

Inelastic versus total nucleon-nucleon cross section at large N_c

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In this paper, the implication of the extreme large N_c limit for nucleon-nucleon cross section is examined. Starting from the N_c scaling of the S -matrix elements, a relation between the total inelastic and total cross sections is derived for the regime in which the momenta are much larger than the QCD scale. A conceptual complication arises from the fact that there is a tower of baryons, such as the Δ , with mass splittings from the nucleon which go to zero at large N_c . Since these baryons are stable at large N_c , the meaning of elasticity must be modified. Processes which only transform nucleons to these baryons are considered elastic; processes in which at least one additional meson is produced are considered inelastic. It is shown that in the extreme large N_c limit, the total inelastic cross section is exactly one eighth of the total cross section. In contrast, in the physical world of $N_c = 3$, the total cross section for high-energy nucleon-nucleon scattering is dominated by inelastic processes.

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I. INTRODUCTION

The approach to quantum chromodynamics (QCD) based on an expansion around the limit of an infinite number of colors has attracted wide interest since its introduction in the 1970s [1,2]. Its attractiveness stems from the fact that many aspects of the QCD simplify in the limiting case of $N_c \rightarrow \infty$. Moreover, to the extent that the physical world of $N_c = 3$ is close to the infinitely colorful world of $N_c \rightarrow \infty$, one can obtain useful phenomenological predictions following this approach. Many qualitative and semiquantitative predictions are possible since certain quantities are analytically calculable in the leading order of the $1/N_c$ expansion. This is remarkable since at present there is no known way to solve QCD even at leading order in the ordinary expansion in powers of the coupling constant. For example, it can be shown that an emergent spin-flavor symmetry [3–8] arises for certain baryon observables at large N_c , and this symmetry allows for predictions of *ratios* of various physical quantities with corrections which are known to be of relative order $1/N_c$ or in favorable cases $1/N_c^2$ [7,9]. The predictions of these tend to describe the physical world remarkably well [10].

Recently, there has been interest in the implications of the large N_c limit on nucleon-nucleon scattering. The spin-flavor dependence of the inclusive differential cross section, that is, the cross section for the nucleons to emerge at a fixed angle with any number of mesons produced, was deduced in Ref. [11] for the kinematic regime in which the incident momenta are large compared to Λ_{QCD} (or more precisely for the regime of Witten kinematics [2] in which the incident velocity of the nucleons is held fixed when N_c is taken large). Since the mass of the nucleon is proportional to N_c , this automatically yields kinematics with large momenta, even large compared to the Λ_{QCD} scale. The analysis of Ref. [11] follows from the emergent symmetry discussed above. It was argued in Ref. [12] that the total cross section should follow the same spin-flavor dependence as the inclusive differential cross section.

The development of an analytic formula for the total nucleon-nucleon cross section in the extreme large N_c limit [in which $\ln(N_c) \gg 1$] in the Witten kinematics, which was done in Ref. [13], represents an important theoretical breakthrough. The result is

$$\sigma_{\text{tot}} = \frac{2\pi}{m_\pi^2} \ln^2(N_c). \quad (1)$$

A general analysis of the assumptions underlying the derivation of Eq. (1) was presented in Ref. [14]; the key issue in that work is the N_c scaling of the real and imaginary parts of nucleon-nucleon S -matrix elements. It was argued that the proper quantity to focus on is the logarithm of the S matrix. A number of arguments strongly suggest that both the real and imaginary parts of the logarithm of the S -matrix elements are proportional to N_c . This work is a followup to these papers; a new interesting result is derived determining a fraction of cross section corresponding to inelastic processes, again restricted to Witten kinematics and to the extreme large N_c limit. In particular, we find the following relation to hold:

$$\frac{\sigma_{\text{in}}}{\sigma_{\text{tot}}} = \frac{1}{8}. \quad (2)$$

There are two important caveats to this result. The first is that at large N_c , there is a tower of baryons with spin equal to isospin which is degenerate with the nucleon, more specifically, with the mass splitting proportional to $1/N_c$. Thus, reactions in which nucleons only transform to other members of this degenerate multiplet of states and *no* other particles (mesons) are emitted are considered elastic at large N_c . This definition of elasticity is natural at large N_c . It is important to note that only a few lowest states in the baryon multiplet are relevant in the presented analysis and therefore we do not need to worry about the fact that the mass splitting is proportional to J^2/N_c and may become non-negligible as J becomes large (comparable to N_c). The reason for this is the fact that in order to excite a baryon by $I = J = 1$, a pion exchange must occur; excitation of a nucleon to a state with large $I = J$ requires also a large number of pion exchanges and therefore occurs at much shorter distance not affecting the

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total cross sections. The second caveat is that, like Eq. (1), this formula only holds in the extreme large N_c limit in which $\ln(N_c) \gg 1$. This is not a computational problem, but rather a problem if one thinks about phenomenological relevance of obtained relations. Note that the world of $N_c = 3$ is far from this limit. Thus, the result (while of considerable theoretical interest) should not be expected to hold even approximately at $N_c = 3$. Indeed, it does not.

II. CROSS SECTION CALCULATION

First, let us briefly summarize the basic properties of nucleon physics in the large N_c limit, and their consequences for our analysis. We also use this opportunity to set the notation used in this paper. According to the standard N_c scaling rules [2], the mass of the nucleon is proportional to N_c . The Witten kinematic regime, which has a smooth large N_c limit, is the one for fixed velocity; thus, the nucleon momentum is also of order N_c . The self-consistency of a Schrödinger-type equation in this regime requires also the strength of the interaction to be of order N_c [15,16] while the range of the interaction is to be kept finite. In summary,

$$M = N_c \tilde{M}, \quad k = N_c \tilde{k}, \quad V(r) = N_c \tilde{V}(r). \quad (3)$$

Characters with a tilde indicate the variables with the N_c scaling removed; plain characters stand for the full quantities (this notation is used throughout the paper). It directly follows from Eq. (3) that the angular momentum l is also proportional to N_c if one considers the semiclassical limit, in which the impact parameter b is held fixed. Recall that the semiclassical limit is appropriate at large N_c and Witten kinematics [13,14].

The key theoretical quantity determining the scattering process is the elastic S matrix, where the meaning of elastic is defined as above. The incident and final states depend on the spin and flavor configurations of the baryons; let us label one such configuration by A . The two-body states also depend on the angular momentum l . Thus, a two-body elastic S -matrix element is specified by $S_{A,l,A'l'}^{\text{elastic}}$. We note that in the collision of nucleons, the total angular momentum j rather than orbital one l is conserved; so, the S matrix is not diagonal in l . This technicality, however, does not alter the leading N_c behavior discussed in this work.

The limit in which we are interested is $N_c \rightarrow \infty$. This limit considered in the kinematic regime specified by scaling (3) corresponds to the semiclassical approximation to quantum mechanics. Of course, the actual problem of interest, the nucleon-nucleon scattering at large momentum, is inherently a field-theoretical problem; it is possible to create other particles (mainly mesons). However, this can be captured in a quantum mechanical language by considering a potential with nonzero imaginary part which is matched to the results of field theory. On the one hand, the imaginary part leads to a violation of unitarity and a loss of flux; on the other hand, it is precisely what one expects during inelastic processes, i.e., the flux is lost from an elastic channel into other channels (for example, a state with nucleons and pions). Since we are not interested in the specifics of inelastic channels, the use of the quantum mechanics with a complex potential is

sufficient. The appropriateness of this approach was discussed more extensively in Ref. [14].

The starting point of our discussion is the observation that the logarithm of the S matrix for the elastic processes is proportional to N_c in the large N_c limit [14] for k , l , and l' of order N_c . A precise way to state this is that eigenvalues of the elastic S matrix can be written as real and imaginary phase shifts. For our purposes, it is useful to write the j th eigenvalue of the S matrix $s_j(\tilde{k})$ in terms of the respective phase shifts:

$$\begin{aligned} s_j(\tilde{k}) &= \exp(-2\delta_j^{\text{Im}}(\tilde{k}) + 2i\delta_j^{\text{Re}}(\tilde{k})) \\ &= \exp(-2N_c \tilde{\delta}_j^{\text{Im}}(\tilde{k}) + 2iN_c \tilde{\delta}_j^{\text{Re}}(\tilde{k})), \end{aligned} \quad (4)$$

where the indices Im and Re indicate the imaginary and real parts of the phase shift, respectively. The imaginary part of the phase shift parametrizes the loss of flux into inelastic channels.

The quantity of interest in this work is the inelastic cross section. We consider the case in which the initial nucleons are in some configuration that we label A . In a partial wave expansion, the general expression for inelastic scattering is given by

$$\begin{aligned} \sigma_{\text{in}}^A &= \frac{\pi}{k^2} \sum_l (2l+1) \left(1 - \sum_{l',A'} |S_{l,A'l'}^{\text{elastic}}|^2 \right) \\ &\approx \frac{\pi}{k^2} \int dl^2 \left(1 - \sum_{l',A'} |S_{l,A'l'}^{\text{elastic}}|^2 \right), \end{aligned} \quad (5)$$

where the second form is valid in the regime where many partial waves contribute, as they do at large N_c in Witten kinematics. Inserting Eq. (4) and changing variables to an integration over an impact parameter $b = l/k$ yields

$$\sigma_{\text{in}}^A = \pi \int db^2 \left(1 - \sum_j |v_{A,N_c,b\tilde{k}}^j|^2 e^{-4N_c \tilde{\delta}_j^{\text{Im}}(\tilde{k})} \right), \quad (6)$$

where $v_{A,N_c,b\tilde{k}}^j$ is the $A, l = N_c \tilde{l}$ component of the j th normalized eigenvector of the elastic S matrix. Since they are normalized,

$$\sum_j |v_{A,N_c,b\tilde{k}}^j|^2 = 1. \quad (7)$$

In the formal large N_c limit, the factor in parentheses in Eq. (6) is equal to one for all values of b . Since the region of integration in b^2 extends to infinity, this implies that the inelastic cross section diverges at large N_c . In a sense, this is not surprising since the classical cross section is always infinite unless the potential has a strictly finite support, and the large N_c regime corresponds to the semiclassical one. The interesting question is as to how the cross section diverges. We know that, for any finite value of N_c , the cross sections are finite. The reason for this concerns an ordering of limits. At sufficiently large impact parameter b , the effective absorptive potential goes to zero, while at sufficiently large N_c it goes to infinity. Thus, for any b at sufficiently large N_c , the factor in parentheses in Eq. (6) is arbitrarily close to unity. However, for any finite N_c at sufficiently large b , it is arbitrarily close to zero. Thus, at very large but finite N_c , one expects the factor

to be essentially unity for the region of small b and essentially zero when $b \rightarrow \infty$ with a crossover somewhere in-between.

For now, let us suppose that the crossover regime is narrow, that is, the characteristic width of the crossover region is parametrically small compared to size of the region for which the factor in parentheses in Eq. (6) is nearly unity. In that case, the factor can be modeled by a Heaviside step function $\theta(b^2 - b_0^2)$ where b_0 parametrizes the location where the crossover takes place: the system looks like a black-disk scattering. Thus,

$$\sigma_{\text{in}}^A \approx \pi b_0^2. \quad (8)$$

It is easy to see that this does in fact happen and that $b_0 \approx \ln(N_c)/(2m_\pi)$ [13,14].

The key point is that the overall strength of the effective absorptive potential scales as N_c and its longest-ranged contribution is of two-pion range [14]; it causes an exponential dropoff which goes as $\exp(-2m_\pi b)$ times some subexponential function. As was shown in Ref. [14], when the imaginary part of the potential is of order N_c , then the scaling of Eq. (4) holds. On the other hand, when the scale of the imaginary potential is small, the Born approximation for the partial wave is valid and the imaginary part of the phase shift is proportional to the imaginary potential itself. Clearly, at large N_c , the transition region is determined by the longest-range part of the interaction since shorter-range contributions will exponentiate away, leaving the interaction strength parametrically of order N_c .

The factor $2m_\pi$ in the exponential dropoff is crucial for the difference between total cross section derived in Ref. [13] and inelastic cross section derived here leading to Eq. (2); let us therefore briefly review the origin of this factor (for a more thorough explanation, we refer the reader to Ref. [14]). In a conventional meson-exchange model, the interaction between baryons is mediated via mesons. Since we are interested only in total (inelastic) cross section, only the longest-range contribution, in other words lightest intermediate meson, is relevant. At large N_c , just as at $N_c = 3$, the lightest mesons are pions. Elastic scattering is dominated by one-pion exchange, however, it is kinematically forbidden to create a new meson when only one-meson exchange is considered. The first order where inelasticity appears is the two-pion exchange; therefore, the exponential dropoff is dominated by a factor of $2m_\pi$. Recall that we are interested only in the longest-range part of the interaction and therefore we can neglect the exchange of other possible mesons (pion and eta, for example).

Returning to the formula for the inelastic cross section, one expects the transition region to be centered on a value of $b = b_0$ for which the effective absorptive potential is of order N_c^0 since in this region the phase shift is neither parametrically large yielding the factor in parentheses in Eq. (6) to be effectively unity nor small yielding it close to zero. Since the dominant scaling at large b goes exponentially, one expects b_0 , the center of the transition region, to satisfy

$$N_c \exp(-2m_\pi b_0) \sim 1. \quad (9)$$

To make this statement more concrete, we will solve for b_0 using $N_c \exp(-2m_\pi b_0) = c$, where c is an arbitrary constant of order unity. This yields $b_0 = -\ln(c)/(2m_\pi) + \ln(N_c)/(2m_\pi)$.

Consider the extreme large N_c limit in which $\ln(N_c) \gg 1$. In that case, $\ln(N_c)/(2m_\pi)$ dominates parametrically over $\ln(c)/(2m_\pi)$ and b_0 is well approximated by $\ln(N_c)/(2m_\pi)$.

What is left to be shown is that the width of the crossover region is parametrically small compared to b_0 and thus the approximation using Heaviside step function in Eq. (6) is justified. This is clearly true in the extreme large N_c limit since the width of the crossover region is of order N_c^0 . To see this, note that the transition region can be defined as a region in which the potential (for a given finite value of N_c) goes from a very large number of order unity (yielding large absorption and black-disk behavior) to a very small number of order unity (yielding essentially no absorption). This happens over a few e-folds since the functional form of the potential is exponential; one e-fold is characterized by changes of b of order $1/(2m_\pi)$, which is independent of N_c . Thus, the ratio of the width of the crossover region to b_0 scales as $1/\ln(N_c)$. In the extreme large N_c limit, this is negligible.

Thus, we see that

$$\sigma_{\text{in}}^A = \frac{\pi}{4m_\pi^2} \ln^2(N_c) \quad (10)$$

for any initial spin-flavor state A . Comparing with Eq. (1) immediately yields Eq. (2). From the derivation it should be apparent that corrections to Eq. (2) are expected to be of relative order $1/\ln(N_c)$.

It is easy to understand why the ratio of the inelastic to the total cross section in Eq. (2) is $1/8$. Note that the characteristic range of the inelastic scattering is $2m_\pi$ while for the elastic scattering it is m_π ; the longest-range elastic interaction is dominated by one-pion exchange whereas inelastic channels emerge at the level of two-pion exchange. Hence, the effective b_0 for elastic scattering is twice that for the inelastic one. Since the cross section depends on b_0^2 , a factor of 4 emerges in the difference between the total (1) and inelastic (10) cross sections. There is an additional factor of 2 coming from extremely forward diffractive scattering which is purely elastic and has no classical analog.

III. DISCUSSION

Naturally, the question of the phenomenological relevance of the obtained results arises since the world is not $N_c = \infty$ but rather $N_c = 3$. The total cross section predicted by Eq. (1) is $\sigma_{\text{tot}} \approx 150$ mb, which is 50% more than the recent observation from LHC [17] at $\sqrt{s} = 7$ TeV: $\sigma_{\text{tot}}^{\text{EXP}} \approx 100$ mb. Considering the crudeness of the extreme large N_c limit, the prediction for the total cross section is in the right ballpark. The situation for the inelastic-to-total ratio is much worse. The total cross section at high energies is dominated by the inelastic processes; at LHC, the ratio is approximately $3/4$. It is in complete disagreement with Eq. (2) since it predicts the dominant portion of the total cross section to be elastic; the fraction of the inelastic cross section is expected to be only $1/8$. The theoretical result does not match the observation even qualitatively. The obvious discrepancy between the large N_c results and the observation is, however, not that surprising. Formally, the relations (1), (2), and (10) hold in the extreme

large N_c limit, specifically for $\ln(N_c) \gg 1$, which is clearly not the case for $N_c = 3$. However, regardless of the lack of immediate phenomenological significance, the ability to calculate the total cross section as well as its inelastic component analytically is very interesting from the point of view of theoretical analysis.

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