Integral transform of the Coulomb Green's function by the Hankel function and off-shell scattering

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(Received 3 October 2013; revised manuscript received 29 October 2013; published 5 December 2013)

In the representation space approach a useful analytical expression for the integral transform of the Coulomb Green's function by the Hankel function is constructed via Sturmian representation of the bound-state Coulomb Green's function. This integral transform is exploited to construct off-shell Jost solutions for motion in Coulomb and Coulomb plus separable interactions in the maximal reduced form. The expressions for the corresponding off-shell T matrices are also constructed by using a modified relation between the off-shell physical solution and the T matrix that does not involve the potential explicitly. Finally, off-shell T matrices are computed to examine the role of the Coulomb interaction in proton-proton scattering in the $^1S_0$ channel.

DOI: 10.1103/PhysRevC.88.064001 PACS number(s): 03.65.Nk, 21.45.Ff, 24.10.−i, 02.30.Uu

I. INTRODUCTION

A short-range local potential can be approximated by a finite rank separable nonlocal potential in a mathematically well-defined sense [1]. These nonlocal potentials can describe a much wider variety of phenomena than that encompassed with short-range local potentials [2]. Thus, the experiments which involve scattering by additive interactions are treated within the framework of a nonlocal separable model which facilitates the construction of closed-form analytical expressions for physical observables. Also these representations produce acceptable values of all physical observables for elastic nucleon-nucleon scattering [3–5]. As the Schrödinger equation for the separable nonlocal interaction is solvable in simple analytical form, it is also used in nuclear matter calculation [6,7] and with the Faddeev equation for three-body systems [8]. In reality a majority of the nuclear reactions involve at least two charged hadrons and therefore the theoretical formalism to describe such processes must include the Coulomb interaction in a convenient way. One of the characteristic features of the Faddeev equations is that they are expressed in terms of the amplitude components, i.e., in terms of the splitting of the three-body entities considered in the Lippmann-Schwinger equations [9]. This is a consequence of the fact that to properly handle the asymptotic structure of the three-body problem, Faddeev [10] introduces the definition of a channel in which only two of the particles are assumed to interact while the third particle is free. As a result the Faddeev equations have as input the channel two-body transition matrices of each two-body subsystem. Because of the kinematics involved, these two-body T matrices appear off-the-energy-shell, that is, the energy parameter corresponds to neither of the momenta arguments in the two-body T matrix. One of the formalisms [11–18] to handle the Coulomb potential in a three-body system was formulated by Alt *et al.* [15–18]. In this case the two-body transition operators are considered in such a form that the effect due to the Coulomb part is isolated, which leads to well-defined scattering amplitudes for two charged particles. Many authors [19,20] treat the three-body system by first considering only the nuclear interactions to calculate nuclear transition amplitudes in an exact way and then the Coulomb effect is included by adding the Coulomb transition amplitude to the nuclear transition amplitude so obtained. Here we shall treat the Coulomb effect rigorously in constructing off-shell transition matrices for the Coulomb-nuclear interaction within the framework of the separable model through the Coulomb-Sturmian expansion method. The essence of this method lies in the term-by-term separable expansion of the Coulomb Green's operator. If the total interaction also contains a Coulomb potential, that potential is kept in the Green's operator, thereby avoiding all difficulties associated with a typical Coulomb problem. In view of the importance of the charged-particle scattering/reaction [21] we shall construct closed-form analytical expressions for the off-shell Jost solutions and T matrices within the framework of a separable model for the Coulomb-nuclear interaction.

Based on a differential equation approach [22] to the T matrix, Fuda and Whiting [23] introduced the concept of the off-shell Jost function $f_\ell(k,q)$. The behavior of the irregular solution of the radial Schrödinger equation near the origin determines the Jost function, which plays an important role in analyzing the analytic properties of partial wave scattering amplitudes. Similar to the on-shell Jost function $f_\ell(k)$, the off-shell Jost function $f_\ell(k,q)$ is obtained from the irregular solution $f_\ell(k,q,r)$ of the inhomogeneous Schrödinger equation. The results for the off-shell Jost functions for motion in Coulomb and Coulomb-separable potentials have been published in a number of publications [24–27]. Relatively recently, we constructed the expressions for the off-shell Jost solutions for the Coulomb and Coulomb-like interactions in the representation space formalism via different approaches to the problem [28–36]. The primary aim of the present work is to look for another straightforward method to derive closed-form expressions for the off-shell Jost solutions based on the judicious exploitation of the term-by-term separability of the Sturmian function representation of the Coulomb Green's function.

The T matrix has an established importance in nuclear physics with respect to its close relation to experiment. The on-shell T matrix elements are related to scattering phase shifts, and the half-off-shell T matrix element is a measurable

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quantity [27]. In the three-body problem, the off-shell T matrix is a direct and transparent link between experimental two-nucleon data and three-nucleon observables. The T matrix is therefore closely related to the experiment. The T-matrix approach presents a logical and useful tool to study off-shell effects which in turn enables us to examine the physical constraints that must be applied to the parametrization of the unknown parts of the nuclear force.

In conventional potential scattering theory the scattering amplitude can be obtained by taking the on-shell limit of the off-shell T matrix. This is not true for Coulomb and Coulomb-like potentials. However, in such a situation, relevant physical information can also be extracted from n off-shell T matrix. The off-shell T matrix can be calculated by exploiting the result that exists between the off-shell physical solution and T matrix. Thus, it is of some importance to have in the literature a relatively uncomplicated mathematical prescription to derive expressions for off-shell physical and Jost solutions as well as T matrices relating to scattering by Coulomb and Coulomb-like interactions which are encountered in atomic, molecular, and nuclear physics.

Section II is devoted to developing an interacting Green's function and related quantities. In Sec. III closed-form expressions for the integral transforms of the Coulomb Green's functions and related off-shell Jost solutions are derived. In Sec. IV off-shell T matrices are constructed by using a modified expression which does not involve the potential explicitly. Finally, results and discussions are presented in Sec. V.

II. INTERACTING GREEN'S FUNCTION AND RELATED QUANTITIES

The off-shell physical solutions $\psi_\ell^{C(+)}(k,q,r)$ and $\psi_\ell^{\mathrm{CS}(+)}(k,q,r)$ for the Coulomb and Coulomb-plus–separable (CS) potentials satisfy the inhomogeneous differential equations [27]

$$\left[\frac{d^2}{dr^2}+k^2-\frac{\ell(\ell+1)}{r^2}-V_C(r)\right]\psi_\ell^{C(+)}(k,q,r) = (k^2-q^2)\hat{j}_\ell(qr) \tag{1}$$

and

$$\left[\frac{d^2}{dr^2}+k^2-\frac{\ell(\ell+1)}{r^2}-V_C(r)\right]\psi_\ell^{\mathrm{CS}(+)}(k,q,r) - \lambda_\ell g_\ell(\beta_\ell,r)d_\ell(\beta_\ell,k,q) = (k^2-q^2)\hat{j}_\ell(qr), \tag{2}$$

with

$$V_C(r)=\frac{2k\eta}{r}, \tag{3a}$$

$$\hat{j}_\ell(x)=\frac{1}{2i}[\hat{h}_\ell^{(+)}(x)-\hat{h}_\ell^{(-)}(x)], \tag{3b}$$

$$\hat{h}_\ell^{(+)}(x)=xh_\ell^{(+)}(x), \tag{3c}$$

$$\hat{h}_\ell^{(-)}(x)=(-1)^\ell\hat{h}_\ell^{(+)}(-x). \tag{3d}$$

The quantity $d_\ell(\beta_\ell,k,q)$ is expressed as

$$d_\ell(\beta_\ell,k,q)=\int_0^\infty ds g_\ell(\beta_\ell,s)\psi_\ell^{\mathrm{CS}(+)}(k,q,s), \tag{4a}$$

where $g_\ell(\beta_\ell,r)$ is the form factor of the Graz separable potential given by

$$g_\ell(\beta_\ell,r)=2^{-\ell}\ell!^{-1}r^\ell e^{-\beta_\ell r}. \tag{4b}$$

The particular solutions of Eqs. (1) and (2) which are of interest to us read as

$$\psi_\ell^{C(+)}(k,q,r)=\frac{(k^2-q^2)}{2i}\left[\tilde{G}_\ell^{C(+)}(r,q)-(-1)^\ell\tilde{G}_\ell^{C(+)}(r,-q)\right] \tag{5}$$

and

$$\psi_\ell^{\mathrm{CS}(+)}(k,q,r) = \frac{(k^2-q^2)}{2i}\left[\tilde{G}_\ell^{\mathrm{CS}(+)}(r,q)-(-1)^\ell\tilde{G}_\ell^{\mathrm{CS}(+)}(r,-q)\right], \tag{6}$$

where the integral transforms of Green's functions are

$$\tilde{G}_\ell^{C(+)}(r,q)=\int_0^\infty dr' G_\ell^{C(+)}(r,r')\hat{h}_\ell^{(+)}(qr'), \tag{7a}$$

$$\tilde{G}_\ell^{\mathrm{CS}(+)}(r,q)=\int_0^\infty dr' G_\ell^{\mathrm{CS}(+)}(r,r')\hat{h}_\ell^{(+)}(qr'), \tag{7b}$$

$$\tilde{G}_\ell^{C(+)}(r,-q)=\left|\tilde{G}_\ell^{C(+)}(r,q)\right|_{q\to -q}, \tag{8a}$$

and

$$\tilde{G}_\ell^{\mathrm{CS}(+)}(r,-q)=\left|\tilde{G}_\ell^{\mathrm{CS}(+)}(r,q)\right|_{q\to -q}. \tag{8b}$$

Green's functions for Coulomb and Coulomb-plus-separable potentials are introduced as [37]

$$G_\ell^{C(+)}(r,r')=-\frac{\varphi_\ell^C(k,r_<)f_\ell^C(k,r_>)}{\Im_\ell^C(k)} \tag{9}$$

and

$$G_\ell^{\mathrm{CS}(+)}(r,r')=-\frac{\varphi_\ell^{\mathrm{CS}}(k,r_<)f_\ell^{\mathrm{CS}}(k,r_>)}{\Im_\ell^{\mathrm{CS}}(k)}. \tag{10}$$

Here $\varphi_\ell^C(k,r)$, $f_\ell^C(k,r)$, $\varphi_\ell^{\mathrm{CS}}(k,r)$, and $f_\ell^{\mathrm{CS}}(k,r)$ are the regular and irregular solutions of the Schrödinger equation corresponding to Eqs. (1) and (2), and $r_<$ and $r_>$ have their usual meaning. Also

$$\Im_\ell^C(k)=(2\ell+1)!!k^{-\ell}e^{i\ell\pi/2}f_\ell^C(k), \tag{11}$$

and

$$\Im_\ell^{\mathrm{CS}}(k)=(2\ell+1)!!k^{-\ell}e^{i\ell\pi/2}f_\ell^{\mathrm{CS}}(k). \tag{12}$$

Substitution of Eq. (10) in Eq. (7b) together with Eq. (3b) involves certain tedious integrals. To circumvent these difficulties in the calculation we shall express the physical Green's function for the Coulomb-plus-separable potential in terms of the pure Coulomb physical Green's function and their integral transforms as follows.

The Lippmann-Schwinger integral equation [38] for the Coulomb-plus-separable Green's function is written as

$$G_\ell^{\mathrm{CS}(+)}(r,r')=G_\ell^{C(+)}(r,r')+\lambda_\ell\int_0^\infty\int_0^\infty dsdt G_\ell^{C(+)}(r,s) \times g_\ell(\beta_\ell,s)g_\ell(\beta_\ell,t)G_\ell^{\mathrm{CS}(+)}(t,r'). \tag{13}$$

Multiplying both sides by $g_\ell(\beta_\ell, r)$ and integrating over the whole range we obtain

$$G_\ell^{CS(+)}(r,r') = G_\ell^{C(+)}(r,r') + \frac{\lambda_\ell}{D_\ell^{(+)}(k)} \tilde{G}_\ell^{C(+)}(r,\beta_\ell)\tilde{G}_\ell^{C(+)}(\beta_\ell,r'), \quad (14)$$

with $D_\ell^{(+)}(k)$ the Fredholm determinant [26,35] associated with the physical boundary condition

$$D_\ell^{(+)}(k) = 1 - \lambda_\ell \int_0^\infty \int_0^\infty ds dt\, g_\ell(\beta_\ell, s) G_\ell^{C(+)}(r,s) g_\ell(\beta_\ell, t), \quad (15)$$

and $\tilde{G}_\ell^{C(+)}(r,\beta_\ell)$ the integral transform of the Coulomb physical Green's function by the form factor of the separable potential under consideration

$$\tilde{G}_\ell^{C(+)}(r,\beta_\ell) = \int_0^\infty ds\, g_\ell(\beta_\ell, s) G_\ell^{C(+)}(r,s). \quad (16)$$

Now the integral transform of the physical Green's function for motion in the Coulomb-plus-separable interaction by the Hankel function as defined in Eq. (7b) is obtained as

$$\tilde{G}_\ell^{CS(+)}(r,q) = \tilde{G}_\ell^{C(+)}(r,q) + \frac{\lambda_\ell}{D_\ell^{(+)}(k)} \tilde{G}_\ell^{C(+)}(r,\beta_\ell)\tilde{G}_\ell^{C(+)}(\beta_\ell,q), \quad (17)$$

with

$$\tilde{G}_\ell^{C(+)}(\beta_\ell,q) = \int_0^\infty \int_0^\infty dr dr'\, g_\ell(\beta_\ell, r) G_\ell^{C(+)}(r,r') \hat{h}_\ell^{(+)}(qr'). \quad (18)$$

Combining Eqs. (5), (6), (8a), and (8b) together Eqs. (9)–(18) one will be in a position to write an explicit expression for off-shell physical solutions related to Coulomb and Coulomb-plus-separable interaction in terms of integral transform of the Coulomb physical Green's function.

Also there exists a relation between off-shell physical solution and Jost solutions [27,35] expressed as

$$\psi_\ell^{(+)}(k,q,r) = \frac{\pi q}{2} e^{-i\ell\pi/2} T_\ell(k,q,k^2) f_\ell(k,r) + \frac{1}{2i}[e^{-i\ell\pi/2} f_\ell(k,q,r) - e^{i\ell\pi/2} f_\ell(k,-q,r)], \quad (19)$$

where $f_\ell(k,r)$ and $f_\ell(k,q,r)$ are the on- and off-shell Jost solutions, and $T_\ell(k,q,k^2)$ is the half off-shell T matrix. The half off-shell T matrix is expressed in terms of appropriate on- and off-shell Jost functions $f_\ell(k)$ and $f_\ell(k,q)$ as

$$T_\ell(k,q,k^2) = \left(\frac{k}{q}\right)^\ell \left[\frac{f_\ell(k,q) - f_\ell(k,-q)}{i\pi q f_\ell(k)}\right]. \quad (20)$$

III. INTEGRAL TRANSFORMS OF COULOMB GREEN'S FUNCTION AND OFF-SHELL SOLUTIONS

In Eq. (19) we have seen that $\psi_\ell^{(+)}(k,q,r)$ can be expressed in terms of appropriate on- and off-shell Jost solutions, respectively. Thus, having the expression for $\psi_\ell^{(+)}(k,q,r)$ one can identify the corresponding off-shell Jost solution. Here we shall derive a closed-form expression for $\psi_\ell^{(+)}(k,q,r)$ by evaluating the integral transforms of the Coulomb Green's function. The integral transforms of the Coulomb Green's function by the form factors of the Graz separable potential have been published in a number of publications [27,39,40]. Relatively recently, we have also derived the Hankel transform of the same [30,33,35] via different approaches to the problems. Here we shall look for another approach to the problem of Hankel transform by exploiting the term-by-term separability of the Sturm series representation [41,42] of the bound-state Coulomb Green's function, and the result thus obtained will be continued analytically to get $\tilde{G}_\ell^{C(+)}(r,q)$.

For the bound-state Coulomb Green's function $G_\ell^C(r,r')$, the Green's operator [41,42] reads as

$$G_\ell^C = \sum_{n=\ell+1}^\infty -\frac{n}{n - s/k} G_{0\ell} |\lambda_n \ell\rangle \langle \lambda_n \ell| G_{0\ell}. \quad (21)$$

Here the Sturm states $|\lambda_n\ell\rangle$ are the eigenstates of $V_\ell G_{0\ell}$ [40,41] represented by

$$V_\ell G_{0\ell}(-k^2)|\lambda_n\ell\rangle = \lambda_n |\lambda_n\ell\rangle, \quad n = \ell+1, \ell+2, \ldots, \quad (22)$$

with energy $-k^2 > 0$. The quantities V_ℓ and $G_{0\ell}$ denote the partially projected potential and free-particle Green's operator. The relation between the Coulomb bound state $|\kappa_n\ell\rangle$ and the Sturmian states $|\lambda_n\ell\rangle$is given by

$$|\kappa_n\ell\rangle = \sqrt{2}\kappa_n G_{0\ell}|\lambda_n\ell\rangle. \quad (23)$$

The integral transform of the physical Coulomb Green's function with the Riccati-Hankel function $\hat{h}_\ell^{(+)}(qr)$ is written as

$$\langle r| G_\ell^C |\hat{h}_\ell^{(+)} q\rangle = \sum_{n=\ell+1}^\infty -\frac{n}{n - s/\kappa_n} \times \langle r| G_{0\ell}|\lambda_n\ell\rangle \langle \lambda_n\ell| G_{0\ell}|\hat{h}_\ell^{(+)} q\rangle. \quad (24)$$

The two integrals involved in the above equation can be evaluated quite easily. The quantity $\langle r| G_{0\ell}|\lambda_n\ell\rangle$ is related to the Coulomb bound-state energy eigenfunction as

$$\langle r|\kappa_n\ell\rangle = \left[\frac{\kappa_n}{n}\left(\frac{\Gamma(n-1)}{\Gamma(n+\ell+1)}\right)\right]^{1/2} \times (2\kappa_n r)^{\ell+1} e^{-\kappa_n r} L_{n-\ell-1}^{2\ell+1}(2\kappa_n r). \quad (25)$$

The associated Laguerre polynomial $L_p^\alpha(z)$ is related to the confluent hypergeometric function by the following relation [43]

$$L_p^\alpha(z) = \frac{\Gamma(p+\alpha+1)}{\Gamma(\alpha+1)\Gamma(p+1)} {}_1F_1(-p;\alpha+1;z). \quad (26)$$

The other integral $\langle\lambda_n\ell|G_{0\ell}|\hat{h}_\ell^{(+)}q\rangle$ is evaluated to get

$$\langle\lambda_n\ell|G_{0\ell}|\hat{h}_\ell^{(+)}q\rangle = \int_0^\infty dr\,\langle\lambda_n\ell|\,G_{0\ell}\,|r\rangle\,\hat{h}_\ell^{(+)}(qr)$$
$$= \left[\frac{2\kappa_n}{n}\left(\frac{\Gamma(n+\ell+1)}{\Gamma(n-\ell)}\right)\right]^{1/2}\frac{(2\kappa_n)^{\ell+2}}{2\Gamma(2\ell+2)}\sum_{L=0}^{\ell}\frac{e^{i(2L-\ell)\pi/2}(\ell-L+1)\Gamma(L+\ell+1)}{(2iq)^L\Gamma(L+1)(\kappa_n-iq)^{(\ell-L+2)}}$$
$$\times{}_2F_1\left(\ell-n+1,\ell-L+2;2\ell+2;\frac{2\kappa_n}{\kappa_n-iq}\right), \tag{27}$$

where the Riccati-Hankel function $\hat{h}_\ell^{(+)}(qr)$ is [43]

$$\hat{h}_\ell^{(+)}(qr) = qrh_\ell^{(+)}(qr) = \sum_{L=0}^{\ell}\frac{e^{i(2L-\ell)\pi/2}\Gamma(L+\ell+1)}{(2iqr)^L\Gamma(L+1)\Gamma(\ell-L+1)}e^{iqr} \tag{28}$$

and

$$\int_0^\infty dze^{-\lambda z}z^\nu{}_1F_1(a;c;pz) = \Gamma(\nu+1)\lambda^{-\nu-1}{}_2F_1(a,\nu+1;c;p/\lambda)\,. \tag{29}$$

Combining Eq. (23) with Eqs. (25)–(29) yields

$$\langle r|G_\ell^C|\hat{h}_\ell^{(+)}q\rangle = -\frac{(2\kappa_n)^{2\ell+5}\,r^{\ell+1}e^{-\kappa_n r}}{[2\Gamma(2\ell+2)]^2}\sum_{L=0}^{\ell}\frac{e^{i(2L-\ell)\pi/2}(\ell-L+1)\Gamma(L+\ell+1)}{\Gamma(L+1)(\kappa_n-iq)^{(\ell-L+2)}}\sum_{j=0}^{\infty}\frac{\Gamma(j+2\ell+2)}{(j+\ell+1-s/\kappa_n)\Gamma(j+1)}$$
$$\times{}_1F_1(-j;2\ell+2;2\kappa_n r)\,{}_2F_1\left(-j,\ell-L+2;2\ell+2;\frac{2\kappa_n}{\kappa_n-iq}\right),\quad j=n-\ell-1. \tag{30}$$

The expression in Eq. (30) may be continued analytically by replacing κ_n by $-ik$ and $-s/\kappa_n$ by $i\eta$ to have

$$\langle r|G_\ell^C|\hat{h}_\ell^{(+)}q\rangle = -\frac{(2k)^{2\ell+5}\,r^{\ell+1}e^{ikr}}{[2\Gamma(2\ell+2)]^2}\sum_{L=0}^{\ell}\frac{e^{i(L-2\ell+1)\pi/2}(\ell-L+1)\Gamma(L+\ell+1)}{\Gamma(L+1)(k+q)^{(\ell-L+2)}}$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(j+2\ell+2)}{(j+\ell+1+i\eta)\Gamma(j+1)}{}_1F_1(-j;2\ell+2;-2ikr)\,{}_2F_1\left(-j,\ell-L+2;2\ell+2;\frac{2k}{k+q}\right). \tag{31}$$

To remove the infinite sum in the above equation we rearranged the terms in the infinite series and made use of the following relations [44–48]

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z)\,, \tag{32}$$
$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c;z/(z-1))\,, \tag{33}$$
$$c\,{}_2F_1(a,b;c;z) - c\,{}_2F_1(a+1,b;c;z) + b\,{}_2F_1(a+1,b+1;c+1;z) = 0, \tag{34}$$
$$\Lambda_{\rho,\sigma}(a;c;z) = \sum_{n=0}^{\infty}\frac{\rho^n}{\Gamma(n+1)}\theta_{\sigma+n}(a;c;z)\,, \tag{35}$$

and

$$\theta_\sigma(a;c;z) = \frac{z^\sigma}{\sigma(\sigma+c-1)}{}_2F_2(1,c+a;\sigma+1,\sigma+c;z)\,, \tag{36}$$

to obtain

$$\tilde{G}_\ell^{C(+)}(r,q) = \langle r|\,G_\ell^C|\hat{h}_\ell^{(+)}q\rangle = -\sum_{L=0}^{\ell}M_L(k,q)\big[A_L(k,q)\varphi_\ell^C(k,r) - (-2ik)^{L-\ell-2}\,r^{\ell+1}e^{ikr}\Lambda_{\rho,\sigma}(\ell+1+i\eta;2\ell+2;-2ikr)\big], \tag{37a}$$

with

$$M_L(k,q) = \frac{(-2ik)^{2\ell+1}\,e^{i(2L-\ell)\pi/2}\Gamma(L+\ell+1)}{(-2iq)^L\,\Gamma(L+1)\Gamma(\ell-L+1)}, \tag{37b}$$
$$A_L(k,q) = \frac{\Gamma(1+i\eta+\ell)\Gamma(1-L-\ell)\Gamma(\ell-L+2)}{[-i(k+q)]^{(\ell-L+2)}\,\Gamma(2\ell+2)\Gamma(2+i\eta-L)}\left(\frac{q+k}{2k}\right)^{2\ell+1}{}_2F_1\left(1-\ell-L,i\eta-\ell;2+i\eta-L;\frac{q-k}{q+k}\right) \tag{37c}$$

and

$$\varphi_\ell^C(k,r) = r^{\ell+1}e^{ikr}{}_1F_1(\ell+1+i\eta;2\ell+2;-2ikr) = r^{\ell+1}e^{-ikr}{}_1F_1(\ell+1-i\eta;2\ell+2;2ikr). \tag{37d}$$

The above expression is in exact agreement with that of Ref. [35]. The integrals in Eqs. (16) and (18) have been expressed earlier in their maximal reduced form by us [35,39,40] and are written as

$$\tilde{G}_\ell^{C(+)}(r,\beta_\ell) = \frac{r^{\ell+1}e^{ikr}}{2ik2^\ell\ell!}\left[\frac{2ik}{(\ell+1+i\eta)(\beta_\ell-ik)}{}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{\beta_\ell+ik}{\beta_\ell-ik}\right)\right.$$
$$\left.\times{}_1F_1(\ell+1+i\eta,2\ell+2;-2ikr) - \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(\ell+1+i\eta,2\ell+2;-2ikr)\right] \tag{38}$$

and

$$\tilde{G}_\ell^{CS(+)}(\beta_\ell,q) = -\frac{1}{2^{2\ell}(\ell!)^2D_\ell^{(+)}(k)}\left\{\frac{q^{-\ell}\Gamma(2\ell+2)}{(k^2-q^2)(\beta_\ell-ik)}\frac{1}{(\ell+1+i\eta)}{}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{\beta_\ell+ik}{\beta_\ell-ik}\right)\right.$$
$$\left.-\sum_{L=0}^{\ell}\frac{(i)^{L-\ell}(\ell+L)!}{(2q)^LL!(\ell-L)!}Z_\ell(\beta_\ell,k,q)\right\}, \tag{39}$$

with

$$Z_\ell(\beta_\ell,k,q) = \frac{(i)^{L-\ell}\Gamma(2\ell+2)}{2^\ell\ell!(2k)^{2\ell+1}}\frac{\partial^{\ell+1-L}}{\partial q^{\ell+1-L}}\left\{\frac{1}{\beta_\ell-ik}\left[\frac{\Gamma(i\eta-\ell)}{\Gamma(\ell+2+i\eta)}\left(\frac{q+k}{q-k}\right)^{i\eta}(q^2-k^2)^\ell\right.\right.$$
$$\times{}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\frac{\beta_\ell+ik}{\beta_\ell-ik}\right) - \frac{(q+k)^{2\ell}}{(i\eta-\ell)\Gamma(2\ell+2)}\left(\frac{2ik}{\beta_\ell+ik}\right)^{2\ell+1}$$
$$\left.\left.\times\left\{{}_2F_1\left(1,i\eta-\ell;i\eta-\ell+1;\frac{(q-k)(\beta_\ell+ik)}{(q+k)(\beta_\ell-ik)}\right)-1\right\}-(q+k)^{2\ell}X_\ell(\beta_\ell,k,q)\right]\right\}, \tag{40}$$

and

$$X_\ell(\beta_\ell,k,q) = \frac{1}{(i\eta-\ell)\Gamma(2\ell+2)} - \frac{(q-k)(\beta_\ell-ik)}{(q+k)(\beta_\ell+ik)(i\eta-\ell+1)}\sum_{n=0}^{2\ell-1}\frac{(-1)^n}{\Gamma(n+3)\Gamma(2\ell-n)}$$
$$\times\left(\frac{\beta_\ell-ik}{\beta_\ell+ik}\right)^n({}_2F_1)_{n+1}\left(1,i\eta-\ell;i\eta-\ell+2;\frac{(q-k)(\beta_\ell+ik)}{(q+k)(\beta_\ell-ik)}\right). \tag{41}$$

Here $({}_2F_1)_{n+1}(^*)$ represents the first $(n+1)$ terms of the hypergeometric series. The Fredholm determinant $D_\ell^{(+)}(k)$ introduced in Eq. (15) reads as [35,39,40]

$$D_\ell^{(+)}(k) = 1+\lambda_\ell\frac{\Gamma(2\ell+2)(\beta_\ell-ik)^{-2}}{2^{2\ell}(\ell!)^2(2\beta_\ell)^{2\ell+1}(\ell+1+i\eta)}{}_2F_1\left(1,i\eta-\ell;\ell+2+i\eta;\left(\frac{\beta_\ell+ik}{\beta_\ell-ik}\right)^2\right). \tag{42}$$

Now combining Eq. (17) together with Eqs. (37a)–(42) one will be in a position to write an expression for $\tilde{G}_\ell^{\mathrm{CS}(+)}(r,q)$. Thus, with the knowledge of $\tilde{G}_\ell^{C(+)}(r,q)$ and $\tilde{G}_\ell^{\mathrm{CS}(+)}(r,q)$ in conjunction with Eqs. (5), (6), (8a), and (8b) one can write explicit expressions for off-shell physical solutions for the interactions under consideration. By comparing our constructed expressions for off-shell physical solutions for Coulomb and Coulomb-plus-Graz-separable potential with Eq. (19) the corresponding off-shell Jost solutions are identified [35] as

$$f_\ell^C(k,q,r) = e^{i\ell\pi/2}(k^2-q^2)\sum_{L=0}^{\ell}M_L(k,q)\left[A_L(k,q)\varphi_\ell^C(k,r)+e^{i(\ell-L)\pi/2}(2k)^{L-\ell-2}\right.$$
$$\left.\times r^{\ell+1}e^{ikr}\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1-\ell-L}(\ell+1+i\eta,2\ell+2;-2ikr)\right] - \frac{e^{-\pi\eta/2}}{\Gamma(\ell+1)}\left(\frac{k}{q}\right)^\ell f_\ell^C(k,q)f_\ell^C(k,r), \tag{43a}$$

where the Coulomb off-shell Jost function $f_\ell^C(k,q)$and on-shell Jost solution $f_\ell^C(k,r)$ are given by [27,35]

$$f_\ell^C(k,q) = \frac{1}{(2\ell+1)!!}\left(\frac{q}{k+q}\right)^\ell\left(\frac{q+k}{q-k}\right)^{i\eta}\sum_{L=0}^{\ell}\frac{(\ell+1-L)(\ell+L)!}{L!}\left(\frac{q-k}{2q}\right)^L{}_2F_1\left(\ell+1-i\eta,\ell+L;2\ell+2;\frac{2k}{(k+q)}\right), \tag{43b}$$

and

$$f_\ell^C(k,r) = -(2kr)^{\ell+1} i e^{i(kr-\ell\pi/2)} e^{\pi\eta/2}\Psi(\ell+1+i\eta, 2\ell+2, -2ikr). \tag{43c}$$

The other one is

$$\begin{aligned} f_\ell^{\rm CS}(k,q,r) = {}& f_\ell^C(k,q,r) + \lambda_\ell \frac{e^{i\ell\pi/2}(k^2-q^2)}{2^\ell \ell! D_\ell(k)} \sum_{L=0}^{\ell} \frac{(i)^{L-\ell}(\ell+L)!}{(2q)^L L!(\ell-L)!} \\ & \times Z_\ell(\beta_\ell,k,q)\left[\frac{-1}{(\beta_\ell - ik)(\ell+1+i\eta)}\, {}_2F_1\left(1, i\eta-\ell; \ell+2+i\eta; \frac{\beta_\ell+ik}{\beta_\ell-ik}\right)\right. \\ & \times \Phi(\ell+1+i\eta, 2\ell+2; -2ikr) + \frac{(-2ik)^{2\ell+1}\Gamma(\ell+1+i\eta)}{(\beta_\ell^2+k^2)^{\ell+1}}\left(\frac{\beta_\ell - ik}{\beta_\ell+ik}\right)^{i\eta} \\ & \left.\times \Psi(\ell+1+i\eta, 2\ell+2; -2ikr) - \frac{1}{2ik}\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(\ell+1+i\eta, 2\ell+2; -2ikr)\right] r^{\ell+1} e^{ikr}. \end{aligned} \tag{44}$$

The s-wave versions of Eqs. (43a) and (44) exactly coincide with the results of Refs. [29–34]. The expressions in (43a) and (44) produce their correct limiting behaviors and on-shell discontinuities. The on-shell limiting behaviors of Eqs. (43a), (43b), and (44) are given by the singular factor $(q-k)^{-i\eta}$, where η is the Sommerfeld parameter. The corresponding on-shell quantities can be obtained from their off-shell expressions by using Coulombian asymptotic states [27].

IV. OFF-SHELL T MATRICES

The off-shell T matrix is related to an off-shell physical solution through the relation

$$T_\ell(p,q,k^2) = \frac{2}{\pi pq}\int_0^\infty dr\, \hat{j}_\ell(pr) V(r) \psi_\ell^{(+)}(k,q,r). \tag{45}$$

Here we shall construct the expressions for T matrices by using a modified expression for the same which does not involve the potential explicitly [33]. The procedure is as follows.

From the differential equation for an off-shell physical solution one has

$$\begin{aligned} V(r)\psi_\ell^{(+)}(k,q,r) = {}& \left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right]\psi_\ell^{(+)}(k,q,r) \\ & - (k^2-q^2)\hat{j}_\ell(qr). \end{aligned} \tag{46}$$

Substituting Eq. (46) in (45) and integrating the resulting integral twice by parts with the conditions

$$\lim_{r\to 0} r^{-\ell-1}\psi_\ell^{(+)}(k,q,r) = 1, \tag{47a}$$

and

$$\lim_{x\to 0} x^{-\ell-1}\hat{j}_\ell(x) = 1, \tag{47b}$$

we get

$$\begin{aligned} T_\ell(p,q,k^2) = {}& \frac{2(k^2-p^2)}{\pi pq}\int_0^\infty dr\, \hat{j}_\ell(pr)\psi_\ell^{(+)}(k,q,r) \\ & - S_\ell(p,q,k^2), \end{aligned} \tag{48a}$$

where

$$\begin{aligned} S_\ell(p,q,k^2) &= \frac{2(k^2-q^2)}{\pi pq}\int_0^\infty dr\, \hat{j}_\ell(pr)\hat{j}_\ell(qr) = 0 \quad \text{for } \ell = 0, \\ &= \frac{(k^2-q^2)}{p^3 q}\delta(p-q) \quad \text{for } \ell > 0. \end{aligned} \tag{48b}$$

In deriving the result in Eq. (48a) we have made use of the differential equation

$$\left[\frac{d^2}{dr^2} + p^2 - \frac{\ell(\ell+1)}{r^2}\right]\hat{j}_\ell(pr) = 0. \tag{49}$$

In the following we present an alternative approach to arrive at the result in Eq. (48a). Combining Eqs. (45) and (46) we obtain

$$\begin{aligned} T_\ell(p,q,k^2) = {}& \frac{2}{\pi pq}\int_0^\infty dr\, \hat{j}_\ell(pr)\left\{\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right]\right. \\ & \left. \times\ \psi_\ell^{(+)}(k,q,r) - (k^2-q^2)\hat{j}_\ell(qr)\right\}. \end{aligned} \tag{50}$$

Using the transposed operator relation $\langle\varphi|\,\hat{O}\,|\psi\rangle = \langle\psi|\,\tilde{O}\,|\varphi\rangle$ [49] in Eq. (50) together with Eqs. (48b) and (49) one arrives at the result in (48a). In view of Eqs. (3b), (3d), (5), and (48a) the Coulomb off-shell T matrix reads as

$$\begin{aligned} T_\ell^C&(p,q,k^2) \\ &= Y(p,q,k^2)\big[I_\ell^C(p,q,k^2) - (-1)^\ell I_\ell^C(p,-q,k^2) \\ &\quad - (-1)^\ell I_\ell^C(-p,q,k^2) + I_\ell^C(-p,-q,k^2)\big] \\ &\quad - S_\ell(p,q,k^2) \end{aligned} \tag{51}$$

with

$$I_\ell^C(p,q,k^2) = \int_0^\infty dr\, \hat{h}_\ell^{(+)}(pr)\tilde{G}_\ell^{C(+)}(r,q), \tag{52}$$

$$I_\ell^C(p,-q,k^2) = I_\ell^C(p,q,k^2)\big|_{q\to -q}, \tag{53a}$$

$$I_\ell^C(-p,q,k^2) = I_\ell^C(p,q,k^2)\big|_{p\to -p}, \tag{53b}$$

$$I_\ell^C(-p,-q,k^2) = I_\ell^C(p,q,k^2)\big|_{q\to -q;\, p\to -p}, \tag{53c}$$

and

$$Y(p,q,k^2) = \frac{(k^2-p^2)(k^2-q^2)}{i\pi pq}. \tag{54}$$

Combination of Eqs. (28), (37a)–(37c), and (52) leads to

$$I_\ell^C(p,q,k^2) = C_{Lj}(p,q)\Bigg[(-1)^{\ell+L}\frac{(i)^{L+j-2}\Gamma(1-\ell-L)\Gamma(\ell-L+2)\Gamma(\ell+1+i\eta)}{\Gamma(2+i\eta-L)\Gamma(2\ell+2)}(k+q)^{\ell+L-1} \times {}_2F_1\left(1-\ell-L, i\eta-\ell; 2+i\eta-L; \frac{q-k}{q+k}\right)\frac{\partial^{\ell-j+1}}{\partial p^{\ell-j+1}}\int_0^\infty dr e^{i(p+k)r}\Phi(\ell+1+i\eta, 2\ell+2; -2ikr) + \frac{(-2ik)^{\ell+L-1}}{(i)^{\ell-j+1}}\sum_{n=0}^{\infty}\left(\frac{k-q}{2k}\right)^n\frac{1}{n!}\frac{\partial^{\ell-j+1}}{\partial p^{\ell-j+1}}\int_0^\infty dr e^{i(p+k)r}\theta_{n+1-\ell-L}(\ell+1+i\eta, 2\ell+2; -2ikr)\Bigg], \tag{55}$$

with

$$C_{Lj}(p,q) = \sum_{j=0}^{\ell}\sum_{L=0}^{\ell}\frac{(i)^{2(j+L-\ell)}(\ell+j)!(\ell+L)!}{(2ip)^j(2iq)^L L!\,j!(\ell-j)!(\ell-L)!}. \tag{56}$$

Evaluation of integrals in Eq. (55) using Eq. (29) and the following standard integral [43–46]

$$\int_0^\infty dz e^{-bz}\theta_\sigma(a,c;pz) = \frac{\Gamma(\sigma)p^\sigma}{(\sigma+c-1)b^{\sigma+1}}{}_2F_1(1,\sigma+a;\sigma+c;p/b), \quad \mathrm{Re}\,\sigma>0, \quad \mathrm{Re}\,(\sigma+c)>1, \quad \mathrm{Re}\,b>\mathrm{Re}\,p, \tag{57}$$

some algebraic manipulation with the help of the transformation relations in Eqs. (32), (33), and

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b-c+1;1-z) + (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}{}_2F_1(c-a,c-b;c-a-b+1;1-z) \tag{58}$$

yields

$$I_\ell^C(p,q,k^2) = C_{Lj}(p,q)\frac{(i)^{L+j-2}}{(-2ik)(\ell-i\eta)}\frac{\partial^{\ell-j+1}}{\partial p^{\ell-j+1}}\left(\frac{p+k}{2k}\right)^{2\ell}\Bigg[\frac{\Gamma(1-\ell-L)\Gamma(\ell-L+2)\Gamma(\ell+1+i\eta)}{\Gamma(2+i\eta-L)\Gamma(2\ell+1)} \times(-1)^{(\ell+L)}(k+q)^{\ell+L-1}{}_2F_1\left(1-\ell-L, i\eta-\ell; 2+i\eta-L; \frac{q-k}{q+k}\right){}_2F_1\left(-2\ell, i\eta-\ell; 1+i\eta-\ell; \frac{p-k}{p+k}\right) + \frac{(-1)^{\ell+L-1}}{(2k)^{1-L-\ell}}\sum_{n=0}^{\infty}\left(\frac{k-q}{2k}\right)^n\frac{\Gamma(n+1-\ell-L)}{n!}{}_2F_1\left(i\eta-\ell, -n-1-\ell+L; 1+i\eta-\ell; \frac{p-k}{p+k}\right)\Bigg] \tag{59}$$

Now equations (51)–(54) and (59) can be combined together to get the desired expression for the off-shell Coulomb T matrix. We have verified that for the s-wave case our expression for off-shell Coulomb T-matrix is in exact agreement with those of Refs. [31,33] which reads as

$$T^C(p,q,k^2) = \frac{e^{i\pi/2}k}{\pi pq}\Bigg[F\left(1, i\eta; 1+i\eta; \frac{(q-k)(p-k)}{(q+k)(p+k)}\right) - F\left(1, i\eta; 1+i\eta; \frac{(q+k)(p-k)}{(q-k)(p+k)}\right) - F\left(1, i\eta; 1+i\eta; \frac{(q-k)(p+k)}{(q+k)(p-k)}\right) + F\left(1, i\eta; 1+i\eta; \frac{(q+k)(p+k)}{(q-k)(p-k)}\right)\Bigg]. \tag{60}$$

At the half-shell point, i.e., $p\to k$ the $T-$matrix element $T^C(k,q,k^2)\to 0$. This can easily be observed by transforming two of the four hypergeometric functions in Eq. (60) by utilizing the relation [44,45]

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}{}_2F_1(a,1-c+a;1-b+a;z^{-1}) + (-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}{}_2F_1(b,1-c+b;1-a+b;z^{-1}). \tag{61}$$

Also there appears a singularity as $p\to q\neq k$ because of the logarithmic singularities of the two hypergeometric functions involved in Eq. (60) with the argument becoming equal to unity [1]. When $p\to k, q\to k$, i.e., at the on-shell point, the $T-$matrix element is also zero.

Combining Eqs. (3b), (3d), (17), and (48a) the off-shell T matrix for motion in the Coulomb-plus-Graz-separable potential reads as

$$T_\ell^{\mathrm{CS}}(p,q,k^2) = T_\ell^C(p,q,k^2) + Y(p,q,k^2)\frac{\lambda_\ell}{D_\ell^{(+)}(k)}\left[\tilde{G}_\ell^{C(+)}(\beta_\ell,q) - (-1)^\ell\tilde{G}_\ell^{C(+)}(\beta_\ell,-q)\right] \times\left[\tilde{G}_\ell^{C(+)}(\beta_\ell,p) - (-1)^\ell\tilde{G}_\ell^{C(+)}(\beta_\ell,-p)\right] - S_\ell(p,q,k^2). \tag{62}$$

Equation (62) together with (39)–(42), (54), and the expression for $T_\ell^C(p,q,k^2)$ produces the expression for the off-shell T matrix for Coulomb-plus-Graz-separable potential that leads to the correct limiting value for $\ell = 0$ [31,33]. For an s-wave the form factors of the Graz-separable potential coincides with that of Yamaguchi [50]. Therefore, the expression for Coulomb-plus-Yamaguchi off-shell T matrix reads as

$$T^{CY}(p,q,k^2) = T^C(p,q,k^2) + \frac{2}{\pi pq} K(\beta,q,k^2)\left[-\frac{p}{(\beta^2+p^2)} + \frac{k}{(\beta^2+k^2)} \times \left\{ {}_2F_1\left(1, i\eta; 1+i\eta; \frac{(p-k)(\beta+ik)}{(p+k)(\beta-ik)}\right) + {}_2F_1\left(1, i\eta; 1+i\eta; \frac{(p+k)(\beta+ik)}{(p-k)(\beta-ik)}\right)\right\}\right], \quad (63)$$

with

$$K(\beta,q,k^2) = \lambda \frac{(\beta+ik)(\beta+iq)}{2D_0^{(+)}(k)(1+i\eta)(\beta^2+k^2)(\beta^2+q^2)} \times \left[(k-q)\,{}_2F_1\left(1, i\eta; 2+i\eta; \frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right) - (k+q)\,{}_2F_1\left(1, i\eta; 2+i\eta; \frac{(q+k)(\beta+ik)}{(q-k)(\beta-ik)}\right)\right]. \quad (64)$$

The off-shell T matrix for motion in the Coulomb-plus-Yamaguchi potential shows the same limiting behaviors as the pure Coulomb case. In the next section we shall study the p-p and n-p off-shell T matrices in the $^1S_0$ channel by using Eqs. (60) and (63) together with (64).

V. RESULTS AND DISCUSSIONS

We have computed $T^{CY}(p,q,k^2)$ and $T^Y(p,q,k^2)$ for $p\,\langle k$ and $p\rangle\,k$ for fixed values of $q = 0.25$ and 0.55 fm^{-1} with the parameters [39,51,52] $\lambda = -2.405$ fm^{-3} and $\beta = 1.1$ fm^{-1}. We have chosen to work with $(2k\eta)^{-1} = 28.80$ fm. This is the

TABLE I. Off-shell T matrices T^{CY} (p,q,k^2) and T^Y (p,q,k^2) as a function of the off-shell momentum p for $q = 0.25$ fm^{-1} at $E_{lab} = 10$ MeV.

p (fm^{-1})	Re T^{CY} (p,q,k^2)	Im T^{CY} (p,q,k^2)	Re T^Y (p,q,k^2)	Im T^Y (p,q,k^2)
0.01	−1.108	−1.307	−0.987	−1.423
0.05	−1.086	−1.303	−0.985	−1.420
0.10	−1.062	−1.292	−0.979	−1.411
0.15	−1.017	−1.273	−0.969	−1.397
0.20	−0.926	−1.247	−0.956	−1.377
0.25	–	–	−0.939	−1.353
0.30	−1.054	−1.152	−0.919	−1.324
0.347	0	0	−0.898	−1.294
0.35	−1.581	−1.109	−0.897	−1.292
0.40	−1.389	−1.283	−0.872	−1.256
0.45	−1.322	−1.275	−0.846	−1.219
0.50	−1.267	−1.245	−0.818	−1.179
0.55	−1.216	−1.208	−0.790	−1.138
0.60	−1.167	−1.168	−0.761	−1.097
0.65	−1.119	−1.125	−0.732	−1.055
0.70	−1.072	−1.082	−0.703	−1.013
0.75	−1.027	−1.040	−0.674	−0.971
0.80	−0.984	−0.998	−0.646	−0.931
0.85	−0.941	−0.957	−0.618	−0.891
0.90	−0.901	−0.917	−0.591	−0.852
0.95	−0.862	−0.878	−0.566	−0.815
1.00	−0.824	−0.840	−0.541	−0.779
2.00	−0.365	−0.374	−0.229	−0.330
3.00	−0.172	−0.203	−0.117	−0.169
4.00	−0.124	−0.128	−0.069	−0.100
5.00	−0.086	−0.089	−0.046	−0.066
6.00	−0.064	−0.066	−0.032	−0.046

TABLE II. Off-shell T matrices T^{CY} (p,q,k^2) and T^Y (p,q,k^2) as a function of the off-shell momentum p for $q = 0.55$ fm^{-1} at $E_{lab} = 10$ MeV.

p (fm^{-1})	Re T^{CY} (p,q,k^2)	Im T^{CY} (p,q,k^2)	Re T^Y (p,q,k^2)	Im T^Y (p,q,k^2)
0.01	−1.266	−1.308	−0.831	−1.197
0.05	−1.250	−1.303	−0.829	−1.195
0.10	−1.243	−1.292	−0.824	−1.187
0.15	−1.235	−1.273	−0.816	−1.175
0.20	−1.22	−1.246	−0.804	−1.159
0.25	−1.216	−1.208	−0.790	−1.138
0.30	−1.217	−1.150	−0.773	−1.114
0.347	0	0	−0.756	−1.090
0.35	−1.535	−1.125	−0.754	−1.087
0.40	−1.279	−1.292	−0.734	−1.057
0.45	−1.171	−1.282	−0.712	−1.025
0.50	−1.078	−1.252	−0.688	−0.992
0.55	–	–	−0.664	−0.957
0.60	−0.978	−1.174	−0.640	−0.923
0.65	−0.956	−1.130	−0.616	−0.887
0.70	−0.924	−1.087	−0.591	−0.852
0.75	−0.889	−1.044	−0.567	−0.817
0.80	−0.854	−1.002	−0.543	−0.783
0.85	−0.819	−0.960	−0.520	−0.750
0.90	−0.785	−0.920	−0.498	−0.717
0.95	−0.752	−0.881	−0.476	−0.686
1.00	−0.720	−0.844	−0.455	−0.655
2.00	−0.320	−0.375	−0.192	−0.278
3.00	−0.173	−0.203	−0.098	−0.142
4.00	−0.109	−0.128	−0.058	−0.084
5.00	−0.076	−0.089	−0.038	−0.055
6.00	−0.057	−0.066	−0.027	−0.039

proton Bohr radius. In Tables I and II we present our results both for $T^{\mathrm{CY}}(p,q,k^2)$ and $T^{Y}(p,q,k^2)$ as a function of p for $E_{\mathrm{lab}} = 10$ MeV with $q = 0.25$ and 0.55 fm^{-1}, respectively. The values of the off-shell T matrix for a pure Yamaguchi interaction [50] have been obtained by turning off the Coulomb interaction in the numerical routine for $T^{\mathrm{CY}}(p,q,k^2)$. Therefore, the two sets of numbers, namely, those for $T^{\mathrm{CY}}(p,q,k^2)$ and $T^{Y}(p,q,k^2)$ are expected to provide a basis for looking into the role of the Coulomb interaction in the p-p off-shell scattering. As expected $T^{Y}(p,q,k^2)$ is a continuous function of the off-shell momentum$p$, and Re $T^{Y}(p,q,k^2)$ and Im $T^{Y}(p,q,k^2)$ increase smoothly as p becomes large. In contrast to this, $T^{\mathrm{CY}}(p,q,k^2)$ exhibits a discontinuity at $p = q$ and becomes exactly zero at the half-shell point $p = k$. Beyond the half-shell point, Re $T^{\mathrm{CY}}(p,q,k^2)$ and Im $T^{\mathrm{CY}}(p,q,k^2)$ increase almost with equal gradient as Re $T^{Y}(p,q,k^2)$ and Im $T^{Y}(p,q,k^2)$. Taking notice of our numbers in Tables I and II we see that the Coulomb interaction affects the low off-shell momentum data more significantly and the two sets of numbers for $T^{\mathrm{CY}}(p,q,k^2)$ and $T^{Y}(p,q,k^2)$ are not practically discernable for high p values. Our results are in order for the $(p,2p)$ reaction in which a single proton is knocked out of the nucleus and the momentum transfer distribution is measured. The momentum transfer distribution is closely correlated with the distribution of momenta which the nuclear proton had before it was knocked out [53]. The characteristic discontinuity of $T^{\mathrm{CY}}(p,q,k^2)$ at the energy shell arises from the fact that the Coulomb potential distorts not only the scattered wave but also the incident plane wave [54]. Sharma and Jain [55] and Kok *et al.* [56] confirmed that off-shell effects are sizeable for the $(\alpha, 2\alpha)$ reaction. The importance of the investigation of the off-shell T matrix consists in the fact that the elements of the T matrix are measurable quantities and if all of them are known from the experimental data, the interaction potential can be constructed. These off-shell elements are generally obtained from the analysis of the p-p bremsstrahlung, $(p,2p)$ reaction, nuclear matter, three-body bound, and scattering problems. Calculation of a direct reaction such as the inelastic scattering of nucleons by nuclei demands the off-shell T-matrix element and also for the calculation of binding energy of nuclear matter as well as finite nuclei. The deuteron photodisintegration cross section was also found to be sensitive to the off-shell T matrix [57]. Thus, it is our belief that our constructed expressions may serve as an efficient and convenient starting point for rigorous calculations on three-body problems with charges.

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