons of $J^P = \frac{1}{2}^\pm$ and $\frac{3}{2}^\pm$. The coefficients multiplying each spin structure have been presented in partial-wave-decomposed form, thus permitting partial-wave analyses, once sufficient data become available for these reactions. The method in Ref. [27] is general and can be applied, in principle, to derive the structure of the reaction amplitude involving higher-spin Ξ production.

Furthermore, a minimal set of independent observables required to determine the reaction amplitude completely has been identified. In addition to the unpolarized cross sections, one also needs single- and double-spin observables, which poses a formidable experimental challenge, in particular, since one needs to measure the polarization of the outgoing Ξ. Note that, for the Ξ of spin $1/2$, there are two complex amplitudes to be determined, whereas for the Ξ of spin $3/2$, there are four complex amplitudes. We then formulated the problem using the SDM approach and expressed the spin observables in terms of the SDM elements. Following Refs. [30] and [31], it was shown that the latter can be extracted from the moments associated with the Ξ decay processes in conjunction with the self-analyzing nature of the hyperon (Λ or Ξ) resulting from the subsequent decay of the Ξ produced in the primary reaction. The moments, in turn, can be extracted from the measurement of the angular distribution of the decay products.

Since the determination of the spin and parity quantum numbers is a fundamental part of any spectroscopy study, reflection symmetry about the reaction plane has been exploited, in particular, to show that, apart from the spin-transfer coefficient K_{yy}, the ratio of the SDM elements given by Eq. (36) determines the parity of a Ξ resonance with an arbitrary spin. Furthermore, the moments given by Eq. (40) determine the spin and parity of the Ξ resonance simultaneously [31].

We also mention that the present analysis applies as given only to Ξ resonances that are sufficiently narrow to permit their being treated like on-shell particles. For broad resonances, a partial-wave analysis would be required to extract them from experimental data.

In summary, the present analysis provides a model-independent framework for developing reliable reaction theories of Ξ production to help in the planning of future experimental efforts in Ξ baryon spectroscopy. This will also help in analyzing the data to understand the production mechanisms of Ξ baryons.

ACKNOWLEDGMENTS

APPENDIX A: PARTIAL-WAVE DECOMPOSITION

In this Appendix, we give the partial-wave decomposition of the coefficients multiplying each spin structure of the reaction amplitudes in Eqs. (4) and (14) for Ξ with spin-parity $J^P = \frac{1}{2}^\pm$ and $\frac{3}{2}^\pm$. The partial-wave expansion of the (plane-wave) matrix element $\hat{M}$ in Eqs. (4) and (14) is

$$
\begin{aligned}
&\langle S' M_{S'} | \hat{M}(\boldsymbol{p}',\boldsymbol{p}) | S M_S \rangle \\
&= \sum i^{L-L'} \langle S M_S L M_L | J M_J \rangle \langle S' M_{S'} L' M_{L'} | J M_J \rangle \\
&\quad \times M_{L'L}^{TJ}(p',p) Y_{L'M_{L'}}(\hat{\boldsymbol{p}}') Y_{LM_L}^*(\hat{\boldsymbol{p}}) \hat{P}_T,
\end{aligned} \tag{A1}
$$

where S, L, J, and T stand for the total spin, orbital angular momentum, total angular momentum, and total isospin, respectively, of the initial $\bar{K}N$ state. The corresponding projection quantum numbers are denoted by M_S, M_L, and M_J. The primed quantities represent the corresponding quantum numbers of the final $K\Xi$ state. The summation runs over all quantum numbers not specified on the left-hand side of Eq. (A1). The relative momenta of the initial $\bar{K}N$ and final $K\Xi$ states are denoted $\boldsymbol{p}$ and $\boldsymbol{p}'$, respectively, and $p = |\boldsymbol{p}|$, $p' = |\boldsymbol{p}'|$. In the following, without loss of generality, we choose $\hat{\boldsymbol{n}}_3$ along the momentum $\boldsymbol{p}$ of the nucleon in the CM system, i.e., $\hat{\boldsymbol{n}}_3 \equiv \hat{\boldsymbol{p}}$ as specified in Fig. 1. In Eq. (A1), $\hat{P}_T$ stands for the total isospin projection operator onto the isospin singlet ($T = 0$) and isospin triplet ($T = 1$) states,

$$
\hat{P}_0 = \tfrac{1}{4}(1 - \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \quad \text{and} \quad \hat{P}_1 = \tfrac{1}{4}(3 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2), \tag{A2}
$$

where the $\boldsymbol{\tau}_i$ ($i = 1,2$) are the usual vectors made out of isospin Pauli matrices.

For a Ξ of $J^P = \frac{1}{2}^\pm$, following Ref. [27], the coefficients M_i in Eq. (4) are given by

$$
M_0 = \frac{1}{4\pi} \sum_{L',T} \left[(L'+1) M_{L'L'}^{TJ_+}(p',p) + L' M_{L'L'}^{TJ_-}(p',p) \right] P_{L'}(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}') \hat{P}_T, \tag{A3a}
$$

$$
M_2 = \frac{i}{4\pi} \sum_{L',T} \left[M_{L'L'}^{TJ_-}(p',p) - M_{L'L'}^{TJ_+}(p',p) \right] P_{L'}^1(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}') \hat{P}_T, \tag{A3b}
$$

$$
M_1 = \frac{i}{4\pi} \sum_{L',T} \left[M_{L'L'-1}^{TJ_-}(p',p) + M_{L'L'+1}^{TJ_+}(p',p) \right] P_{L'}^1(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}') \hat{P}_T, \tag{A3c}
$$

$$
M_3 = \frac{i}{4\pi} \sum_{L',T} \left[(L'+1) M_{L'L'+1}^{TJ_+}(p',p) - L' M_{L'L'-1}^{TJ_-}(p',p) \right] P_{L'}(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}') \hat{P}_T, \tag{A3d}
$$

where $J_\pm \equiv L' \pm \frac{1}{2}$, and $P_{L'}(x)$ and $P_{L'}^1(x)$ denote the Legendre and associated Legendre functions, respectively.[2] The amplitudes M_i here are operators in isospin space whose actions are specified by the projectors $\hat{P}_T$ defined in Eq. (A2).

Likewise, for a Ξ of $J^P = \frac{3}{2}^\pm$, following Ref. [27], the coefficients F_i and G_i in Eq. (14) are given by

$$
F_1 = i\frac{3}{8\pi} \sum_{J,L',T} (-1)^{L'+J+\frac{3}{2}} [J]^2 \frac{[L']}{\sqrt{L'(L'+1)}} \begin{Bmatrix} \frac{1}{2} & L' & J \\ L' & \frac{3}{2} & 1 \end{Bmatrix} M_{L'L'}^{JT}(p',p) P_{L'}^1(\hat{\boldsymbol{p}}' \cdot \hat{\boldsymbol{p}}) \hat{P}_T, \tag{A4a}
$$

$$
F_2 = \frac{1}{\sqrt{2}} \sum_{J,L',L,T} i^{L-L'} (-1)^{J+\frac{1}{2}} [J]^2 \begin{Bmatrix} \frac{1}{2} & L & J \\ L' & \frac{3}{2} & 2 \end{Bmatrix} M_{L'L}^{JT}(p',p) a_{L'L} \hat{P}_T, \tag{A4b}
$$

$$
F_3 = \frac{1}{2\sqrt{2}} \sum_{J,L',L,T} i^{L-L'} (-1)^{J+\frac{1}{2}} [J]^2 \begin{Bmatrix} \frac{1}{2} & L & J \\ L' & \frac{3}{2} & 2 \end{Bmatrix} M_{L'L}^{JT}(p',p) b_{L'L} \hat{P}_T, \tag{A4c}
$$

$$
F_4 = \frac{1}{\sqrt{2}} \sum_{J,L',L,T} i^{L-L'} (-1)^{J+\frac{1}{2}} [J]^2 \begin{Bmatrix} \frac{1}{2} & L & J \\ L' & \frac{3}{2} & 2 \end{Bmatrix} M_{L'L}^{JT}(p',p) c_{L'L} \hat{P}_T \tag{A4d}
$$

and

$$
G_1 = \frac{1}{2\sqrt{2}} \sum_{J,L',L,T} i^{L-L'} (-1)^{J+\frac{1}{2}} [J]^2 \begin{Bmatrix} \frac{1}{2} & L & J \\ L' & \frac{3}{2} & 2 \end{Bmatrix} M_{L'L}^{JT}(p',p) a'_{L'L} \hat{P}_T, \tag{A5a}
$$

---

[2]Here, the phase convention for the associated Legendre function is such that $P_1^1(x) = +\sin(x)$.

$$G_2 = \frac{1}{2\sqrt{2}} \sum_{J,L',L,T} i^{L-L'} (-1)^{J+\frac{1}{2}} [J]^2 \begin{Bmatrix} \frac{1}{2} & L & J \\ L' & \frac{3}{2} & 2 \end{Bmatrix} M_{L'L}^{JT}(p',p)\, b'_{L'L}\, \hat{P}_T, \tag{A5b}$$

$$G_3 = \sqrt{\frac{3}{2}}\frac{1}{4\pi} \sum_{J,L',L,T} i^{L-L'} (-1)^{J+\frac{1}{2}} [J]^2 \frac{[LL']}{\sqrt{L'(L'+1)}} \langle L\,0\,L'\,1 | 1\,1 \rangle \begin{Bmatrix} \frac{1}{2} & L & J \\ L' & \frac{3}{2} & 1 \end{Bmatrix} M_{L'L}^{JT}(p',p)\, P_{L'}^1(\hat{\boldsymbol{p}}'\cdot\hat{\boldsymbol{p}})\, \hat{P}_T, \tag{A5c}$$

$$G_4 = \frac{\sqrt{3}}{8\pi} \sum_{J,L',L,T} i^{L-L'} (-1)^{J+\frac{3}{2}} [J]^2 [LL'] \langle L\,0\,L'\,0 | 1\,0 \rangle \begin{Bmatrix} \frac{1}{2} & L & J \\ L' & \frac{3}{2} & 1 \end{Bmatrix} M_{L'L}^{JT}(p',p)\, P_{L'}(\hat{\boldsymbol{p}}'\cdot\hat{\boldsymbol{p}})\, \hat{P}_T, \tag{A5d}$$

where we have introduced the notation $[J] \equiv \sqrt{2J+1}$ and $[j_1 j_2] \equiv [j_1][j_2]$. The summations extend over all the quantum numbers, J, L', L, and T. Note that total parity conservation imposes the condition $(-1)^{L'+L} = \pm 1$, as the parity of the Ξ baryon is $\pi_\Xi = \pm 1$. The coefficients $a_{L'L}$, $b_{L'L}$, etc., are given by

$$a_{L',L} = 2\frac{[LL']}{4\pi} \langle L\,0\,L'\,2 | 2\,2 \rangle \sqrt{\frac{(L'-2)!}{(L'+2)!}}\, P_{L'}^2(\hat{\boldsymbol{p}}'\cdot\hat{\boldsymbol{p}}), \tag{A6a}$$

$$b_{L',L} = 2\frac{[LL']}{4\pi} \langle L\,0\,L'\,1 | 2\,1 \rangle \sqrt{\frac{(L'-1)!}{(L'+1)!}}\, P_{L'}^1(\hat{\boldsymbol{p}}'\cdot\hat{\boldsymbol{p}}), \tag{A6b}$$

$$c_{L',L} = \frac{[LL']}{4\pi} \left[\langle L\,0\,L'\,2 | 2\,2 \rangle \sqrt{\frac{(L'-2)!}{(L'+2)!}}\, P_{L'}^2(\hat{\boldsymbol{p}}'\cdot\hat{\boldsymbol{p}}) \right.$$
$$\left. + \sqrt{\frac{3}{2}} \langle L\,0\,L'\,0 | 2\,0 \rangle\, P_{L'}(\hat{\boldsymbol{p}}'\cdot\hat{\boldsymbol{p}}) \right], \tag{A6c}$$

$$a'_{L',L} = i b_{L',L}, \tag{A6d}$$

$$b'_{L',L} = -i a_{L',L}. \tag{A6e}$$

APPENDIX B: SDM FORMALISM

A density operator can be used to describe an ensemble of quantum states. It is defined as

$$\hat{\rho} \equiv \sum_\psi I_\psi\, |\psi\rangle\langle\psi|, \tag{B1}$$

where I_ψ denotes the probability of finding an element of the ensemble in state ψ, subject to the condition $\sum_\psi I_\psi = 1$. For the present application, the states $|\psi\rangle$ are the spin states of the initial state, N, or the final state, Ξ. For the initial nucleon state, the spin-density operator reads

$$\hat{\rho} \to \hat{\rho}^N \equiv \sum_{\psi_N} I_{\psi_N}\, |\psi_N\rangle\langle\psi_N|$$
$$= \frac{1}{2}\left(\mathbb{1} + \bar{\boldsymbol{P}}\cdot\boldsymbol{\sigma} \right), \tag{B2}$$

where $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ denotes the vector formed of Pauli spin matrices and $\bar{\boldsymbol{P}}$ is the polarization vector of the nucleon, which is the difference between the probability of finding the nucleon in the $m_N = +\frac{1}{2}$ spin state and the probability of finding the nucleon in the $m_N = -\frac{1}{2}$ state (m_N is the spin projection

quantum number along the $\bar{\boldsymbol{P}}$ direction) or, symbolically, $|I_{\psi_+} - I_{\psi_-}| = |\bar{\boldsymbol{P}}|$.

An unpolarized ensemble has $\bar{\boldsymbol{P}} = 0$. The trace of this spin-$\frac{1}{2}$ density matrix is normalized to 1. By introducing the notation $\sigma_0 \equiv \mathbb{1}$ and $P_0 \equiv 1$, the nucleon SDM in Eq. (B2) can be rewritten as

$$\hat{\rho}^N = \frac{\mathbb{1} + \bar{\boldsymbol{P}}\cdot\boldsymbol{\sigma}}{2} = \sum_{i=0}^3 \bar{P}_i\, \hat{\rho}^{N,i}, \tag{B3}$$

with

$$\hat{\rho}^{N,i} \equiv \tfrac{1}{2}\sigma_i \tag{B4}$$

for $i = 0,\ldots,3$.

The spin-density operator for a produced Ξ particle, $\hat{\rho}^\Xi$, can be expressed in terms of the production amplitude $\hat{M}$, which is an operator that maps the initial nucleon spin state ψ_N into a spin state of the Ξ. In the helicity basis for the produced Ξ, the corresponding SDM elements read

$$\rho_{\lambda\lambda'}^\Xi(\psi_N) \equiv \langle\lambda| \hat{M} |\psi_N\rangle \langle\psi_N| \hat{M}^\dagger |\lambda'\rangle, \tag{B5}$$

where λ and λ' enumerate the Ξ's helicities.

When the beam of antikaons scatters off an ensemble of nucleons, one needs to average over all nucleon spin states with their appropriate probability weights; i.e.,

$$\rho_{\lambda,\lambda'}^\Xi \equiv \sum_{\psi_N} I_{\psi_N} \rho_{\lambda\lambda'}^\Xi(\psi_N)$$
$$= \sum_{\psi_N} I_{\psi_N} \langle\lambda| \hat{M} |\psi_N\rangle \langle\psi_N| \hat{M}^\dagger |\lambda'\rangle$$
$$= \langle\lambda| \hat{M}\hat{\rho}^N \hat{M}^\dagger |\lambda'\rangle$$
$$= \langle\lambda| \hat{\rho}^\Xi |\lambda'\rangle, \tag{B6}$$

where Eq. (B1) was used to show that the Ξ spin-density operator is given by [40]

$$\hat{\rho}^\Xi = \hat{M}\hat{\rho}^N \hat{M}^\dagger. \tag{B7}$$

Using Eq. (B3), we may write

$$\hat{\rho}^\Xi = \sum_{i=0}^3 \bar{P}_i\, \hat{\rho}^{\Xi,i}, \tag{B8}$$

where

$$\hat{\rho}^{\Xi,i} \equiv \tfrac{1}{2}\hat{M}\sigma_i \hat{M}^\dagger \tag{B9}$$

for $i = 0,\ldots,3$. Here, $\hat{\rho}^{\Xi,0}$ and $\hat{\rho}^{\Xi,j}$ ($j = 1,2,3$) provide the respective contributions for unpolarized and polarized initial nucleons.

For a Ξ baryon of spin 1/2, comparing Eqs. (7) and (B9) gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\,\text{Tr}[\hat{M}\hat{M}^\dagger] = \text{Tr}[\hat{\rho}^{\Xi,0}], \tag{B10a}$$

$$\frac{d\sigma}{d\Omega}T_i = \frac{1}{2}\,\text{Tr}[\hat{M}\sigma_i\hat{M}^\dagger] = \text{Tr}[\hat{\rho}^{\Xi,i}], \tag{B10b}$$

$$\frac{d\sigma}{d\Omega}P_i = \frac{1}{2}\,\text{Tr}[\hat{M}\hat{M}^\dagger\sigma_i] = \text{Tr}[\hat{\rho}^{\Xi,0}\sigma_i], \tag{B10c}$$

$$\frac{d\sigma}{d\Omega}K_{ij} = \frac{1}{2}\,\text{Tr}[\hat{M}\sigma_i\hat{M}^\dagger\sigma_j] = \text{Tr}[\hat{\rho}^{\Xi,i}\sigma_j]. \tag{B10d}$$

When a Ξ baryon of spin 3/2 (or higher-spin) is involved, there will be many more possible degrees of polarization than in the spin-1/2 case. For the particular case of the spin-transfer coefficient, K_{ij}, discussed in connection to the parity of Ξ, its definition given in Eq. (6d) has to be generalized. For this purpose, we first introduce the operator $\Omega(\boldsymbol{J}\cdot\hat{\boldsymbol{n}})$ as

$$\Omega(\boldsymbol{J}\cdot\hat{\boldsymbol{n}}) \equiv \sum_{M=-J}^{+J}(-1)^{\frac{1}{2}-M}\,\mathsf{P}_{\hat{\boldsymbol{n}}}^{J,M}, \tag{B11}$$

where $\mathsf{P}_{\hat{\boldsymbol{n}}}^{J,M}$ denotes the spin-projection operator onto an arbitrary direction $\hat{\boldsymbol{n}}$ for an arbitrary half-integer spin J. It can be explicitly calculated as

$$\mathsf{P}_{\hat{\boldsymbol{n}}}^{J,M} = \prod_{m=-J}^{+J}{}'\,\frac{m-\boldsymbol{J}\cdot\hat{\boldsymbol{n}}}{m-M}, \tag{B12}$$

where the prime indicates that the factor with $m = M$ is omitted. Here, $\boldsymbol{J} \equiv (J_1, J_2, J_3)$ stands for the generator of spin-J rotation. This expression provides a rotationally invariant polynomial of order $2J$ in $\boldsymbol{J}\cdot\hat{\boldsymbol{n}}$ that is a generalization to arbitrary spin of the usual $(\mathbb{1} \pm \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})/2$ projectors for spin 1/2.

With the spin-projection operator defined above, the spin-transfer coefficient involving a Ξ baryon with an arbitrary spin J is now generalized to

$$\frac{d\sigma}{d\Omega}K_{ba} = \frac{1}{2}\,\text{Tr}[\hat{M}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{b}}\,\hat{M}^\dagger\,\Omega(\boldsymbol{J}\cdot\hat{\boldsymbol{a}})], \tag{B13}$$

where $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{a}}$ are the spin directions. For $J = 1/2$, in view of $\Omega(\boldsymbol{J}\cdot\hat{\boldsymbol{a}}) \to \boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}}$, this reduces to the familiar expression (6d), of course. For Cartesian directions $\hat{\boldsymbol{b}} = \hat{\boldsymbol{n}}_i$ and $\hat{\boldsymbol{a}} = \hat{\boldsymbol{n}}'_j$, in particular, Eq. (B13) may be written as

$$\frac{d\sigma}{d\Omega}K_{ij'} = \text{Tr}\left[\hat{\rho}^{\Xi,i}\Omega_{j'}^J\right], \tag{B14}$$

where $\Omega_{j'}^J \equiv \Omega(\boldsymbol{J}\cdot\hat{\boldsymbol{n}}'_j)$.

For the Cartesian frame $\{\hat{\boldsymbol{n}}'_1, \hat{\boldsymbol{n}}'_2 \equiv \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}'_3 \equiv \hat{\boldsymbol{p}}'\}$ aligned with the momentum $\boldsymbol{p}'$ of the outgoing Ξ (see Fig. 1), explicit expressions for $\Omega_{j'}^J$ are found as

$$\Omega_{x'}^{\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{B15a}$$

$$\Omega_{y'}^{\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \tag{B15b}$$

$$\Omega_{z'}^{\frac{3}{2}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B15c}$$

which were derived with the help of the spin-3/2 generators in their spinor representation,

$$J_1 = \frac{\sqrt{3}}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{2}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{B16a}$$

$$J_2 = i\frac{\sqrt{3}}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & \frac{-2}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{B16b}$$

$$J_3 = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{B16c}$$

For arbitrary spin of Ξ, K_{ba} of Eq. (B13) becomes

$$K_{ba} = \frac{\Sigma_{ba}^{\text{even}} - \Sigma_{ba}^{\text{odd}}}{\Sigma_{ba}^{\text{even}} + \Sigma_{ba}^{\text{odd}}}, \tag{B17}$$

where

$$\Sigma_{ba}^{\text{even/odd}} = \sum_{m_a,m_b}\frac{d\sigma_{m_b,m_a}}{d\Omega}, \quad m_a - m_b = \text{even/odd}, \tag{B18}$$

denotes the sum of all polarized differential cross sections such that the differences of all possible combinations of initial and final spin projections m_a and m_b along $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$, respectively, are an even or an odd number.

APPENDIX C: EXPLICIT FORM OF THE SDMs

In this section, we list the SDM elements of each ρ^i ($i = 0, \ldots, 3$) in terms of the helicity amplitudes, $\mathcal{H}_i$. The Hermitian ρ^i matrices are arranged according to

$$\hat{\rho}^i = \begin{pmatrix} \rho_{J,J}^i & \rho_{J,J-1}^i & \cdots & \rho_{J,-J}^i \\ \rho_{J-1,J}^i & \rho_{J-1,J-1}^i & \cdots & \rho_{J-1,-J}^i \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{-J,J}^i & \rho_{-J,J-1}^i & \cdots & \rho_{-J,-J}^i \end{pmatrix}. \tag{C1}$$

For $J = \frac{1}{2}$, the matrices read explicitly

$$\hat{\rho}^0 = \frac{1}{2}\begin{pmatrix} |\mathcal{H}_1|^2 + |\mathcal{H}_2|^2 & 2i\pi_\Xi\,\text{Im}[\mathcal{H}_2\mathcal{H}_1^*] \\ -2i\pi_\Xi\,\text{Im}[\mathcal{H}_2\mathcal{H}_1^*] & |\mathcal{H}_1|^2 + |\mathcal{H}_2|^2 \end{pmatrix}, \tag{C2a}$$

$$\hat{\rho}^1 = \frac{1}{2}\begin{pmatrix} 2\,\text{Re}[\mathcal{H}_2\mathcal{H}_1^*] & \pi_\Xi(|\mathcal{H}_1|^2 - |\mathcal{H}_2|^2) \\ \pi_\Xi(|\mathcal{H}_1|^2 - |\mathcal{H}_2|^2) & -2\,\text{Re}[\mathcal{H}_2\mathcal{H}_1^*] \end{pmatrix}, \tag{C2b}$$

$$\hat{\rho}^2 = \frac{1}{2}\begin{pmatrix} -2\,\mathrm{Im}[\mathcal{H}_2\mathcal{H}_1^*] & -i\pi_\Xi(|\mathcal{H}_1|^2 + |\mathcal{H}_2|^2) \\ i\pi_\Xi(|\mathcal{H}_1|^2 + |\mathcal{H}_2|^2) & -2\,\mathrm{Im}[\mathcal{H}_2\mathcal{H}_1^*] \end{pmatrix},$$

(C2c)

$$\hat{\rho}^3 = \frac{1}{2}\begin{pmatrix} |\mathcal{H}_1|^2 - |\mathcal{H}_2|^2 & -2\pi_\Xi\,\mathrm{Re}[\mathcal{H}_2\mathcal{H}_1^*] \\ -2\pi_\Xi\,\mathrm{Re}[\mathcal{H}_2\mathcal{H}_1^*] & -|\mathcal{H}_1|^2 + |\mathcal{H}_2|^2 \end{pmatrix},$$

(C2d)

where π_Ξ is the parity of the Ξ.

For a $J = \frac{3}{2}$ resonance,

$$\hat{\rho}^0 = \frac{1}{2}\begin{pmatrix} |\mathcal{H}_2|^2 + |\mathcal{H}_1|^2 & \mathcal{H}_2\mathcal{H}_4^* + \mathcal{H}_1\mathcal{H}_3^* & \pi_\Xi(\mathcal{H}_1\mathcal{H}_4^* - \mathcal{H}_2\mathcal{H}_3^*) & 2i\pi_\Xi\,\mathrm{Im}[\mathcal{H}_2\mathcal{H}_1^*] \\ \mathcal{H}_4\mathcal{H}_2^* + \mathcal{H}_3\mathcal{H}_1^* & |\mathcal{H}_4|^2 + |\mathcal{H}_3|^2 & 2i\pi_\Xi\,\mathrm{Im}[\mathcal{H}_3\mathcal{H}_4^*] & \pi_\Xi(-\mathcal{H}_3\mathcal{H}_2^* + \mathcal{H}_4\mathcal{H}_1^*) \\ \pi_\Xi(\mathcal{H}_4\mathcal{H}_1^* - \mathcal{H}_3\mathcal{H}_2^*) & -2i\pi_\Xi\,\mathrm{Im}[\mathcal{H}_3\mathcal{H}_4^*] & |\mathcal{H}_4|^2 + |\mathcal{H}_3|^2 & -\mathcal{H}_4\mathcal{H}_2^* - \mathcal{H}_3\mathcal{H}_1^* \\ -2i\pi_\Xi\,\mathrm{Im}[\mathcal{H}_2\mathcal{H}_1^*] & \pi_\Xi(-\mathcal{H}_2\mathcal{H}_3^* + \mathcal{H}_1\mathcal{H}_4^*) & -\mathcal{H}_2\mathcal{H}_4^* - \mathcal{H}_1\mathcal{H}_3^* & |\mathcal{H}_2|^2 + |\mathcal{H}_1|^2 \end{pmatrix},$$

(C3a)

$$\hat{\rho}^1 = \frac{1}{2}\begin{pmatrix} 2\,\mathrm{Re}[\mathcal{H}_2\mathcal{H}_1^*] & \mathcal{H}_1\mathcal{H}_4^* + \mathcal{H}_2\mathcal{H}_3^* & \pi_\Xi(\mathcal{H}_2\mathcal{H}_4^* - \mathcal{H}_1\mathcal{H}_3^*) & \pi_\Xi(-|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2) \\ \mathcal{H}_4\mathcal{H}_1^* + \mathcal{H}_3\mathcal{H}_2^* & 2\,\mathrm{Re}[\mathcal{H}_4\mathcal{H}_3^*] & \pi_\Xi(|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2) & \pi_\Xi(-\mathcal{H}_4\mathcal{H}_2^* + \mathcal{H}_3\mathcal{H}_1^*) \\ \pi_\Xi(\mathcal{H}_4\mathcal{H}_2^* - \mathcal{H}_3\mathcal{H}_1^*) & \pi_\Xi(|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2) & -2\,\mathrm{Re}[\mathcal{H}_4\mathcal{H}_3^*] & \mathcal{H}_3\mathcal{H}_2^* + \mathcal{H}_4\mathcal{H}_1^* \\ \pi_\Xi(-|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2) & \pi_\Xi(-\mathcal{H}_2\mathcal{H}_4^* + \mathcal{H}_1\mathcal{H}_3^*) & \mathcal{H}_2\mathcal{H}_3^* + \mathcal{H}_1\mathcal{H}_4^* & -2\,\mathrm{Re}[\mathcal{H}_2\mathcal{H}_1^*] \end{pmatrix},$$

(C3b)

$$\hat{\rho}^2 = \frac{1}{2}\begin{pmatrix} 2\,\mathrm{Im}[\mathcal{H}_1\mathcal{H}_2^*] & i(-\mathcal{H}_1\mathcal{H}_4^* + \mathcal{H}_2\mathcal{H}_3^*) & i\pi_\Xi(\mathcal{H}_2\mathcal{H}_4^* + \mathcal{H}_1\mathcal{H}_3^*) & -i\pi_\Xi(|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2) \\ i(\mathcal{H}_4\mathcal{H}_1^* - \mathcal{H}_3\mathcal{H}_2^*) & 2\,\mathrm{Im}[\mathcal{H}_3\mathcal{H}_4^*] & i\pi_\Xi(|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & -i\pi_\Xi(\mathcal{H}_4\mathcal{H}_2^* + \mathcal{H}_3\mathcal{H}_1^*) \\ -i\pi_\Xi(\mathcal{H}_4\mathcal{H}_2^* + \mathcal{H}_3\mathcal{H}_1^*) & -i\pi_\Xi(|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & 2\,\mathrm{Im}[\mathcal{H}_3\mathcal{H}_4^*] & i(\mathcal{H}_3\mathcal{H}_2^* - \mathcal{H}_4\mathcal{H}_1^*) \\ i\pi_\Xi(|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2) & i\pi_\Xi(\mathcal{H}_2\mathcal{H}_4^* + \mathcal{H}_1\mathcal{H}_3^*) & i(-\mathcal{H}_2\mathcal{H}_3^* + \mathcal{H}_1\mathcal{H}_4^*) & 2\,\mathrm{Im}[\mathcal{H}_1\mathcal{H}_2^*] \end{pmatrix},$$

(C3c)

$$\hat{\rho}^3 = \frac{1}{2}\begin{pmatrix} -|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2 & -\mathcal{H}_2\mathcal{H}_4^* + \mathcal{H}_1\mathcal{H}_3^* & \pi_\Xi(\mathcal{H}_1\mathcal{H}_4^* + \mathcal{H}_2\mathcal{H}_3^*) & -2\pi_\Xi\,\mathrm{Re}[\mathcal{H}_2\mathcal{H}_1^*] \\ -\mathcal{H}_4\mathcal{H}_2^* + \mathcal{H}_3\mathcal{H}_1^* & -|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2 & 2\pi_\Xi\,\mathrm{Re}[\mathcal{H}_4\mathcal{H}_3^*] & \pi_\Xi(-\mathcal{H}_3\mathcal{H}_2^* - \mathcal{H}_4\mathcal{H}_1^*) \\ [\pi_\Xi(\mathcal{H}_4\mathcal{H}_1^* + \mathcal{H}_3\mathcal{H}_2^*) & 2\pi_\Xi\,\mathrm{Re}[\mathcal{H}_4\mathcal{H}_3^*] & |\mathcal{H}_4|^2 - |\mathcal{H}_3|^2 & -\mathcal{H}_4\mathcal{H}_2^* + \mathcal{H}_3\mathcal{H}_1^* \\ -2\pi_\Xi\,\mathrm{Re}[\mathcal{H}_2\mathcal{H}_1^*] & \pi_\Xi(-\mathcal{H}_2\mathcal{H}_3^* - \mathcal{H}_1\mathcal{H}_4^*) & -\mathcal{H}_2\mathcal{H}_4^* + \mathcal{H}_1\mathcal{H}_3^* & |\mathcal{H}_2|^2 - |\mathcal{H}_1|^2 \end{pmatrix}.$$

(C3d)

APPENDIX D: MEASURING THE SDM ELEMENTS

In Sec. III, we have identified a set of SDM elements that determines the reaction amplitude completely. A standard way of measuring the SDM elements is via the subsequent decay of the produced particle in a primary reaction by exploiting the self-analyzing property of the decay-product particle.

In the present work, the reaction in Eq. (1) is the primary (or production) reaction, where the Ξ baryon is produced. If the produced Ξ is a ground state Ξ, then it decays via a single weak-decay process into

$$\Xi \to \Lambda + \pi,$$

(D1)

whose associated Ξ decay-asymmetry parameters, α_Ξ, are known to be [2]

$$\alpha_{\Xi^0} = -0.406 \pm 0.013,$$

(D2a)

$$\alpha_{\Xi^-} = -0.458 \pm 0.012.$$

(D2b)

An excited Ξ resonance, Ξ^*, on the other hand, may undergo a double-decay process,

$$\Xi^* \to \Xi + \pi$$
$$\hookrightarrow \Lambda + \pi$$

(D3a)

or

$$\Xi^* \to \Lambda + \bar{K}$$
$$\hookrightarrow N + \pi.$$

(D3b)

The associated Λ decay-asymmetry parameter for the second-step process $\Lambda \to N + \pi$ is [2]

$$\alpha_{\Lambda-} = +0.642 \pm 0.013 \quad (\Lambda^0 \to p + \pi^-),$$
$$\alpha_{\Lambda 0} = +0.650 \pm 0.015 \quad (\Lambda^0 \to n + \pi^0).$$

(D4)

The Ξ production process of Eq. (1), is described in the CM frame of the reaction. The Ξ decay processes of Eqs. (D1) and (D3), on the other hand, are described in the rest frame of the produced Ξ, whose right-handed Cartesian coordinate system $\{\hat{\boldsymbol{n}}_1', \hat{\boldsymbol{n}}_2' \equiv \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3' \equiv \hat{\boldsymbol{p}}'\}$ is fully specified in Fig. 1.

In the double-decay processes, (D3), the subsequent Λ decay process is described in the rest frame of the decaying Λ denoted by $\{\hat{\boldsymbol{n}}_1'', \hat{\boldsymbol{n}}_2'', \hat{\boldsymbol{n}}_3''\}$, with $\hat{\boldsymbol{n}}_3'' \equiv \hat{\boldsymbol{p}}_\Lambda$, where $\hat{\boldsymbol{p}}_\Lambda$ describes the direction of the Λ's momentum in the $\{\hat{\boldsymbol{n}}_1', \hat{\boldsymbol{n}}_2', \hat{\boldsymbol{n}}_3'\}$ frame (see Fig. 1); the other two axes are given by $\hat{\boldsymbol{n}}_2'' = (\hat{\boldsymbol{n}}_3' \times \hat{\boldsymbol{p}}_\Lambda)/|\hat{\boldsymbol{n}}_3' \times \hat{\boldsymbol{p}}_\Lambda|$ and $\hat{\boldsymbol{n}}_1'' = \hat{\boldsymbol{n}}_2'' \times \hat{\boldsymbol{n}}_3''$.

1. Single-decay process: Ground-state Ξ

The ground-state Ξ decays weakly almost entirely into $\Lambda + \pi$. We define the amplitude describing the Ξ production process $\bar{K} + N \to K + \Xi$, followed by the subsequent weak decay of the produced Ξ, $\Xi \to \Lambda + \pi$, as [30,31]

$$A \equiv A(\Omega_\Xi, \Omega_\Lambda, \lambda_\Xi, \lambda_\Lambda, \lambda_N)$$
$$= \langle \Omega_\Lambda, \lambda_\Lambda | \hat{M}_D | \lambda_\Xi \rangle \langle \Omega_\Xi, \lambda_\Xi | \hat{M} | \lambda_N \rangle,$$

(D5)

with $\langle \Omega_\Xi, \lambda_\Xi | \hat{M} | \lambda_N \rangle$ denoting the production reaction amplitude (in the corresponding CM frame) and

$$\langle \Omega_\Lambda, \lambda_\Lambda | \hat{M}_D | \lambda_\Xi \rangle \equiv \sqrt{\frac{2J+1}{4\pi}} F_{\lambda_\Lambda}^\Xi D_{\lambda_\Xi, \lambda_\Lambda}^{J*}(\Omega_\Lambda) \qquad (D6)$$

denoting the subsequent Ξ decay amplitude (in the rest frame of Ξ). Here, Ω_Ξ and Ω_Λ are short-hand notations, respectively, for the polar and azimuthal angles of the produced Ξ in the CM frame of the production, $\Omega_\Xi = (\theta, \phi = 0)$, and for the polar and azimuthal angles of the Λ in the rest frame of the produced Ξ, $\Omega_\Lambda = (\theta_\Lambda, \phi_\Lambda)$. $\lambda_\Xi(\lambda_\Lambda)$ is the helicity of the $\Xi(\Lambda)$ in the respective frame, while J denotes the spin of the

decaying Ξ. $F_{\lambda_\Lambda}^\Xi$ stands for the helicity Ξ decay amplitude and $D_{\lambda_\Xi, \lambda_\Lambda}^J(\Omega_\Lambda)$ is the usual Wigner rotation matrix. Here, the argument Ω_Λ in $D_{\lambda_\Xi, \lambda_\Lambda}^J(\Omega_\Lambda)$ is to be understood as the set of Euler angles $\{\alpha, \beta, \gamma\}$, such that $D_{\lambda_\Xi, \lambda_\Lambda}^J(\Omega_\Lambda) \equiv D_{\lambda_\Xi, \lambda_\Lambda}^J(\alpha = \phi_\Lambda, \beta = \theta_\Lambda, \gamma = 0)$ in the conventions defined in Ref. [41].

The angular distribution of the Λ hyperon, $I(\Omega_\Lambda)$, in the $\Xi \to \Lambda + \pi$ decay (for a fixed Ξ angle Ω_Ξ) is given by

$$I(\Omega_\Lambda) = \sum_{i=0}^3 \bar{P}_i I^i(\Omega_\Lambda), \qquad (D7)$$

where

$$
\begin{aligned}
I^i(\Omega_\Lambda) &\equiv \sum_{\text{all } \lambda's} A(\Omega_\Xi, \Omega_\Lambda, \lambda_\Xi, \lambda_\Lambda, \lambda_N) \rho_{\lambda_N, \lambda_N'}^{N,i} A^*(\Omega_\Xi, \Omega_\Lambda, \lambda_\Xi', \lambda_\Lambda, \lambda_N') \\
&= \frac{(2J+1)}{4\pi} \sum_{\text{all } \lambda's} F_{\lambda_\Lambda}^\Xi F_{\lambda_\Lambda}^{\Xi*} M_{\lambda_\Xi, \lambda_N} \rho_{\lambda_N, \lambda_N'}^{N,i} M_{\lambda_\Xi', \lambda_N'}^* D_{\lambda_\Xi, \lambda_\Lambda}^{J*}(\Omega_\Lambda) D_{\lambda_\Xi', \lambda_\Lambda'}^{J}(\Omega_\Lambda) \\
&= \frac{(2J+1)}{4\pi} \sum_{\text{all } \lambda's} F_{\lambda_\Lambda}^\Xi F_{\lambda_\Lambda}^{\Xi*} \rho_{\lambda_\Xi, \lambda_\Xi'}^{\Xi,i} D_{\lambda_\Xi, \lambda_\Lambda}^{J*}(\Omega_\Lambda) D_{\lambda_\Xi', \lambda_\Lambda'}^{J}(\Omega_\Lambda).
\end{aligned} \qquad (D8)
$$

Here, $\rho_{\lambda_N, \lambda_N'}^{N,i} = \langle \lambda_N | \hat{\rho}^{N,i} | \lambda_N' \rangle$ denotes the target nucleon SDM element with $\hat{\rho}^{N,i}$ given by Eq. (B3), and Eq. (B7) was used in the last step. Also, we note that the explicit reference to the Ω_Ξ dependence of the angular distribution I^i in Eqs. (D7) and (D8) has been suppressed for the sake of simplicity of notation. The same holds for the angular distribution in Eqs. (D19) and (D20) in the next subsection.

We now define the moments, $H^i(L, M)$, of this distribution as

$$
\begin{aligned}
H^i(L, M) &\equiv \int d\Omega_\Lambda I^i(\Omega_\Lambda) D_{M,0}^L(\Omega_\Lambda) \\
&= t_{LM}^{J,i} \sum_{\lambda_\Lambda} F_{\lambda_\Lambda}^\Xi F_{\lambda_\Lambda}^{\Xi*} \langle J \lambda_\Lambda L 0 | J \lambda_\Lambda \rangle,
\end{aligned} \qquad (D9)
$$

where $d\Omega_\Lambda \equiv \sin\theta_\Lambda d\theta_\Lambda d\phi_\Lambda$. The quantity $t_{LM}^{J,i}$ here is related to the SDM elements of the Ξ by [30]

$$t_{LM}^{J,i} \equiv \sum_{\lambda_\Xi, \lambda_\Xi'} \rho_{\lambda_\Xi, \lambda_\Xi'}^{\Xi,i} \langle J \lambda_\Xi' L M | J \lambda_\Xi \rangle, \qquad (D10)$$

whose inversion produces

$$\rho_{\lambda_\Xi, \lambda_\Xi'}^{\Xi,i} = \sum_L \frac{2L+1}{2J+1} \langle J \lambda_\Xi' L M | J \lambda_\Xi \rangle t_{LM}^{J,i}, \qquad (D11)$$

where $M = \lambda_\Xi - \lambda_\Xi'$.

Introducing, further, the quantities

$$g_\pm^\Xi \equiv F_{\pm\frac{1}{2}}^\Xi F_{\pm\frac{1}{2}}^{\Xi*}, \qquad (D12)$$

we can re-express the moments in Eq. (D9) as

$$H^i(L, M) = t_{LM}^{J,i} \langle J \tfrac{1}{2} L 0 | J \tfrac{1}{2} \rangle [g_+^\Xi + (-1)^L g_-^\Xi]. \qquad (D13)$$

We note that the $g_\pm^\Xi$ introduced in Eq. (D12) are related to the Ξ decay-asymmetry parameter, α_Ξ, given in Eq. (D2) by

[42,43]

$$\frac{g_+^\Xi - g_-^\Xi}{g_+^\Xi + g_-^\Xi} = \alpha_\Xi. \qquad (D14)$$

Then, taking the ratio of the moments $H^i(L, M)$ ($i = 1, 2, 3$) and $H^0(0, 0)$, we obtain

$$\frac{H^i(L, M)}{H^0(0, 0)} = \zeta_L \frac{t_{LM}^{J,i}}{t_{00}^{J,0}} \langle J \tfrac{1}{2} L 0 | J \tfrac{1}{2} \rangle, \qquad (D15)$$

where $\zeta_L = 1$ for even L and $\zeta_L = \alpha_\Xi$ for odd L.

Now, from Eq. (D10) and the definition of $\hat{\rho}^{\Xi,0}$ in Eq. (B9), we get

$$t_{00}^{J,0} = \text{Tr}[\hat{\rho}^{\Xi,0}] = \frac{d\sigma}{d\Omega}. \qquad (D16)$$

One can now use Eq. (D15) to extract $t_{LM}^{J,i}$. Once $t_{LM}^{J,i}$ is known, the SDM elements $\rho_{\lambda_\Xi, \lambda_\Xi'}^{\Xi,i}$ are obtained by making use of Eq. (D11). Note that the nonvanishing moments $H^i(LM)$ are restricted to $L \leqslant 2J$ and $|M| \leqslant L$.

2. Double-decay process: Excited Ξ resonance

The double-decay processes shown in Eq. (D3) are treated analogously to the single-decay process in the previous subsection. Here, we discuss the decay chain with the subsequent decay of the Λ hyperon, $\Lambda \to p + \pi^-$, but the results apply to any decay chain that is a strong decay followed by a weak decay and containing a single pseudoscalar meson at each step of the decay.

As for the single-decay process case discussed in the previous subsection, we begin by defining the amplitude describing the Ξ^* production process, $\bar{K} + N \to K + \Xi^*$, followed by the strong decay of the produced Ξ^*, $\Xi^* \to \Lambda + K$, and the subsequent weak decay of Λ, $\Lambda \to N + \pi$,

as [30,31]

$$A \equiv A(\Omega_\Xi, \Omega_\Lambda, \Omega_p, \lambda_N, \lambda_\Xi, \lambda_\Lambda, \lambda_p)$$
$$= \langle \Omega_p, \lambda_p | \hat{M}'_D | \lambda_\Lambda \rangle \langle \Omega_\Lambda, \lambda_\Lambda | \hat{M}_D | \lambda_\Xi \rangle \langle \Omega_\Xi, \lambda_\Xi | \hat{M} | \lambda_N \rangle, \tag{D17}$$

where $\langle \Omega_\Xi, \lambda_\Xi | \hat{M} | \lambda_N \rangle$ stands for the Ξ^* production amplitude and

$$\langle \Omega_\Lambda, \lambda_\Lambda | \hat{M}_D | \lambda_\Xi \rangle \equiv \sqrt{\frac{2J+1}{4\pi}} F_{\lambda_\Lambda}^\Xi D_{\lambda_\Xi, \lambda_\Lambda}^{J*}(\Omega_\Lambda), \tag{D18a}$$

$$\langle \Omega_p, \lambda_p | \hat{M}'_D | \lambda_\Lambda \rangle \equiv \sqrt{\frac{2}{4\pi}} F_{\lambda_p}^\Lambda D_{\lambda_\Lambda, \lambda_p}^{\frac{1}{2}*}(\Omega_p) \tag{D18b}$$

denote the subsequent Ξ^* strong-decay and Λ weak-decay amplitudes, respectively. We note that the Ξ production and

$$I^i(\Omega_\Lambda, \Omega_p) \equiv \sum_{\text{all } \lambda's} A(\Omega_\Xi, \Omega_\Lambda, \Omega_p, \lambda_N, \Lambda_\Xi, \lambda_\Lambda, \lambda_p) \rho_{\lambda_N, \lambda'_N}^{N,i} A^*(\Omega_\Xi, \Omega_\Lambda, \Omega_p, \lambda'_N, \lambda'_\Xi, \lambda'_\Lambda, \lambda_p)$$
$$= \frac{2(2J+1)}{16\pi^2} \sum_{\text{all } \lambda's} \rho_{\lambda_\Xi, \lambda'_\Xi}^{\Xi,i} g_{\lambda_p}^\Lambda g_{\lambda_\Lambda, \lambda'_\Lambda}^\Xi D_{\lambda_\Lambda, \lambda_p}^{\frac{1}{2}*}(\Omega_p) D_{\lambda'_\Lambda, \lambda_p}^{\frac{1}{2}}(\Omega_p) D_{\lambda_\Xi, \lambda_\Lambda}^{J*}(\Omega_\Lambda) D_{\lambda'_\Xi, \lambda'_\Lambda}^{J}(\Omega_\Lambda), \tag{D20}$$

with

$$g_{\lambda_p}^\Lambda \equiv F_{\lambda_p}^\Lambda F_{\lambda_p}^{\Lambda*} \quad \text{and} \quad g_{\lambda_\Lambda, \lambda'_\Lambda}^\Xi \equiv F_{\lambda_\Lambda}^\Xi F_{\lambda'_\Lambda}^{\Xi*}. \tag{D21}$$

To arrive at the last equality in Eq. (D20), we have made use of Eq. (B7).

We now define the moments $H^i(l, m, L, M)$ as

$$H^i(l, m, L, M) \equiv \int d\Omega_\Lambda d\Omega_p I^i(\Omega_\Lambda, \Omega_p) D_{M,m}^L(\Omega_\Lambda) D_{m,0}^l(\Omega_p)$$
$$= t_{LM}^{J,i} \sum_{\lambda_\Lambda, \lambda'_\Lambda} g_{\lambda_\Lambda, \lambda'_\Lambda}^\Xi \langle J \lambda'_\Lambda L m | J \lambda_\Lambda \rangle \langle \tfrac{1}{2} \lambda'_\Lambda l m | \tfrac{1}{2} \lambda_\Lambda \rangle \sum_{\lambda_p} g_{\lambda_p}^\Lambda \langle \tfrac{1}{2} \lambda_p l 0 | \tfrac{1}{2} \lambda_p \rangle, \tag{D22}$$

with $t_{LM}^{J,i}$ given by Eq. (D10).

The different $g_{\lambda_\Lambda, \lambda'_\Lambda}^\Xi$ are related to each other by

$$g_{--}^\Xi = g_{++}^\Xi, \tag{D23a}$$

$$g_{+-}^\Xi = g_{-+}^\Xi = \pi_\Xi (-1)^{J+\frac{1}{2}} g_{++}^\Xi. \tag{D23b}$$

The $g_{\lambda_p}^\Lambda$ terms can be related to the Λ decay-asymmetry parameter, α_Λ, by [42,43]

$$\frac{g_+^\Lambda - g_-^\Lambda}{g_+^\Lambda + g_-^\Lambda} = \alpha_\Lambda. \tag{D24}$$

Note that the nonvanishing moments $H^i(l, m, L, M)$ are restricted to $|m| \leqslant l$, $l \leqslant 1$, $|M| \leqslant L$, and $L \leqslant 2J$, as can be read off from Eq. (D22). The moments $H^i(0, 0, L, M)$

decay amplitudes are calculated in the CM frame of the production reaction and the rest frame of the produced Ξ, respectively, in exactly the same way as for the single-decay case discussed in the previous subsection. The subsequent Λ decay amplitude, $\langle \Omega_p, \lambda_p | \hat{M}'_D | \lambda_\Lambda \rangle$, is calculated in the rest frame of the decaying Λ denoted $\{\hat{n}''_1, \hat{n}''_2, \hat{n}''_3\}$ [cf. the second paragraph following Eq. (D4)], where $\Omega_p = (\theta_p, \phi_p)$ is short-hand notation for the polar and azimuthal angles θ_p and ϕ_p, respectively, of the decay-product proton measured in the Λ rest frame.

The angular distribution of the entire double-decay process (for a fixed Ξ production angle Ω_Ξ) is given as

$$I(\Omega_\Lambda, \Omega_p) = \sum_{i=0}^3 \bar{P}_i I^i(\Omega_\Lambda, \Omega_p), \tag{D19}$$

where

and $H^i(1, m, L, M)$ vanish identically for odd and even L, respectively, due to Eqs. (D22) and (D23). Analogously to the single-decay case, the ratios of the moments

$$\frac{H^i(0, 0, L, M)}{H^0(0, 0, 0, 0)} = \frac{t_{LM}^{J,i}}{t_{00}^{J,0}} \langle J \tfrac{1}{2} L 0 | J \tfrac{1}{2} \rangle \tag{D25a}$$

for even L and

$$\frac{H^i(1, 0, L, M)}{H^0(0, 0, 0, 0)} = \frac{\alpha_\Lambda}{3} \frac{t_{LM}^{J,i}}{t_{00}^{J,0}} \langle J \tfrac{1}{2} L 0 | J \tfrac{1}{2} \rangle \tag{D25b}$$

for odd L allow us to determine $t_{LM}^{J,i}$. Since $t_{00}^{J,0} = d\sigma/d\Omega$, once $t_{LM}^{J,i}$ is extracted, the SDM elements $\rho_{\lambda_\Xi, \lambda'_\Xi}^{\Xi,i}$ can be determined via Eq. (D11).

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