Field theory of the $d + t \to n + \alpha$ reaction dominated by a ^{5}He* unstable particle

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(Received 9 August 2013; published 31 January 2014)

An effective, nonrelativistic quantum field theory for the $dt \to n\alpha$ fusion reaction in the low-energy, resonance region is presented. The theory assumes that the reaction is dominated by an intermediate ^{5}He* unstable spin-$3/2^+$ resonance. It involves two parameters in the coupling of the dt and $n\alpha$ particles to the unstable resonant state and the resonance energy level—only three real parameters in all. All Coulomb corrections to this process are computed. The resultant field theory is exactly solvable and provides an excellent description of the dt fusion process.

DOI: 10.1103/PhysRevC.89.014622 PACS number(s): 24.30.−v, 25.40.−h, 03.70.+k, 11.10.−z

I. INTRODUCTION

A. Motivation and purpose

In this paper, we examine the reaction

$$d + t \to n + \alpha \tag{1.1}$$

from an effective field theory point of view. We employ modern techniques of many-body, nonrelativistic quantum field theory[1] to describe this reaction and also make use of the contemporary ideas of effective field theory. In the modern effective field theory approach, stable nuclei (which are treated as particles) and resonant nuclear states (which are treated as unstable particles) are described by individual fields. The fields that correspond to asymptotic states produce particles when they act on the vacuum (no-particle) state. But fields that correspond to resonances have no corresponding single-particle states.[2] For the reaction that we consider in this paper, we shall assume that only a single intermediate resonant state, corresponding to a spin-$3/2^+$ ^{5}He*, is needed. Thus we shall have creation and annihilation fields for this unstable intermediate resonance as well as such fields for deuteron, triton, alpha, and neutron particles.

The effective field theory is a generalization of the pseudopotential method introduced by Fermi [2] in 1936 for low-energy neutron scattering on molecules. Fermi used a $\delta(\mathbf{r})$ function taken in the Born approximation. The constant multiplying the δ function was chosen to give the correct scattering length on a nucleus. The use of a field to describe a composite nucleus was done as early as 1973 by Schwinger [3] when he described the deuteron and used this description to rederive the effective range formula for the deuteron's form factor and for its photodisintegration cross section. The modern use of field theory methods in nuclear physics was advocated by Weinberg [4] in 1990. The review of Epelbaum, Hammer, and Meißner [5] describes the application of effective quantum field theory methods to nuclear problems in detail and contains an extensive list of references to work in this area.

The effective field theory may be viewed as the simplest mathematical method to implement a "black box" description of nuclear reactions at low energies. This is a theoretical description that uses the fewest number of parameters. If the process involves a resonant intermediate state, then an unstable field is needed in addition to the fields that describe the propagation of the initial and final particles. As the energy of the reacting particles is increased, additional parameters must be included that correspond to coupling constants for field interactions involving spatial gradients, which correspond to interactions that give higher momentum dependence in the reaction amplitudes. The number of parameters required increases rapidly with increasing energy.

Here we are concerned with reactions in the low-energy limit but with a resonant intermediate state, the ^{5}He* state. This introduces three parameters: two constants g_{dt} and $g_{n\alpha}$ for the coupling of the dt and $n\alpha$ fields to the unstable ^{5}He* field and the resonant energy of this unstable field.

A traditional method to compute coupled-channels nuclear reactions is to use R-matrix theory. This theory entails nuclear channel radii as well as excited-state energies and channel couplings. The subsequent companion paper [6] describes the one-level R-matrix theory for the two dt and $n\alpha$ channels. This paper explains in detail how the zero-channel radii limit of this R-matrix theory is precisely the result (1.12) below for the effective field theory with the coupling to the unstable ^{5}He* particle.

There are several motivations for this work. It provides a detailed example of how effective field theory methods work for a nontrivial example that involves a higher spin unstable field. Coulomb corrections appear not only in the initial state but also in the unstable field's self-energy involving the dt loop. Field theory methods of angular-momentum coupling simplify the computation. The result provides a very accurate description of the dt fusion reaction that involves only three real parameters. Our simple description may be employed in calculations of plasma screening effects that employ field theory methods and thus requires a field theory of the fusion process.[3]

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[1]These methods are explained in detail in, for example, the first two chapters of Ref. [1].

[2]Fields describing unstable, resonance particles are described at some length in Sec. 6.3 of Ref. [1].

[3]This was the initial motivation for our work on this theory. A preliminary sketch of the field theory method of computing nuclear fusion rates in a plasma appears in Ref. [7].

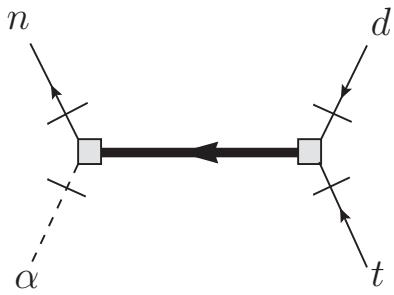

FIG. 1. Graphical representation of the $dt \to n\alpha$ transition amplitude. The thick directed line is the interacting ^{5}He* Green's function. Solid, directed lines are nonzero spin particles, and the dashed line is the spin-zero α particle. Shaded boxes are $g_{n\alpha}$ and g_{dt} couplings. The dt shaded box includes the Coulomb wave function factor $\psi^{(C)}_{\mathbf{p}_{dt}}(0)$. Hash marks on the external lines indicate that they correspond to on-shell asymptotic particles, not propagators.

B. Heuristic description of the method

Before proceeding to a description of our results followed by the details of our computations, we pause to provide a pictorial description of our work.

Figure 1 represents the $dt \to n\alpha$ fusion amplitude. The order (as also in all subsequent graphs) from right to left follows the usual convention for quantum transition amplitudes. The absolute square of this amplitude, multiplied by appropriate phase-space factors, is the fusion cross section shown in Eq. (1.6). The ^{5}He* propagator obeys the algebraic equation represented in Fig. 2, with the self-energy function denoted by the shaded region on the far right described in Fig. 3. The fact that the free-particle propagator appears with the wrong minus sign is discussed in the final paragraph of the following Results subsection and also in detail in the subsequent companion paper [6]. In Fig. 3, the $n\alpha$ self-energy function is presented explicitly in Eq. (3.27). The sum of all the dt graphs is explicitly given by Eq. (4.17). The contribution of these self-energy functions appears in the absolute square of the inverse ^{5}He* Green's function (1.8).

C. Results

Since the paper contains lengthy calculations, it is worthwhile to present the dt fusion result here before plunging into all the details, including Coulomb corrections. A major effect of these is provided by the familiar Gamow barrier penetration factor for the initial charged deuteron and triton particles. It is the square of the Coulomb wave function evaluated at zero

FIG. 2. Diagrammatic structure of the momentum space algebraic equation for the interacting ^{5}He* Green's function. The thick, directed line represents the interacting Green's function with all its self-energy corrections. The double line stands for the wrong-sign free-particle propagator. The shaded region immediately to the right of the free propagator represents the $n\alpha$ and dt self-energies contained in $\Sigma(W)$.

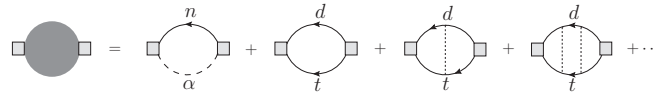

FIG. 3. Self-energy of the ^{5}He* Green's function, including the instantaneous Coulomb interaction. The first graph stands for the $n\alpha$ self-energy function. The infinite set of terms (as indicated by the final ellipsis) include all the corrections to the dt self-energy function due to instantaneous Coulomb exchanges between the charged d and t particles.

particle separation, $\psi^{(C)}_{\mathbf{p}_{dt}}(0)$,

$$\left|\psi^{(C)}_{\mathbf{p}_{dt}}(0)\right|^2 = \frac{2\pi\eta}{\exp\{2\pi\eta\} - 1}, \tag{1.2}$$

in which for the deuteron and triton, each with a single electron charge e, in ordinary cgs units (but with the $\hbar = 1$ convention that we usually follow),

$$\eta = \frac{e^2}{v_{dt}}, \tag{1.3}$$

where v_{dt} is the relative velocity of the deuteron and triton. We use m_{ab} to denote the reduced mass of a pair of particles a,b. So the dt momentum in the center-of-mass (c.m.) system is $p_{dt} = m_{dt}\, v_{dt}$, with the energy of this relative motion in the center-of-mass system given by

$$E = \frac{1}{2} m_{dt}\, v_{dt}^2 = \frac{\mathbf{p}_{dt}^2}{2m_{dt}}. \tag{1.4}$$

With our convention for the zero point of the energy W in the center-of-mass system,

$$\frac{\mathbf{p}_{dt}^2}{2m_{dt}} = W + \epsilon_d + \epsilon_t, \tag{1.5}$$

in which ϵ_d and ϵ_t are the deuteron and triton binding energies. Thus, at a dt threshold where the dt relative momentum vanishes, $W = -\epsilon_d - \epsilon_t < 0$. In our approximation in which the particles interact only with an unstable intermediate field, in addition to the external propagation barrier penetration, the only other Coulomb corrections are to the dt "bubble graphs" that appear in the unstable ^{5}He* propagator. The inclusion of all Coulomb effects is detailed in the work leading to Eq. (4.5), which reads:

$$\sigma_{dt\to n\alpha} = \frac{8}{9} 4\pi m_{n\alpha} \frac{p_{n\alpha}^5}{v_{dt}} \frac{g_{dt}^2}{4\pi} \frac{g_{n\alpha}^2}{4\pi} \left|\psi^{(C)}_{\mathbf{p}_{dt}}(0)\right|^2 |G^{(C)}_*(W)|^2. \tag{1.6}$$

Here $p_{n\alpha}$ is the relative momentum in the center-of-mass frame of the produced n,α particles. By energy conservation, it is given by

$$\frac{p_{n\alpha}^2}{2m_{n\alpha}} = \frac{p_{dt}^2}{2m_{dt}} + Q\,, \tag{1.7}$$

where $Q \simeq 17.59$ MeV is the energy release of the reaction. Since the $n\alpha$ pair is produced in a D wave, the amplitude contains a factor of $p_{n\alpha}^2$ and the squared amplitude $p_{n\alpha}^4$. Phase space of the produced particles gives an additional factor of

$p_{n\alpha}$, so an overall factor of $p_{n\alpha}^5$ appears. The couplings of the initial d,t fields and the final n,α fields to the unstable ^{5}He* field are denoted by g_{dt} and $g_{n\alpha}$.

The unstable interacting Green's function that appears in the fusion cross section is given by

$$|G_*^{(C)}(W)|^{-2} = \left[\frac{p_{dt}^2}{2m_{dt}} - E_* - \frac{g_{dt}^2}{4\pi}\,\Delta(W)\right]^2 + \left[\frac{g_{dt}^2}{4\pi}\,2\,m_{dt}\,p_{dt}\,\left|\psi_{p_{dt}}^{(C)}(0)\right|^2 + \frac{g_{n\alpha}^2}{4\pi}\,\frac{2}{3}\,m_{n\alpha}\,p_{n\alpha}^5\right]^2. \tag{1.8}$$

This is the function given in Eq. (4.27) whose derivation and description precedes Eq. (4.27). The energy E_* along with the coupling parameters g_{dt}^2 and $g_{n\alpha}^2$ are determined by fitting the $dt \to n\alpha$ fusion cross section.

The function Δ is a Coulomb-modified dt loop function that is given by [see Eq. (4.19)]

$$\Delta(W) = \frac{4m_{dt}}{b_0}\,[\mathrm{Re}\,\psi(i\eta) - \ln\eta]\,, \tag{1.9}$$

where $b_0 = 1/e^2\,m_{dt} = 24.04\,\mathrm{fm}$ and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Γ function. Although the peak in the fusion cross section, or the maximum of the modified astrophysical factor $\overline{S}_{dt\to n\alpha}$ shown in Fig. 4, are determined by E_*, the positions of these are not directly related to the value of E_* since there are shifts brought about by the Coulomb self-energy correction $\Delta(W)$ and by the variation of the factors that involve p_{dt} and $p_{n\alpha}$.

The astrophysical S factor is conventionally defined by

$$S = E\,e^{2\pi\eta}\,\sigma. \tag{1.10}$$

The multiplication by $E = mv^2/2$ removes the two factors of $1/v$ that naturally appear in the cross section: the $1/v$ arising from the division of the reaction rate by the incident flux and the $1/v$ factor that appears in the overall η factor in the squared Coulomb wave function (1.2). The factor $\exp\{2\pi\eta\}$ removes the major exponential factor (the factor which appears in the Gamow barrier penetration approximation) in the squared Coulomb wave function (1.2). We prefer, however, to use a slightly modified astrophysical $\overline{S}$ factor that we define by

$$\overline{S}_{dt\to n\alpha} = \frac{p_{dt}^2}{\hbar^2}\,[e^{2\pi\eta} - 1]\,\sigma_{dt\to n\alpha} = \frac{2\,m_{dt}}{\hbar^2}\,E\,[e^{2\pi\eta} - 1]\,\sigma_{dt\to n\alpha}. \tag{1.11}$$

Here we have multiplied by p^2 rather than by $E = p^2/2m$ because this makes $\overline{S}$ dimensionless.[4] Moreover, we have multiplied by $[\exp\{2\pi\eta\} - 1]$ rather than by only the Gamow barrier penetration factor $\exp\{2\pi\eta\}$ to remove the complete energy dependence of the squared Coulomb wave function.[5]

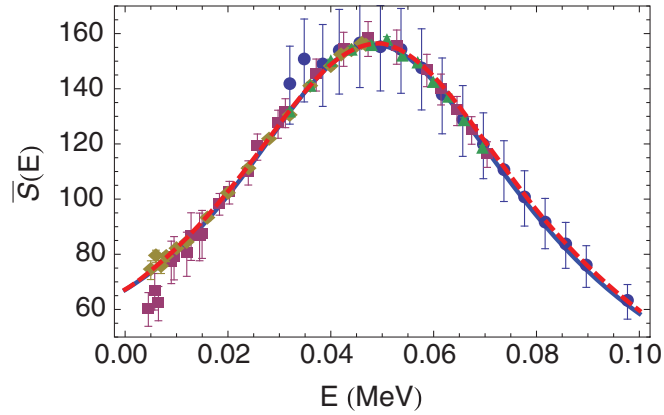

FIG. 4. (Color online) Dimensionless version of the astrophysical factor $\overline{S}_{dt\to n\alpha}$ determined by the definition (1.11) for the dt reaction compared with the experimental data as a function of the deuteron center-of-mass energy E. The solid (blue) curve is the best fit of the simple effective field theory result (1.12). It has a χ^2 per degree of freedom of 0.784. The dashed (red) curve is based on the cross section of Bosch and Hale [8]. The multilevel, multichannel R-matrix analysis of the ^{5}He system on which the Bosch and Hale cross sections are based includes data for $n\alpha$ and dt elastic scattering, in addition to those for the associated inelastic reactions, at energies equivalent to a laboratory deuteron energy up to 11 MeV. It fits the 2665 experimental data points included using 117 free parameters with a χ^2 per degree of freedom of 1.56. The (magenta) squares are the data of Arnold *et al.* [9]; the (olive) diamonds are the data of Jarmie *et al.* [10] renormalized by a factor of 1.017; the (green) triangles are the relative data of Brown *et al.* [11] renormalized by a factor of 1.025. The necessity of these renormalizations of the experimental data is discussed in the text. The (blue) circles are the older data of Argo *et al.* [12], which we show for completeness but which we do not use in our fit.

In terms of this notation, our result becomes

$$\overline{S}_{dt\to n\alpha} = \frac{8}{9}\,4\pi\,m_{dt}\,m_{n\alpha}\,p_{n\alpha}^5\,\frac{g_{dt}^2}{4\pi}\,\frac{g_{n\alpha}^2}{4\pi}\,\frac{2\pi}{b_0}\,|G_*^{(C)}(W)|^2. \tag{1.12}$$

A fit of this result to the data reduced to construct $\overline{S}_{dt\to n\alpha}$ is presented in Fig. 4. The fit to the dt fusion cross section with our formula gives the parameter values

$$E_* = -154 \pm 8\ \mathrm{keV},\qquad \frac{g_{dt}^2}{4\pi} = 199 \pm 8\ \mathrm{fm}^3\,\mathrm{MeV}^2,\qquad \frac{g_{n\alpha}^2}{4\pi} = 16.4 \pm 1.0\ \mathrm{fm}^7\,\mathrm{MeV}^2. \tag{1.13}$$

The early cross-section measurements [9,12] used to determine the parameters of the fit were reported with rather

[4]We have previously displayed the $\hbar$ factors explicitly to emphasize that we are multiplying by a wave number squared, $(p/\hbar)^2 \sim (\mathrm{length})^{-2}$, although in general we use quantum units in which $\hbar = 1$.

[5]We are interested in the energy range $0 < E < 300$ keV with includes the resonance at $E \simeq 50$ keV. The change from S to $\overline{S}$ is of relative order $\exp\{-2\pi\eta\}$ and increases as the energy increases. At $E = 300$ keV, the change is about 10%.

large uncertainties (typically ∼10%), which combined relative and normalization (scale) uncertainties. However, in the more recent measurement of Jarmie *et al*. [10], the relative errors were much smaller (∼0.5%) and were reported separately from the larger scale uncertainty of 1.26%. The subsequent measurement of Brown *et al*. [11] likewise had small relative errors, but no absolute normalization was determined in this experiment. For the purpose of reporting the data, Brown *et al*. determined an approximate scale by matching in the region of overlap to the earlier absolute measurement of Jarmie *et al*.

When fitting these data in the comprehensive ^{5}He R-matrix analysis that was used to produce the reaction cross sections of Bosch and Hale [8], separate normalization parameters were allowed to vary for each data set, the one for the Jarmie data being constrained in the total χ^2 by its 1.26% uncertainty and the one for the Brown *et al*. data unconstrained, since it was purely a relative measurement. The values of the renormalization factors found from that analysis, 1.017 for Jarmie *et al*. [10] and 1.025 for Brown *et al*. [11], were applied to the experimental data sets (cross sections and uncertainties) prior to performing the more limited fitting of effective field theory result $\bar{S}_{dt\to n\alpha}$ [Eq. (1.12)] over the resonance described here.[6]

Our result, which entails only three parameters, fits the data very well. To achieve this, it is necessary to start with a free-particle Lagrangian for the unstable ^{5}He* field with the "wrong" sign. This would not be acceptable if the theory were taken to be more fundamental with an extended region of validity rather than a effective theory whose applicability is only to the low-energy regime. It is easy to show that the simple theory with two initial spin-zero particles which interact via an intermediate (s-channel) field (the simple scalar-particle analog of our theory) produces an effective range formula with a negative effective range parameter [6]. A positive effective range parameter is achieved in this theory if the intermediate field has a wrong-sign free-particle Lagrangian. Thus the restricted validity of this simple effective field theory should be acceptable just as is that of the effective range theory.[7]

[6]We also tried letting the normalizations on these data sets float in this latter analysis, and they varied from the values given above by about 0.25%, well within the expected variance of these numbers, so there was no need to employ this different scale.

[7]Kaplan [13] has obtained a good fit to the neutron-proton singlet S-wave scattering phase shift out to a laboratory energy of 340 MeV with a theory that contains pion exchange, a local point (contact) interaction, and an s-channel intermediate field with a wrong-sign free-particle Lagrangian. He was motivated to use this minus sign because of his analysis in Sec. II of the quantum-mechanical delta-shell potential which led to his Eq. (2.11). Schwinger [3] described the deuteron as an effective field and derived the effective range formula for the neutron-proton triplet S-wave scattering as well as the corresponding approximation for the deuteron form factor and the low-energy deuteron photodisintegration. A careful reading of Ref. [3] reveals that, hidden in the nonrelativistic reduction of a relativistic theory that involves a wave function renormalization for the deuteron field, the resulting free-particle deuteron propagator corresponds to a wrong-sign Lagrangian, a sign change brought about by a negative sign in the wave-function renormalization.

D. Outline

Section II A explains our conventions for the fields and their free-particle Lagrange functions. Section II B defines the interaction Lagrange functions for the coupling of the initial dt and the final $n\alpha$ to the unstable ^{5}He* field and the appropriate spin-orbit combination of the $n\alpha$ fields that enter into their interaction.

Section III describes the dynamics of our theory in the absence of the Coulomb interactions. This is done in some detail because this underlying theory, which involves higher spin fields, has some complexity, and it clarifies the development to proceed with simpler stages. Section III A presents the calculation of the self-energy functions for the unstable ^{5}He* propagator using dimensional continuation to define their needed regularization and express the intermediate expressions in term of quantum-mechanical transformation functions that simplifies the subsequent computation of Coulomb corrections. Section III B describes our result for the $dt \to n\alpha$ fusion cross section in the absence of Coulomb interactions.

Section IV displays the Coulomb corrections to the fusion process. The dt particles initially interact at a point, thereby bringing about a factor of the squared Coulomb wave function at the origin $|\psi^{(C)}(0)|^2$. The dt piece of the resonant-state propagator also has Coulomb corrections. Using the formalism developed in Sec. III A, these corrections become expressed in a dispersion relation form that is a representation of the logarithmic derivative of the Γ function $\psi(z)$.

After a brief summary of our work in the concluding Sec. V, unsuccessful approaches to avoid the introduction of the wrong-sign ^{5}He* free propagator are mentioned. We then note how extensions of the effective field theory method to multichannel descriptions of light nuclear reactions might be obtained without great effort.

Appendix A contains a short account of Galilean invariance that provides results needed in the text. Efficient quantum field theory methods that couple spins are explained in Appendix B. The theory of Coulomb corrections is discussed in Appendix C.

II. EFFECTIVE FIELD THEORY: INGREDIENTS

A. Fields and kinematics

As discussed in the Introduction, each particle in our reaction system is described by creation and annihilation fields. The free-field part of the Lagrange function for each of these fields has the generic form

$$\mathcal{L}^{(0)}_{\text{A}} = \chi^\dagger_{\text{A}}\, i\,\frac{\partial}{\partial t}\,\chi_{\text{A}} - \mathcal{H}^{(0)}_{\text{A}}, \tag{2.1}$$

with

$$\mathcal{H}^{(0)}_{\text{A}} = \chi^\dagger_{\text{A}}\left[\frac{-\nabla^2}{2m_{\text{A}}} - \epsilon_{\text{A}}\right]\chi_{\text{A}}. \tag{2.2}$$

As shown in Appendix A, Galilean invariance requires that the inertial mass m_{A} of a composite nucleus is the sum of the masses of the neutrons and protons of which it is composed. We have written $\epsilon_{\text{A}} > 0$ for the binding energy of the composite

TABLE I. Bosonic fields and their properties.

Particle	Spin	Operators	Mass	Binding
Alpha	0^+	$\phi_\alpha^\dagger(\mathbf{r},t)$, $\phi_\alpha(\mathbf{r},t)$	$m_\alpha = 2m_p + 2m_n$	ϵ_α
Deuteron	1^+	$\boldsymbol{\phi}_d^\dagger(\mathbf{r},t)$, $\boldsymbol{\phi}_d(\mathbf{r},t)$	$m_d = m_p + m_n$	ϵ_d

TABLE II. Fermionic fields and their properties.

Particle	Spin	Operators	Mass	Binding
Neutron	$\frac{1}{2}^+$	$\psi_n^\dagger(\mathbf{r},t)$, $\psi_n(\mathbf{r},t)$	m_n	$\epsilon_n \equiv 0$
Triton	$\frac{1}{2}^+$	$\psi_t^\dagger(\mathbf{r},t)$, $\psi_t(\mathbf{r},t)$	$m_t = m_p + 2m_n$	ϵ_t
^{5}He*	$\frac{3}{2}^+$	$\boldsymbol{\psi}_*^\dagger(\mathbf{r},t)$, $\boldsymbol{\psi}_*(\mathbf{r},t)$	$m_* = 2m_p + 3m_n$	ϵ_*

particle A. The total free particle Hamiltonian,

$$H^{(0)} = \sum\nolimits_{\text{A}} \int (d^3\mathbf{r})\, \mathcal{H}_{\text{A}}^{(0)}, \tag{2.3}$$

measures the energy of an asymptotic state where all the particles are separated by large distances.

In view of the structure of the total Hamiltonian (2.3) with the pieces (2.2), the total energy E_{ab} of a pair a,b of stable particles separated by large distances is

$$E_{ab} = \frac{\mathbf{p}_a^2}{2m_a} - \epsilon_a + \frac{\mathbf{p}_b^2}{2m_b} - \epsilon_b = \frac{\mathbf{P}_{ba}^2}{2M_{ab}} + \frac{\mathbf{p}_{ba}^2}{2m_{ab}} - \epsilon_a - \epsilon_b, \tag{2.4}$$

in which M_{ab} and m_{ab} are the total and reduced masses of the a,b system; $\mathbf{P}_{ba}$ and $\mathbf{p}_{ba}$ are the total and relative momenta. The energy W_{ab} in the center-of-mass system is the Galilean invariant

$$W_{ab} = E_{ab} - \frac{\mathbf{P}_{ba}^2}{2M_{ab}} = \frac{\mathbf{p}_{ba}^2}{2m_{ab}} - \epsilon_a - \epsilon_b. \tag{2.5}$$

Our description of the $dt \to n\alpha$ reaction will employ only the initial deuteron (d), neutron (n), and the final triton (t), alpha (α) particles, and a single unstable ^{5}He* nucleus. The corresponding bosonic fields and their properties, spin-parity, masses, and binding energies, are listed in Table I. Note that we conveniently describe the deuteron spin states as a vector representing "linear polarization"; the usual $J_z' = m = \{\pm 1,0\}$ states are the $\{(x \pm iy)/\sqrt{2}, z\}$ components of this vector. Properties of the fermionic fields are given in Table II.

As explained in Appendix B, the condition that the unstable ^{5}He* field carry only spin $3/2$ (with no additional spin-$1/2$ piece) can be conveyed in the requirement that this vector-spinor field obeys[8]

$$\boldsymbol{\sigma} \cdot \boldsymbol{\psi}_*(\mathbf{r},t) = 0 = \boldsymbol{\psi}_*^\dagger(\mathbf{r},t) \cdot \boldsymbol{\sigma}, \tag{2.6}$$

in which $\boldsymbol{\sigma}$ are the Pauli spin matrices.

The unstable, resonant ^{5}He* state has a "binding" energy ϵ_* that is negative so it can decay into a deuteron plus a triton. The conservation of total energy W in the center-of-mass system for the dt fusion reaction gives

$$\frac{\mathbf{p}_{dt}^2}{2m_{dt}} - \epsilon_d - \epsilon_t = \frac{\mathbf{p}_{n\alpha}^2}{2m_{n\alpha}} - \epsilon_n - \epsilon_\alpha. \tag{2.7}$$

At threshold, $\mathbf{p}_{dt} = 0$, and the produced n,α pair has a kinetic energy $\mathbf{p}_{n\alpha}^2/2m_{n\alpha} = Q$. Here Q is the conventional notation for the energy liberated by the reaction. Since by our convention $\epsilon_n = 0$,

$$Q = \epsilon_\alpha - \epsilon_d - \epsilon_t \simeq 17.59 \text{ MeV}. \tag{2.8}$$

B. Unstable particle interactions

As discussed in the Introduction, the interaction Lagrange function describes the coupling of the reacting particles to an intermediate unstable field that describes a $3/2^+$ resonance ^{5}He* in the intermediate state as follows:

$$\begin{aligned}\mathcal{L}_1 = {} & g_{dt}[\boldsymbol{\psi}_*^\dagger \psi_t \cdot \boldsymbol{\phi}_d + \boldsymbol{\phi}_d{}^\dagger \cdot \psi_t^\dagger \boldsymbol{\psi}_*] \\ & + g_{n\alpha}[\boldsymbol{\psi}_*^\dagger \cdot \boldsymbol{\Psi}_{\alpha n} + \boldsymbol{\Psi}_{\alpha n}^\dagger \cdot \boldsymbol{\psi}_*]\end{aligned} \tag{2.9}$$

Here the dt field pair contains spin $1/2$ as well as spin $3/2$. However, the coupling of this pair to the unstable particle field with spin $3/2$ projects out only the spin-$3/2$ part of the dt pair. The coupling of the unstable particle field to the αn pair is more complicated since, as discussed in detail in Appendix B, it involves an internal D-wave angular momentum in this pair. This $l = 2$ internal angular momentum combines with the spin $1/2$ in the neutron to produce the spin-$3/2^+$ field $\boldsymbol{\Psi}_{\alpha n}$. As explained in Appendix B, this field is given by

$$\Psi_{\alpha n}^l(\mathbf{r},t) = \phi_\alpha(\mathbf{r},t)\, \mathcal{T}_{\alpha n}^{lm}\, \sigma^m\, \psi_n(\mathbf{r},t), \tag{2.10}$$

in which a sum over repeated vector or tensor indices is implied, and with

$$\mathcal{T}_{\alpha n}^{lm} = \mathcal{P}_{\alpha n}^l \mathcal{P}_{\alpha n}^m - \tfrac{1}{3}\delta^{lm}\, \mathcal{P}_{\alpha n}^k \mathcal{P}_{\alpha n}^k, \tag{2.11}$$

where

$$\mathcal{P}_{\alpha n}^k = \frac{m_{n\alpha}}{m_n}\frac{1}{i}\overrightarrow{\nabla}^k - \frac{m_{n\alpha}}{m_\alpha}\frac{1}{i}\overleftarrow{\nabla}^k, \tag{2.12}$$

with $m_{n\alpha}$ the reduced mass of the alpha-neutron system. The arrow over a derivative indicates whether the derivative acts to the left or to the right. As we shall see, the differential operator $\mathcal{P}_{\alpha n}^k$ reduces to the relative momenta of the $n\alpha$ pair when the reaction amplitudes are computed.

III. EFFECTIVE FIELD THEORY: DYNAMICS

The reaction amplitudes may be described by the interaction picture which involves free-field matrix elements of the time-ordered, unitary evolution operator,

$$U = \left(\exp\left\{i\int dt \int (d^3\mathbf{r})\, \mathcal{L}_1\right\}\right)_+. \tag{3.1}$$

The propagators of nonrelativistic fields are retarded functions in time: Particles are created at an earlier time and then destroyed at a later time. Hence, on expanding the interaction picture time evolution operator (3.1), it is easy to see that the

[8]This is just the nonrelativistic version of the Rarita-Schwinger description of spin-$3/2$ fields [14].

only contributions to a two-body reaction with an interaction of the form of Eq. (2.9) are as follows. Starting at an early time, the initial particle pair is destroyed and the unstable ^{5}He* particle is created. In the leading order expansion of the evolution operator, this unstable particle decays into the particles in the final state. In the next-to-leading order, the unstable particle propagates for some time and then decays into a particle pair. Each particle in this pair now propagates for another period of time until the pair is destroyed with the creation of another unstable particle. This chain of "bubbles" goes on until the final two-particle state is reached. The first terms in the expansion of the evolution operator give a single "bubble" surrounded by two unstable free-field propagators. The next set of expansion terms give two "bubbles" joined by three unstable free-field propagators, and so forth, to result in an infinite set of graphs consisting of alternating lines and "bubbles."

We consider in the next subsection, Sec. III A, the unstable particle Green's function neglecting the Coulomb interaction. This allows us to focus on the evaluation of the $n\alpha$ and dt contributions to the self-energy, $\Sigma_{n\alpha}(W)$ and $\Sigma_{dt}(W)$. These are sufficiently complex computations, even with the neglect of the instantaneous Coulomb interactions, to warrant a dedicated discussion. Then, in following Sec. IV, we include corrections due to the instantaneous Coulomb interactions for these particles.

A. Unstable particle Green's function

The unstable particle's Green's function may be expressed as

$$G_*^{lm}(\mathbf{r}-\mathbf{r}',t-t') = \int \frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{dE}{2\pi}\, e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-iE(t-t')} P_{3/2}^{lm}\, G_*(W). \tag{3.2}$$

Here $P_{3/2}^{lm}$ is the projection matrix (B18) into the spin-3/2 subspace. It is a matrix in the 2×2 spinor space and a second-rank tensor in the vector indices exhibited. Because of Galilean invariance, the scalar factor $G_*(W)$ is a function only of the energy in the center-of-mass frame

$$W = E - \frac{p^2}{2m_*}. \tag{3.3}$$

The unstable ^{5}He* inverse Green's function scalar factor has the form

$$G_*^{-1}(W) = -(W+\epsilon_*) - \Sigma(W+i\eta), \tag{3.4}$$

where, as discussed at some length in the Introduction above, the free-particle piece is taken corresponding to a wrong-sign Lagrangian. The structure of the corresponding Green's function is described by an integral equation in space-time which reduces to an algebraic equation in momentum-frequency space, as indicated by the diagram in Fig. 2. The self-energy $\Sigma(W)$ is the sum of a dt part and an $n\alpha$ part:

$$\Sigma(W) = \Sigma_{dt}(W) + \Sigma_{n\alpha}(W). \tag{3.5}$$

The self-energy functions, corresponding to dt and $n\alpha$ loop graphs, is displayed in Fig. 5.

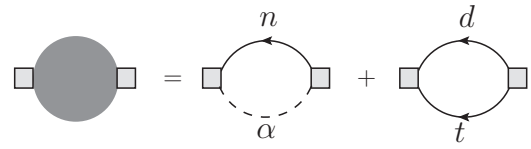

FIG. 5. The self-energy diagrams of the ^{5}He*, neglecting the instantaneous Coulomb interaction, corresponding to Eq. (3.5). Shaded boxes indicate the $g_{n\alpha}$ and g_{dt} vertices appropriate to each graph. The first loop graph on the right-hand side describes the $n\alpha$ contribution Eq. (3.27); the second loop graph corresponds to the dt contribution Eq. (3.26).

The dt contribution involves the propagator or Green's function for the spin-1 deuteron, which has the simple tensor structure $G_d^{kl} = \delta^{kl}\, G_d$, where G_d is a scalar function, and the Green's function for the triton, which is a unit matrix in the 2×2 spinor space times a scalar function G_t. The unit tensor δ^{kl} describing the deuteron spin and the unit matrix in the deuteron spin space both act as unity when acting upon the components of the unstable field Green's function. Hence, the dt loop can be written as the scalar function

$$\Sigma_{dt}(W) = i g_{dt}^2 \int (d^3\bar{\mathbf{r}})\, dt\, e^{-i\mathbf{p}\cdot\bar{\mathbf{r}}+iEt}\, G_d(\bar{\mathbf{r}},t)\, G_t(\bar{\mathbf{r}},t). \tag{3.6}$$

The scalar part of the Green's functions have the generic form

$$\begin{aligned} G(\mathbf{r}-\mathbf{r}',t-t') &= -i\langle 0|(\chi(\mathbf{r},t)\chi^\dagger(\mathbf{r}',t'))_+|0\rangle \\ &= -i\theta(t-t')\int \frac{(d^3\mathbf{p})}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} e^{-iE(p)(t-t')}, \end{aligned} \tag{3.7}$$

in which θ is the unit step function and

$$E(p) = \frac{\mathbf{p}^2}{2m} - \epsilon \tag{3.8}$$

is the energy which includes the binding energy $-\epsilon$ as well as the kinetic energy with a mass m that is the sum of the nucleon masses that make up the nucleus described by the field χ.

Such Green's functions have the structure of a time-dependent, quantum-mechanical transformation function of a free particle. It will prove convenient to write them in this form, namely as

$$G(\mathbf{r}-\mathbf{r}',t-t') = -i\langle \mathbf{r},t|\mathbf{r}',t'\rangle^{(0)}\,\theta(t-t'). \tag{3.9}$$

Here we have included the superscript to indicate that we are evaluating the free-particle transformation function with no Coulomb interactions. Later, in Sec. IV, when we turn to the Coulomb corrections, this superscript will indicate the evaluation of the transformation function in the presence of the Coulomb interaction with $(0)\to(C)$. It is useful to use this relation because it is then natural to pass to center-of-mass and relative coordinates and write

$$\begin{aligned} &G_b(\mathbf{r}_b-\mathbf{r}_b',t-t')\, G_a(\mathbf{r}_a-\mathbf{r}_a',t-t') \\ &\quad = -\langle \mathbf{R},t|\mathbf{R}',t'\rangle^{(0)}_{ba\ \text{c.m.}}\ \langle \mathbf{r},t|\mathbf{r}',t'\rangle^{(0)}_{ba\ \text{rel}}\ \theta(t-t'). \end{aligned} \tag{3.10}$$

Here, as usual,

$$\mathbf{R} = \frac{m_a \mathbf{r}_a + m_b \mathbf{r}_b}{m_a + m_b}, \quad \mathbf{r} = \mathbf{r}_b - \mathbf{r}_a, \tag{3.11}$$

are the center-of-mass and relative coordinates. The free-particle dynamics in the transformation function of the relative motion $\langle \mathbf{r},t|\mathbf{r}',0\rangle^{(0)}_{ba\ \mathrm{rel}}$ is described by the Hamiltonian

$$H_{ba\ \mathrm{rel}} = \frac{\mathbf{p}^2_{ba}}{2m_{ab}} - \epsilon_b - \epsilon_a, \tag{3.12}$$

which contains the binding energies displayed in Eq. (3.8) so as to provide the correct reference energy. We shall find that the representation (3.10) is extremely convenient when we turn to compute the Coulomb corrections to the self-energy function.[9] To return to the evaluation of the self-energy function Σ_{dt}, we note that at the coincident points $\mathbf{r}_d = \mathbf{r}_t = \bar{\mathbf{r}}$, $\mathbf{R} = \bar{\mathbf{r}}$, and we encounter

$$\int (d^3\bar{\mathbf{r}})\, e^{-i\mathbf{p}\cdot\bar{\mathbf{r}}} \langle \bar{\mathbf{r}},t|\mathbf{0},0\rangle^{(0)}_{ba\ \mathrm{c.m.}} = \exp\left\{-i\frac{\mathbf{p}^2}{2M_{ba}}\,t\right\}. \tag{3.13}$$

In the present case, $M_{ba} = m_d + m_t = 2m_p + 3m_n = m_*$. Hence, the self-energy function involves a Fourier transform in time with a single energy variable $W = E - \mathbf{p}^2/2m_*$, as must be the case in virtue of the Galilean invariance of the theory, and we have

$$\Sigma_{dt}(W) = -ig^2_{dt}\int_0^\infty dt\, e^{iWt}\langle \mathbf{0},t|\mathbf{0},0\rangle^{(0)}_{dt\ \mathrm{rel}}. \tag{3.14}$$

To evaluate the loop function that appears in the self-energy $\Sigma_{n\alpha}(w)$, we note that it entails

$$\begin{aligned}
&\langle 0|\Psi^k_{\alpha n}(\bar{\mathbf{r}},t)\Psi^l_{\alpha n}(\bar{\mathbf{r}}',t)|0\rangle \\
&\quad = \langle 0|\phi_\alpha(\bar{\mathbf{r}},t)\,\mathcal{T}^{km}_{\alpha n}\sigma^m\psi_n(\bar{\mathbf{r}},t)\psi^\dagger_n(\bar{\mathbf{r}}',t)\sigma^n\mathcal{T}^{nl}_{\alpha n}\phi^\dagger_\alpha(\bar{\mathbf{r}}',t)|0\rangle \\
&\quad = \langle \bar{\mathbf{r}},t|\bar{\mathbf{r}}',t'\rangle^{(0)}_{ba\ \mathrm{c.m.}}\sigma^m\sigma^n\left[\nabla^k\nabla^m - \tfrac{1}{3}\delta^{km}\nabla^2\right] \\
&\qquad \times \left[\nabla^n\nabla^l - \tfrac{1}{3}\delta^{nl}\nabla^2\right]\langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{ba\ \mathrm{rel}}\Big|_{\mathbf{r}=\mathbf{0}}.
\end{aligned} \tag{3.15}$$

Here we have made use of the translational invariance of the free-particle transformation function $\langle \mathbf{r},t|\mathbf{r}',0\rangle^{(0)}_{ba\ \mathrm{rel}} = \langle \mathbf{r} - \mathbf{r}',t|\mathbf{0},0\rangle^{(0)}_{ba\ \mathrm{rel}}$ to write all the derivatives on the left as shown. Since the $\mathbf{r} \to \mathbf{0}$ limit yields a rotationally invariant function whose tensor structure can only involve the unit tensor $\delta^{\cdot\cdot}$, we have

$$\begin{aligned}
&\left[\nabla^k\nabla^m - \tfrac{1}{3}\delta^{km}\nabla^2\right]\left[\nabla^n\nabla^l - \tfrac{1}{3}\delta^{nl}\nabla^2\right]\langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{ba\ \mathrm{rel}}\Big|_{\mathbf{r}=\mathbf{0}} \\
&\quad = \tfrac{1}{15}\left[\delta^{kn}\delta^{ml} + \delta^{kl}\delta^{mn} - \tfrac{2}{3}\delta^{km}\delta^{nl}\right](\nabla^2)^2\langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{ba\ \mathrm{rel}}\Big|_{\mathbf{r}=\mathbf{0}},
\end{aligned} \tag{3.16}$$

as contractions with various $\delta^{\cdot\cdot}$ establishes. Placing this result in Eq. (3.15), using the Pauli matrix composition law (B9) and the form (B18) of the spin-3/2 projection matrix, we conclude that

$$\begin{aligned}
&\langle 0|\Psi^k_{\alpha n}(\bar{\mathbf{r}},t)\,\Psi^l_{\alpha n}(\mathbf{0},0)|0\rangle \\
&\quad = \langle \bar{\mathbf{r}},t|\mathbf{0},0\rangle^{(0)}_{\alpha n\ \mathrm{c.m.}}\tfrac{1}{3}\,P^{kl}_{3/2}(\nabla^2)^2\,\langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{\alpha n\ \mathrm{rel}}\Big|_{\mathbf{r}=\mathbf{0}}.
\end{aligned} \tag{3.17}$$

The projection matrix $P^{kl}_{3/2}$ that appears here can be omitted. It either acts upon the ^{5}He* propagator that only contains spin $3/2^+$. Hence, it can be replaced by unity, and just as in the previous evaluation of the dt self-energy function, we have

$$\Sigma_{n\alpha}(W) = -ig^2_{n\alpha}\int_0^\infty dt e^{iWt}(\nabla^2)^2\langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{\alpha n\ \mathrm{rel}}\Big|_{\mathbf{r}=\mathbf{0}}. \tag{3.18}$$

To complete the computation, we need to evaluate expressions that are divergent in three spatial dimensions. These divergences can be removed by "subtractions"—by deleting the divergent pieces and replacing them by appropriate mass and wave function renormalizations. In our case, with the highly divergent $n\alpha$ piece, this subtraction method is a cumbersome method. Moreover, it is not a proper, well-defined mathematical procedure. The proper procedure is to, first, regulate the theory to make it well defined and then perform whatever renormalizations that are needed. Any regularization scheme must make the theory unphysical in some sense because if it were not, one could have a well-defined, finite theory, and one should use this new theory rather than that which one started with. One regularization scheme is that of Pauli and Villars. It produces a regularized theory with sectors that have negative probabilities until the renormalizations are made and the regularization removed. Here we shall find it very convenient to use dimensional regularization where the three spatial dimensions are continued to an arbitrary ν dimensional space, and the limit $\nu \to 3$ is performed only after all computations have been performed.

In ν spatial dimensions

$$\begin{aligned}
&\left\{\frac{1}{(\nabla^2)^2}\right\}\langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{ba\,\mathrm{rel}}\Big|_{\mathbf{r}=\mathbf{0}} \\
&\quad = e^{i(\epsilon_b+\epsilon_a)t}\int\frac{(d^\nu\mathbf{p})}{(2\pi)^\nu}\left\{\frac{1}{(\mathbf{p}^2)^2}\right\}\exp\left\{-it\,\frac{\mathbf{p}^2}{2m_{ba}}\right\} \\
&\quad = e^{i(\epsilon_b+\epsilon_a)t}\,\frac{\Omega_{\nu-1}}{(2\pi)^\nu}\int_0^\infty p^{\nu-1}dp\left\{\frac{1}{p^4}\right\}\exp\left\{-it\frac{p^2}{2m_{ba}}\right\},
\end{aligned} \tag{3.19}$$

where in the second line we have passed to hyperspherical coordinates with $\Omega_{\nu-1}$ the area of a unit $\nu-1$ sphere embedded in a ν-dimensional space. We change variables, writing explicitly $i = \exp\{\pi i/2\}$ to be able to carefully keep

[9] With Coulomb interaction present, the dt self-energy illustrated by the second loop diagram in Fig. 5 does not factor into the product of two simple d and t propagators. Instead, it involves a four-point Green's function with Coulomb interactions between the d and t fields. However, the self-energy function entails this four-point Green's function with common initial space-time coordinates and common final coordinates which has exactly the decomposition into center-of-mass and relative coordinates displayed on the right-hand side of Eq. (3.10) with the difference that the dynamics of the transformation function for the relative motion involves Coulomb forces. We shall make use of this in Sec. IV regarding Couloumb corrections.

track of the phase, to get

$$\left\{\begin{matrix}1\\(\nabla^2)^2\end{matrix}\right\}\langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{ba\ \mathrm{rel}}\Big|_{\mathbf{r}=\mathbf{0}} = e^{i(\epsilon_b+\epsilon_a)t}\ \frac{\Omega_{\nu-1}}{(2\pi)^\nu}\left(\frac{2m_{ba}}{t}\right)^{\nu/2} e^{-\nu\pi i/4}\ \frac{1}{2}\int_0^\infty u^{\nu/2}\,\frac{du}{u}\left\{\begin{matrix}1\\-(2m_{ba}/t)^2\ u^2\end{matrix}\right\} e^{-u}$$
$$= e^{i(\epsilon_b+\epsilon_a)t}\ \frac{\Omega_{\nu-1}}{(2\pi)^\nu}\left(\frac{2m_{ba}}{t}\right)^{\nu/2} e^{-\nu\pi i/4}\ \frac{1}{2}\left\{\begin{matrix}\Gamma\left(\frac{\nu}{2}\right)\\-(2m_{ba}/t)^2\ \Gamma\left(\frac{\nu}{2}+2\right)\end{matrix}\right\}. \tag{3.20}$$

We may now complete the evaluation of the one-loop self-energy functions as follows:

$$\left\{\begin{matrix}\Sigma_{dt}(W)\\ \Sigma_{n\alpha}(W)\end{matrix}\right\} = -i\int_0^\infty dt\, e^{iWt}\left\{\begin{matrix}1\\(1/3)\ (\nabla^2)^2\end{matrix}\right\}\left\{\begin{matrix}g^2_{dt}\ \langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{dt\ \mathrm{rel}}\big|_{\mathbf{r}=\mathbf{0}}\\ g^2_{n\alpha}\ \langle \mathbf{r},t|\mathbf{0},0\rangle^{(0)}_{n\alpha\ \mathrm{rel}}\big|_{\mathbf{r}=\mathbf{0}}\end{matrix}\right\}. \tag{3.21}$$

The a,b center-of-mass channel momentum is defined by

$$\frac{p^2_{ba}}{2m_{ab}} = W+\epsilon_a+\epsilon_b, \tag{3.22}$$

and so we encounter integrals of the form

$$-i\int_0^\infty dt\, e^{iwt}\left\{\begin{matrix}t^{-\nu/2}\\ t^{-((\nu/2)+2)}\end{matrix}\right\} = e^{-\pi i\nu/4}\int_0^\infty dv\, e^{-v}\left\{\begin{matrix}w^{(\nu/2)-1}\ v^{-\nu/2}\\ -w^{(\nu/2)+1}\ v^{-((\nu/2)+2)}\end{matrix}\right\} = e^{-\pi i\nu/4}\left\{\begin{matrix}w^{(\nu/2)-1}\,\Gamma(1-(\nu/2))\\ -w^{(\nu/2)+1}\,\Gamma(-1-(\nu/2))\end{matrix}\right\}, \tag{3.23}$$

in which

$$w = \left\{\begin{matrix}\frac{p^2_{dt}}{2m_{dt}}\\ \frac{p^2_{n\alpha}}{2m_{n\alpha}}\end{matrix}\right\} \tag{3.24}$$

for the two cases. Hence,

$$\left\{\begin{matrix}\Sigma_{dt}(W)\\ \Sigma_{n\alpha}(W)\end{matrix}\right\} = \frac{\Omega_{\nu-1}}{(2\pi)^\nu}\ e^{-\nu\pi i/2}\ \frac{1}{2}\left\{\begin{matrix}g^2_{dt}\ p^\nu_{dt}\ \frac{2m_{dt}}{p^2_{dt}}\,\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)\\ g^2_{n\alpha}\ (1/3)\ p^{\nu+4}_{n\alpha}\ \frac{2m_{n\alpha}}{p^2_{n\alpha}}\,\Gamma\left(\frac{\nu}{2}+2\right)\Gamma\left(1-\left(2+\frac{\nu}{2}\right)\right)\end{matrix}\right\}$$
$$= \frac{\Omega_{\nu-1}}{(2\pi)^\nu}\ e^{-\nu\pi i/2}\ \frac{1}{2}\left\{\begin{matrix}g^2_{dt}\ p^\nu_{dt}\ \frac{2m_{dt}}{p^2_{dt}}\ \frac{\pi}{\sin\pi\left(\frac{\nu}{2}\right)}\\ g^2_{n\alpha}\ (1/3)\ p^{\nu+4}_{n\alpha}\ \frac{2m_{n\alpha}}{p^2_{n\alpha}}\ \frac{\pi}{\sin\pi\left(\frac{\nu}{2}+2\right)}\end{matrix}\right\}. \tag{3.25}$$

We now find that there is no impediment to taking the limit $\nu\to 3$. This is a great advantage of the application of the dimensional continuation[10] method of regularization in our work—other methods would require the introduction of counter terms to cancel divergent quantities. We have captured the $\nu\to 3$ limit

$$\Sigma_{dt}(W) = -\frac{g^2_{dt}}{4\pi}\ 2m_{dt}\, i\ p_{dt} \tag{3.26}$$

and

$$\Sigma_{n\alpha}(W) = -\frac{1}{3}\ \frac{g^2_{n\alpha}}{4\pi}\ p^4_{n\alpha}\ 2m_{n\alpha}\, i\ p_{n\alpha}. \tag{3.27}$$

[10]The previous exposition is meant to be self-contained. If the reader needs a more extended discussion, a full, pedagogical description of the dimensional method is presented, for example, in chapters 3 and 4 of Ref. [1]. We should note that the tensor algebra used before the extension to $\nu\neq 3$ spatial dimensions that commenced in Eq. (3.19) was restricted to $\nu=3$. This use of the $\nu=3$ tensor algebra is justified because, with the dimensional continuation method, no divergence appears in the $\nu\to 3$ limit.

With these self-energy functions, the unstable particle Green's functions reads

$$G^{-1}_*(W) = -(W+\epsilon_*) + i\,\frac{g^2_{dt}}{4\pi}2m_{dt}\,p_{dt} + i\,\frac{g^2_{n\alpha}}{4\pi}\,\frac{2}{3}m_{n\alpha}\,p^5_{n\alpha}. \tag{3.28}$$

B. $dt\to n\alpha$ reaction amplitude and cross section

Expanding out the interaction picture time-ordered evolution operator (3.1) in powers of the interaction Lagrange function (2.9) and resumming the resulting bubble chains expresses the $dt\to n\alpha$ fusion amplitude as

$$T^l_{n\alpha\,dt}(\mathbf{p}_{n\alpha};\mathbf{p}_{dt}) = \mathcal{Q}^l_{n\alpha\,dt}(\mathbf{p}_{n\alpha})\tilde{T}_{n\alpha\,dt}(W), \tag{3.29}$$

with

$$\tilde{T}_{n\alpha\,dt}(W) = g_{n\alpha}\,G_*(W)\,g_{dt}, \tag{3.30}$$

as was indicated diagrammatically in Fig. 1. The kinematical structure of the reaction amplitudes has the following ingredients. The relative momentum $\mathbf{p}_{n\alpha}$ is constrained by the energy conservation equation

$$\frac{\mathbf{p}^2_{n\alpha}}{2m_{n\alpha}} = W+\epsilon_n+\epsilon_\alpha. \tag{3.31}$$

The tensor (2.11), together with σ^k, is sandwiched between the $n\alpha$ fields to form the composite field $\Psi^{\dagger l}_{\alpha n}(\mathbf{r},t)$ that initiates the reaction via the expansion of the time-ordered interaction operator (3.1). The differential operations that appear in $\Psi^{\dagger l}_{\alpha n}(\mathbf{r},t)$ become replaced by momenta in constructing the scattering amplitude, and these momenta combine to produce $\mathbf{p}_{n\alpha}$—as they must to keep the theory Galilean invariant. The construction entails the angular-momentum $l=2$ tensor

$$\mathcal{T}^{mk}(\mathbf{p}) = p^m p^k - \tfrac{1}{3}\mathbf{p}^2\delta^{mk} \tag{3.32}$$

contracted with the projection matrix $P^{kl}_{3/2}$ of the unstable ^{5}He* propagator [leaving the scalar part $G_*(W)$ displayed in the scalar amplitude (3.30)] to produce the factor shown in Eq. (3.29),

$$\mathcal{Q}^l_{n\alpha dt}(\mathbf{p}_{n\alpha}) = \sigma^m \mathcal{T}^{mk}(\mathbf{p}_{n\alpha}) P^{kl}_{3/2}. \tag{3.33}$$

The total cross section involves the solid angle integration

$$\begin{aligned}
&\int \frac{d\Omega}{4\pi} \mathcal{Q}^k_{n\alpha dt}(\mathbf{p}_{n\alpha}) \mathcal{Q}^l_{n\alpha dt}(\mathbf{p}_{n\alpha}) \\
&= \int \frac{d\Omega}{4\pi} P^{km}_{3/2} \mathcal{T}^{mn}(\mathbf{p}_{n\alpha}) \sigma^n \sigma^r \mathcal{T}^{rs}(\mathbf{p}_{n\alpha}) P^{sl}_{3/2} \\
&= \frac{1}{15}\left(\mathbf{p}^2_{n\alpha}\right)^2 \left[\delta^{mr}\delta^{ns} + \delta^{ms}\delta^{nr} - \frac{2}{3}\delta^{mn}\delta^{rs}\right] \\
&\quad \times P^{km}_{3/2}[\delta^{nr} + i\epsilon^{nrq}\sigma^q] P^{sl}_{3/2} \\
&= \frac{1}{3}\left(\mathbf{p}^2_{n\alpha}\right)^2 P^{kl}_{3/2}.
\end{aligned} \tag{3.34}$$

The calculation of the last line of Eq. (3.34) from that preceding it is facilitated by using a matrix notation and noting that the form $\vec{\sigma}\cdot\vec{\mathcal{S}}$ enters, of which $P_{3/2}$ is an eigenvector with eigenvalue $+1$.

Thus the $dt \to n\alpha$ reaction total cross section in our approximation that has an unstable, $3/2^+$ resonant intermediate ^{5}He state is given by

$$\begin{aligned}
\sigma_{dt\to n\alpha} = \frac{1}{6}\frac{m_{n\alpha}}{(2\pi)^2}\frac{p_{n\alpha}}{v_{dt}} \int d\Omega \ \mathrm{tr} \\
\times T^l_{n\alpha dt}(\mathbf{p}_{n\alpha};\mathbf{p}_{dt})^\dagger T^l_{n\alpha dt}(\mathbf{p}_{n\alpha};\mathbf{p}_{dt}),
\end{aligned} \tag{3.35}$$

where now tr denotes the trace over the spin-1/2 parts. Using the result (3.34) with the trace formula $\mathrm{tr}\, P^{ll}_{3/2} = 2(3/2)+1 = 4$ which simply counts the number of spin-3/2 states, we obtain

$$\sigma_{dt\to n\alpha} = \frac{8}{9} 4\pi m_{n\alpha} \frac{p^5_{n\alpha}}{v_{dt}} \frac{g^2_{dt}}{4\pi}\frac{g^2_{n\alpha}}{4\pi}|G_*(W)|^2. \tag{3.36}$$

The energy dependence of the cross section is best revealed if we use

$$\begin{aligned}
W &= \frac{p^2_{dt}}{2m_{dt}} - \epsilon_d - \epsilon_t \\
&= \frac{p^2_{dt}}{2m_{dt}} + Q - \epsilon_\alpha \\
&= \frac{p^2_{n\alpha}}{2m_{n\alpha}} - \epsilon_\alpha,
\end{aligned} \tag{3.37}$$

where, we recall, $\epsilon_n \equiv 0$ sets the energy scale, and the energy release in the reaction Q is given by the binding energy difference, $Q = \epsilon_\alpha - \epsilon_d - \epsilon_t$. Thus

$$\begin{aligned}
p_{n\alpha} &= \left[2m_{n\alpha}\left(\frac{p^2_{dt}}{2m_{dt}} + Q\right)\right]^{1/2} \\
&= (2m_{n\alpha}Q)^{1/2}\left[1 + \frac{p^2_{dt}}{2m_{dt}Q}\right]^{1/2},
\end{aligned} \tag{3.38}$$

where we write the second equality to emphasize that, since in the energy region of interest $p^2_{dt}/2m_{dt} \ll Q$, the momenta $p_{n\alpha}$ is nearly a constant determined by the energy release Q. Thus we write the squared unstable particle's Green's function as

$$\begin{aligned}
&|G_*(W)|^{-2} \\
&= \left[-\left(\frac{p^2_{dt}}{2m_{dt}} - E_*\right) - \mathrm{Re}\,\Sigma_{dt}(W) - \mathrm{Re}\,\Sigma_{n\alpha}(W)\right]^2 \\
&\quad + [\mathrm{Im}\,\Sigma_{dt}(W) + \mathrm{Im}\,\Sigma_{n\alpha}(W)]^2 \\
&= \left[\frac{p^2_{dt}}{2m_{dt}} - E_*\right]^2 + \left[\frac{g^2_{dt}}{2\pi}m_{dt}p_{dt} + \frac{g^2_{n\alpha}}{6\pi}m_{n\alpha}p^5_{n\alpha}\right]^2.
\end{aligned} \tag{3.39}$$

in which we write the unrenormalized energy ϵ_* in terms of the initial dt energy as

$$E_* = \epsilon_d + \epsilon_t - \epsilon_*. \tag{3.40}$$

The first equality in Eq. (3.39) is in a form that we shall shortly make use of when we take account of Coulomb corrections.

IV. COULOMB CORRECTIONS

The $dt \to n\alpha$ reaction was described by the diagram in Fig. 1, including Coulomb corrections. These Coulomb corrections make two changes to the work that was just performed. The dt entrance channel that connects these particles to the unstable, interacting ^{5}He* resonant Green's function involves a point interaction. Hence, one effect of the Coulomb force between the dt in the fusion process is to multiply the cross section by the square of the Coulomb wave function $\psi^{(C)}_{\mathbf{p}_{dt}}(0)$ at the origin. Thus the initial shaded box at the right in Fig. 1 must now contain $\psi^{(C)}_{\mathbf{p}_{dt}}(0)$ multiplying the coupling constant g_{dt}. The other effect of the Coulomb interactions is to modify the dt loop graphs in the ^{5}He* resonant Green's function by including arbitrary numbers of instantaneous Coulomb interactions as depicted in Fig. 3. These heuristic remarks are substantiated in Appendix C. Here we shall simply state and discuss the results of these Coulomb corrections.

The cross section involves the square of the amplitude and thus the square of the Coulomb wave function at the origin,

$$\left|\psi^{(C)}_{\mathbf{p}_{dt}}(0)\right|^2 = \frac{2\pi\eta\exp\{-2\pi\eta\}}{1-\exp\{-2\pi\eta\}}. \tag{4.1}$$

This is essentially the familiar Gamow barrier penetration factor. For our deuteron-triton system, each with a single

electron charge e, in ordinary cgs units,

$$\eta = \frac{e^2}{v_{dt}} = \frac{e^2 m_{dt}}{p_{dt}}. \tag{4.2}$$

It is sometimes convenient to write

$$\eta = \frac{1}{b_0 p_{dt}}, \tag{4.3}$$

with, in our units in which $\hbar = 1$, b_0 is the Bohr radius for the dt system,

$$b_0 = \frac{1}{e^2 m_{dt}} = 24.04 \text{ fm} = 0.1218 \frac{c}{\text{MeV}}, \tag{4.4}$$

where 1 fm $= 10^{-13}$ cm, and we have made use of $\hbar c = 197.33$ MeV fm in writing the last equality. In our theory, the Coulomb corrections to the intermediate-state nuclear interactions appear only in the unstable field propagator. Thus, including all the Coulomb effects, the previous fusion cross section (3.36) becomes

$$\sigma_{dt\to n\alpha} = \frac{8}{9} 4\pi m_{n\alpha} \frac{p_{n\alpha}^5}{v_{dt}} \frac{g_{dt}^2}{4\pi} \frac{g_{n\alpha}^2}{4\pi} \left|\psi^{(C)}_{\mathbf{p}_{dt}}(0)\right|^2 |G_*^{(C)}(W)|^2. \tag{4.5}$$

Here we use an additional superscript on the unstable particle's interacting Green's function $G_*^{(C)}(W)$ to note that it now includes the effect of the Coulomb interaction. The only effect of this interaction is on the previous dt loop function (3.14) that contains charged particles, with

$$\begin{aligned}\Sigma_{dt}(W) &\to \Sigma_{dt}^{(C)}(W) \\ &= -i g_{dt}^2 \int_0^\infty dt e^{iWt} \langle \mathbf{0},t|\mathbf{0},0\rangle^{(C)}_{dt\,\text{rel}},\end{aligned} \tag{4.6}$$

corresponding to the infinite series of diagrams indicated in Fig. 3 containing the dt intermediate state. Introducing a complete set of incoming wave intermediate eigenstates gives

$$\begin{aligned}\langle \mathbf{0},t|\mathbf{0},0\rangle^{(C)}_{dt\,\text{rel}} &= \int \frac{(d^3\mathbf{p}')}{(2\pi)^3} \langle \mathbf{0},t|\mathbf{p}'\text{in}\rangle^{(C)}_{dt\,\text{rel}} \langle \mathbf{p}'\text{in}|\mathbf{0},0\rangle^{(C)}_{dt\,\text{rel}} \\ &= e^{i(\epsilon_d+\epsilon_t)t} \int \frac{(d^3\mathbf{p}')}{(2\pi)^3} \exp\left\{-i\frac{p'^2}{2m_{dt}}t\right\} \left|\psi^{(C)}_{p'}(0)\right|^2,\end{aligned} \tag{4.7}$$

in which m_{dt} is the reduced mass of the dt system and $\psi^{(C)}_{p'}(0)$ is the Coulomb wave function (4.1). Using the dt relative momentum p_{dt} so the energy in the center of mass is given by

$$W + \epsilon_d + \epsilon_t = \frac{p_{dt}^2}{2m_{dt}}, \tag{4.8}$$

performing the time integration in Eq. (4.6) with the decomposition (4.7), and performing the angular part of the momentum integral produces

$$\overline{\Sigma^{(C)}}_{dt}(W) = g_{dt}^2 \frac{m_{dt}}{\pi^2} \int_0^\infty dp' \left|\psi^{(C)}_{p'}(0)\right|^2 \frac{p'^2}{p_{dt}^2 - p'^2 + i0^+}. \tag{4.9}$$

At large momenta, the Coulomb wave function approaches the free-particle limit, $p' \to \infty : \psi^{(C)}_{p'}(0) \to 1$. Hence, the integral here does not converge at large momenta. We have noted this divergence by temporarily placing an overline on the function. Previously, we dealt with this convergence problem by employing the dimensional continuation method, which simply removes this divergence in three dimensions. Here, however, we have an integrand involving the square of the Coulomb wave function and the application of the dimensional continuation method is more complex. There is no real problem here because the divergence produces an additional constant that is simply removed by an additive renormalization of the energy ϵ_*, a renormalization which we shall assume has been implicitly performed. Thus we simply subtract the asymptotic limit and replace

$$\frac{p'^2}{p_{dt}^2 - p'^2 + i0^+} \to \frac{p'^2}{p_{dt}^2 - p'^2 + i0^+} + 1 = \frac{p_{dt}^2}{p_{dt}^2 - p'^2 + i0^+}, \tag{4.10}$$

remove the overline from the self-energy function, and write

$$\Sigma_{dt}^{(C)}(W) = g_{dt}^2 \frac{m_{dt} p_{dt}^2}{\pi^2} \int_0^\infty dp' \frac{\left|\psi^{(C)}_{p'}(0)\right|^2}{p_{dt}^2 - p'^2 + i0^+}. \tag{4.11}$$

This shows explicitly that the Coulomb-corrected self-energy function $\Sigma_{dt}^{(C)}(W)$ is the boundary value of a function that is analytic in the upper half complex $p_{dt}^2/2m_{dt}$ plane. This is because it is the Fourier transform of a retarded, causal response function. We conclude that the real part of $\Sigma_{dt}^{(C)}(W)$ must accompany its imaginary part to keep the proper, complete analytic function. For physical, real energies we may use

$$\text{Im}\frac{1}{p_{dt}^2 - p'^2 + i0^+} = -\pi\delta\left(p_{dt}^2 - p'^2\right) = -\frac{\pi}{2p_{dt}}\delta(p_{dt} - p') \tag{4.12}$$

to compute the imaginary part,

$$\text{Im}\Sigma_{dt}^{(C)}(W) = -g_{dt}^2 \frac{m_{dt} p_{dt}}{2\pi} \left|\psi^{(C)}_{p_{dt}}(0)\right|^2. \tag{4.13}$$

To have an explicit representation of the real part, we use

$$\left|\psi^{(C)}_{p'}(0)\right|^2 = \frac{2\pi\eta'}{\exp\{2\pi\eta'\} - 1} \tag{4.14}$$

and change the integration variable to $t = \eta' = 1/(p'b_0)$, obtaining

$$\Sigma_{dt}^{(C)}(W) = g_{dt}^2 \frac{2m_{dt}}{\pi b_0} \int_0^\infty dt t \frac{1}{t^2 - \eta^2 + i0^+} \frac{1}{\exp\{2\pi t\} - 1}, \tag{4.15}$$

in which $\eta = 1/(p_{dt}b_0)$. This new form of the self-energy function is simply related to the ψ function—the logarithmic derivative of the Γ function—because of the integral representation[11]

$$\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty dt t \frac{1}{t^2 + z^2} \frac{1}{\exp\{2\pi t\} - 1}, \tag{4.16}$$

[11] See, for example, Sec. 1.7.2, Eq. (27), of Ref. [15] or Sec. 8.361, Eq. (3), of Ref. [16].

where it is assumed that $\mathrm{Re}z > 0$. Hence,[12]

$$\Sigma_{dt}^{(C)}(W) = -g_{dt}^2 \frac{m_{dt}}{\pi b_0}\left[\psi(i\eta) - \ln\eta - \frac{\pi}{2}i - \frac{i}{2\eta}\right], \quad (4.17)$$

which gives the real part

$$\mathrm{Re}\Sigma_{dt}^{(C)}(W) = -\frac{g_{dt}^2}{4\pi}\Delta(W), \quad (4.18)$$

where

$$\Delta(W) = \frac{4m_{dt}}{b_0}\left[\mathrm{Re}\psi(i\eta) - \ln\eta\right]. \quad (4.19)$$

We pause to record some formulas that can prove to be useful checks on numerical computations. It follows from the standard series development[13] of the ψ function that

$$\mathrm{Re}\psi(i\eta) = -\gamma + \sum_{k=1}^{\infty}\frac{1}{k}\frac{\eta^2}{k^2+\eta^2}, \quad (4.20)$$

which gives the small η behavior

$$\begin{aligned}\mathrm{Re}\psi(i\eta) &= -\gamma + \zeta(3)\eta^2 + \cdots \\ &\simeq -0.5772157 + 1.20206\eta^2. \end{aligned} \quad (4.21)$$

Moreover,

$$\mathrm{Re}\psi(i\eta) = -\gamma + \frac{\eta^2}{1+\eta^2} + \frac{1}{2}\frac{\eta^2}{4+\eta^2} + R(\eta), \quad (4.22)$$

where we have the bound

$$\begin{aligned}|R(\eta)| \leqslant B(\eta) &= \eta^2\sum_{k=3}^{\infty}\frac{1}{k^3} = \eta^2\left[\zeta(3) - 1 - \frac{1}{8}\right] \\ &= \eta^2\,[0.07706]\,. \end{aligned} \quad (4.23)$$

The large η limit[14] gives

$$\mathrm{Re}\psi(i\eta) - \ln\eta = \frac{1}{12\eta^2} - \frac{1}{120}\frac{1}{\eta^4} + \cdots. \quad (4.24)$$

The inverse unstable field Green's function now reads

$$G_*^{(C)-1}(W) = -(W+\epsilon_*) - \Sigma_{dt}^{(C)}(W) - \Sigma_{n\alpha}(W), \quad (4.25)$$

with $\Sigma_{dt}^{(C)}(W)$ the function that we have just computed, while

$$\Sigma_{n\alpha}(W) = -i\frac{2}{3}\frac{g_{n\alpha}^2}{4\pi}m_{n\alpha}p_{n\alpha}^5 \quad (4.26)$$

is as before.

The absolute square of the unstable field Green's function that is needed for the fusion cross section (4.5) is produced by the trivial notational change $\Sigma_{dt} \to \Sigma_{dt}^{(C)}$ in the formula (3.39), producing

$$\begin{aligned}|G_*^{(C)}(W)|^{-2} = &\left[\frac{p_{dt}^2}{2m_{dt}} - E_* - \frac{g_{dt}^2}{4\pi}\Delta(W)\right]^2 \\ &+\left[\frac{g_{dt}^2}{2\pi}m_{dt}p_{dt}\left|\psi_{p_{dt}}^{(C)}(0)\right|^2 + \frac{g_{n\alpha}^2}{6\pi}m_{n\alpha}p_{n\alpha}^5\right]^2, \end{aligned} \quad (4.27)$$

in which we again write the unrenormalized energy of the unstable particle in terms of the initial dt energy as

$$E_* = \epsilon_d + \epsilon_t - \epsilon_*. \quad (4.28)$$

V. CONCLUSION AND DISCUSSION

We have demonstrated that an excellent description of the $dt \to n\alpha$ reaction in the resonance region is obtained with an effective quantum field theory that entails only the interaction of the initial and final particles dt and $n\alpha$ with an unstable ^{5}He* $3/2^+$ unstable field. The fit, with a χ^2 per degree-of-freedom less than unity, is achieved with just three parameters: the energy of the ^{5}He* resonance E_* and the two coupling parameters g_{dt} and $g_{n\alpha}$. We have calculated the Coulomb corrections exactly in Sec. IV and taken into account their effect on both the incoming dt particles and on the strong interactions which transmute these particles into the final $n\alpha$ particles.

It is worthwhile noting that in a previous draft of our work, we found that the $dt \to n\alpha$ resonance could not be fit by using an intermediate $3/2^+$ unstable field which had the right sign free-particle Lagrangian. The fit was so poor that a χ^2 value to describe it is not meaningful. Roughly, if the parameters were adjusted to fit the maximum of the cross-section resonance, then the data around half maximum were about 30% above the fit. This result led us to add an additional contact interaction that coupled the initial and final particles in a $3/2^+$ state. We chose this form of the contact interaction so it would enter into the self-energy function of the $3/2^+$ field's Green's function and would be a candidate to alter the resonance shape produced by the theory. As described in Eq. (2.10), the final $n\alpha$ particles are produced in a $3/2^+$ state by the field combination $\mathbf{\Psi}_{\alpha n}$ and, hence, the contact interaction was chosen to have the form

$$\lambda[\mathbf{\Psi}_{\alpha n}^\dagger \cdot \boldsymbol{\phi}_d\psi_t + \psi_t^\dagger\boldsymbol{\phi}_d^\dagger \cdot \mathbf{\Psi}_{\alpha n}]. \quad (5.1)$$

As should be expected, the introduction of the additional free parameter λ improved the fit. However, the improvement was not significant. With the parameters again chosen to have the resonance maximum fitted, the data around half maximum were now about 20% above the fit. Therefore, we reverted to the simple, previous single interaction with an intermediate unstable field and changed the sign of its free-particle propagation to achieve the excellent description of the dt fusion reaction that is presented in this paper. The quality of this fit, about 1% deviation for most of the data points, was a dramatic improvement over the other work just discussed.

The relationship between the effective field theory applied here and the R-matrix approach is presented in the following paper [6]. It establishes an identity between the effective field

[12] After the completion of this derivation of the dt self-energy function (4.17) with Coulomb corrections, we became aware of a similar result for the proton-proton self-energy function given by Kong and Ravndal [17]. Although our two derivations share some similarities, they differ in detail.

[13] See, for example, Sec. 1.7, Eq. (3), of Ref. [15].

[14] See, for example, Sec. 1.18, Eq. (7), of Ref. [15].

theory approach and that of the R matrix, in the limit that the R-matrix channel radii go to zero. In this limit, the R-matrix parameterization provides an excellent fit of the data when generalized to allow for "unphysical" negative values of the reduced widths. These unphysical couplings are directly related to the wrong-sign free-particle Lagrange function used in the present work. This is a promising indication that carrying out a multichannel, many-resonance effective field theory description of light nuclear reaction data is possible by suitably generalizing current R-matrix methods and codes that are already in use.

ACKNOWLEDGMENTS

We are indebted to Mark Paris for his detailed comments that have improved the presentation of this paper. We are grateful to David Kaplan and Michael Birse for useful comments, some of which occurred during discussions when one of us (G.M.H.) attended a workshop at the Institute for Nuclear Theory at the University of Washington. This work was carried out under the auspices of the National Nuclear Security Administration of the US Department of Energy at Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396.

APPENDIX A: GALILEAN INVARIANCE

The consequences of translational and rotational invariance are obvious. The consequences of Galilean invariance—the invariance of the theory under boosts to a moving coordinate frame—are less obvious, although perhaps they are clear once they have been derived. Here we sketch out the derivation of some of these consequences.[15]

We denote the generator of boosts to moving frames by $\mathbf{G}$. It has the construction

$$\mathbf{G} = \mathbf{P}t - M\mathbf{R}(t), \tag{A1}$$

where, for a general set $\{\chi_a, \chi_a^\dagger\}$ of nonrelativistic fields,

$$\mathbf{P} = \sum_a \int (d^3\mathbf{r}) \chi_a^\dagger(\mathbf{r},t) \frac{1}{i} \nabla \chi_a(\mathbf{r},t) \tag{A2}$$

is the total momentum operator of the system, and

$$M\mathbf{R}(t) = \sum_a \int (d^3\mathbf{r}) \chi_a^\dagger(\mathbf{r},t) m_a \mathbf{r} \chi_a(\mathbf{r},t) \tag{A3}$$

defines a the center-of-mass operator $\mathbf{R}$, with m_a the kinematical mass of the particle, bound state, or resonance created and annihilated by the a-th field and $M = \sum_a m_a$ is the total mass of all these (perhaps quasi-) particles. The kinematical mass m_a is the mass that appears in the kinetic energy part of the Hamiltonian density $\chi_a^\dagger(-\nabla^2/2m_a)\chi_a$ for the a-th field.

The continued iteration of the infinitesimal Galilean boosts yields the unitary transformation

$$U(\mathbf{v}) = \exp\{i\mathbf{G}\cdot\mathbf{v}\} = \exp\{i\,[\mathbf{P}t - M\mathbf{R}(t)]\cdot\mathbf{v}\} \tag{A4}$$

to a frame moving with the finite velocity $\mathbf{v}$. Standard methods now show that the response of a field χ_a to a finite Galilean boost is given by

$$\begin{aligned}\chi_a(\mathbf{r},t) &\to U^{-1}(\mathbf{v})\chi_a(\mathbf{r},t)U(\mathbf{v}) \\ &= \exp\left\{-im_a\left[\mathbf{r}\cdot\mathbf{v} + \tfrac{1}{2}\mathbf{v}^2 t\right]\right\}\chi_a(\mathbf{r}+\mathbf{v}t,t).\end{aligned} \tag{A5}$$

We shall make use of three implications of this transformation.

Nucleons can be put into bound or resonant states. Hence, the effective field theory can contain interactions of the schematic form

$$H_{\rm int} \sim \int (d^3\mathbf{r}) \chi_{b_n}^\dagger \cdots \chi_{b_1}^\dagger \chi_{a_m} \cdots \chi_{a_1}. \tag{A6}$$

In view of the response of the individual fields given by Eq. (A5), Galilean invariance requires that

$$m_{b_n} + \cdots + m_{b_1} = m_{a_m} + \cdots + m_{a_1}. \tag{A7}$$

Thus, for example, the inertial mass of the deuteron is the sum of the neutron and proton masses, $m_d = m_n + m_p$, the triton mass is $m_t = 2m_n + m_p$, and the α mass is $m_\alpha = 2m_n + 2m_p$.

The second consequence of Galilean invariance that we mention is that a generic free-particle Lagrangian must be of the form

$$\begin{aligned}L_{\rm A} = \int &(d^3\mathbf{r}) \chi_{\rm A}^\dagger(\mathbf{r},t) \\ &\times \left[i\frac{\partial}{\partial t} - \frac{1}{2m_{\rm A}}\left(\frac{1}{i}\nabla\right)^2 - \epsilon_{\rm A}\right]\chi_{\rm A}(\mathbf{r},t).\end{aligned} \tag{A8}$$

Galilean invariance does not require the overall sign of the Lagrangian that we have written here. As discussed in the text, placing an overall minus sign factor in front of the free-particle Lagrange function may be useful for an effective theory that describes only low-energy processes. However, Galilean invariance requires does that the relative signs of the time and spatial derivatives must appear as they are shown here.[16]

The final application that we need involves field derivatives. To obtain Galilean invariant interactions, we need field combinations with derivatives that undergo a simple phase change under Galilean transformations. This does not happen with a single derivative of a single field. However, with a pair of fields χ_a, χ_b, we can define a derivative operation,

$$\mathcal{P}_{ba}^k = \frac{m_{ab}}{m_a}\frac{1}{i}\overset{\rightarrow k}{\nabla} - \frac{m_{ab}}{m_b}\frac{1}{i}\overset{\leftarrow k}{\nabla}, \tag{A9}$$

[15] A detailed exposition of the Galilean invariance of a nonrelativistic field theory is presented, for example, in the discussion of Problem 1 on Page 118 of Ref. [1]. The explicit solution of this problem has been done in the MIT 8.323 Second homework solutions by J. Goldstone (February, 1995), but this may not be readily accessible.

[16] In this regard, we note that the free-particle Lagrangians for the fields t_i and s_a presented in Eq. (2.3) of Ref. [5] violate Galilean invariance. This emphasizes the need for paying attention to fundamental principles when constructing effective field theories as we are doing here.

in which the $\rightarrow$ calls for the derivative to act to the right and the $\leftarrow$ calls for the derivative to act to the left. Here

$$m_{ab} = \frac{m_a m_b}{M_{ab}} \tag{A10}$$

is the reduced mass of the a,b pair with

$$M_{ab} = m_a + m_b \tag{A11}$$

the total mass. It is now easy to see that

$$U^{-1}(\mathbf{v})\chi_b(\mathbf{r},t)\mathcal{P}^k_{ba}\chi_a(\mathbf{r},t)U(\mathbf{v}) = e^{-i(m_b+m_a)[\mathbf{r}\cdot\mathbf{v}+\frac{1}{2}\mathbf{v}^2 t]}\chi_b(\mathbf{r}+\mathbf{v}t,t)\mathcal{P}^k_{ba}\chi_a(\mathbf{r}+\mathbf{v}t,t), \tag{A12}$$

which is the desired transformation law.

APPENDIX B: SPIN STRUCTURE

Here we shall describe some algebraic properties of the spin matrices that are needed in the text. We shall keep to our $\hbar = 1$ convention to simplify the notation. The action on any field χ of an infinitesimal rotation generated by the field operator angular momentum $\mathbf{J}$ is given by

$$\{\mathbf{L}+\mathbf{S}\}\,\chi(\mathbf{r},t) = [\chi(\mathbf{r},t),\mathbf{J}]\,, \tag{B1}$$

in which

$$\mathbf{L} = \mathbf{r} \times \frac{1}{i}\nabla \tag{B2}$$

are the orbital angular-momentum differential operators and $\mathbf{S}$ are the spin matrices. The structure of the rotation group is conveyed by

$$[J^k, J^l] = i\epsilon^{klm}J^m, \tag{B3}$$

where ϵ^{klm} is the completely antisymmetrical numerical tensor associated with vector cross product, with $\epsilon^{123} = 1$. This numerical tensor satisfies the "double cross" relation

$$\epsilon^{klm}\epsilon^{kpq} = \delta^{lp}\delta^{mq} - \delta^{lq}\delta^{mp}. \tag{B4}$$

Since L^k acts on the coordinates of the field while S^k acts on the (notationally suppressed) components of the fields, L^k and S^k commute among each other, while

$$[L^k, L^l] = i\epsilon^{klm}L^m \tag{B5}$$

and

$$[S^k, S^l] = i\epsilon^{klm}S^m. \tag{B6}$$

We turn now to examine the spin matrices $\mathbf{S}$ for spin s with $s = 1/2, 1, 3/2$. These matrices must obey the commutator (B6) and have the square

$$\mathbf{S}^2 = s(s+1). \tag{B7}$$

1. Spin 1/2

For spin $1/2$,

$$\boldsymbol{S} = \tfrac{1}{2}\boldsymbol{\sigma}, \tag{B8}$$

in which σ^k are the familiar Hermitian Pauli matrices that obey

$$\sigma^k\sigma^l = \delta^{kl} + i\epsilon^{klm}\sigma^m. \tag{B9}$$

The antisymmetrical part of this constraint implies that the spin-$1/2$ matrices S^k satisfy the angular-momentum commutation relation (B6) while setting $k = l$ and summing over these identified indices from 1 to 3 shows that

$$\mathbf{S}^2 = \tfrac{3}{4} = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right), \tag{B10}$$

which identifies the spin value $1/2$.

2. Spin 1

We write the spin matrices for spin 1 as

$$\boldsymbol{S} = \boldsymbol{\mathcal{S}}. \tag{B11}$$

The latter are given by

$$(\mathcal{S}^k)^{lm} = i\epsilon^{lkm}, \tag{B12}$$

since one can easily verify from the relation (B4) that the angular-momentum commutation relation (B6) holds and that

$$\boldsymbol{\mathcal{S}}^2 = 1(1+1) = 2. \tag{B13}$$

3. Spin 3/2

The combination of spin $1/2$ and spin 1 is described by the spin matrix

$$\boldsymbol{S} = \tfrac{1}{2}\boldsymbol{\sigma} + \boldsymbol{\mathcal{S}}. \tag{B14}$$

This matrix obviously obeys the angular-momentum commutator (B6). But we need a constraint to keep to the spin-$3/2$ subspace. To obtain this constraint, we note that the Pauli result (B9) and the properties of the spin-1 matrix noted above imply the characteristic equation

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}})^2 + (\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}}) - 2 = 0. \tag{B15}$$

The eigenvalues of the matrix, which we denote by a prime, must obey this characteristic matrix equation, and, hence, they are given by

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}})' = \begin{cases} -2, \\ +1. \end{cases} \tag{B16}$$

Therefore we have

$$\mathbf{S}^2 = \tfrac{3}{4} + 2 + \boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}} = \begin{cases} \frac{1}{2}\left(\frac{1}{2}+1\right), & \text{for} \quad (\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}})' = -2, \\ \frac{3}{2}\left(\frac{3}{2}+1\right), & \text{for} \quad (\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}})' = 1; \end{cases} \tag{B17}$$

the eigenvalues -2 and 1 of the matrix $\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}}$ correspond to spins $s = 1/2$ and $s = 3/2$.

We need the matrix $P_{3/2}$ that projects into the $s = 3/2$ subspace. A little computation utilizing the characteristic equation (B15) shows that

$$P_{3/2} = \tfrac{1}{3}\{\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}} + 2\} \tag{B18}$$

obeys

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}})\,P_{3/2} = P_{3/2} = P_{3/2}\,(\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}}) \tag{B19}$$

and

$$P^2_{3/2} = P_{3/2}, \tag{B20}$$

so it is indeed the correct projection matrix. Writing out the components gives

$$P^{lm}_{3/2} = \tfrac{1}{3}\{\sigma^k i\epsilon^{lkm} + 2\delta^{lm}\}. \tag{B21}$$

In view of the result (B17), a spin-3/2 field $\psi_{3/2}$ must obey the constraint

$$\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}}\psi_{3/2} = \psi_{3/2}. \tag{B22}$$

and if this constraint is obeyed, the field contains only spin 3/2. To obtain an equivalent constraint that is algebraically simpler, we note that

$$\sigma^l\,(\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}})^{lm} = -\epsilon^{lkn}\epsilon^{lkm}\sigma^n = -2\sigma^m. \tag{B23}$$

Hence, in view of Eq. (B19),

$$\sigma^l\,(\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{S}})^{lm}\,P^{mn}_{3/2} = -2\sigma^m P^{nm}_{3/2} = \sigma^m P^{nm}_{3/2}; \tag{B24}$$

whence

$$\sigma^l P^{lm}_{3/2} = 0, \tag{B25}$$

and, similarly,

$$P^{lm}_{3/2}\sigma^m = 0. \tag{B26}$$

Therefore,

$$\boldsymbol{\sigma}\cdot\boldsymbol{\psi}_{3/2}(\mathbf{r},t) = 0 = \boldsymbol{\psi}^\dagger_{3/2}(\mathbf{r},t)\cdot\boldsymbol{\sigma}, \tag{B27}$$

constrain $\boldsymbol{\psi}_{3/2}$ and $\boldsymbol{\psi}^\dagger_{3/2}$ to contain only spin 3/2.

In the text, we need the spin-3/2 combination of n,α fields that couple to the $3/2^+$ ^{5}He* resonant state. The composition of the $3/2^+$ state from the $1/2^+$ neutron and 0^+ alpha requires that the parity of the relative orbital state of the n,α pair be even, which is to say that the orbital angular momentum l be an even integer, with $l = 2$, the only possibility in virtue of the rules of angular-momentum addition. The differential operation

$$\mathcal{T}^{kl}_{\alpha n} = \mathcal{P}^k_{\alpha n}\mathcal{P}^l_{\alpha n} - \tfrac{1}{3}\delta^{kl}\mathcal{P}^m_{\alpha n}\mathcal{P}^m_{\alpha n} \tag{B28}$$

transforms as $l = 2$. Here, $\mathcal{P}^k_{\alpha n}$ is defined by Eq. (A9) and a proof akin to that with Eq. (A12) shows that

$$\Psi^l_{\alpha n}(\mathbf{r},t) = \phi_\alpha(\mathbf{r},t)\mathcal{T}^{lm}_{\alpha n}\sigma^m\psi_n(\mathbf{r},t) \tag{B29}$$

has the correct Galilean transformation law. Using the Pauli spin formula (B9) and the symmetry $\mathcal{T}^{lk}_{\alpha n} = +\mathcal{T}^{kl}_{\alpha n}$ we compute

$$\sigma^k\Psi^k_{\alpha n}(\mathbf{r},t) = \phi_\alpha(\mathbf{r},t)\delta^{kl}\mathcal{T}^{kl}_{\alpha n}\psi_n(\mathbf{r},t) = 0. \tag{B30}$$

Hence, in view of Eq. (B27) and the discussion following it, we conclude that $\Psi^k(\mathbf{r},t)$ does indeed contain only spin 3/2.

APPENDIX C: COULOMB CORRECTIONS

To place our work in context, we first briefly review the case in which there are only strong interactions with no Coulomb corrections. The four-point $n\alpha dt$ Green's function is the vacuum expectation value of a time-ordered product,

$$G_{n\alpha dt}(x'_n,x'_\alpha;x_d,x_t) = i^2\langle 0|(\psi_n(x'_n)\phi_\alpha(x'_\alpha)\phi^\dagger_d(x_d)\psi^\dagger_t(x_t))_+|0\rangle, \tag{C1}$$

in which the space time coordinate x is a short-hand notation for $\mathbf{r},t$. It has the decomposition

$$G_{n\alpha dt}(x'_n,x'_\alpha;x_d,x_t) = \int(d^4\bar{x}_n)(d^4\bar{x}_\alpha)(d^4\bar{x}_d)(d^4\bar{x}_t)G_n(x'_n-\bar{x}_n)G_\alpha(x'_\alpha-\bar{x}_\alpha) \times\Gamma_{n\alpha dt}(\bar{x}_n,\bar{x}_\alpha,\bar{x}_d,\bar{x}_t)G_d(\bar{x}_d-x_d)G_t(\bar{x}_t-x_t). \tag{C2}$$

Here $\Gamma_{n\alpha dt}(\bar{x}_n,\bar{x}_\alpha,\bar{x}_d,\bar{x}_t)$ contains only connected graphs. In our nonrelativistic theory, the single-particle propagators contain no self-energy corrections and thus have the generic form of the time-retarded functions,

$$G(x-x') = -i\theta(t-t')\int\frac{(d^3\mathbf{p})}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-iE(\mathbf{p})(t-t')} = -i\theta(t-t')\langle\mathbf{r},t|\mathbf{p}',t'\rangle^{(0)}, \tag{C3}$$

where in the second line we denote the free-particle quantum-mechanical transformation function by the superscript (0). The reaction amplitude involves the asymptotic early and late time limits $t_d = t_t \to -\infty$ and $t_n = t_\alpha \to +\infty$ and the identification of the propagator momenta $\mathbf{p}$ with the initial momenta, $\mathbf{p}_d$ and $\mathbf{p}_t$, and final momenta, $\mathbf{p}_n$ and $\mathbf{p}_\alpha$, by the appropriate Fourier transformation. Therefore, aside from conventional factors and overall δ functions of energy and momentum conservation, the reaction amplitude $T_{n\alpha dt}(\mathbf{p}_n,\mathbf{p}_\alpha;\mathbf{p}_d,\mathbf{p}_t)$ involves a Fourier transform[17] of $\Gamma_{n\alpha dt}(\bar{x}_n,\bar{x}_\alpha,\bar{x}_d,\bar{x}_t)$.

We now consider effect that Coulomb corrections have on the charged initial dt state. With Coulomb interactions present, there are Coulomb exchanges in the propagation of the initial dt particles before strong interactions operate. Thus the product of the two initial free-particle propagators $G_d(\bar{x}_d - x_d)G_t(\bar{x}_t - x_t)$ in Eq. (C2) must be replaced by the four-point dt Green's function that includes the Coulomb interaction. The general reaction amplitude involves the identification of the two initial times, $t_d = t_t$. Our theory, in which the strong interactions are represented by the unstable, s-channel intermediate field $\psi^\dagger_*$ that has a local coupling to the d and t fields, forces the identification $\bar{x}_d = \bar{x}_t$ or $\bar{\mathbf{r}}_d = \bar{\mathbf{r}}_t$ and $\bar{t}_d = \bar{t}_t$, with $\bar{t}_d > t_d$. Hence, the required four-point Coulomb Green's appears in the restricted form,

$$-G^{(C)}_{dt}(\mathbf{r}_d,\bar{t}_d,\mathbf{r}_t=\mathbf{r}_d,\bar{t}_t=\bar{t}_d;\mathbf{r}_d,t_d,\mathbf{r}_t,t_t=t_t) = \langle 0|\psi_d(\bar{\mathbf{r}}_d,\bar{t})\psi_t(\mathbf{r}_d,t_d)\psi^\dagger_t(\mathbf{r}_t,t_d)\psi^\dagger_d(\mathbf{r}_d,t_d)|0\rangle^{(C)} = \langle\bar{\mathbf{r}}_d=\bar{\mathbf{r}}_t,\bar{t}_d|\mathbf{r}_d,\mathbf{r}_t,t_d\rangle^{(C)}. \tag{C4}$$

The second line here follows from the usual nonrelativistic theory where a field operator $\psi^\dagger$ acting to the right adds a single particle to the state, and an operator ψ acting to the left adds a particle.

We have noted that the transition amplitude involves taking the spatial Fourier transform. Hence, passing to the transition

[17] Here we are sketching the nonrelativistic analog of the relativistic reduction formula that is discussed, for example, in Sec. 2 of chapter 6 of Brown [1].

amplitude involves $\langle \bar{\mathbf{r}}_d = \bar{\mathbf{r}}_t, \bar{t}_d | \mathbf{p}_d, \mathbf{p}_t, t_d \rangle^{(C)}$. On introducing relative and center-of-mass coordinates, this becomes

$$\begin{aligned} &\langle \bar{\mathbf{r}}_d = \bar{\mathbf{r}}_t, \bar{t}_d | \mathbf{p}_d, \mathbf{p}_t, t_d \rangle^{(C)} \\ &\quad = \langle \mathbf{r}_{\rm rel} = 0 | \mathbf{p}_{dt} \rangle^{(C)} \langle \mathbf{r}_d | \mathbf{P}_{dt} \rangle^{(0)} e^{-iE(\bar{t}_d - t_d)}. \end{aligned} \tag{C5}$$

Here $\mathbf{p}_{dt}$ is the relative momentum of the dt (which is the same notation as used in the text), $\mathbf{P}_{dt}$ is the total momentum of this pair of particles, and E is the total energy in the initial state. Now

$$\langle \mathbf{r}_{\rm rel} = 0 | \mathbf{p}_{dt} \rangle^{(C)} = \psi^{(C)}_{\mathbf{p}_{dt}}(0) \tag{C6}$$

is precisely the Coulomb wave function at the origin introduced in the text. Only this factor alters the Coulomb transformation function (C5) from that of a free particle,

$$\begin{aligned} &\langle \bar{\mathbf{r}}_d = \bar{\mathbf{r}}_t, \bar{t}_d | \mathbf{p}_d, \mathbf{p}_t, t_d \rangle^{(C)} \\ &\quad = \psi^{(C)}_{\mathbf{p}_{dt}}(0) \langle \bar{\mathbf{r}}_d = \bar{\mathbf{r}}_t, \bar{t}_d | \mathbf{p}_d, \mathbf{p}_t, t_d \rangle^{(0)}. \end{aligned} \tag{C7}$$

Hence, the reduction formula for the transition amplitude with Coulomb as well as strong interactions is the same as that for the amplitude with only strong interactions and no Coulomb corrections except for the overall multiplication of the $\psi^{(C)}_{\mathbf{p}_{dt}}(0)$ factor corresponding to initial Coulomb interactions previous to the first strong interaction. Of course, there are additional internal Coulomb corrections to the strong interactions. For our theory, these appear as multiple instantaneous Coulomb exchanges between the d and the t in the dt loop that contributes to the unstable ^{5}He* Green's function as shown in Fig. 3.

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