

# Universality in similarity renormalization group evolved potential matrix elements and $T$ -matrix equivalence

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We examine how the universality of two-nucleon interactions evolved using similarity renormalization group (SRG) transformations correlates with  $T$ -matrix equivalence, with the ultimate goal of gaining insight into universality for three-nucleon forces. With sufficient running of the SRG flow equations, the low-energy matrix elements of different realistic potentials evolve to a universal form. Because these potentials are fit to low-energy data, they are (approximately) phase equivalent only up to a certain energy, and we find universality in evolved potentials up to the corresponding momentum. More generally we find universality in *local* energy regions, reflecting a local decoupling by the SRG. The further requirements for universality in evolved potential matrix elements are explored using two simple alternative potentials. We see evidence that in addition to predicting the same observables, common long-range potentials (i.e., explicit pion physics) is required for universality in the potential matrix elements after SRG flow. In agreement with observations made previously for  $V_{\text{low } k}$  evolution, regions of universal potential matrix elements are restricted to where half-on-shell  $T$ -matrix equivalence holds.

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## I. INTRODUCTION

A wide variety of realistic potentials are available for the low-energy nuclear two-body problem, including both phenomenological interactions [1,2] and interactions motivated from systematic expansions, such as chiral effective field theory ( $\chi$ EFT) [3–5]. The ability of these different potentials to reproduce the same low-energy observables (e.g., see the phase shifts in Fig. 1) is one type of universality. However, this universality does not generally extend to their Hamiltonian matrix elements, which can vary drastically, reflecting the broad freedom to redefine interactions without changing  $S$ -matrix elements.

These realistic potentials lead to computational difficulties in most many-body calculations, because requiring them to reproduce elastic phase shifts up to the pion-production threshold leads to strong coupling between low- and high-momentum matrix elements. (The exceptions are  $\chi$ EFT potentials with low cutoffs and  $J$ -matrix-based inverse scattering potentials [6].) For many-body methods using basis expansions, for example, this coupling requires matrix sizes that become prohibitively large for accurate microscopic calculations of nuclei. Thus we must face the problem of restrictions to smaller Hamiltonian matrices while maintaining the accuracy of predicted observables.

To address this problem, Lee-Suzuki transformations were applied in free space to integrate out high-energy degrees of freedom and soften an initial realistic potential, generating phase-equivalent “low-momentum” or “ $V_{\text{low } k}$ ” potentials [7,8]. This can be done in one step or using a renormalization group (RG) equation for the potential [9]. Bogner and collaborators observed that a wide variety of realistic potentials have very similar low-momentum matrix elements after softening, which they termed the *model independence* of  $V_{\text{low } k}$  potentials [7,10]. The diagonal  $V_{\text{low } k}$  potential matrix elements were found to match in regions of phase equivalence of the realistic potentials while the off-diagonal matrix elements matched in

regions of half-on-shell (HOS)  $T$ -matrix equivalence [7]. They suggested that differences in the HOS  $T$ -matrix and thus the off-diagonal  $V_{\text{low } k}$  potential matrix elements occur because of different treatments of pion physics [7].

Subsequently, similarity renormalization group (SRG) unitary transformations have been used to soften nuclear potentials while preserving observables [8,11–15]. Like  $V_{\text{low } k}$  transformations, the SRG decouples high-energy from low-energy physics, allowing one to truncate the matrices above some decoupling scale [8,16,17]. Further, the low-energy matrix elements of initial realistic potentials also flow to the same form, but differ in detail from  $V_{\text{low } k}$  transformations. There is preliminary evidence that the SRG flow to common matrix elements extends to three-body forces [18,19], which are important ingredients for consistent treatments of nuclei with RG methods [15,20].

In analogy to the behavior of other Hamiltonians under RG transformations, this model independence is naturally interpreted as a flow to universality in the evolved potential matrix elements. This form of universality can have powerful consequences if it can be understood and exploited. It suggests that for low-energy problems, a broad class of starting potentials that fits data will be equally effective after evolution [21,22]. If realized for many-body forces, it may be possible to more easily construct accurate potentials (choosing operators based solely on the ease of use, then fitting constants to data), if they flow to a universal form after running the SRG.

In applications of RG to *local* quantum field theory, universality is a proven tool. When different theories are decomposed into relevant, marginal, and irrelevant interactions according to their behavior under RG flow, universality arises naturally among theories that share the same relevant and marginal local interactions. That is, if they differ only in the strength of their irrelevant couplings, RG transformations reveal the universality as the RG flow rapidly eliminates any irrelevant differences. In nonrelativistic many-body theories that employ *nonlocal* interactions, the possibility of universality in the form

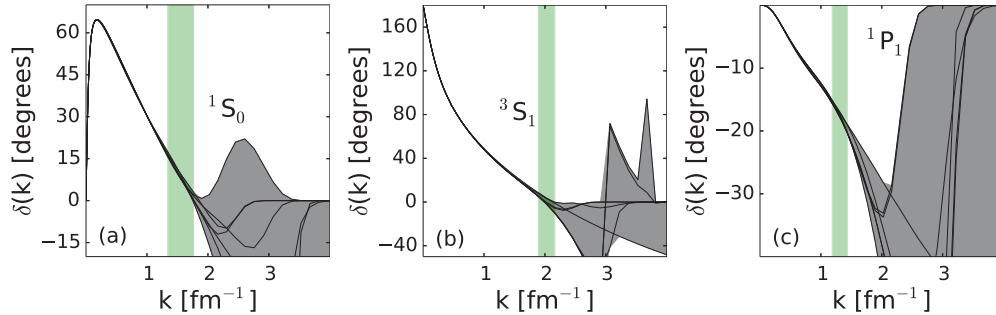


FIG. 1. (Color online) Phase shifts of various realistic potentials (see text) in the (a)  $^1S_0$ , (b)  $^3S_1$ , and (c)  $^1P_1$  partial waves. The shaded regions show the range between the largest and smallest phase shifts. The vertical bands indicate the region where phase shift equivalence between the potentials ends, as defined by Eqs. (3) and (4).

of phase equivalence is not a surprise; inverse scattering theory and effective field theory imply that infinitely many potentials will yield the same low-energy results. But the emergence of universal Hamiltonians (i.e., universal matrix elements) from RG flow is not obvious without an operator classification that isolates irrelevant differences between potentials. To make progress in the absence of such a classification, we focus here on understanding the prerequisites for universality in SRG-evolved matrix elements, starting with two-body interactions.

We re-examine for the SRG the conclusions of Ref. [7] for  $V_{\text{low } k}$  potentials, that the potentials must be phase equivalent up to a certain resolution scale but also have consistent, explicit handling of the long-range pion physics [8]. We use an inverse scattering separable potential (ISSP) to test if universality in potential matrix elements emerges at high energies and without explicit pion-exchange terms. The ISSP can reproduce all observables in the two-nucleon problem, and we will see explicitly that this is not enough for all low-momentum matrix elements to flow towards a universal form at finite cutoff. Also, when creating the ISSP we are free to choose a binding energy independent of the phase shifts; thus we can see the effect of differences in the binding energy on evolved low-momentum matrix elements.

To test the idea that the same explicit long-range treatment is required for flow to a universal form, we introduce a second simple potential that is phase equivalent at low energies and includes explicit one-pion exchange (OPE). We use the model proposed by Navarro Pérez *et al.* [23,24], which combines the OPE potential with a sum of  $\delta$ -shell potentials. This potential replaces the short-range physics with simple terms to be fit to phase shifts, while preserving the long-range force.

In Sec. II, we briefly review the SRG and decoupling, and comment on similarities with the  $V_{\text{low } k}$  RG. We discuss universality in matrix elements of modern realistic potentials in Sec. III. The main focus will emerge in Sec. IV, where we provide a working description of the ISSP formalism, examine universality in ISSP's, and discuss the resulting insight into the prerequisites for universality. Section V gives a description of the  $\delta$ -shell plus OPE potential and examines the SRG flow of this potential to a universal form. We also comment on the SRG flow of the JISP16 potential. Finally, we conclude in Sec. VI with a summary and the outlook for the three-body problem. Although this is a study of universality only for two-nucleon

interactions, it serves as a step toward more efficient handling of the three- and many-nucleon interactions.

## II. SIMILARITY RENORMALIZATION GROUP

The similarity renormalization group is a continuous series of infinitesimal unitary transformations acting on the Hamiltonian. The simplest SRG transformations can be expressed in differential form as a flow equation:

$$\frac{dH_s}{ds} = [\eta_s, H_s] = [[G_s, H_s], H_s], \quad (1)$$

where  $s$  is a flow parameter [8,11,12]. For most nuclear applications to date, the operator  $G_s$  is the kinetic energy operator, denoted  $T$ . (We will refer to  $G_s$  in this work as the SRG “generator.”) The most commonly used diagonalizing generator for non-nuclear applications is known as the Wegner generator [25]. It uses the diagonal of the Hamiltonian,  $H_s^d$ , instead of  $T$  for  $G_s$ . Flows using the Wegner generator are indistinguishable from  $T$  for the range of evolution in the present study but can differ drastically if the SRG cutoff becomes very low [26] or if a large-cutoff chiral potential is used [27].

The goal of the SRG is to decouple high-energy from low-energy degrees of freedom in the Hamiltonian by driving far off-diagonal matrix elements to zero. Instead of  $s$ , we usually refer to the decoupling scale,  $\lambda = s^{-1/4}$  for  $T$  and  $H_s^d$ , where  $\lambda$  is chosen to have the same units as momentum. In the SRG flow with the  $T$  generator, the dominant term of Eq. (1) for far off-diagonal matrix elements is the term linear in the potential,  $[[T, V_s], T]$ , where  $V_s \equiv H_s - T$ . If we keep just this term, the flow equation is immediately solved for these matrix elements, yielding (with mass  $m = 1$ )

$$V_s(k, k') \simeq V_{s=0}(k, k') e^{-\left(\frac{k^2 - k'^2}{\lambda^2}\right)^2}. \quad (2)$$

Thus  $\lambda^2$  is roughly the maximum difference between kinetic energies of nonzero matrix elements. Once the Hamiltonian is sufficiently evolved to exhibit decoupling, low-energy observables can be obtained from a truncated Hamiltonian [16] or one finds naturally that a smaller expansion basis is needed for a desired degree of convergence.

A *nondiagonalizing* alternative for  $G(s)$  is the block generator,  $H_s^{bd}$ , defined in Ref [28].  $H_s^{bd}$  matrix elements are the block diagonal elements of the evolved Hamiltonian  $H_s$ ,

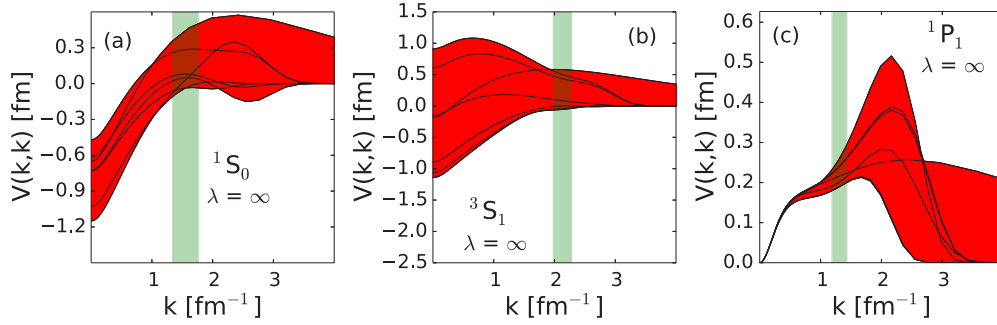


FIG. 2. (Color online) Diagonal matrix elements  $V(k,k)$  of various unevolved realistic potentials (see text) in the (a)  $^1S_0$ , (b)  $^3S_1$ , and (c)  $^1P_1$  partial waves. The shaded regions show the range of values and the vertical bands are from Fig. 1.

separated at a fixed chosen cutoff parameter  $\Lambda$ . (That is, the generator  $H_s^{bd}$  in a momentum basis is obtained from  $H_s(k,k')$  by setting to zero the matrix elements where  $k < \Lambda$  and  $k' > \Lambda$  or  $k > \Lambda$  and  $k' < \Lambda$ .) This is the same pattern of decoupling achieved with  $V_{lowk}$  Lee-Suzuki transformations [7,8,23]. In fact, the  $V_{lowk}$  and SRG block diagonal transformations have been shown to result in very similar Hamiltonians for the lower energy block if the SRG transformation is run to  $\lambda \ll \Lambda$  [28]. For  $H_s^{bd}$ ,  $\lambda = s^{-\frac{1}{2}}$  and represents the maximum difference in energy for coupling between the blocks above and below  $\Lambda$ . It has been shown that SRG with the  $T$  generator [17] and  $V_{lowk}$  [7,8,23] each drive realistic potentials to separate low-energy universal forms, and we will show that  $H_s^{bd}$  also drives potential matrix elements to a different universal form. Because a nondiagonalizing transformation exhibits universality in low-energy potential matrix elements, universality cannot simply be a consequence of the generator  $\eta_s = [T, H_s]$  driving potentials toward the diagonal.

### III. MODERN REALISTIC POTENTIALS

We have chosen a representative phenomenological potential and a set of  $\chi$ EFT potentials to evolve and examine in various partial waves. The phenomenological potential is Argonne  $v_{18}$  (AV18), which employs basis operators in position representation and fits the coupling constants to elastic scattering data [1,2]. We use the next-to-next-to-next-to-leading order ( $N^3$ LO)  $\chi$ EFT potential from Entem and Machleidt with a cutoff of 500 MeV [3] and then five  $N^3$ LO

$\chi$ EFT potentials with various cutoffs from Epelbaum *et al.* [4]. These  $\chi$ EFT potentials have different regularization and phase shift fitting schemes, which creates differences in the matrix elements of the potentials.

From Fig. 2 one can see that the diagonals of the initial potentials in momentum representation are quite different (the differences are particularly evident in lower partial waves, so we focus on those). In making these comparisons, we do not single out individual potentials but use a shaded region to highlight the range of matrix element variation. As advertised, after evolution the matrix elements collapse at low momentum to a universal dependence on momentum (the result at fixed  $\lambda = 1.5 \text{ fm}^{-1}$  is shown in Fig. 3). This feature is not restricted to the diagonal elements; low-energy off-diagonal matrix elements of the potentials also evolve to universal values (see Fig. 15 below). At higher momentum, the potential matrix elements deviate.

Following Ref. [7], we compare phase shift and matrix element deviations to identify the correlations between phase equivalence and matrix element universality. In Fig. 1, we have identified vertical bands within which the phase shift equivalence among the various potentials ends and significant deviation begins. While identifying an exact point marking this deviation will be somewhat arbitrary, we can roughly choose a normalized width description that is consistent with visual assessments of the phase shift plots. In particular, for each partial wave, the vertical band represents the region characterized by

$$0.03 < \epsilon(k) < 0.1, \quad (3)$$

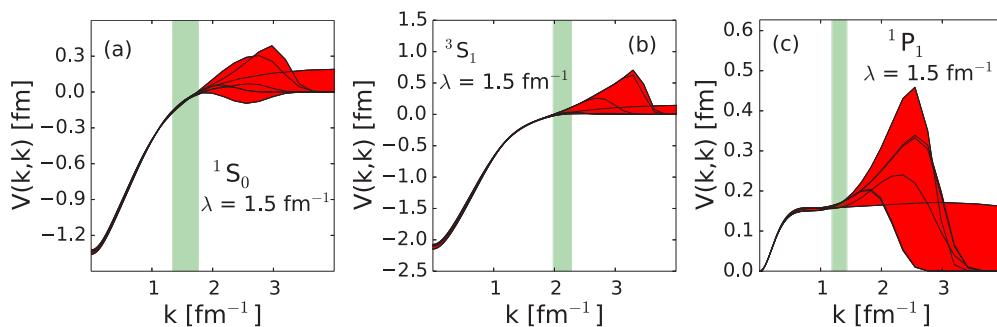


FIG. 3. (Color online) Diagonal matrix elements of various realistic potentials in the (a)  $^1S_0$ , (b)  $^3S_1$ , and (c)  $^1P_1$  partial waves evolved by the SRG to  $\lambda = 1.5 \text{ fm}^{-1}$ . The shaded regions show the range of values and the vertical bands are from Fig. 1.

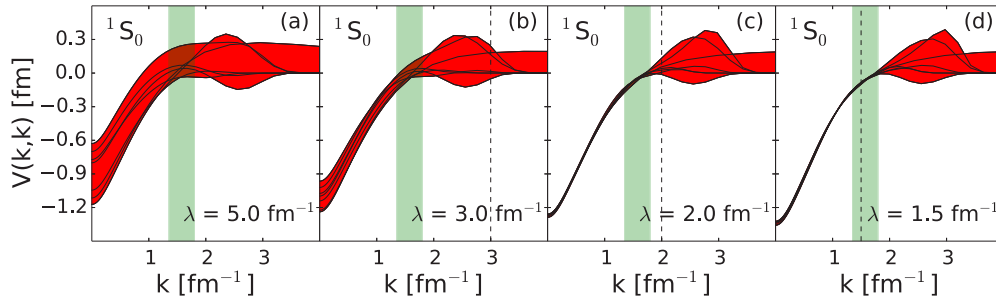


FIG. 4. (Color online) Diagonal matrix elements of various realistic potentials in the  $^1S_0$  partial wave evolved by the SRG to  $\lambda =$  (a)  $5.0 \text{ fm}^{-1}$ , (b)  $3.0 \text{ fm}^{-1}$ , (c)  $2.0 \text{ fm}^{-1}$ , and (d)  $1.5 \text{ fm}^{-1}$  (marked by the vertical dashed line). The shaded regions show the range of values and the vertical bands are from Fig. 1.

where

$$\epsilon(k) \equiv \frac{\delta_{\text{high}}(k) - \delta_{\text{low}}(k)}{\Delta}. \quad (4)$$

The numerator is the range of phase shifts at a fixed  $k$  while  $\Delta$  is the range of phase shifts for the entire universality region. Our studies imply that the precise definition of  $\epsilon$  is not important; as long as it consistently identifies the regions where phase equivalence ends it can be used to consistently compare to the regions where the universality of matrix elements end.

By comparing Figs. 2 and 3, we see that while diagonal matrix elements of the initial potentials differ significantly in the region where phase equivalence ends, this same region corresponds to where the matrix elements have collapsed to universal values by  $\lambda = 1.5 \text{ fm}^{-1}$ . This suggests the hypothesis that *a prerequisite for matrix element universality is phase equivalence*. Namely, if there are *local* regions in energy in which potentials are not phase equivalent, then there is no universality in those regions (this is tested further in Sec. IV). Examining the diagonals of the potentials more closely, we observe that for the  $^1S_0$  and  $^3S_1$  channels, the lowest matrix elements are not exactly the same. This may be a consequence of not evolving  $\lambda$  further. From the  $T$  generator curves in Fig. 5, we can see that the slight width of the band decreases as we evolve chiral potentials to  $\lambda = 0.5 \text{ fm}^{-1}$ . Also, as we will see below, differences in the binding energy of the deuteron play an important role in the low-energy matrix elements of the  $^3S_1$  potential.

How low must  $\lambda$  be before we see universality? Figure 4 shows the diagonals of the  $^1S_0$  potential evolved to four different  $\lambda$  values. The vertical bands correspond to the same region where phase equivalence ends for the  $^1S_0$  channel as in Fig. 1, while the vertical dashed line shows the value of  $\lambda$ . We see in this partial wave (and in others not shown as well) that universality in the matrix elements does not occur until  $\lambda$  approaches the vertical band. A natural hypothesis is that the matrix elements will not fully collapse to universal form until  $\lambda$  reaches the region of phase equivalence. There may be an intrinsic low-energy scale common to each of these potentials that determines at which  $\lambda$  universality in potential matrix elements will appear. A possibility is that this scale is a consequence of explicit treatment of pion physics in each of the modern realistic potentials. To test the latter explanation, a potential with phase equivalence at much higher momenta

and no explicit pion physics is required, which we consider in the next section.

As described earlier, the block-diagonalizing generator  $H_s^{bd}$  will drive the potential matrix elements to a different universal form than  $T$ . This is illustrated in Fig. 5 with a set of  $\chi$ EFT potentials in the  $^1S_0$  channel. When evolved to  $\lambda \leq 2 \text{ fm}^{-1}$  with the  $T$  generator, the universal form of diagonal potential matrix elements emerges over the full region of phase equivalence. For the block diagonal generator with  $\Lambda \leq 2 \text{ fm}^{-1}$ , however, only diagonal matrix elements below  $\Lambda$  become universal and with a different flow than the matrix elements evolved with  $T$ . The universality is only up to  $\Lambda$  because this SRG only decouples one block from the other, so matrix elements at momenta above  $\Lambda$  still couple to matrix elements in phase inequivalent regions and therefore do not collapse to a universal form. (Note that in the  $V_{\text{low}k}$  RG, the higher block is set to zero.) We will discuss only  $G_s = T$  in the rest of this study but emphasize that the ideas about universality apply to both generators, although only in the low-momentum block for the  $H_s^{bd}$  SRG.

The region of phase equivalence for the realistic potentials is limited by the energies to which they can be fit to elastic scattering phase shifts. Because of this, if we wish to investigate different regions of universality, we must use a method that can “fit” the phase shifts in a controlled range of energies. One of the simplest approaches is solving the inverse scattering problem with a separable potential, which we consider in the next section.

#### IV. SEPARABLE INVERSE SCATTERING POTENTIAL

Instead of fitting coupling constants for predetermined operators to the phase shifts, an inverse scattering procedure constructs a potential directly from the phase shifts. Separability is just a constraint to define a unique potential, chosen here due to its simplicity. For instance, when solving the Lippmann-Schwinger equation, a separable potential reduces the problem of solving an integral equation to simply evaluating an integral. The three-body Faddeev equations also simplify for a separable potential, as one of the integrals over internal momenta becomes trivial. A key feature of the ISSP for this study is that the potential is entirely created from the phase shifts and binding energy of the deuteron; no *explicit* pion exchange or other physics is imposed. This allows us to determine



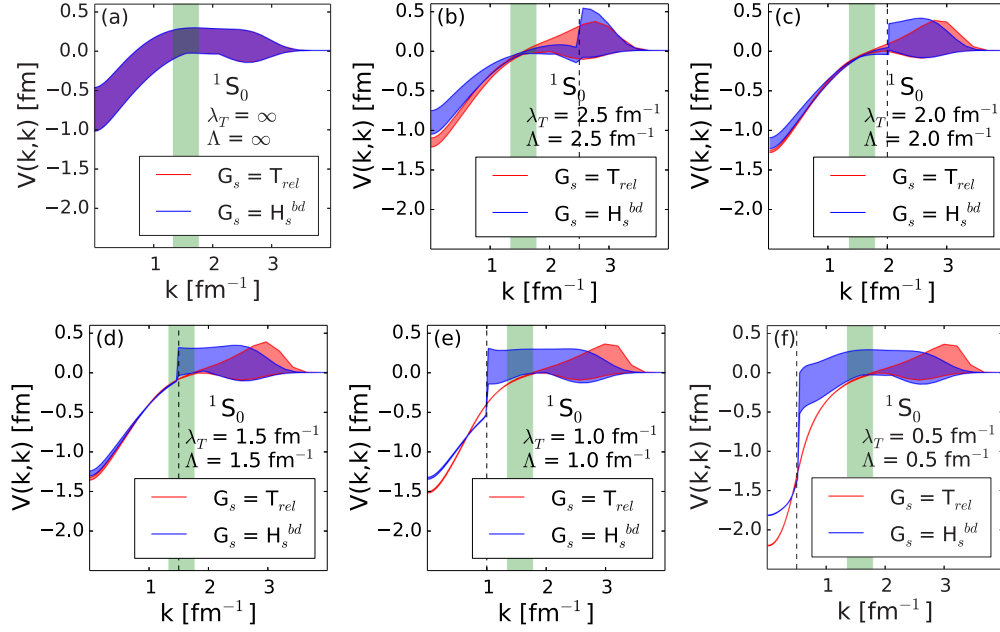


FIG. 5. (Color online) The spread of diagonal matrix elements of various  $\chi$ EFT potentials (see text) in the  $^1S_0$  partial wave are shown as shaded regions for the unevolved potential and then after evolution to  $\lambda_T = 2.5, 2.0, 1.5, 1.0,$  and  $0.5 \text{ fm}^{-1}$  with the  $T$  generator (red [light gray]). These are compared to the spread of the corresponding matrix elements for the  $H_s^{bd}$  generator with  $\Lambda = 2.5, 2.0, 1.5, 1.0,$  and  $0.5 \text{ fm}^{-1}$ , all evolved to  $\lambda = 0.5 \text{ fm}^{-1}$  (blue [medium gray]). The vertical bands are from Fig. 1 and the vertical dashed lines mark  $\lambda_T$  or  $\Lambda$ .

whether or not universality requires extra physics, such as explicit long-range pion terms or other phenomenological considerations. We start with a brief summary of the inverse scattering separable potential for two nucleons.

### A. Formalism

The form of a rank- $n$  separable potential is

$$V = \sum_{i,j=0}^{n-1} |v_i\rangle \Lambda_{ij} \langle v_j|. \quad (5)$$

For our purposes a rank-1 separable potential will be sufficient, but future studies may benefit from a higher-rank potential. A rank-1 potential in momentum representation takes the form

$$V(k, k') = \sigma v(k)v(k'), \quad (6)$$

where  $\sigma$  is simply  $\pm 1$ . Details of the rank-1 separable inverse scattering problem are well documented [29,30]; here, we simply state the main results, some limitations, and how to work around the limitations. The solution to the separable inverse scattering problem is [29]

$$\sigma v^2(k) = -\frac{k^2 - k_b^2}{k^2} \frac{\sin[\delta(k)]}{k} e^{-\Delta(k)}, \quad (7)$$

$$\Delta(k) = \frac{1}{\pi} P \int_0^\infty \frac{dk' \delta(k')}{k' - k}, \quad (8)$$

$$E_b = \frac{\hbar^2 k_b^2}{2m}, \quad (9)$$

where  $k_b$  is zero if there is no bound state and equal to the binding momentum for a single bound state with binding

energy  $E_b$  (for a rank-1 separable potential there can be at most one bound state).

Once  $v(k)$  is determined, the entire potential is known from Eq. (6). The binding energy  $E_b$  can be tuned independently of the phase shifts. A limitation of rank-1 separable potentials is that if the phase shift as a function of momentum cross zero, then so too must the potential, and a rank-1 ISSP as defined thus far can never change signs if  $v$  is real. This point is clear from Eq. (10), which follows from the Lippmann-Schwinger equation for a separable potential (with standing wave boundary conditions):

$$\frac{1}{k} \tan[\delta_l(k)] = -\frac{V_l(k, k)}{1 + \frac{2}{\pi} P \int \frac{dp p^2 V_l(p, p)}{p^2 - k^2}}. \quad (10)$$

A zero-crossing in  $\delta(k)$  corresponds to a zero crossing on the right side of this equation, which can only be achieved by the numerator crossing zero if the denominator remains finite.

Because some of the phase shifts for nucleon-nucleon partial waves exhibit zero crossings, we need an inverse scattering potential that allows this feature. We can still use the same rank-1 formalism, however, if we split the problem into two energy regimes, above and below the zero crossing [30]. Then we can define

$$\delta_{<}(k) \equiv \delta(k)\theta(k_0 - k), \quad (11)$$

$$\delta_{>}(k) \equiv \delta(k)\theta(k - k_0), \quad (12)$$

$$V(k, k') = V_{<}(k, k') + V_{>}(k, k'), \quad (13)$$

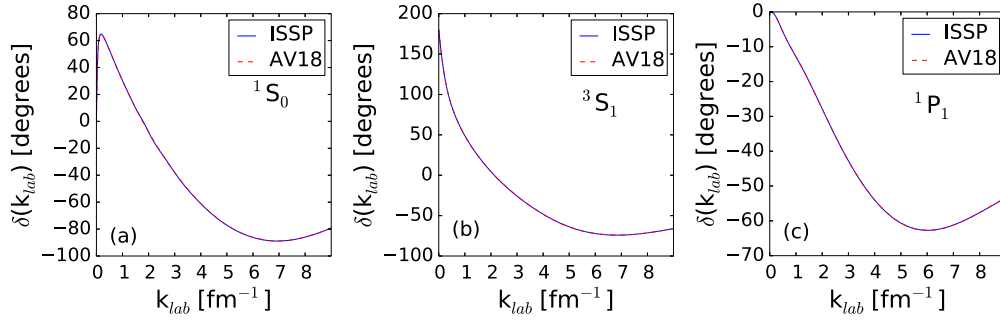


FIG. 6. (Color online) Phase shifts using the AV18 potential and the ISSP up to high laboratory momentum  $k_{lab}$  in the (a)  $^1S_0$ , (b)  $^3S_1$ , and (c)  $^1P_1$  partial waves.

and determine  $V_<$  and  $V_>$  separately using the rank-1 formalism with  $\delta_<$  and  $\delta_>$  as input, respectively. We have confirmed numerically that potentials created with this prescription accurately reproduce the input phase shifts.

The method described thus far works directly for uncoupled channels, but for NN scattering we must also account for coupled channels, where some further formalism is required. For this purpose, we use the Blatt-Beidenharn (BB) convention for phase shifts in the coupled channel [30,31]. (In the plots we employ the more typically used Stapp- $\bar{N}$  convention for the phase shifts [32].) The BB convention can be summarized as

$$\mathbf{S}(k) = \mathbf{U}^\dagger(k) \hat{\Delta}(k) \mathbf{U}(k), \quad (14)$$

$$\hat{\Delta}(k) = \begin{pmatrix} e^{2i\delta_0(k)} & 0 \\ 0 & e^{2i\delta_1(k)} \end{pmatrix}, \quad (15)$$

$$\mathbf{U}(k) = \begin{pmatrix} \cos[\epsilon(k)] & \sin[\epsilon(k)] \\ -\sin[\epsilon(k)] & \cos[\epsilon(k)] \end{pmatrix}. \quad (16)$$

Here,  $\mathbf{S}(k)$  is the scattering matrix define in Ref. [31], with  $k$  the momentum corresponding to the interaction energy. Then the inverse scattering potential can be written as

$$\mathbf{V}(k, k') = \mathbf{U}^\dagger(k) \hat{\mathbf{V}}(k, k') \mathbf{U}(k'), \quad (17)$$

where

$$\hat{\mathbf{V}}(k, k') = \begin{pmatrix} \hat{V}_0(k, k') & 0 \\ 0 & \hat{V}_1(k, k') \end{pmatrix}. \quad (18)$$

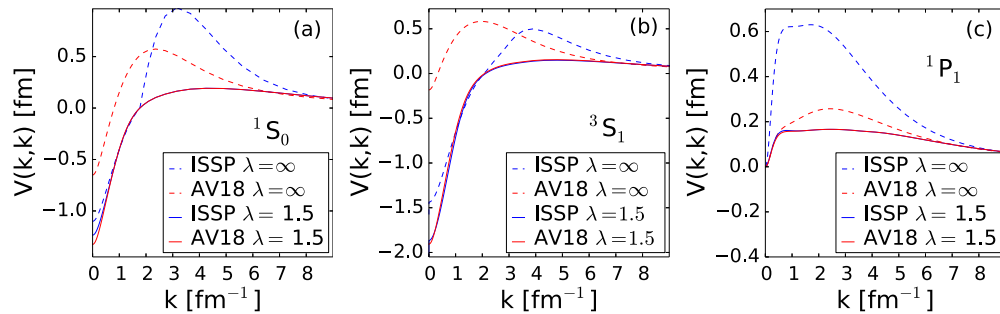


FIG. 7. (Color online) Diagonal matrix elements of the AV18 potential and the ISSP up to high laboratory momentum  $k_{lab}$  in the (a)  $^1S_0$ , (b)  $^3S_1$ , and (c)  $^1P_1$  partial waves. Cutoff  $\lambda$  is in units of  $\text{fm}^{-1}$ .

To proceed, one uses the inverse scattering method for uncoupled channels to find  $\hat{V}_0(k, k')$  from  $\delta_0(k)$  and  $\hat{V}_1(k, k')$  from  $\delta_1(k)$ . The complete potential is then found by a rotation by the mixing parameter,  $\epsilon(k)$ . With this complete separable inverse scattering formalism, we can now create a phase-equivalent potential at all energies in any given partial-wave channel.

## B. Universality in separable inverse scattering potentials

We use phase shifts from Argonne  $v_{18}$  to create the phase-equivalent ISSP. In Fig. 6, we see that the elastic phase shifts are quantitatively reproduced well above the inelastic threshold. We choose Argonne  $v_{18}$  specifically because it has phase shifts that extend to this high energy, but any realistic potential could be used for starting phase shifts. (Note: for simplicity we treat the problem nonrelativistically with only elastic scattering because we are interested in testing universality and low-energy effects, not to have a realistic description of high-energy physics.) The ISSP's from chiral potentials exhibit similar behavior, except that the internal cutoffs drive matrix elements and phase shifts to zero at high energies, which is less useful for the present investigations. The accuracy of the ISSP in reproducing phase shifts can be further increased simply by using more grid points and increasing the maximum momentum if the phase shifts are nonzero above this momentum.

Figure 7 shows the diagonal matrix elements of Argonne  $v_{18}$  and the ISSP for three different partial waves before and after SRG evolution. We observe that after SRG evolution to  $\lambda = 1.5 \text{ fm}^{-1}$ , universality in the diagonal matrix elements also extends to the full range of energies. In fact, the only

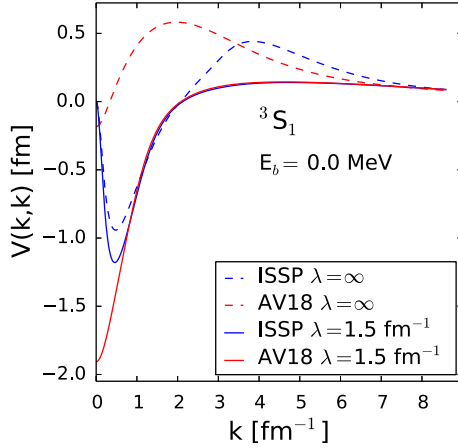


FIG. 8. (Color online) Initial and evolved diagonal matrix elements in the  ${}^3S_1$  channel for AV18 and an ISSP with a binding energy of 0 MeV.

discernible difference in the evolved potential diagonals is below the SRG cutoff. Above  $\lambda$  the matrix elements in the region shown are completely collapsed to universal values.

Because the binding energy in the ISSP formalism is independently tuned from the phase shifts, we can investigate in the deuteron  ${}^3S_1$ - ${}^3D_1$  coupled channel how universality in potential matrix elements is affected by differences in the bound-state energy. Figures 8 and 9 show the effects of phase-equivalent potentials having the wrong binding energy. In Fig. 8, the ISSP is created from the phase shifts of the Argonne  $v_{18}$  potential in the deuteron channel, but with a binding energy of 0 MeV instead of 2.224 MeV. It is evident that the effect on diagonal matrix elements is substantial. The low-energy matrix elements of the bare ISSP tend towards zero as the momentum decreases. As the potentials evolve, the diagonal matrix elements are driven to universal values except that the ISSP is constrained by its binding energy to approach zero as momentum approaches zero.

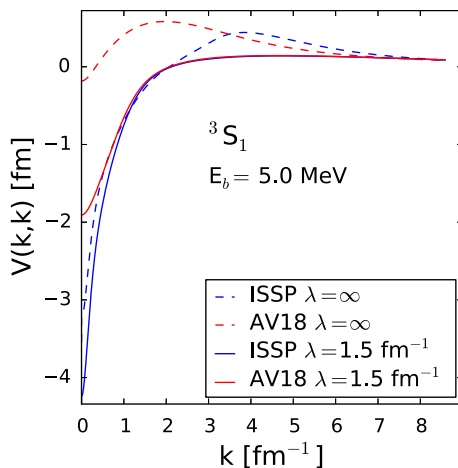


FIG. 9. (Color online) Initial and evolved diagonal matrix elements in the  ${}^3S_1$  channel for AV18 and an ISSP with a binding energy of 5 MeV.

A similar effect can be seen in Fig. 9 where instead of 0 MeV as input binding energy, the ISSP is created with input binding energy of 5 MeV. The ISSP reproduces this energy better than 100 eV. This potential is overbound and its lowest momentum matrix elements are forced lower than if it had the physical deuteron binding energy. Again, the higher momentum matrix elements flow towards a universal form because of phase equivalence. Together these plots show that phase equivalence is not the only prerequisite for universality in the diagonal potential matrix elements, but a correct binding energy is also necessary. (That is, we need  $S$ -matrix equivalence for negative energies as well.) This may account for the small deviations in the potentials at lowest momenta in Fig. 2. The  ${}^3D_1$  partial wave plots of the corresponding ISSP potentials with different binding energies are indistinguishable. This effect only appears in the  ${}^3S_1$  potentials. It is possible that a virtual bound state in the  ${}^1S_0$  partial wave has a similar effect on the evolved low-momentum potential matrix elements, but the ISSP cannot tune virtual bound states and residues in the same way it accommodates bound states; thus we do not investigate this point further.

Next we turn to off-diagonal matrix elements. Figure 10 shows the potential matrix elements  $V(k_0, k)$  for  $k_0 = 0.1 \text{ fm}^{-1}$  as a function of  $k$  for the ISSP and all of the realistic potentials evolved to  $\lambda = 1.5 \text{ fm}^{-1}$ . We can see that although these off-diagonal cuts for the modern potentials agree at  $\lambda = 1.5 \text{ fm}^{-1}$ , the ISSP matrix elements do not. By using a diagonalizing SRG transformation (that is,  $G_s = T$ ), the off-diagonal potential matrix elements are exponentially suppressed. Because of this, it appears that the ISSP approaches a universal form, but unlike the realistic potentials, there is no finite  $\lambda$  at which the ISSP collapses to universal form. Figure 10 shows low-energy half-on-shell (HOS)  $T$  matrices from each of the unevaluated realistic potentials and the ISSP. We observe that the realistic potentials, which will evolve to a universal form, have essentially the same low-momentum, low-energy HOS  $T$ -matrix elements, while the ISSP does not. This is consistent with carrying over to the SRG the suggestion from Ref. [7] that HOS  $T$ -matrix equivalence is required for off-diagonal universality in  $V_{\text{low } k}$  RG-evolved matrix elements, much like phase shift equivalence is required for universality of diagonals. We only show the  ${}^1S_0$  partial waves, but the same pattern holds for all partial waves. Clearly, matching observables is not enough to produce fully universal potentials after evolution, and in the next section we will examine if matching observables and also including the same explicit one-pion exchange potential will be enough for potentials to evolve to a low-energy universal form.

As a further test, we created ISSP's using altered phase shifts in *localized* regions of energy to see if the flow to universal diagonal matrix elements is disturbed only locally. We use the  ${}^1P_1$  channel for clarity. Figure 11(a) shows the  ${}^1P_1$  phase shifts for Argonne  $v_{18}$  and for an ISSP that is phase equivalent *except* for a Gaussian bump that we impose by hand at low energy. In Fig. 11(b) we see that the potentials evolve to the same diagonal values everywhere but at low energy. Another potential was constructed by creating low- and high-energy regions of phase equivalence, and imposing a Gaussian bump (around  $k_{\text{lab}} = 4.0 \text{ fm}^{-1}$ ) to create a difference

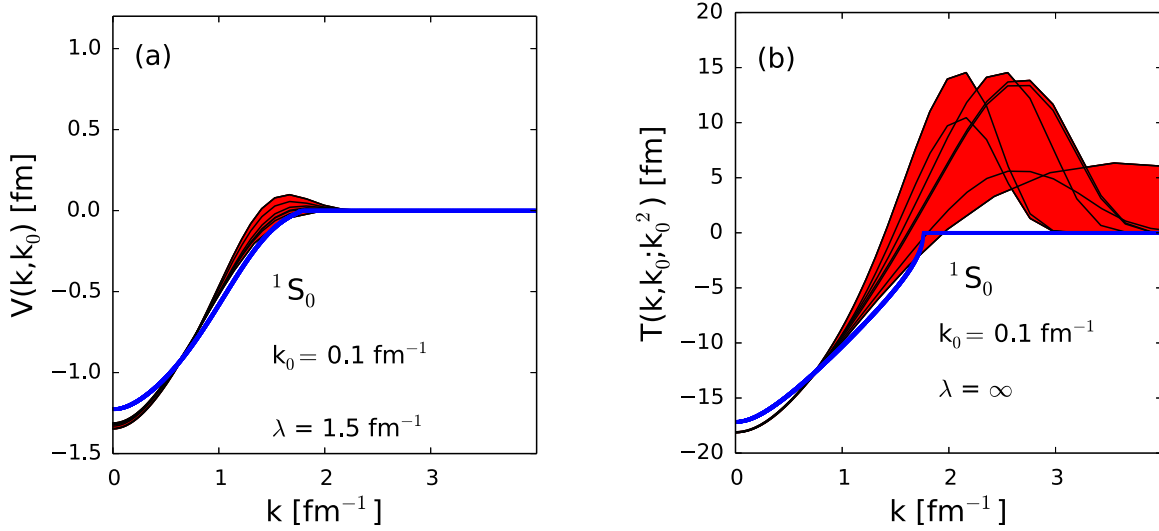


FIG. 10. (Color online) (a) Off-diagonal SRG evolved potential matrix elements  $V(k, k_0)$  with  $k_0 = 0.1 \text{ fm}^{-1}$  and (b) unevolved half-on-shell  $T$  matrices  $T(k, k_0; k_0^2)$ . In both figures, the thick line is from the ISSP while the bands are various realistic potentials.

in the intermediate energy phase shifts; see Fig. 12(a). In Fig. 12(b) the evolution to common diagonal values again works everywhere except near where the phase shifts disagree.

We conclude from these figures (and other tests not shown) that the SRG evolved diagonal potential matrix elements are altered only in a region localized near the altered phase shifts. This suggests that an SRG softened potential is *locally decoupled* such that the integral in the Lippmann-Schwinger (LS) equation for the on-shell  $T$  matrix can be truncated as

$$T_l(k, k; k^2) = V_l(k, k) + \frac{2}{\pi} P \int_{k-\Lambda}^{k+\Lambda} dp p^2 \frac{V_l(k, p) T_l(p, k; k^2)}{k^2 - p^2}, \quad (19)$$

where the lower limit of the integral is taken to be zero if  $k - \Lambda < 0$ . In Eq. (19),  $\Lambda$  represents the local decoupling scale, which we will set to SRG  $\lambda$ . (In fact  $\lambda$  appears to be a conservative upper bound for  $\Lambda$  to quantitatively reproduce phase shifts.)

Figure 13 shows phase shifts calculated from Eq. (19) with  $\Lambda = 4.0 \text{ fm}^{-1}$  in the  $^1S_0$  channel for the Argonne  $v_{18}$  potential evolved to three different SRG  $\lambda$ 's. These are compared to the actual phase shifts of the unevolved potential. We see that with this large value of  $\Lambda$ , the truncated phase shifts for even the unevolved potential are largely reproduced and the low-momentum phase shifts from evolved potentials are indistinguishable from the actual phase shifts. (The periodicity at high momentum for  $\lambda = 1.5 \text{ fm}^{-1}$  is a numerical grid artifact.) In Fig. 14 we more severely truncate the integral

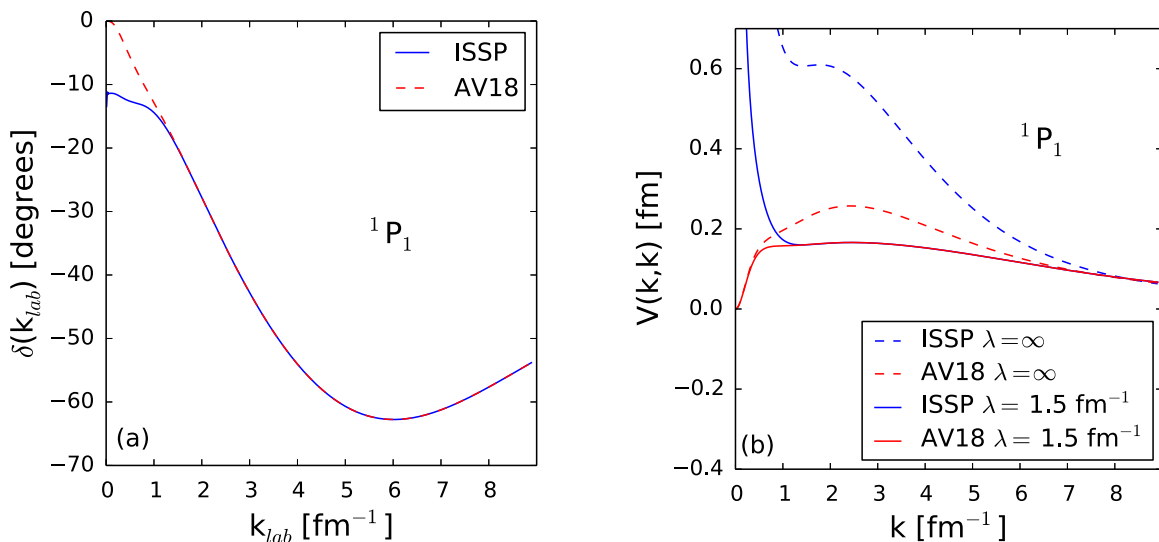


FIG. 11. (Color online) Low-energy phase difference effects on universality: (a) phase shifts and (b) diagonals of potentials.



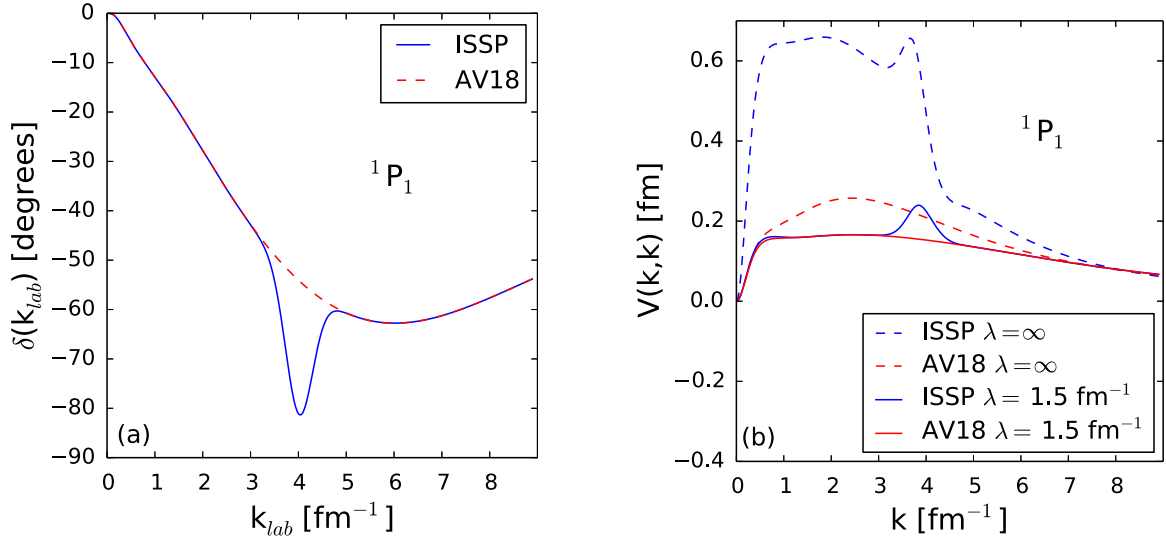


FIG. 12. (Color online) Intermediate-energy phase difference effects on universality: (a) phase shifts and (b) diagonals of potentials.

in the LS equation to  $\Lambda = 1.5 \text{ fm}^{-1}$ . We see clearly that the potential evolved to  $\lambda = 4.0 \text{ fm}^{-1}$  is not decoupled enough to reproduce the original phase shifts, but the potential evolved to  $\lambda = 1.5 \text{ fm}^{-1}$  has phase shifts identical to the previous plot. This suggests that evolution with  $T$  does locally decouple energy scales.

### V. OPE PLUS $\delta$ SHELL

Here we further test the suggestion that explicit treatment of the longest-ranged physics is a requirement for potentials to evolve to a universal form [7]. In particular, we develop a simple test potential that is (approximately) phase equivalent in the same momentum regions as the realistic potentials but also has the same explicit long-range forces. We use the model

from Pérez *et al.* that combines the one-pion exchange (OPE) potential with a sum of  $N$   $\delta$ -shell potentials [23,24] in each partial wave:

$$V_l(r) = V_l^{\text{OPE}}(r) + \sum_{i=1}^N g_i^l \delta(r - r_i). \quad (20)$$

The explicit form of the OPE potential can be found in Ref. [33]. We choose the  $\{r_i\}$  as short-range lengths (under  $2 \text{ fm}^{-1}$ ) and fit the  $\{g_i^l\}$  to match low-momentum phase shifts. We choose a different regulator than Ref. [23,24], instead regulating the potential in momentum representation with a separable form factor:

$$f_{\text{reg}}(k, k') = e^{-(k/\Lambda)^4} e^{-(k'/\Lambda)^4}, \quad (21)$$

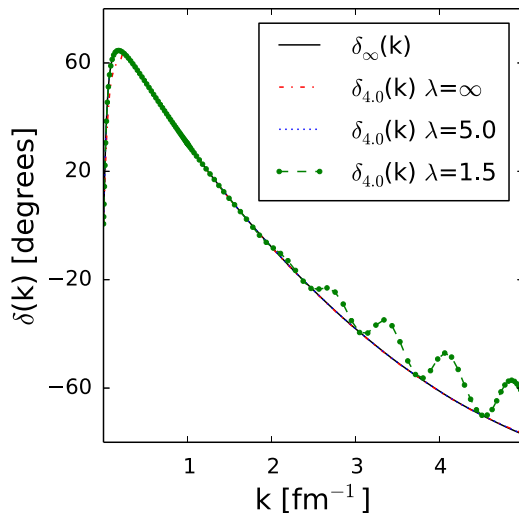


FIG. 13. (Color online) Phase shifts of Argonne  $v_{18}$  potential and truncated phase shifts of evolved potentials with  $\Lambda = 4.0 \text{ fm}^{-1}$ . Cutoff  $\lambda$  is in units of  $\text{fm}^{-1}$ .

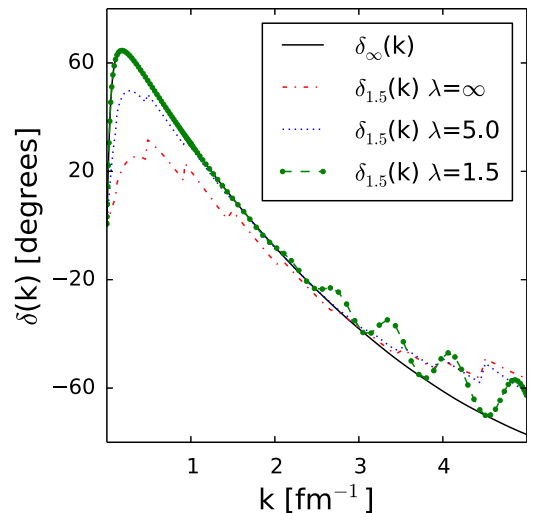


FIG. 14. (Color online) Phase shifts of Argonne  $v_{18}$  potential and truncated phase shifts of evolved potentials with  $\Lambda = 1.5 \text{ fm}^{-1}$ . Cutoff  $\lambda$  is in units of  $\text{fm}^{-1}$ .

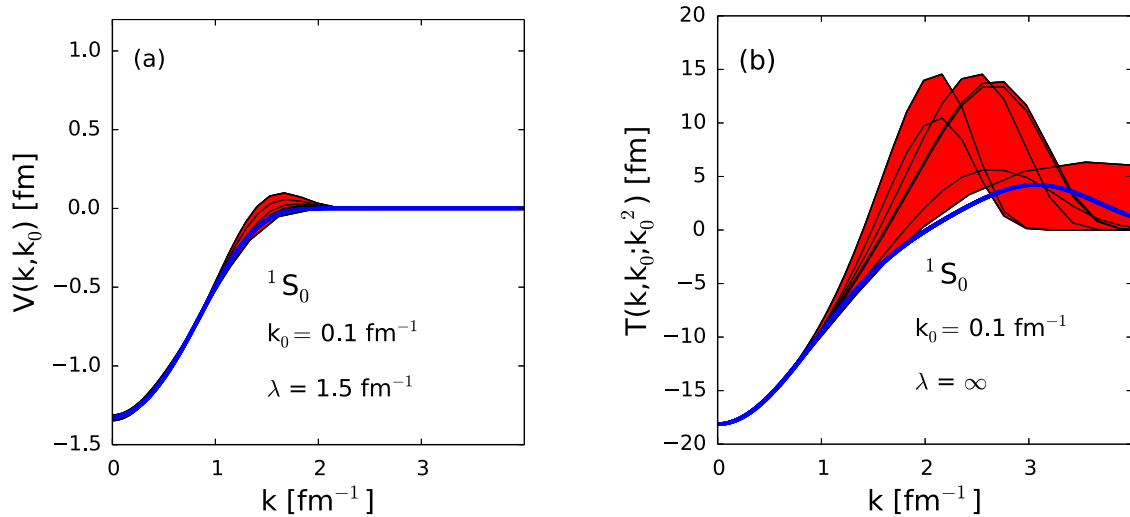


FIG. 15. (Color online) (a) Off-diagonal SRG evolved potential matrix elements. (b) Unevolved half-on-shell  $T$  matrices. In both figures, the thick line is the  $\delta$  shell plus OPE potential while the bands are from realistic modern potentials.

for which we choose  $\Lambda = 3 \text{ fm}^{-1}$ . We now have a potential with explicit long-range pion terms and adjustable short-range terms, which is phase equivalent at low momentum to the realistic potentials.

#### A. Universality in OPE plus $\delta$ shell

We can see from Fig. 15 that the OPE plus  $\delta$ -shell off-diagonal potential elements evolve to the same universal form as the modern realistic potentials. Also, Fig. 15 shows the corresponding unevolved HOS  $T$  matrices. We see that the OPE plus  $\delta$ -shell potential has the same low-energy low-momentum HOS  $T$  matrix and shows a corresponding low-momentum universality in off-diagonal matrix elements. This behavior is not unique to the  $^1S_0$  partial wave, but appears for all partial waves. This simple potential explicitly contains only the longest range OPE potential and has very simple short-range terms, but it collapses to the same universal low-momentum potential after SRG evolution. Combined with the ISSP results, this is strong evidence that the same explicit inclusion of the longest-range contributions to the potential,

which is reflected in low-energy HOS  $T$ -matrix equivalence, is required for collapse to a universal form.

#### B. JISP potential

In principle, a good test of our observations about universality is the JISP16 potential, which is a realistic potential constructed using the  $J$ -matrix version of inverse scattering theory [6,34]. Because there is no explicit incorporation of a pion-exchange tail in the functional form of the potential, we might expect the Hamiltonian to exhibit nonuniversal evolution with the SRG for off-diagonal matrix elements. In fact, the unevolved JISP potential is already soft and changes only slightly under SRG evolution. But as shown in Fig. 16 in the  $^1S_0$  channel for a set of off-diagonal matrix elements (and true for the diagonal and other partial waves), JISP16 is already close to the universal form reached by the chiral  $N^3\text{LO}$  potentials. There are still differences, but they are small. However the JISP HOS  $T$  matrix is also close to the others (perhaps as the result of additional adjustments of the potential

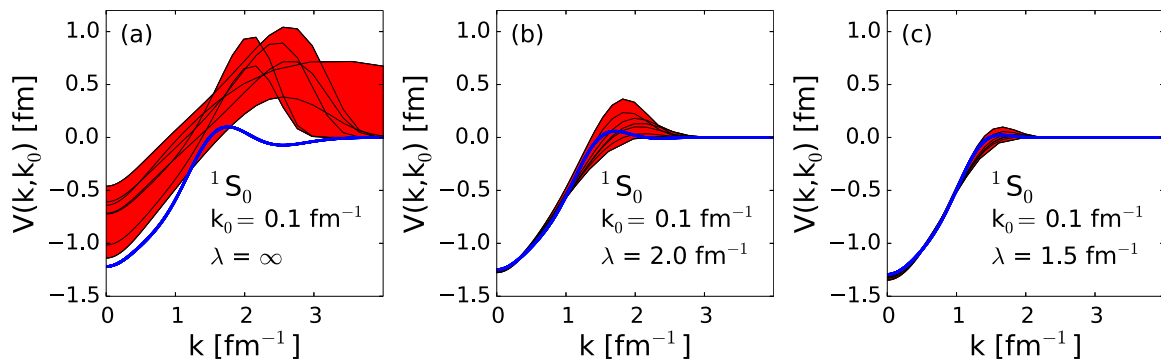


FIG. 16. (Color online) Off-diagonal matrix elements of (a) chiral  $N^3\text{LO}$  potentials and the JISP16 potential in the  $^1S_0$  channel and the same potential evolved by the SRG to (b)  $\lambda = 2.0 \text{ fm}^{-1}$  and (c)  $\lambda = 1.5 \text{ fm}^{-1}$ .

using the freedom of the inverse scattering framework [34]), so there is no inconsistency with our general conclusions.

## VI. CONCLUSION

Modern realistic two-nucleon potentials exhibit a flow to universal potential matrix elements under the similarity RG. High and low momenta are decoupled in this universal matrix, allowing us to truncate the matrix and drastically simplify low-energy bound state and reaction calculations. Any initial interaction that yields this universal matrix after SRG evolution is equally effective. This is of little practical importance for the two-nucleon potential, but it could be extremely useful if many-nucleon potentials display this same type of universality. Producing accurate realistic many-nucleon potentials is extremely difficult. Our results suggest that any convenient potential that includes long-range pion exchange interactions can be used to produce universal many-nucleon interactions when evolved with an SRG transformation.

Our study of universality for two-body potentials yields the following observations:

- (i) Inverse scattering separable potentials, with no explicit consideration of long-range pion exchange, exhibit a universal collapse of diagonal matrix elements after evolution in regions of phase equivalence.
- (ii) If an intermediate region of phase inequivalence is imposed, the collapse does not occur in this region but still occurs in every region of phase equivalence. This implies that SRG softened potentials are actually *locally* decoupled in energy and momentum.
- (iii) An incorrect binding energy has a strong effect on the lowest potential matrix elements and will prevent flow towards a universal form.
- (iv) While phase equivalence and correct binding energies (i.e.,  $S$ -matrix equivalence) are apparently requirements for universality in two-body potential matrix elements, the ISSP example shows that these are not sufficient to guarantee a potential that will flow to

the same off-diagonal values as conventional realistic potentials.

- (v) However, a potential that reproduces low-energy observables and contains explicit long-range (OPE) terms does flow to universal form, which is consistent with observations made for  $V_{\text{low } k}$  evolution in Ref. [7].
- (vi) To the extent that low-energy HOS  $T$ -matrix equivalence indicates long-range equivalence of potentials, it signals off-diagonal universality in evolved potential matrix elements.
- (vii) For universality to appear, the SRG decoupling parameter must be sufficiently low so that potential matrix elements in the low-momentum region of HOS  $T$ -matrix equivalence are decoupled from high-momentum matrix elements.

These considerations address the onset of universality for the two-body part of the inter-nucleon potential but for a complete discussion we have to consider the full many-body Hamiltonian. It is well established that the evolution to smaller values of  $\lambda$  induces many-body forces of increasing importance [8,15] and the SRG transformations will only be approximately unitary if they are omitted. This entails a lower limit to the region of universality in practical applications. In future work we will test whether our observations of universality for two-nucleon interactions carry over to three-body forces and seek a practical operator classification procedure that will allow the full power of the RG to be applied to nuclear problems.

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