PHYSICAL REVIEW C **89**, 012201(R) (2014)

Proton root-mean-square radii and electron scattering

Ingo Sick* and Dirk Trautmann†
Departement Physik, Universität Basel, CH4056 Basel, Switzerland
(Received 13 November 2013; published 6 January 2014)

The standard procedure of extracting the proton root-mean-square radii from models for the Sachs form factors $G_e(q)$ and $G_m(q)$ fitted to elastic electron-proton scattering data is more uncertain than traditionally assumed. The extrapolation of $G(q)$, from the region $q_{\min} < q < q_{\max}$ covered by data to momentum transfer $q = 0$ where the rms radius is obtained, often depends on uncontrolled properties of the parametrization used. Only when ensuring that the corresponding densities have a physical behavior at large radii r can reliable rms radii be determined.

DOI: 10.1103/PhysRevC.89.012201 PACS number(s): 14.20.Dh, 21.10.Ft, 21.10.Ky, 25.30.Bf

Introduction. Accurate knowledge of the proton root-mean-square (rms) radii is important for both an understanding of proton structure and the interpretation of the extremely precise data on transition energies in the hydrogen atom. Traditionally, the rms radii have been derived from data on elastic electron-proton scattering at low momentum transfer q. The best determinations are based on parametrizations of the Sachs form factors $G_e(q)$ and $G_m(q)$ with the parameters directly fitted to the observables, i.e., cross sections and polarization transfer data. The slope of the parameterized form factors at $q = 0$ yields the charge and magnetic rms radii, respectively.

This topic of the proton radii has recently become a subject of intense discussion with the publication of the charge rms radius determined from the Lamb shift in muonic hydrogen [1]. This radius, 0.8409 ± 0.0004 fm, disagrees by many standard deviations with the value 0.8775 ± 0.005 fm from the *world* data on electron scattering [2], and it also disagrees with the recent value extracted from the transition energies in electronic hydrogen [3], 0.8758 ± 0.0077 fm. This disagreement has led to a large number of studies dealing with problems with the determination of the radii, or new physics [4–12].

The discrepancy has also led to renewed scrutiny of the procedure used to extract rms radii from electron-scattering data. Problematic in particular is the fact that the (e,e) data sensitive to proton finite size are the ones in the region of momentum transfer $q = 0.6$–1.2 fm^{-1} [13]; the determination of the rms radius involves an (implicit or explicit) extrapolation to $q = 0$.

Difficulties due to large-r tail. The extrapolation from finite q to $q = 0$ is much more difficult for the proton than for nuclei with mass number $A > 2$. The proton charge and magnetization distributions are of roughly exponential shape, as the form factors $G_e(q)$ and $G_m(q)$ are roughly described by dipole distributions. The exponential tail at large radii r leads to a very slow convergence of the proton rms radius as a function of the upper cutoff $r_{\rm cut}$ when calculated from the integral over the charge density $\rho(r)$. This is demonstrated in Fig. 1, which compares the convergence to the one obtained for a heavy nucleus. In order to get 98% of the proton radius, one has to integrate out to an $r_{\rm cut}$ of three times the rms radius. The uncertainty of this large-r tail, which is poorly fixed by the (e,e) data, corresponds to uncertainties in the shape of $G(q)$ at very low q; this in turn complicates the standard extrapolation of $G(q)$ from the momentum transfers covered by data to $q = 0$ where the radius is determined from the $q = 0$ slope.

We illustrate the difficulties of this extrapolation with several recent results of form-factor fits. For this *qualitative* discussion—carried out to understand what happens rather than to derive numerical results—we ignore complications such as relativistic effects or two-photon exchange and take the density (charge or magnetic) $\rho(r)$ as the Fourier transform of the corresponding Sachs form factor $G(q)$ and vice versa.

As a first example, we consider fits to the (e,e) data recently measured by Bernauer *et al.* [14,15] covering the region between $q_{\min} = 0.55$ fm^{-1} and $q_{\max} = 5.1$ fm^{-1}. These authors used different parametrizations of $G_e(q)$ and $G_m(q)$ to fit the data and extract the $q = 0$ slopes. They showed in particular the results for fits using an inverse polynomial (IP) parameterization and different orders N of the polynomial. A curious result was found for the magnetic rms radius, which jumped between order $N = 7$ and $N = 10$ from 0.76 to 0.96 fm [14]. The strange behavior of the $N = 10$ result (which, incidentally, has the best χ^2 per degree of freedom) turns out to be due to the fact that at $q > q_{\max}$ the $N = 10$ IP form factor has a pole. The density corresponding to such a $G(q)$ has an (oscillatory) tail that extends to extremely large radii ($r = \infty$). This large-r density implies structure of $G(q)$ at very low q, below $q = q_{\min}$, and the curvature at very small q falsifies the extrapolation to $q = 0$ where the radius is extracted.

We must point out that the situation for the rms radius from the IP fit of order $N = 7$ (selected by Ref. [15]) is not much better; G_m has a pole as well, although at larger momentum transfer with consequences for the density and radius that are less severe.

As a second example we mention the form factor fits to the same data [15] made by Lorenz *et al.* [16] using a continued fraction (CF) parametrization. It turns out that the five-parameter CF fit [17] to the full data set also has a pole with correspondingly large values of the density at large r, so the value of the rms radius cannot be relied upon either.

As a third example, we discuss a fit we have made while exploring the role of the cutoff $q_{\max}$ in the determination of the rms radius. For sake of easier comparison, we use the same data, but only up to $q_{\max} = 2$ fm^{-1}; this is entirely sufficient

*ingo.sick@unibas.ch
†Dirk.Trautmann@unibas.ch

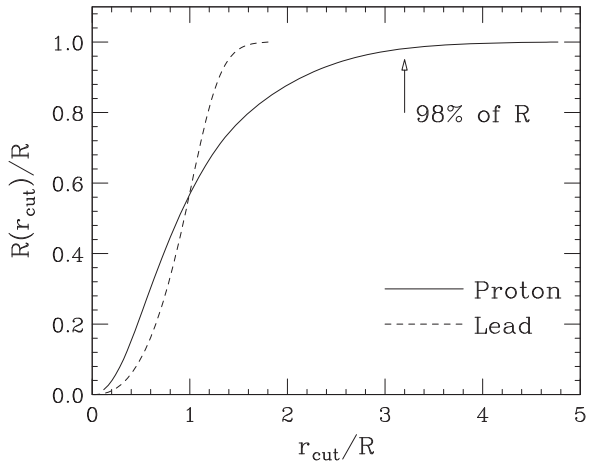

FIG. 1. $[\int_0^{r_{\rm cut}} \rho(r) r^4 dr / \int_0^{\infty} \rho(r) r^4 dr]^{1/2}$ as a function of the upper integration limit $r_{\rm cut}$. Both axes are normalized to the rms radius R.

as the data below 2 fm^{-1} are the only ones sensitive to the rms radii [13]. The data have been fitted with a four-parameter Pade parametrization for G_e and G_m. The fit gives an excellent χ^2 of 1.06 per degree of freedom, as low as a spline fit to the data [14]. The fitted $G(q)$ has no pole but yields a charge rms radius of 1.48 fm! As for the previous examples, the problem is a consequence of an uncontrolled behavior of the form factor for $q > q_{\rm max}$ which, together with $G(q_{\rm min} < q < q_{\rm max})$, implies a shape of $\rho(r)$ that has a tail to extremely large r.

This unreasonable result can be understood by looking at Fig. 2, which shows the form factor $G_e(q)$ in the region $q < q_{\rm min}$. The solid curve represents the Pade fit, and the dotted curve corresponds to a usual parameterization yielding an rms

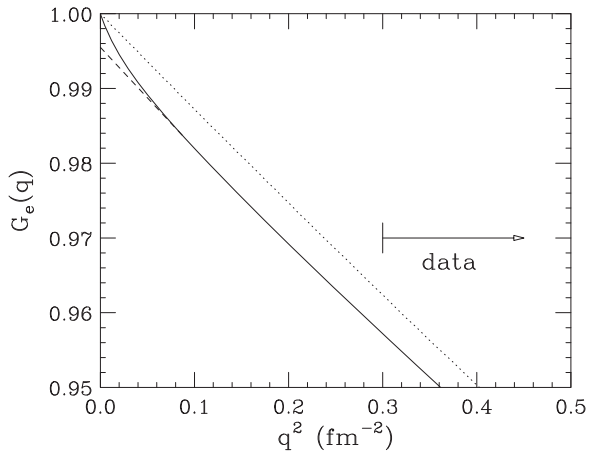

FIG. 2. Pade fit to the data $q < 2$ fm^{-1} of Ref. [15] (solid curve) yielding an rms radius of 1.48 fm. A "standard" fit to the same data, producing an rms radius of ∼0.88 fm, is given by the dotted curve. [Note that the normalization of the data (not shown) is floating, so both fits give excellent χ^2 when compared to data normalized to the respective fit.] The form factor $G_1(q)$ corresponds to the solid curve for $q^2 > 0.06$ fm^{-2}, supplemented for $q^2 < 0.06$ fm^{-2} by the dashed curve. $G_2(q)$ corresponds to the difference between the solid curve and $G_1(q)$.

radius of ∼0.88 fm from a correspondingly smaller $q = 0$ slope. One can conceptually split the Pade curve into two additive contributions, G_1 and G_2. $G_1(q)$ corresponds to the Pade fit for $q^2 > 0.06$ fm^{-2}, supplemented for $q^2 < 0.06$ fm^{-2} by the dashed curve; $G_2(q)$ corresponds to the difference of the Pade form factor and $G_1(q)$. The form factor $G_1(q)$ has a norm of ∼0.995 and a "normal" slope at $q = 0$ corresponding to an rms radius of ∼0.88 fm. The $G_2(q)$ term looks, roughly speaking, like a Gaussian e^{-q^2/a^2} with $a^2 \sim 0.02$ fm^{-2} and a norm of ∼0.005. In r space, this G_2 corresponds to a density of Gaussian shape, proportional to $e^{-r^2/(200\,{\rm fm}^2)}$. This term extends to extremely large values of r. Its contribution at large r leads, despite the small overall norm, to the unreasonable rms radius of 1.48 fm.

From these examples it becomes clear that the usual way of determining the proton rms radii by extrapolating the data from $q_{\rm min} < q < q_{\rm max}$ to $q = 0$ via a parameterization of $G(q)$ is unreliable. Even for standard parameterizations such as the Pade parametrization, with no obvious faults such as poles at $q > q_{\rm max}$ or a divergence at large q such as present in the popular expansions of $G(q)$ in terms of powers of q^2, unreasonable results can be generated. Such procedures cannot be trusted to yield the physical rms radius.

Density at large r. In order to better understand the origin of the difficulty, it is helpful to confront the approaches used for the determination of rms radii for nuclei with mass number $A > 2$ and $A \leqslant 2$. For $A > 2$ the density is parameterized, the observables are calculated via a solution of the Dirac equation, and the parameters are adjusted for good χ^2. The rms radius is obtained from an integral over $\rho(r)$ or, equivalently, from the $q = 0$ slope of the corresponding Born approximation form factor. For $A \leqslant 2$, on the other hand, the form factors $G(q)$ in general are parameterized directly in q space, the observables are calculated with or without corrections beyond first Born approximation and the parameters adjusted for the best χ^2. The slope of $G(q)$ at $q = 0$ gives the rms radius.

These two approaches are *not* equivalent. When parameterizing $\rho(r)$ using, e.g., Fermi densities, sum of Gaussians (SOG), or Fourier-Bessel (FB) series, an implicit assumption is made that $\rho(r) = 0$ outside some radius r_0, or that $\rho(r)$ falls off exponentially at large r. This assumption reflects our physics understanding of nucleon and nuclear wave functions: The density at large enough r must approach zero.

When parameterizing $G(q)$ directly in q space, this condition on $\rho(r)$ at large r is *not* used. Depending on the parametrization chosen for $G(q)$ totally unreasonable behaviors of $G(q > q_{\rm max})$ and the corresponding $\rho(r)$ at large r may occur, and these are responsible for the erroneous rms radii. This is clearly a serious deficiency that calls for a different approach.

The problem induced by an uncontrolled behavior of $G(q > q_{\rm max})$ is less severe for fits that use the data up to the largest momentum transfers where cross sections and polarization transfer data are available. In these cases, the values of the fitted $G(q > q_{\rm max})$ are strongly constrained by the small values of the $G(q)$'s near the top end of the q range covered by experiment. Parameterizations that enforce in addition a fall of at least $1/q^4$ as required to get a regular density at the origin then help to

ensure that the values of $G(q > q_{\max})$ remain small and that the data constrain the shape of $\rho(r)$ (including its tail) as much as possible.

From the parametrization of $G(q)$ alone it is very difficult to judge the behavior $\rho(r)$ other than by looking simultaneously at the corresponding density [provided the parametrization of $G(q)$ does have a Fourier transform]. As a matter of fact, most published fits to e-p data have not been checked for the large-r behavior of ρ and often yield unreasonable behavior, as the above examples show.

Of the parametrizations that have been used in the past to fit the e-p scattering data, there are basically only two types that constrain the large-r behavior of the density: On the one hand side there are the fits based on the vector dominance model (VDM) supplemented by 2π exchange [16–22]. They implicitly constrain the large-r falloff through the masses of the exchange particles that are assumed to mediate the photon-nucleon interaction. These VDM-type fits, however, have in general produced values of χ^2 that are significantly larger than those achieved by more phenomenological parametrizations and they show systematic deviations from the data at low momentum transfer; the model seems to lack the freedom required for a good fit of the e-p data. The radii extracted from a poor fit obviously cannot be trusted.

The other approach that ensures a good behavior of the large-r density is the one involving fits with SOG densities fitted to both the e-p data and a calculated shape of the large-r density [23].

To remedy the unsatisfactory situation with extrapolation of $G(q)$ to $q = 0$ one can proceed as described below. This procedure, unfortunately, is more involved than a simple fit of (e,e) data using a convenient parameterization for $G(q)$, but it helps to avoid the pitfalls discussed above.

In order to guarantee a sensible behavior of the density at large radii, one should employ parametrizations that have an easily accessible form in both r and q space. One then can easily check the r space behavior while fitting the parameters directly in q space without taking too literally the form factor as a Fourier transform of the density.

For the proton (and the same is true for the deuteron; see Ref. [24]) the pronounced sensitivity of the rms radius to the tail of the density can be reduced by using a physical model for the large-r density. At large distances, the density of any bound multiconstituent system is dominated by the least-bound Fock component, in the case of the proton the π^+ from the neutron + pion ($n + \pi^+$) configuration. The corresponding (relative) density can be easily calculated; sophistications such as inclusion of relativistic effects or two-photon exchange (which affect the relation between density and form factor), pion finite size, $\Delta + \pi$ components, etc., can be incorporated (their numerical effect on the tail shape has been found to be relatively small [23]). The resulting density can be used in the fit to constrain the shape of $\rho(r)$ at very large r where the density is safely in the asymptotic regime. In this approach the most dangerous aspect of the model dependence—implicit assumptions on the large-r behavior due to the choice of the model density or model form factor which affect the curvature of $G(q)$ below $q_{\min}$—is replaced by a tail constraint that is explicitly stated and can be included with appropriately specified uncertainties.

When using this type of approach one can employ rather general multiparameter expansions of the density and form factors such as Hermite and Laguerre polynomials times their weight function, or SOG densities; these fit with good χ^2 the data over the full q range where data are available. As we have discussed above, this is most desirable in order to also constrain as much as possible the shape of the density, including the large-r tail, by the e-p data themselves.

A fit of the *world* data following the above procedure has been carried out in Ref. [23] and yielded an rms radius of 0.886 ± 0.008 fm.

Conclusions. While the standard parametrizations of the nucleon form factors $G_e(q)$ and $G_m(q)$ fitted to e-p scattering data are valid representations of the data in the q region where they have been measured, they are not suitable for an extrapolation to $q = 0$ where the proton rms radii are extracted. These parametrizations of the $G(q)$'s lack the constraint that the corresponding densities must approach zero at large r in a way that is compatible with our physics understanding of the proton. Most published fits do not respect this constraint and often produce unreliable radii.

Constraining the large-radius tail of the density using physical arguments is bound to yield a more reliable value for the physical rms radius.

[1] R. Pohl, A. Antognini, F. Nez, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. A. Covita, A. Dax, S. Dhawan, L. M. P. Fernandes, A. Giesen, T. Rraf, T. W. Hänsch, P. Indelicato, L. Julien, C.-Y. Kao, P. Knowles, J. A. M. Lopes, E.-O. Le Bigot, Y.-W. Liu, L. Ludhova, C. M. B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J. M. F. dos Santos, L. Schaller, K. Schuhmann, C. Schwob, T. Taqqu, J. F. C. A. Veloso, and F. Kottmann, Nature (London) **466**, 213 (2010).

[2] P. Mohr, B. Taylor, and D. B. Newell, Rev. Mod. Phys. **84**, 1527 (2012).

[3] C. G. Parthey, A. Matveev, J. Alnis, B. Bernhardt, A. Beyer, R. Holzwarth, A. Maistrou, R. Pohl, K. Predehl, T. Udem, T. Wilken, N. Kolachevsky, M. Abgrall, D. Rovera, C. Salomon, P. Laurent, and T. W. Hänsch, Phys. Rev. Lett. **107**, 203001 (2011).

[4] P. G. Blunden, W. Melnitchouk, and J. A. Tjon, Phys. Rev. C **72**, 034612 (2005).

[5] P. G. Blunden and I. Sick, Phys. Rev. C **72**, 057601 (2005).

[6] A. de Rujula, Phys. Lett. B **697**, 26 (2011).

[7] I. C. Cloët and G. A. Miller, Phys. Rev. C **83**, 012201 (2011).

[8] U. D. Jentschura, Ann. Phys. **326**, 516 (2011).

[9] U. D. Jentschura, Ann. Phys. **326**, 500 (2011).

[10] U. D. Jentschura, arXiv:1107.1737.

[11] J. D. Carroll, A. W. Thomas, J. Rafelski, and G. A. Miller, Phys. Rev. A **84**, 012506 (2011).

[12] C. E. Carlson and M. Vanderhaeghen, arXiv:1109.3779.

[13] I. Sick, Can. J. Phys. **85**, 409 (2007).

[14] J. C. Bernauer, Ph.D. Thesis, University of Mainz, Germany, 2010.
[15] J. C. Bernauer, P. Achenbach, C. Ayerbe Gayoso, R. Böhm, D. Bosnar, L. Debenjak, M. O. Distler, L. Doria, A. Esser, H. Fonvieille, J. M. Friedrich, M. Gómez Rodríguez de la Paz, M. Makek, H. Merkel, D. G. Middleton, U. Müller, L. Nungesser, J. Pochodzalla, M. Potokar, S. Sánchez Majos, B. S. Schlimme, S. Širca, T. Walcher, and M. Weinriefer, Phys. Rev. Lett. **105**, 242001 (2010).
[16] I. T. Lorenz, H.-W. Hammer, and U.-G. Meissner, Eur. Phys. J. A **48**, 151 (2012).
[17] I. T. Lorenz (private communication).
[18] F. Iachello, A. D. Jackson, and A. Lande, Phys. Lett. B **43**, 191 (1973).
[19] G. Hoehler, E. Pietarinen, I. Sabba-Stefanescu, F. Borkowski, G. G. Simon, V. H. Walther, and R. D. Wendling, Nucl. Phys. B **114**, 505 (1976).
[20] P. Mergell, U.-G. Meissner, and D. Drechsel, Nucl. Phys. A **596**, 367 (1996).
[21] E. L. Lomon, Phys. Rev. C **64**, 035204 (2001).
[22] H.-W. Hammer, D. Drechsel, and U.-G. Meissner, Phys. Lett. B **586**, 291 (2004).
[23] I. Sick, Prog. Part. Nucl. Phys. **67**, 473 (2012).
[24] I. Sick, Prog. Nucl. Part. Phys. **47**, 245 (2001).