Global phenomenological descriptions of nuclear odd-even mass staggering

D. Hove, A. S. Jensen, and K. Riisager
Department of Physics and Astronomy, Aarhus University, DK-8000 Aarhus C, Denmark
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We examine the general nature of nuclear odd-even mass differences by employing neutron and proton mass relations that emphasize these effects. The most recent mass tables are used. The possibility of a neutron excess dependence of the staggering is examined in detail in separate regions defined by the main nuclear shells, and a clear change in this dependency is found at $Z = 50$ for both neutrons and protons. A further separation into odd and even neutron (proton) number produces very accurate local descriptions of the mass differences for each type of nucleons. These odd-even effects are combined into a global phenomenological expression, ready to use in a binding energy formula. The results deviate from previous parametrizations, and in particular found to be significantly superior to a recent two term, A^{-1} dependence.

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I. INTRODUCTION

The systematic odd-even mass difference in nuclei was recognized early on, see [1] for a brief historical overview. Several effects will contribute to the experimentally observed differences, in particular pairing and the twofold degeneracy of orbits, see [2–4] for an overview of recent work. Traditionally the odd-even mass staggering has been parametrized by a power law in the mass number A (both $A^{-1/2}$ and $A^{-1/3}$ have been used), but other functional forms have been used, e.g., a constant and a A^{-1} term in [3]. We emphasize that these smoothly varying parametrizations only were meant to reproduce values of staggering averaged over many neighboring nuclei. The odd-even mass differences has traditionally been attributed to nucleon pairing (the largest part) and breaking of the time reversal double degeneracy of the single particle levels (the smallest part). Other effects may contribute as well.

Both experimentally and theoretically one observes significant, systematic deviations from the simple laws [4] and it therefore seems worthwhile to make use of the recently updated, extensive and accurate, nuclear mass table [5] to look more carefully for trends in the experimental odd-even staggering. We shall in particular reinvestigate the suggestion of an explicit dependence on neutron excess [6,7] that was not supported by nuclear pairing models [8]. Our aim is to find an improved phenomenological description of the odd-even mass differences that may be used in combination with semiempirical mass models, and perhaps reveal trends that could inspire future more basic theoretical work.

The relevant theoretical considerations are presented in Sec. II. In particular, the relevant mass relations designed specifically to isolate odd-even effects are introduced. Mass relations of this nature were investigated in detail by Jensen *et al*. [7], but the relations used here are more compact, and are applied with the sole purpose of analyzing odd-even effects in general.

Section III contains the initial examination of odd-even mass differences. The focus is on the general structure of the staggering effects, and to that end the results, free of any manipulations, are presented in this section. To provide a more general overview of this structure Sec. III B presents a three dimensional illustration of neutron staggering as a function of both N and Z. Also included in Sec. III is a short evaluation of the extent of the shell effects.

Section IV contains the examination of staggering effects as a function of isospin projection. This includes both a separation according to odd-even neutron and proton configurations, as well as separation into regions defined by nuclear shells. Having established the effect of each separation the results are combined into one global expression, which describes the collective odd-even staggering effect and includes neutron-proton pairing explicitly. Finally, in the results are compared to a very accurate recent two-term description with an A^{-1} dependency.

II. THEORETICAL FOUNDATION

Fundamental to all following examinations is a rather general separation of the binding energy into three parts [9]

$$B(N,Z) = B_{LD}(N,Z) + B_{sh}(N,Z) + \Delta(N,Z). \quad (1)$$

All smooth aspects are contained in the liquid drop term, B_{LD}, whereas B_{sh} accounts for the smaller, but faster oscillating localized, nuclear shells. The last term, Δ, has to include all other contributions to the nuclear binding energy, that is various types of correlations and in particular variations depending on the parity of the nucleon numbers. The intent in the present paper is to analyze the neutron and proton staggering included in Δ looking for possible global dependencies. To that end any influence from both B_{LD} and B_{sh} must be eliminated. Since these terms by far are the largest some care must be exercised to isolate the desired effects from the measured binding energies.

The smooth contributions (mainly the liquid drop term, B_{LD}) is in principle easily eliminated to any order by appropriately constructed binding energy differences [7]. Removal of first order is achieved by use of the double difference

$$Q(n,z) = -B(N-n, Z-z) + 2B(N,Z) - B(N+n, Z+z). \quad (2)$$

Taylor expansion of the smooth part, B_{LD}, of each of the three B terms define Q_{LD} as the smooth part of Q, that is

$$Q_{LD} = -n^2 \frac{\partial^2 B_{LD}}{\partial N^2} - 2nz \frac{\partial^2 B_{LD}}{\partial N \partial Z} - z^2 \frac{\partial^2 B_{LD}}{\partial Z^2}. \quad (3)$$

This reduction, to only second-order contributions of smooth part through Eq. (2), is very significant but it might not provide sufficient accuracy. Extension to more elaborate mass relations would formally improve this accuracy, but introduce other uncertainties. We shall instead use the liquid drop model itself to eliminate remaining smooth contributions. This amounts to use of $Q - Q_{LD}$ instead of Q, where Q_{LD} can be taken from Eq. (3) or directly from a liquid drop model without any expansion. Our method to extract the odd-even staggering eliminates almost all liquid drop smooth background contributions. As our intention is to study the odd-even staggering specifically, and not the liquid drop model itself, a simple version is employed containing only the most fundamental terms

$$B_{LD} = -a_v A + a_s A^{2/3} + a_c \frac{Z^2}{A^{1/3}} + a_a \frac{(Z - A/2)^2}{A}, \quad (4)$$

where $a_v = 15.56$, $a_s = 17{,}23$, $a_c = 0.697$, and $a_a = 93.14$, all in units of MeV. The accuracy with this expression and those parameter values is sufficient for our purpose.

Any mass relation based on Eq. (2), and with the liquid drop terms subtracted, will then dramatically reduce contributions from the unwanted smooth parts of the binding energy. We also need to eliminate unwanted contributions from B_{sh}. Shell effects vary rather discontinuously across magic numbers while relatively smooth by moving small steps from magic number to either side of it. Therefore the three-point mass relation in Eq. (2) would have only a very small contribution from B_{sh} provided magic numbers for both N and Z are excluded. In general only one side of magic numbers should be allowed to enter the mass relations employed. These expectations will be examined in greater detail in Sec. III C.

Choosing a specific mass relation which emphasizes either neutron or proton odd-even staggering now allows detailed and accurate investigations of this effect contained in Δ from Eq. (1). Two mass relations are in particular ideally suited to study the neutron and proton staggering. They are given as

$$\Delta_n = \tfrac{1}{2}\pi_n(Q(1,0) - Q_{LD}(1,0)), \quad (5)$$

$$\Delta_p = \tfrac{1}{2}\pi_p(Q(0,1) - Q_{LD}(0,1)), \quad (6)$$

where π_n and π_p assure a positive result, when defined as

$$\pi_n = (-1)^N, \pi_p = (-1)^Z. \quad (7)$$

The definition and normalization of Δ_n corresponds to an additional binding energy of Δ_n for even compared to odd values of N. The nuclei contributing to Δ_n and Δ_p in the NZ plane are seen in Fig. 1. They are ideal for isolating odd-even effects as either horizontal or vertical with alternating sign for each isotope. When we have extracted Δ_n and Δ_p from experimental masses, either as numbers or as parametrized analytic expressions, the corresponding contribution to Δ in

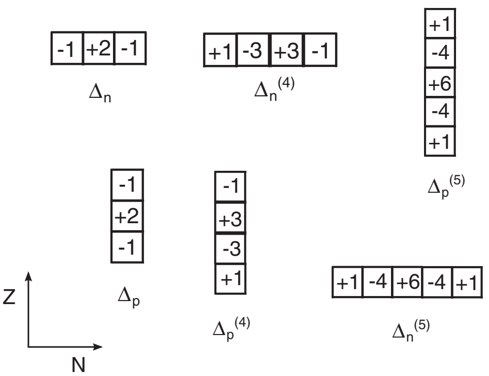

FIG. 1. The structure of Δ_n, Δ_p, $\Delta_n^{(4)}$, $\Delta_p^{(4)}$, $\Delta_n^{(5)}$, and $\Delta_p^{(5)}$ given in the NZ plane. The relative weight of each individual isotope, not including normalizing factors, is also included.

Eq. (1) could be expected to be

$$\Delta = \tfrac{1}{2}(\pi_n \Delta_n + \pi_p \Delta_p). \quad (8)$$

This expression has the classical form for the odd-even mass difference, often referred to as pairing effects although other substantial contributions also can be included.

It may appear as if Δ_n and Δ_p are expressions related solely to either neutron or proton odd-even mass differences. However, the three-point mass relation in Eq. (2) also includes a contribution from neutron-proton pairing effects that are known to be sizable [7,10]. Assuming the neutron-proton pairing effect results in a term, $C\pi_{np} = C(1 - \pi_n)(1 - \pi_p)/4$ (i.e., only a contribution for odd-odd nuclei), then Δ_n would receive a contribution $-C(1 - \pi_p)/2$, and analogously for Δ_p and Eq. (8) must be corrected for this. Equations (5) and (6) therefore reflect odd-even effects in general, including terms arising from possible neutron-proton couplings.

It is a choice to use three-point mass relations to study odd-even effects. In fact, any number of neighboring masses can be combined to provide information about similar effects. Still in any case, unwanted contributions must be eliminated. Replacing the binding energies in Eq. (2) with separation energies of the form $S(N,Z) = B(N,Z) - B(N - n', Z - z')$ would eliminate the smooth terms to second order, as demonstrated by Jensen *et al.* [7]. The corresponding mass relations with $n = z = n' = z' = 1$ result in a combination of four nuclei, denoted $\Delta_n^{(4)}$ and $\Delta_p^{(4)}$ as shown in Fig. 1. This type of four-point nuclear mass relation contains unequal weights on even and odd nuclei which only results in minor inaccuracies. It is ideal for studying the neutron and proton pairing, but it will average the results for odd and even particle numbers and is therefore not recommended [3,11].

Alternatively, a structure eliminating smooth terms to third order could be constructed. This would combine five nuclei as demonstrated by $\Delta_{n,p}^{(5)}$ in Fig. 1. The main drawback is the extent of the structure which implies averaging over more nuclei further apart. By combining five nuclei relatively far apart the odd-even effects would be significantly diminished as a result of the implicit averaging, see also the detailed comparison between Δ and $\Delta^{(5)}$ in Ref. [12].

In summary, for investigation of odd-even mass staggering the most suitable structure is the compact, three nuclei structure with the liquid drop contribution subtracted as presented in Eqs. (5) and (6).

In general, the term Δ in B from Eq. (1) by definition contains all the effects beyond those included in B_{LD} and B_{sh}. Extraction of specific contributions to the nuclear binding energy is done by construction of mass relations dedicated to isolate the desired effects and simultaneously remove all significant contributions from B_{LD} and B_{sh}. These two requirements are not altogether mutually compatible. Removal of smooth parts in B cannot distinguish between the different terms, B_{LD}, B_{sh} or Δ. Only the form of the parts of the assumed Δ contribution is distinguishable, but within such a form still smooth contributions would vanish, even when it is desirable to know them.

We emphasize that extraction of each type of correlation contribution has to be done with a precisely corresponding mass relation. The result is an additive piece (not everything) to the Δ term in B which then should be included in future binding energy expressions. A number of different terms can then accumulate.

III. EXAMINING ODD-EVEN NEUTRON AND PROTON STAGGERING

In the following section the mass relations defined in Eqs. (5) and (6) are used to examine neutron and proton mass staggering respectively. In Sec. III B a detailed look at the neutron effect in three dimensions is presented. This should provide a more general insight into the nature and structure of the odd-even effects. Finally, in Sec. III C the extent of the shell effects is examined based on work by Dieperink and Van Isacker [13]. All measurements used are taken from the recent compilation by Audi *et al.* [5].

A. Isolated odd-even effects

Before any description of general tendencies is attempted the odd-even effects in isolation are presented. Applying the fundamental Δ_n and Δ_p relations from Eqs. (5) and (6) results in Figs. 2(a) and 2(b), respectively. These figures very directly show the odd-even effects in almost complete isolation. All available nuclei have been included as these figures' main purpose is to offer a general impression of the effects. Later, when making more quantitative examinations, some nuclei influenced by unaccountable effects are excluded.

The global behavior seen in the figures is well known. Included in both figures are just over 2100 nuclei, and their conformity immediately suggests a deeper lying structure. Initially, the decline in energy could suggest a power-law dependence on A, but there is a considerable scatter and also clear substructures, most noticeably for Δ_n with $A > 100$ as we shall see later. These structures will be examined in detail in the following sections.

It is also important to note the similarities between Figs. 2(a) and 2(b). Both the scale and the general structure is essentially identical. This similarity between the results for neutron and proton mass relations occurs in all following examinations.

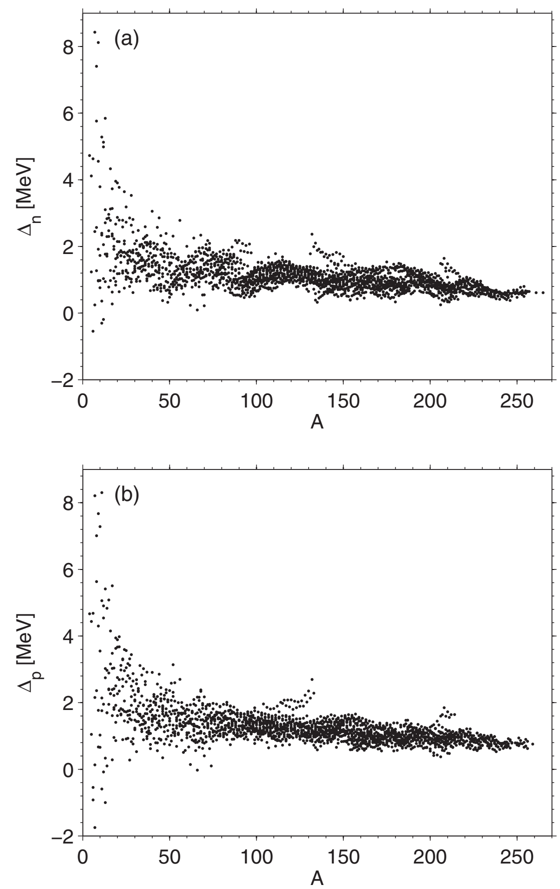

FIG. 2. The odd-even staggering Δ_n and Δ_p as a function of A evaluated using Eqs. (5) and (6).

To avoid tedious repetitions, figures displaying proton mass relations will not be included.

Although a few very light isotopes ($A < 10$) have rather large values, the general scale is almost constant around ~ 0.5–1.5 MeV. This change in structure from the light to the heavy nuclei has led to the suggestion of more sophisticated models in place of the simplest power law dependence used traditionally. Friedman and Bertsch suggested [3] a two term expression given by $\Delta = c_1 + c_2/A$, which provides a more than reasonable global description of the odd-even staggering. This description will be examined more closely in Sec. IV B.

B. General structure of the odd-even effect

A more detailed view of the odd-even effects could be useful when attempting to identify general trends. Trying to describe the effects as functions of only one variable is an unfounded restriction. The possibilities are limited, even if expressing Δ_n and Δ_p as a function of A or $N - Z$. In Fig. 3 Δ_n is shown as a function of both N and Z, which provides a detailed look at the actual structure of the staggering effect in the table of nuclides.

The purpose of these three dimensional figures is to study the substructures visible in Fig. 2 more closely, and to determine which kind of odd-even effects are immediately visible. Inspired by Fig. 2 the three-dimensional figures are limited to the heavier isotopes. The effect of shells are clearly

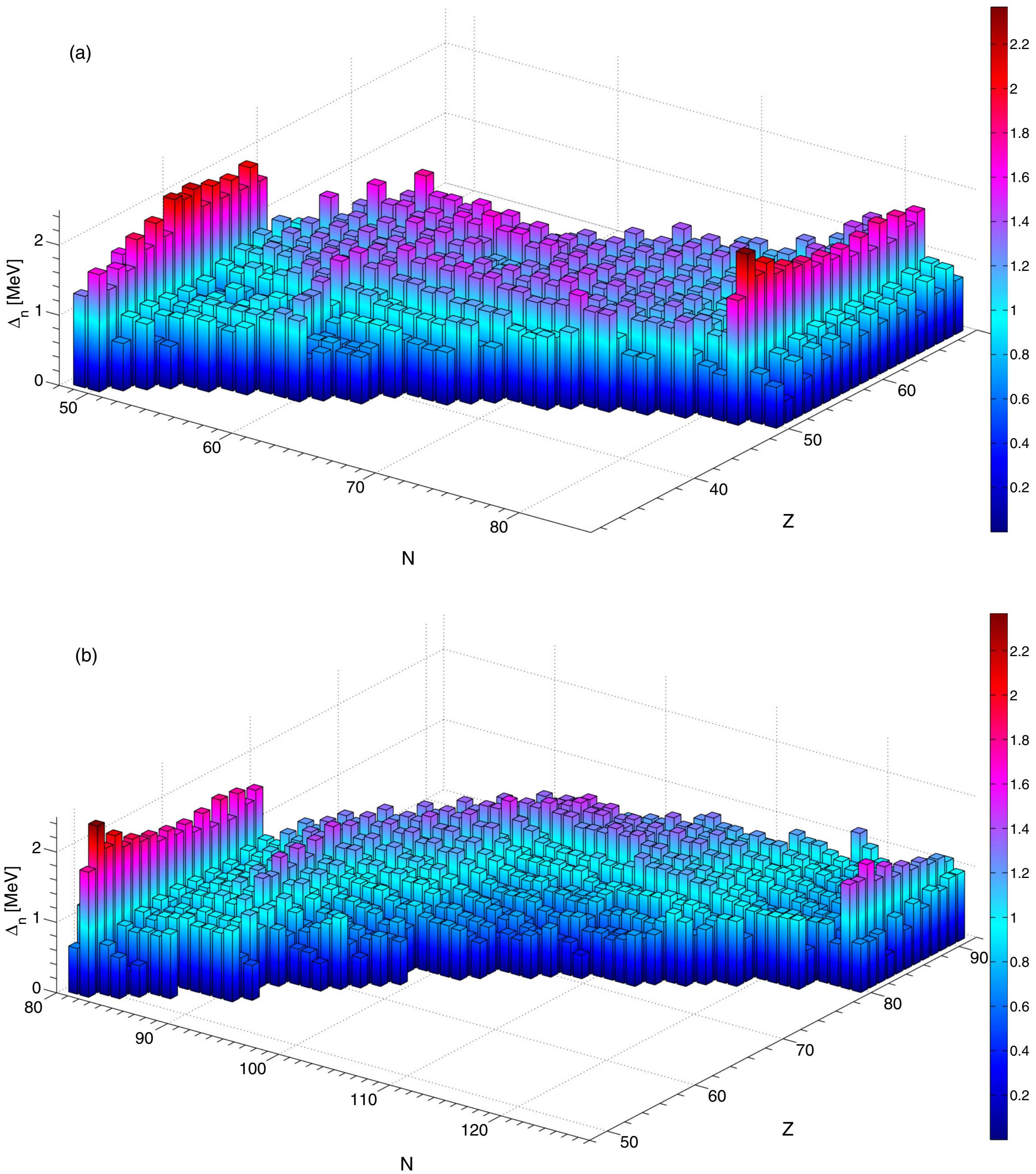

FIG. 3. (Color online) The (N,Z) dependence of Δ_n. Figure (a) shows $49 < N < 84$ and $30 < Z < 69$, and Figure (b) shows $81 < N < 127$ and $49 < Z < 91$.

seen around magic numbers, but as expected they are very localized. Figure 3(a) shows the isotopes between the neutron shells at $N = 50$ and $N = 82$ and Fig. 3(b) the isotopes between the neutron shells at $N = 82$ and $N = 126$. The proton number has an obvious effect on Δ_n: when Z is odd Δ_n is significantly lower compared to the neighboring even-Z nuclei. In both figures another, albeit smaller, odd-even effect is also seen when changing neutron number. When N is even Δ_n is slightly smaller for the neighboring odd-N nuclei.

In addition to the neutron shells a smaller effect at the magic proton numbers $Z = 50$ and $Z = 82$ is also seen. Given that a correlation between Δ_n and proton number has already been established, this result is only somewhat surprising. The effect is also less sharply defined compared to the effect of neutron shells.

Generally, Δ_n has a clear tendency to decrease away from $N = Z$, and the tendency is more pronounced for heavier nuclei. This could indicate a neutron excess dependency, which will be examined in detail in Sec. IV.

C. Extent of shell effects

As seen in Fig. 3 the effect of nuclear shells around magic numbers is very distinct. Although it appears to be

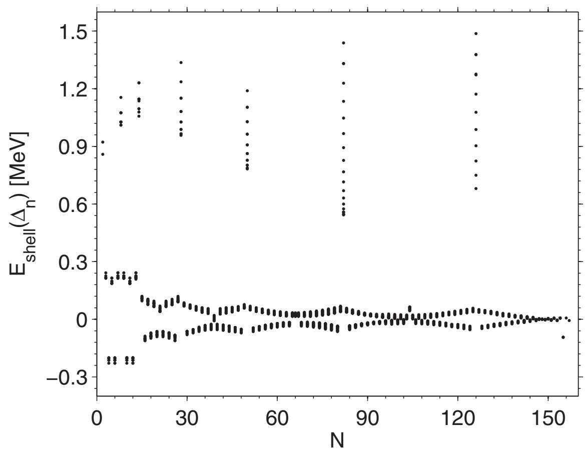

FIG. 4. The extent of shell effects. Values calculated using Dieperink and Van Isacker's expression Eq. (9) are combined as in Eq. (2) using $Q(1,0)$ to estimate the remaining shell contribution for Δ_n.

very localized, a more precise estimation of the actual extent would be useful. The extent can be estimated by applying the fundamental relations from Eqs. (5) and (6) to a quantitative expression of the shell effect. To this end Dieperink and Van Isacker's [13] general expression is used:

$$E_{\text{shell}}(N,Z) = (-1.39S_2 + 0.020(S_2)^2 + 0.003S_3 + 0.075S_{np})\,\text{MeV}, \tag{9}$$

where

$$S_2 = \frac{n_v\bar{n}_v}{D_n} + \frac{z_v\bar{z}_v}{D_z},\quad S_3 = \frac{n_v\bar{n}_v(n_v-\bar{n}_v)}{D_n} + \frac{z_v\bar{z}_v(z_v-\bar{z}_v)}{D_z},\quad S_{np} = \frac{n_v\bar{n}_v z_v\bar{z}_v}{D_n D_z}. \tag{10}$$

Here n_v and z_v are the number of valence nucleons or holes, and $D_{n,z}$ is the degeneracy of the shell. Finally, $\bar{n}_v \equiv D_n - n_v$ and $\bar{z}_v \equiv D_z - z_v$. The magic numbers used by Dieperink and Van Isacker are 2, 8, 14, 28, 50, 82, 126, 184.

The result is presented in Fig. 4, and as expected the effect is extremely localized. A contribution on the scale of 1 MeV around magic numbers seems reasonable when compared to Fig. 3, but otherwise the effect is less than ~ 0.1 MeV, and essentially negligible. The same localized result with respect to Z is found when using Eq. (6) instead of Eq. (5).

Based on these results the exclusion of mass relation evaluations which includes magic numbers should be justified. Unless otherwise stated no corrections are made for the shell effects, the relevant nuclei are merely excluded from the calculations.

Alternatively, the nuclei, which deviates because of shell effects could be determined by using a mass relation like Eq. (2) with $n = 2$ and $z = 2$. Then the odd-even effects would be canceled, and the shell effects would be left in relative isolation. This is done successfully in detail in [14] with a four nucleus mass relation.

IV. DETAILED SEGMENTED ANALYSIS

Section IV A looks at a possible neutron-excess dependency for Δ_n and Δ_p. The nuclei are separated into groups according to even and odd neutron and proton numbers. In addition, the nuclei are divided into regions defined by shells in a similar manner as in Ref. [15] in order to see possible related structure changes. In Sec. IV B the separated nuclei are combined into a globally valid model of the odd-even effects. This is in Sec. IV C compared to a global, two term description, with an A^{-1} dependence.

A. Neutron-excess dependency

As indicated by Figs. 3(b) and 3(a) the general nature of the staggering effects seems to change around $Z = 50$. In order to examine this possibility closer the following results are divided into areas defined by shells.

The structures in question given by Eqs. (5) and (6) involves an odd number of nuclei as seen in Fig. 1. Either a change in proton number or neutron number should then result in a staggering effect, as a result of Pauli's principle. To thoroughly explore both possible staggering effects we consider separately nuclei with (N,Z) being even-even, even-odd, odd-even, and odd-odd.

The isospin projection dependency employed here is, as in [6,7], a scaled $(A^{1/3})$ quadratic neutron-excess dependency

$$A^{1/3}\Delta_{n,p} = a\left(\frac{N-Z}{A}\right)^2 + b, \tag{11}$$

where a and b are constants.

A linear neutron excess dependence was suggested in connection with the Duflo-Zuker mass formula [16]. The corresponding global parametrization with a single parameter was in [17] added to a liquid drop formula to describe the odd-even staggering. As is seen in Fig. 5 the data do not clearly distinguish the different functional forms of the neutron excess dependence. The dashed, blue line is the best fit with a linear (absolute value) neutron excess dependence, as used in the Duflo-Zuker mass formula [16]. The difference between the curves is minute compared to the scatter of the points. When evaluating the root mean squared error of $\Delta_{n,p}$ a difference of 0.01 MeV out of a value of 0.16 MeV is found for the two functional forms. Comparing fits for a series of different regions in the isotopic map we find differences amounting only by at most 0.01 MeV for all cases shown in Table I for the quadratic relation.

We shall focus on nuclei with $A > 50$, where the one-parameter, linear neutron excess dependence is less suited for extrapolation into unknown mass regions. This is mainly due to the linear form which would increase Δ_p away from stability for the proton rich nuclei with $N > Z$. The global value of the single parameter would produce less accurate results for any specific region of nuclei. This global versus local parametrizations will be demonstrated in Table I in connection with descriptions based on Eq. (11). Thus, we choose to use the quadratic structure as it is naturally obtained, when expanding with respect to nucleon number and neutron excess as in

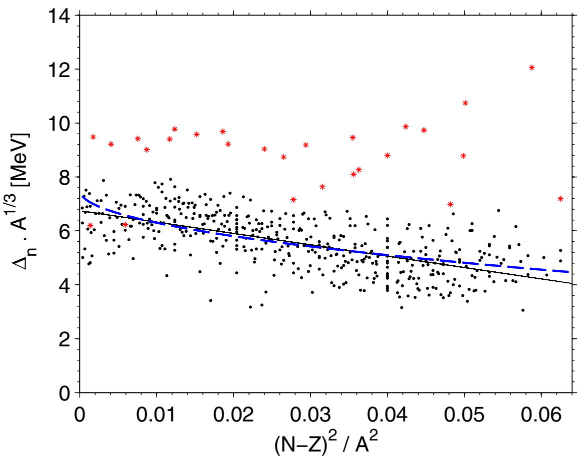

FIG. 5. (Color online) The $(\frac{N-Z}{A})^2$ dependency for all even-even nuclei with $A > 50$. The line is the best possible fit based on Eq. (11). The nuclei indicated in red (fat and full) are influenced either by the Wigner effect or by shell effects and are not included in the calculations. The dashed, blue line is the best result when using a model with a linear dependence on the absolute value of neutron excess, $|\frac{N-Z}{A}|$.

droplet models [18]. We shall briefly return to this question in Sec. IV C.

We have also explored whether other functional forms could reproduce the dependency seen in the two-dimensional plots. In particular, fits have been made replacing the $(N-Z)^2/A^2$ term in Eq. (11) with $P = n_v z_v/(n_z + z_v)$ that has been used to trace the transition to collective behavior [19]. Such transitions depend on the distance from closed shells precisely as the shell effects which as well might be a plausible reasons for the decreasing $\Delta_{n,p}$ values from magic numbers towards the middle of the shells. The resulting fits for $50 < Z$ are better than with constant terms, but not as good as with the $(N-Z)^2$ dependence. Furthermore, the sign of the coefficient in front of P differs for different regions whereas the isospin term has a more consistent behavior. Fits have also been performed replacing the $A^{1/3}$ in Eq. (11) with $A^{1/2}$ and A. Rather similar overall quality of fits were obtained, but with a slight preference for $A^{1/3}$ over $A^{1/2}$, and somewhat better than the $1/A$ dependence.

In all following calculations nuclei influenced by shell effects or the Wigner effect [18] are excluded. They are, however, indicated in red (fat and full) in the figures. All light nuclei with $A < 50$ are also excluded.

Figure 5 shows the result of applying Eq. (11) to all relevant even-even nuclei for $A^{1/3}\Delta_n$. As expected the excluded nuclei indicated in red (fat and full) deviate significantly from the observed tendencies. These tendencies are otherwise reasonable well described by Eq. (11), but upon closer inspection Fig. 5 appears to be a combination of two straight lines; an almost constant line around ~6 MeV, and a slightly decreasing line superimposed on the first.

This is examined more closely by separating the nuclei in three regions. The data for even-even nuclei are shown in Fig. 6 and the parameters of the fitted lines are given in

TABLE I. Fit parameters for Eq. (11) divided into regions defined by shells, and separated for even-even, even-odd, odd-even, and odd-odd nuclei. RMSE is the root mean squared error of $A^{1/3}\Delta_{n,p}$, and RMSE(Δ) is the root mean squared error of $\Delta_{n,p}$ only.

Type	Nuclei (N,Z)	Region limits N	Z	a	b [MeV]	RMSE	RMSE (Δ)
$A^{1/3}\Delta_n$	(e, e)	All	All	−42(5)	6.7(2)	0.81	0.16
$A^{1/3}\Delta_n$	(o, e)	All	All	−58(6)	7.6(2)	0.96	0.19
$A^{1/3}\Delta_n$	(e, o)	All	All	−30(5)	5.1(2)	0.80	0.16
$A^{1/3}\Delta_n$	(o, o)	All	All	−44(6)	5.9(2)	0.91	0.18
$A^{1/3}\Delta_n$	(e, e)	28,82	28,50	−28(10)	6.5(3)	0.93	0.20
$A^{1/3}\Delta_n$	(o, e)	28,82	28,50	−41(12)	7.4(4)	1.06	0.23
$A^{1/3}\Delta_n$	(e, o)	28,82	28,50	−15(8)	4.6(2)	0.76	0.16
$A^{1/3}\Delta_n$	(o, o)	28,82	28,50	−26(9)	5.4(3)	0.87	0.19
$A^{1/3}\Delta_n$	(e, e)	50,82	50,–	−32(12)	6.7(2)	0.48	0.09
$A^{1/3}\Delta_n$	(o, e)	50,82	50,–	−34(13)	7.4(2)	0.47	0.09
$A^{1/3}\Delta_n$	(e, o)	50,82	50,–	2(14)	4.8(3)	0.59	0.12
$A^{1/3}\Delta_n$	(o, o)	50,82	50,–	6(15)	5.4(3)	0.57	0.11
$A^{1/3}\Delta_n$	(e, e)	82,–	50,–	−60(7)	7.3(2)	0.71	0.12
$A^{1/3}\Delta_n$	(o, e)	82,–	50,–	−69(8)	7.8(3)	0.91	0.16
$A^{1/3}\Delta_n$	(e, o)	82,–	50,–	−50(8)	5.8(3)	0.74	0.13
$A^{1/3}\Delta_n$	(o, o)	82,–	50,–	−64(9)	6.5(3)	0.86	0.15
$A^{1/3}\Delta_p$	(e, e)	All	All	−27(5)	6.9(1)	0.73	0.15
$A^{1/3}\Delta_p$	(o, e)	All	All	−13(5)	5.1(2)	0.76	0.16
$A^{1/3}\Delta_p$	(e, o)	All	All	−43(6)	7.9(2)	0.86	0.17
$A^{1/3}\Delta_p$	(o, o)	All	All	−26(6)	6.1(2)	0.85	0.17
$A^{1/3}\Delta_p$	(e, e)	28,82	28,50	−6(7)	6.5(2)	0.62	0.14
$A^{1/3}\Delta_p$	(o, e)	28,82	28,50	4(8)	4.6(2)	0.72	0.16
$A^{1/3}\Delta_p$	(e, o)	28,82	28,50	−16(9)	7.5(3)	0.74	0.16
$A^{1/3}\Delta_p$	(o, o)	28,82	28,50	−2(9)	5.6(3)	0.78	0.17
$A^{1/3}\Delta_p$	(e, e)	50,82	50,–	−43(16)	7.2(3)	0.58	0.11
$A^{1/3}\Delta_p$	(o, e)	50,82	50,–	−13(18)	5.3(4)	0.74	0.15
$A^{1/3}\Delta_p$	(e, o)	50,82	50,–	−39(14)	7.9(3)	0.46	0.09
$A^{1/3}\Delta_p$	(o, o)	50,82	50,–	−10(19)	6.0(4)	0.64	0.12
$A^{1/3}\Delta_p$	(e, e)	82,–	50,–	−42(7)	7.4(2)	0.70	0.13
$A^{1/3}\Delta_p$	(o, e)	82,–	50,–	−35(7)	6.0(2)	0.64	0.11
$A^{1/3}\Delta_p$	(e, o)	82,–	50,–	−53(9)	8.0(3)	0.89	0.16
$A^{1/3}\Delta_p$	(o, o)	82,–	50,–	−45(9)	6.6(3)	0.84	0.15

Table I. Figure 6(a) shows the result for the region given by $28 < N < 82$, and $28 < Z < 50$. Likewise, Fig. 6(b) is for $50 < N < 82$, and $50 < Z$, and Fig. 6(c) is for $82 < N$, and $50 < Z$. It is clear that the neutron-excess dependency is less pronounced for lighter nuclei. In Fig. 6(a) the results are almost constant, when considering the scattering, despite the fact that the a-coefficient indicates a small decrease. In Fig. 6(b), and in particular in Fig. 6(c) there is an unmistakable decrease as a function of isospin.

This change around $Z = 50$ occurs for both Δ_n and Δ_p with even-even nuclei. The change is less definitive, but still observable, when considering odd-even nuclei.

The effect of separating the nuclei according to even-odd, odd-even, and odd-odd is seen in Fig. 7, where the region with $50 < N < 82$ and $50 < Z$ is presented for the three configurations. Figure 7(a) shows the result for even-odd, Fig. 7(b) for odd-even, and Fig. 7(c) for odd-odd. The even-even nuclei are shown in Fig. 6(b).

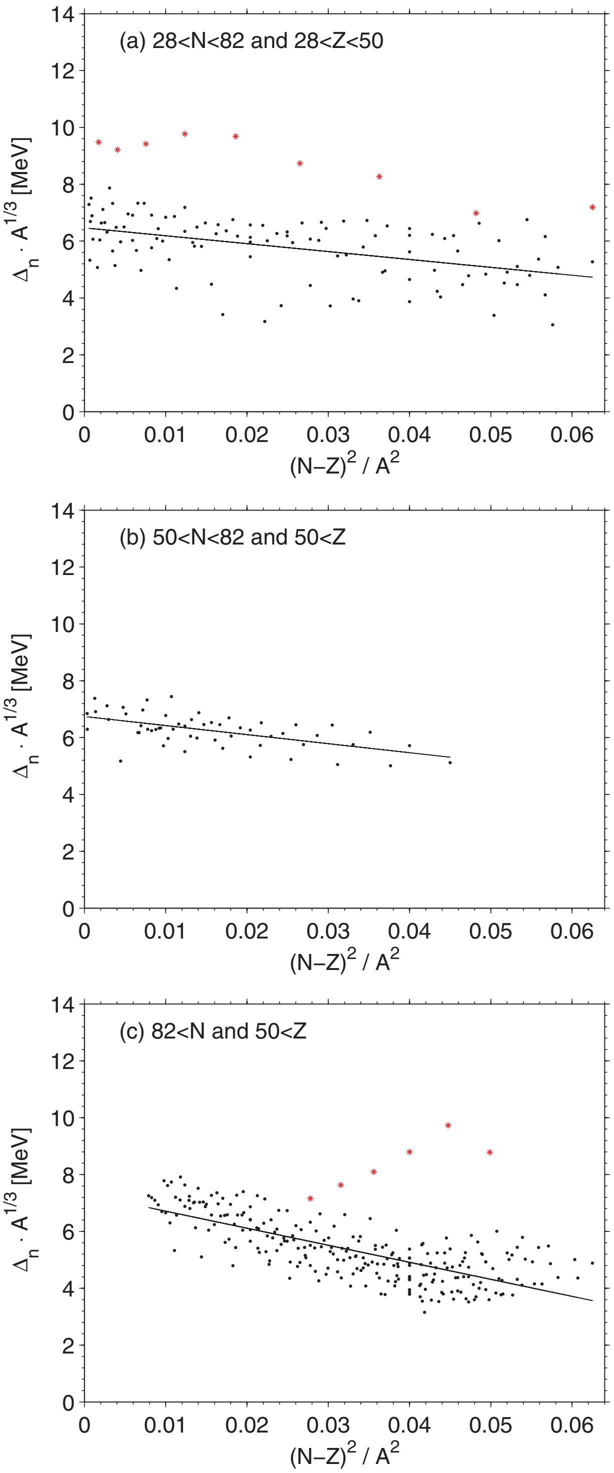

FIG. 6. (Color online) The $(\frac{N-Z}{A})^2$ dependency of even-even nuclei divided into regions defined by major nuclear shells as given in the figures.

The results for even-odd and odd-odd when accounting for scattering are constant, whereas even-even and odd-even very clearly decrease. This superficially indicates that neutrons and protons behave differently. However, their numbers and the valence shells are different as well. For Δ_p in the region where

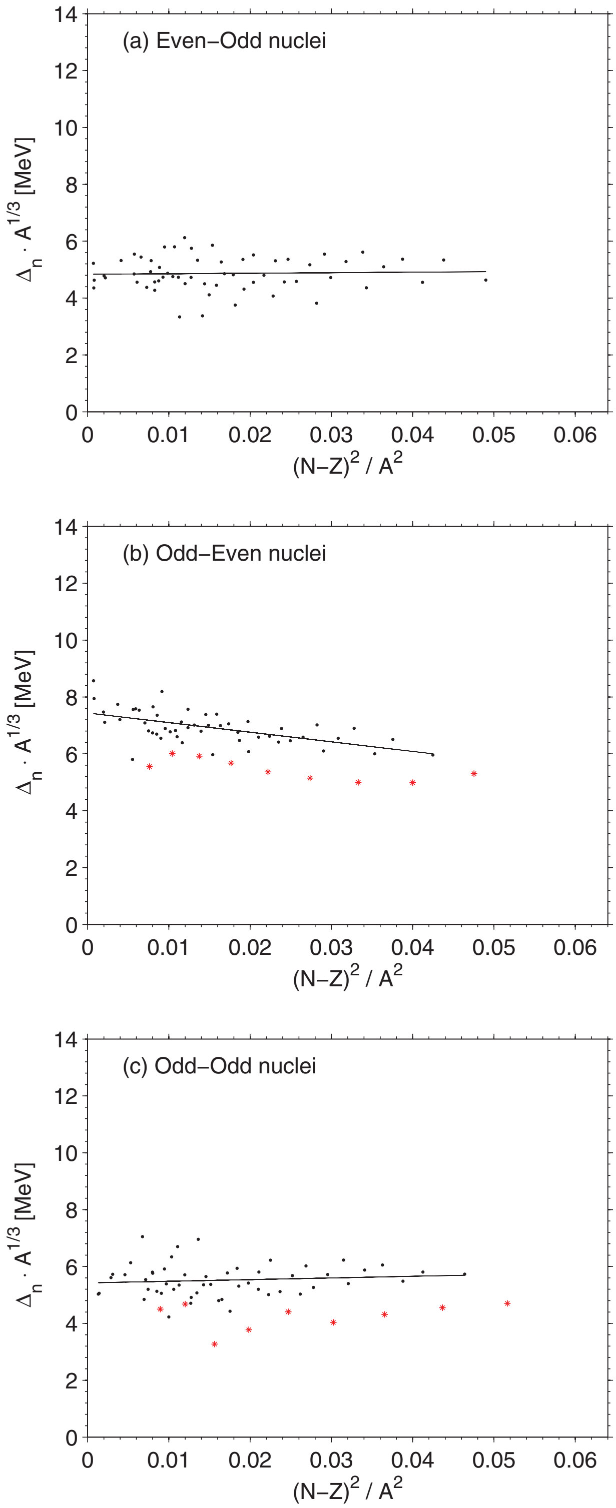

FIG. 7. (Color online) The $(\frac{N-Z}{A})^2$ dependency for nuclei with $50 < N < 82$ and $50 < Z$ divided according to whether (N,Z) is even-odd, odd-even or odd-odd. The even-even nuclei are included in Fig. 6.

$50 < N < 82$ and $50 < Z$ a "symmetric" result is found: even-even and even-odd have a clear $N - Z$ dependence, while odd-odd and odd-even are constant. However, all (even-even, even-odd, odd-even, odd-odd for both Δ_n and

Δ_p) show a clear $N-Z$ dependency for the heavier nuclei where $82 < N$ and $50 < Z$.

The change in neutron-excess dependency at $Z = 50$ for the staggering effect of one type of nucleon is clearly connected to the other nucleon type.

The parameters of the best linear fits in $(N-Z)^2/A^2$ are collected in Table I that includes the results for even-even, even-odd, odd-even, and odd-odd nuclei for both Δ_n and Δ_p. The overall results are presented as well as results for three regions in the nuclear chart. The structure changes seen at $Z = 50$ are even more pronounced for Δ_p. Also included in Table I is the root mean squared error (RMSE) for all four quantities, $A^{1/3}\Delta_{n,p}$ and $\Delta_{n,p}$.

At first glance the fitted parameters seems to reflect rather different dependencies, simply because of the large variation of the parameters. However, all the constant shifts, b, are between 5 and 8 MeV each with uncertainties of about 0.2 MeV. The slope parameter, a, has a much larger range of variation but also determined with much larger uncertainty. In the region where $28 < N < 82$ and $28 < Z < 50$, the neutron shell at $N = 50$ does not signal change of behavior and therefore the full region is always included. The a-parameter is sometimes consistent with zero or very small reflecting the flat behavior discussed in connection with the Figs. 6 and 7.

It is highly significant that the overall uncertainty of the $\Delta_{n,p}$-values are very small. The RMSE is always (significantly) less than 0.20 MeV demonstrating that the parametrization reproduce the observed values very well. These absolute uncertainties are comparable to the underlying chaotic component of $0.1 - 0.2$ MeV in nuclear masses [20]. This can usually be considered as a lower limit for systematic reproduction of nuclear binding energies. In addition, this suggests that the division into individually fitted regions are unnecessary. We shall return and explore this avenue in the next subsection.

The current data reach out to $(N-Z)^2/A^2$ around 0.05–0.06 and Δ_n has for the heaviest region, where $82 < N$ and $50 < Z$, by then decreased by a factor two. A naive extrapolation would give zero odd-even staggering for $(N-Z)^2/A^2$ around 0.10–0.12. This value is obtained for $N \simeq 2Z$ which is not too far from estimates of the neutron dripline position. There is no physical basis for extrapolating this far, but let us briefly discuss the implications this has.

In the traditional interpretation zero odd-even staggering implies that the cost, d, of one lifted particle at the drip line precisely has to be compensated by the pairing gain, $d \approx 0.5\Delta^2/d$, where Δ is the usual pairing gap, that is $d = \Delta/\sqrt{2}$. However, looking further into the basic meaning quickly reveal inconsistencies, because also Δ, as proportional to the odd-even mass difference, has to vanish. Speculations about small d due to small binding energy and/or vicinity to the continuum is not convincing, since close-lying levels usually produce larger pairing gap and pairing energy gain. Therefore, first the indication of small odd-even mass difference at the drip line is based on an extrapolation and therefore not in itself sufficient evidence. Second, we emphasize that the reason for vanishing gaps is due to coupling between neutrons and protons simply because the decrease is as function of the neutron excess.

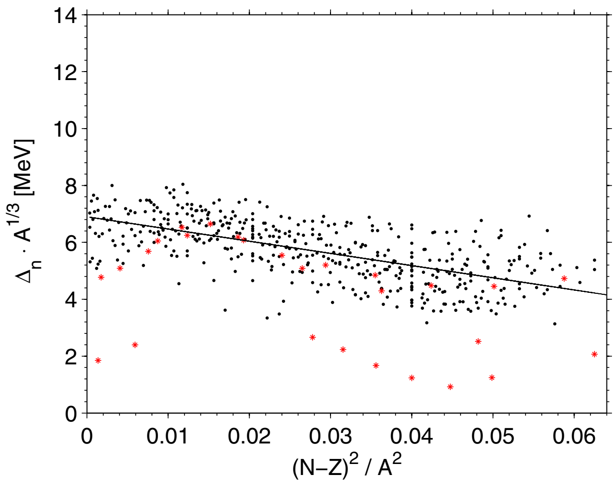

FIG. 8. (Color online) The $(\frac{N-Z}{A})^2$ dependency for all even-even nuclei with $A > 50$ with a shell effect correction given by Eq. (9).

The results in Table I still include remnants of the shell effect. If Eq. (9) is used to correct for shell effects one obtains Fig. 8 that can be compared to Fig. 5. The best linear fit is now $A^{1/3}\Delta_n = -43(5)\,\text{MeV}(\frac{N-Z}{A})^2 + 6.9(2)\,\text{MeV}$ with $\text{RMSE}(\Delta_n) = 0.16\,\text{MeV}$. The only nuclei significantly influenced by the correction are the ones marked in red (fat and full). Based on Fig. 8, Eq. (9) seems to overcompensate for about half of the affected nuclei. This is most likely a result of the different liquid drop parameters used here and in Dieperink and Van Isacker's paper [13].

B. Global descriptions

Although the neutron-excess parametrization when separating into regions defined by shells allows for a very accurate description of the odd-even staggering, it also results in a rather cumbersome model. The scale of the difference when changing between even and odd isotopes can be inferred from Table I. This can be used to combine some of the separated nuclei.

Changing from odd to even Z with Δ_p only changes the constant term b by less than 1 MeV, and the scaling factors a are mostly overlapping. As a very reasonable approximation the separation of Δ_p into even and odd Z can therefore be ignored. The best linear fits based on Eq. (11) when neglecting the separation according to even and odd Z are presented in Table II.

The effect of changing from odd to even N with Δ_p can most easily be inferred from Table II. This is by no means negligible. Instead the effect can be viewed as a constant addition, and as such it can be accounted for. This would result in an expression for the odd-even proton staggering of the form

$$A^{1/3}\Delta_p = a\left(\frac{N-Z}{A}\right)^2 + b - (1-\pi_n)c/2, \qquad (12)$$

TABLE II. Similar to Table I, but the nuclei are separated according to odd and even nucleon number.

Type	Nuclei	Region limits N	Z	a	b	RMSE [MeV]	RMSE (Δ)
$A^{1/3}\Delta_n$	Even Z	All	All	−49(4)	7.1(1)	0.91	0.18
$A^{1/3}\Delta_n$	Odd Z	All	All	−37(4)	5.5(1)	0.89	0.18
$A^{1/3}\Delta_n$	Even Z	28,82	28,50	−33(8)	6.9(2)	1.04	0.23
$A^{1/3}\Delta_n$	Odd Z	28,82	28,50	−20(7)	5.0(2)	0.87	0.19
$A^{1/3}\Delta_n$	Even Z	50,82	50,−	−33(11)	7.1(2)	0.58	0.11
$A^{1/3}\Delta_n$	Odd Z	50,82	50,−	3(11)	5.1(2)	0.67	0.13
$A^{1/3}\Delta_n$	Even Z	82,−	50,−	−65(5)	7.6(2)	0.83	0.14
$A^{1/3}\Delta_n$	Odd Z	82,−	50,−	−57(6)	6.2(2)	0.82	0.14
$A^{1/3}\Delta_p$	Even N	All	All	−35(4)	7.3(1)	0.86	0.17
$A^{1/3}\Delta_p$	Odd N	All	All	−19(4)	5.6(1)	0.86	0.18
$A^{1/3}\Delta_p$	Even N	28,82	28,50	−10(6)	7.0(2)	0.79	0.18
$A^{1/3}\Delta_p$	Odd N	28,82	28,50	1(7)	5.1(2)	0.85	0.19
$A^{1/3}\Delta_p$	Even N	50,82	50,−	−44(14)	7.6(3)	0.67	0.13
$A^{1/3}\Delta_p$	Odd N	50,82	50,−	−13(15)	5.6(3)	0.80	0.16
$A^{1/3}\Delta_p$	Even N	82,−	50,−	−48(6)	7.7(2)	0.82	0.15
$A^{1/3}\Delta_p$	Odd N	82,−	50,−	−40(6)	6.3(2)	0.76	0.13

where a and b are the parameters from Eq. (11), and c is the difference between the constants b for even and odd N.

The region with $50 < N < 82$ and $50 < Z$ is the most problematic, as even and odd N have conflicting tendencies. However, any global description based on a combination of local descriptions must necessarily be an approximation. The result of displacing odd-N nuclei, and then finding the best linear fit based on Eq. (11) is given in Table III. Also included is the size of the displacement, c.

The most interesting result is the combined expression for all nuclei. A noticeable improvement is the reduction in uncertainty for the constants a and b. Though more important is the size of the root mean squared error, which is comparable to RMSE of Δ_p for even-even nuclei in Table I. This global expression should then have almost the same overall accuracy as the former subdivided expressions, while being much more practicable.

A completely analogous combination can be made for Δ_n. Here the separation into odd and even N is neglected, and the calculated displacement is from odd to even Z. Neglecting the first separation is a less good approximation for Δ_n than for Δ_p. The result is also included in Table II.

TABLE III. Similar to Tables I and II, but without separation according to odd-even configuration. The value of c in Eq. (12) signifies the displacement of odd nuclei.

Type	Region limits N	Z	c	a	b	RMSE [MeV]	RMSE (Δ)
$A^{1/3}\Delta_n$	All	All	1.7	−44(3)	7.2(1)	0.92	0.18
$A^{1/3}\Delta_n$	28,82	28,50	1.9	−27(5)	6.9(2)	0.97	0.21
$A^{1/3}\Delta_n$	50,82	50,−	2.0	−11(9)	7.0(2)	0.72	0.14
$A^{1/3}\Delta_n$	82,−	50,−	1.4	−62(4)	7.6(1)	0.83	0.15
$A^{1/3}\Delta_p$	All	All	1.8	−27(3)	7.3(1)	0.89	0.19
$A^{1/3}\Delta_p$	28,82	28,50	1.9	−5(5)	7.0(1)	0.84	0.19
$A^{1/3}\Delta_p$	50,82	50,−	1.9	−26(11)	7.6(2)	0.80	0.16
$A^{1/3}\Delta_p$	82,−	50,−	1.4	−44(4)	7.7(2)	0.81	0.14

The result of displacing odd Z nuclei is presented in Table III, and the value of the displacement is almost identical to the result for Δ_p. Based on RMSE Δ_n is as valid as Δ_p.

It might initially appear as if the $N - Z$ dependency of the proton staggering is less pronounced than the neutron staggering. As stated earlier this dependency increases for heavier nuclei. In other words, the $N - Z$ dependency of the neutron staggering effect is larger, when neutrons are abundant and analogously for protons. The nuclei examined generally have a majority of neutrons, and the $N - Z$ dependence of the neutron staggering is therefore seemingly greater. In the region where $50 < N < 82$ and $50 < Z$ the $N - Z$ dependence is seen to be greater for the proton staggering.

To obtain the term which has to be added to the liquid drop model, Eq. (12) is combined with Eq. (8). However, we have a term in Δ_n proportional to $-(1-\pi_p)/2$ and one in Δ_p proportional to $-(1-\pi_n)/2$ but otherwise of the same magnitude, this is, as remarked in Sec. II, indicative of a neutron-proton pairing term that must be taken out before the two separate expressions are added. Noting that the b coefficients for neutrons and protons in Table III are also essentially identical we obtain the following final relations, for $Z < 50$:

$$\Delta = A^{-1/3}\left(\left(\frac{N-Z}{A}\right)^2(-13\pi_n - 2.5\pi_p) + 3.4(\pi_n+\pi_p) + 1.85\pi_{np}\right)\ \text{MeV}, \tag{13}$$

and for $Z > 50$

$$\Delta = A^{-1/3}\left(\left(\frac{N-Z}{A}\right)^2(-28\pi_n - 20.5\pi_p) + 3.8(\pi_n+\pi_p) + 1.55\pi_{np}\right)\ \text{MeV}. \tag{14}$$

The RMSE is still slightly below 0.2 MeV in these fits. Note that the neutron-proton pairing term here is taken to have a $A^{-1/3}$ mass dependence as the other terms in contrast to earlier works [10]. Since all terms depend on N and Z one should in principle correct for higher-order effects when going from the mass differences $\Delta_{n,p}$ to the Δ that should be included in mass formulas. However, the correction terms are at most of order 10^{-3} and have been neglected. The effects would be larger for mass relations involving more nuclei, as an example we find that the values obtained for $\Delta^{(5)}_{n,p}$ are systematically 10–20 % smaller than the ones for $\Delta_{n,p}$.

We could take seriously the observation that for $Z < 50$ the $N - Z$ dependence is either zero or very small. A fit for these nuclei to Eq. (12) with $a = 0$ gives

$$\Delta = A^{-1/3}(5.9\pi_n + 6.6\pi_p + 1.6\pi_{np})\ \text{MeV}, \tag{15}$$

corresponding to RMSE = 0.21 and 0.25 MeV for Δ_p and Δ_n, respectively. As expected these uncertainties are not as good as obtained by maintaining the $N - Z$ dependent term. They are, however, reasonably close, as well as somewhat simpler.

TABLE IV. The results for the two-term model from Eq. (16), where the RMSE in each region is calculated based on the best local fit.

Type	Nuclei	Region limits N	Z	c_2	c_1 [MeV]	RMSE
Δ_n	Even Z	All	All	58(5)	0.7(0)	0.24
Δ_n	Odd Z	All	All	32(4)	0.6(0)	0.21
Δ_n	Even Z	50,82	50,–	168(34)	0.0(3)	0.11
Δ_n	Odd Z	50,82	50,–	107(36)	0.2(3)	0.12
Δ_n	Even Z	82,–	50,–	123(21)	0.3(1)	0.20
Δ_n	Odd Z	82,–	50,–	54(23)	0.5(1)	0.20
Δ_p	Even N	All	All	60(4)	0.8(0)	0.20
Δ_p	Odd N	All	All	34(4)	0.7(0)	0.19
Δ_p	Even N	50,82	50,–	−13(51)	1.5(4)	0.15
Δ_p	Odd N	50,82	50,–	−76(46)	1.6(4)	0.14
Δ_p	Even N	82,–	50,–	143(18)	0.3(1)	0.17
Δ_p	Odd N	82,–	50,–	86(18)	0.4(1)	0.16

C. Comparisons

To determine the viability of our expression a comparison with other more established models is useful. The previously mentioned two term model suggested by Friedman and Bertsch [3] yields very accurate results, and is well suited as a comparison. We shall also briefly compare to the very similar model presented by Jensen *et al*. [7].

The two term expression suggested by Friedman and Bertsch on the basis of a more detailed physical modeling is

$$\Delta_{n,p} = c_1 + c_2/A, \tag{16}$$

where c_1 and c_2 are constants to be determined. In our fits to this expression we again exclude light nuclei, and nuclei otherwise influenced by shell effects or the Wigner effect. The remains of the smooth aspects are also removed, as in Eqs. (5) and (6). Based on the tendencies observed in Fig. 3, Δ_n is separated according to odd and even Z, and vice versa for Δ_p. The relevant nuclei are also examined both collectively and separated into the two heaviest regions given by $50 < N < 82$ and $50 < Z$, and $82 < N$ and $50 < Z$.

The RMSE for the results is calculated in two ways. First, the error will be calculated as the RMSE of the nuclei in a given region in relation to the best local fit based on Eq. (16). The result of this calculation is presented in Table IV. The intent with the two term model was never to separate it according to shells, so the most interesting results are those which covers all regions. Comparing Tables IV and II the error for Δ_n and Δ_p is seen to be noticeably larger for the two-term model in Eq. (16). For most regions the isospin dependence also gives a smaller RMSE.

In a second step, the error is calculated as the RMSE of the nuclei in a region relative to the best global fit again based on Eq. (16). The result is presented in Table V. The collective result for all the nuclei would be the same as in Table IV, and is not included. The RMSE is inevitably larger than in Table IV, but it is still very reasonable.

This is illustrated in Fig. 9, which also shows the neutron staggering for even nuclei as a function of A^{-1}. The line is the best global fit based on Eq. (16), and the nuclei indicated in blue belongs to the region with $82 < N$ and $50 < Z$. The RMSE in Table V is calculated based on a group of nuclei, such as those marked in blue, with respect to the global fit indicated by the line. As such it will always be larger than the RMSE from Table IV. From Fig. 9 it is clear that Eq. (16) does not reproduce the general tendencies observed in the odd-even staggering. It can, however, still be used as a very accurate approximation.

TABLE V. The same as in Table IV, but the RMSE in each region is calculated based on the best global fit.

Type	Nuclei	Region limits N	Z	c_2	c_1 [MeV]	RMSE
Δ_n	Even Z	50,82	50,–	58(5)	0.7(0)	0.22
Δ_n	Odd Z	50,82	50,–	32(4)	0.6(0)	0.21
Δ_n	Even Z	82,–	50,–	58(5)	0.7(0)	0.21
Δ_n	Odd Z	82,–	50,–	32(4)	0.6(0)	0.20
Δ_p	Even N	50,82	50,–	61(4)	0.8(0)	0.19
Δ_p	Odd N	50,82	50,–	35(4)	0.7(0)	0.18
Δ_p	Even N	82,–	50,–	61(4)	0.8(0)	0.19
Δ_p	Odd N	82,–	50,–	35(4)	0.7(0)	0.17

Let us also compare briefly to the final expression obtained by Jensen *et al*. [7] that, in addition to the smooth terms, is given by

$$\begin{aligned}\tilde{\Delta}(N,Z) = A^{-1/3}\Bigg(3.68\pi_n\left(1-8.15\left(\frac{N-Z}{A}\right)^2\right)\\ + 3.78\pi_p\left(1-6.07\left(\frac{N-Z}{A}\right)^2\right)\Bigg)\\ -43\frac{|N-Z|}{A}+\frac{\pi_{np}(34-24\delta_{N,Z})}{A},\end{aligned} \tag{17}$$

where $\delta_{N,Z}$ is a Kronecker delta. All constants are in units of MeV. The last two terms account for Wigner effects, and are

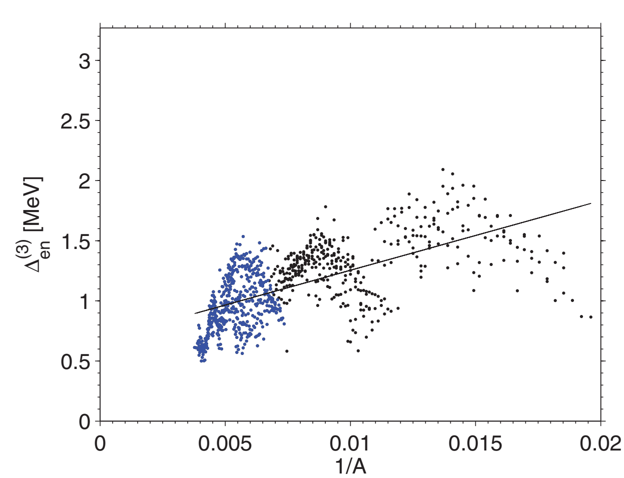

FIG. 9. (Color online) Best global fit with Eq. (16) for even Z nuclei and $A > 50$. The nuclei marked in blue are restricted by $82 < N$ and $50 < Z$. Nuclei influenced by shells or the Wigner effect are not shown.

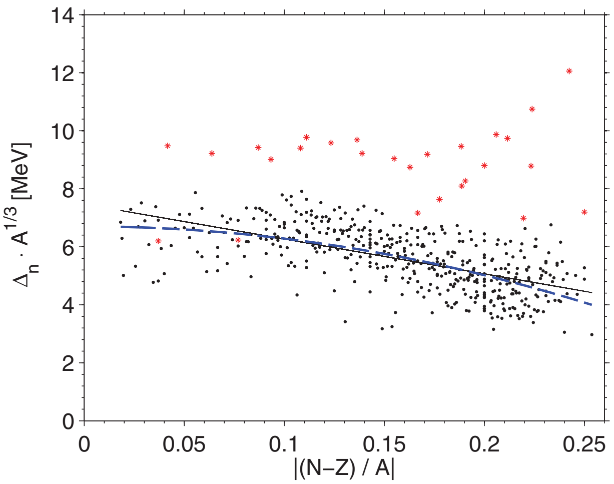

FIG. 10. (Color online) The equivalent to Fig. 5 only with respect to $|\frac{N-Z}{A}|$. Here the dashed, blue line is the best fit quadratic in neutron excess.

not relevant in the comparison since we do not include nuclei with $N = Z$. The first terms have the same dependencies as our result in Eq. (14) and can be compared directly, the main difference being that we have used a three-point mass formula whereas [7] used four-point mass formulas and evaluated neutron pairing, proton pairing and neutron-proton pairing separately. The π_n and π_p terms are very similar, and the coefficients on the $(N - Z)^2$ terms are 5–10 % larger than in the new fits. This is presumably due to our substantially larger data set where most added nuclei have relatively large neutron excess. The neutron-proton pairing term has a similar magnitude for mid to heavy mass nuclei in spite of the different assumed mass dependence.

Finally, the question of whether to use a model which is linear or quadratic in neutron excess deserves some attention. In Fig. 5 the results were shown in relation to a quadratic neutron excess for all even-even nuclei with $A > 50$, and the best linear fit was shown with a dashed, blue line. The same figure, but in relation to a linear neutron excess can be seen in Fig. 10. The result of the best linear fit is $A^{1/3}\Delta_n = -12|\frac{N-Z}{A}| + 7.5$ with RMSE = 0.83, and RMSE(Δ_n) = 0.17 all in units of MeV. The dashed, blue line in Fig. 10 is the best quadratic fit, the values of which can be found in Table I.

The difference between the two descriptions is perhaps surprisingly small. Possible reasons could be that both forms only are approximations to a better generic dependence, or the range of nuclei is too small to distinguish, or the individual scatter of points arise from a chaotic behavior prohibiting substantial improvements in simple fits [20]. A significant increase in the number of measurements of far off stability nuclei would be very helpful to address these questions.

V. CONCLUSION

Our phenomenological study of odd-even staggering terms in the nuclear binding energy makes use of the three-point mass relations, Δ_n and Δ_p. This second order difference eliminates most smooth aspects of the binding energy and the liquid drop model was used to eliminate the remains of the smooth variations. By also avoiding isotopes with magic numbers, on the $N = Z$ line, or generally very light only non-mooth, odd-even contributions remains.

The starting point of our description is the trends seen in Fig. 3. The odd-even configuration of both neutrons and protons was seen to have an influence on the general scale of the staggering. The region in question also influenced the result. To examine, and possibly account, for these observations the nuclei were separated according to odd-even configuration, and into regions defined by nuclear shells. We find that the difference in Δ_n for odd and even Z and the corresponding differences in Δ_p is naturally described in terms of a common neutron-proton pairing term. This is in line with the findings [7,10] made using second order mass differences. By construction we end up with the same overall mass dependence for the neutron-proton pairing term as for the other terms, namely $A^{-1/3}$, where the other works employ A^{-1} and $A^{-2/3}$. There is as yet no basic theoretical framework that can explain the neutron-proton pairing systematics [10] so a closer look at the data may be warranted.

It is well known that the data on odd-even staggering fluctuate systematically around a power-law fit, and odd-even mass differences calculated from self-consistent mean field theory [4] displays a similar behavior. These systematic deviations are also clearly visible in Fig. 3 and suggest a description in terms of a dependence on $N - Z$, as attempted earlier [6,7]. It turns out that the $N - Z$ dependent terms vary in importance as we go from light to heavy nuclei. For $Z < 50$ the results were almost constant, when considering scattering, but for $50 < Z$ there was a clear decrease as a function of neutron excess. This transition was seen for both neutrons and protons.

To make the model globally applicable the separated results were recombined. Since the odd-even Z configuration had a very modest influence on Δ_p it was neglected. The neutron-proton pairing term accounted for the displacement of the odd-N nuclei relative to the even-N nuclei. Similarly, odd-Z nuclei were displaced for Δ_n. This resulted in two global expressions for the odd-even staggering effect as given in Eqs. (13) and (14). These odd-even terms have to be included in phenomenological expressions of the nuclear binding energy where the largest contributions, liquid drop and shell effects, can be maintained from previous studies.

The separated and the combined expressions were compared to a two-term model, with a A^{-1} dependency, and our $(N - Z)^2$ dependence showed greater accuracy both locally and globally. The overall root mean square deviations in all our fits are always (significantly) less than 0.2 MeV. There is as yet no theoretical explanation for a systematic neutron excess dependence of the odd-even staggering for heavy nuclei, i.e., when nuclei fill the large shells, but the fact that the data follow the fit curves suggests that a—direct or indirect—dependence on isospin projection should be considered on top of the previously included physical explanations for odd-even staggering [1,3]. Finally, we note that calculations including three-nucleon forces have now been used [21] to study the variation of the odd-even staggering in the heavy Ca isotopes. The rapid progress in nuclear theory these years may give a new perspective on this old problem.

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