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Pion-nucleus elastic scattering, double-charge-exchange reactions, and subthreshold resonances

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In the scattering of positive pions by nuclei the double-charge-exchange (DCX) reaction creates the possibility of the formation of pionic atom states in the vicinity of the threshold of the reaction. These quasistationary states can manifest themselves as resonances in the elastic scattering cross section. The strength and shape of these resonances are strongly affected by the instability of the nucleus created in the DCX channel. It is shown that this mechanism can explain oscillation structures in the excitation function observed in scattering of low-energy positive pions from ^{12}C.

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I. INTRODUCTION

Understanding the interaction of low-energy (below 50–60 MeV) pions with nuclei has always been a challenge for the theory of pion-nucleus interactions (see, e.g., Refs. [1] and [2]). Due to rather weak π-nucleon interaction at these energies a pion can penetrate relatively deeply into a nucleus and interact with several nucleons or nucleon clusters. It increases the probability of pion absorption and the DCX reaction, as either of these processes requires participation of at least two nucleons. The pion isospin degree of freedom increases the number of possible reaction channels significantly. In addition, one cannot neglect the role of the Coulomb effects in this energy region.

The present paper is devoted to a discussion of narrow resonance structures in the elastic scattering of positive pions from ^{12}C observed in Ref. [3]. Similar resonance structures in the production and absorption of low-energy positive pions were found in Refs. [4–6]. In Ref. [3], the differential cross sections were measured at six scattering angles (37°, 65°, 83°, 103°, 118°, and 142°) in the energy range of 18–44 MeV, with an increment in the incident energy of 2 MeV. Here and throughout the present paper the scattering angles and differential cross sections are given in the center-of-mass reference frame. The differential cross sections were compared to the calculations made within the framework of the unitary scattering theory (UST) of the pion-nucleus scattering developed in Ref. [1] (see Sec. II). Despite a good description of the measured differential cross sections, the experimental data presented in terms of the excitation function (differential cross section at a given scattering angle as a function of energy) showed oscillation structures which the UST approach did not reproduce. These oscillations become more pronounced at angles around 90°.

A typical disagreement between theory and experiment for the excitation function at 83° is shown in Fig. 1 by the solid line. For completeness, the experimental data of other groups [7–11] are presented. It should be noted that these data correspond to slightly different scattering angles. For 13.9 MeV [7] and 34.7 MeV [10], the closest angle is 80.8°. For 20 MeV [8], 30.3 MeV [9], and 40 MeV [11] there are two close angles: 80.9° and 85.9°. To account for variations of the differential cross section with respect to the scattering angle, in Fig. 1 the mean values of the differential cross section measured at these two angles are presented. One can see that the greatest difference between the experimental data is observed at energies around 35 MeV. In Ref. [12] the authors conducted a comprehensive energy-dependent analysis of the elastic scattering of positive pions from ^{12}C, ^{16}O, and ^{40}Ca within the framework of the optical model formalism developed in Refs. [2] and [13]. For ^{12}C, they observed a similar disagreement between the theory and the data in Ref. [3] at angles near 90° and excluded those points from the analysis.

One of the possible explanations of these oscillations could be the formation of a diproton resonance in the πNN system proposed in [4]. However, a systematic experimental study [14] of the $\pi^+ + d \to p + p$ reaction in the same energy range of 18–44 MeV with an increment of 1–2 MeV did not show any oscillations in either the total or the differential cross sections. A narrow πNN resonance (d') proposed in Refs. [15] and [16] may be seen at pion energies around 50 MeV but not at ∼30–35 MeV. Also, this dibarion resonance does not decay into the nucleon-nucleon channel. In Ref. [17], the explanation of narrow resonance structures in the ^{12}C $(\pi^+, pp)X$ reaction as the manifestation of threshold cusp phenomena in pion-nucleus reactions is proposed.

In the present paper we explore the possibility that these oscillations [3] result from the formation of quasistationary pionic atom states in the vicinity of the threshold of the DCX reaction channel. The DCX ^{12}C$(\pi^+,\pi^-)^{12}$O reaction creates two oppositely charged particles that can form pionic atom states below the threshold of this reaction. If a π^- forms a bound state with ^{12}O, it cannot escape below the threshold owing to lack of energy. It is known that such quasibound states can manifest themselves as resonances in the elastic cross section. This type of subthreshold resonance was first investigated and described systematically by Baz' and coworkers [18,19]. Another argument in favor of this mechanism comes from the fact that the mass excess of ^{12}O is 32.06 MeV [20]. This brings the Q value of the DCX channel in π^+-^{12}C scattering into the ∼30-MeV energy region. It is noteworthy that one of the first experimental observations of ^{12}O was obtained in the DCX reaction ^{12}C$(\pi^+,\pi^-)^{12}$O [21].

Resonant formation of pionic atoms in nuclear reactions initiated by different projectiles (see, e.g., Ref. [22]) was

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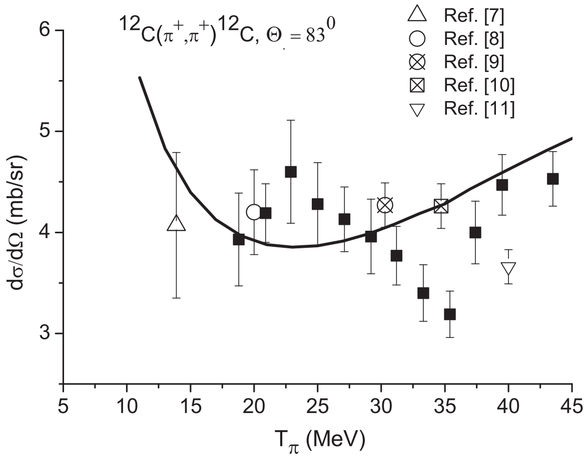

FIG. 1. Excitation function at 83°. The data from Ref. [3] (filled squares) are presented along with the data from Refs. [7–11]. The solid line presents the UST calculations without taking into account the effect of subthreshold resonances.

investigated with regard to the search for deeply bound pionic states [23,24]. The cross sections of most of these reactions should show narrow resonance oscillations in the vicinity of the threshold of these reactions. However, as shown in this paper it might be very difficult to observe these structures unless a pionic atom is formed by an unstable nucleus with the decay width much bigger than the corresponding elastic width.

The paper is organized as follows. In Sec. II we present a short overview of the UST approach which is used for the description of pion-nucleus interaction. Section III is devoted to the theory of subthreshold resonances owing to formation of quasibound pionic atom states. In Sec. IV, a systematic comparison of theoretical calculations with the data is presented. In Sec. V we discuss the main results of the paper. In the Appendix, a brief derivation of the main formulas in Sec. III within the framework of the *R*-matrix formalism is presented.

II. THEORY

In this section we briefly summarize the UST formalism [1] that we use in the description of pion-nucleus scattering. For simplicity we consider the scattering of pions by nuclei with zero spin. The pion-nucleus scattering amplitude is presented in a standard way as

$$f_{\pi A} = f_C + f_{\rm sc}, \tag{1}$$

where f_C is the Coulomb amplitude, and $f_{\rm sc}$ is the nuclear-Coulomb amplitude,

$$f_{\rm sc} = \frac{1}{2ik}\sum_{l=0}^{\infty} e^{2i\sigma_l^{\pm}}\left(S_l e^{2i\delta^{\pm}_{R,l}} - 1\right), \tag{2}$$

where $\sigma_l^{\pm}$ are the Coulomb phases and $\delta^{\pm}_{R,l}$ the Coulomb corrections caused by the effects of the Coulomb distortion of the pion wave. A detailed procedure for calculating these corrections is given in Ref. [1]. The hadronic part is presented by the S matrix,

$$S_l = e^{2i\delta_{\pi A,l}}, \tag{3}$$

where $\delta_{\pi A,l}$ are pure hadronic phase shifts which we calculate within the framework of the UST approach [1].

The UST approach is based on the method of evolution with respect to the coupling constant [25,26]. The consistency theory along with the unitarity provides a correct separation of the potential effects from the nonpotential (true pion absorption) effect. The basic equations are formulated for calculation of the pion-nucleus phase shifts:

$$\delta_{\pi A}(k) = \delta^{\rm pot}_{\pi A}(k) + \delta^{\rm abs}_{\pi A}(k). \tag{4}$$

Here, $\delta^{\rm pot}$ is the part of the pion-nucleus phase shift that is formed by the multiple scattering of a pion by the nuclear nucleons, and $\delta^{\rm abs}$ is the absorption correction. The "potential" part is expressed in terms of the pion-nucleon phase shifts and the nuclear ground-state characteristics such as nuclear form factor and correlation functions. The absorption part is expressed in terms of the absorption parameters $\tilde{B}_0$ and $\tilde{C}_0$,

$$\delta^{\rm abs}_{\pi A}(k) = A(A-1)\frac{1+\epsilon}{1+2\epsilon/A}\hat{\rho}^2(\vec{q})[\tilde{B}_0(k) + \tilde{C}_0(k)(\vec{\kappa}'\cdot\vec{\kappa})], \tag{5}$$

where $\epsilon = \omega_\pi(k)/2M$, ω_π is the pion energy, M is the mass of a nucleon, $\hat{\rho}^2(\vec{q})$ is the Fourier transform of the square of nuclear density $\rho(r)$ normalized to unity, $\vec{q} = \vec{k}' - \vec{k}$ is the momentum transfer, and $\vec{\kappa}$ and $\vec{\kappa}'$ are the pion momenta in the $\pi - 2N$ center-of-mass system. The absorption parameters determined from the pionic atom data are [1]

$$\begin{aligned}\tilde{B}_0(k) &= (-0.1 + i0.1)\ {\rm fm}^4,\\ \tilde{C}_0(k) &= (-2.8 + i1.0)\ {\rm fm}^6.\end{aligned} \tag{6}$$

In its standard form the UST formalism does not take into account the possibility of formation of subthreshold resonances in pion-nucleus interactions.

III. DCX AND SUBTHRESHOLD RESONANCES

The opening of the DCX reaction channel,

$$\pi^+ + {}^{12}{\rm C} \rightarrow (\pi^-, {}^{12}{\rm O})^* \rightarrow \pi^+ + {}^{12}{\rm C}, \tag{7}$$

creates the possibility of formation of bound pionic atom states below the threshold of this reaction. As shown by Baz' and coworkers [18,19], the formation of such subthreshold bound states is reflected by creating resonances in the elastic cross section. In Refs. [18] and [19], the general case of elastic scattering of two particles $X(a,a)X$ below the threshold of the inelastic channel $X(a,b)Y$ when particles b and Y can form bound states was considered. The main idea of the theoretical description of the effect of subthreshold resonances is based on the fact that one can neglect the energy dependence of the wave functions of (a,X) and (b,Y) systems arising from the strong interaction and focus on the analysis of energy dependence of the Coulomb wave function of the bound (b,Y) system.

The formation of the quasibound states owing to opening of the DCX channel modifies the partial S matrix, (3), in the following way (see the Appendix):

$$S_l = e^{2i(\delta_{\pi A,l}+\delta_l^{\rm res})}, \tag{8}$$

where the resonance part below the threshold is given by

$$\delta_l^{\rm res} = \tan^{-1}\frac{\alpha_2+\beta_2(-1)^{l+1}\zeta_l\cot\pi\eta}{\alpha_1+\beta_1(-1)^l\zeta_l\cot\pi\eta}, \tag{9}$$

where parameters $\alpha_{1,2}$ and $\beta_{1,2}$ are the real and imaginary parts of energy independent complex parameters $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ in the vicinity of the threshold energy $E_{\rm thr}$. These constants are expressed in terms of the logarithmic derivatives of pion-nucleus wave functions in the strong interaction region as shown in the Appendix. The parameter

$$\zeta_l = \frac{\pi(2k_2\eta R)^{2l+1}}{(2l+1)\Gamma^2(2l+1)}, \tag{10}$$

where k_2 is the pion momentum in the c.m.s. of the π^+-^{12}O system, $R = r_0A^{1/3}$ is the radius of the strong interaction region, A is the atomic mass number, and

$$\eta = (Z+2)e^2\sqrt{\frac{\mu_{\pi A}}{2|E-E_{\rm thr}|}}. \tag{11}$$

Here, $E \equiv T_\pi$ is the pion kinetic energy, $Z+2$ is the charge of the nucleus created owing to DCX, Z is the charge of the nucleus in the initial elastic channel, and $\mu_{\pi A}$ is the pion-nucleus reduced mass. The parameter ζ_l is energy independent, as $k_1\eta = -(Z+2)e^2\mu_{\pi A}$.

The resonance energies are determined by the condition of $\delta^{\rm res} = n\pi + \frac{\pi}{2}$, which gives the equation

$$\cot\pi\eta = (-1)^l\frac{\alpha_1}{\zeta_l\beta_1}, \tag{12}$$

where ζ_l is defined in (10). The solution of this equation can be written as

$$E_{nl}^{\rm res} = E_{\rm thr} + E_{nl}, \quad E_{nl} = -\frac{(Z+2)^2e^4\mu_{\pi A}}{2n^2\xi_{nl}^2},$$
$$\xi_{nl} \equiv 1 + \frac{1}{n\pi}\tan^{-1}(-1)^l\frac{\zeta_l\beta_1}{\alpha_1}, \quad n = 1,2,3,\ldots. \tag{13}$$

Here, ξ_{nl} represents the strong interaction shift of pure Coulomb energy levels,

$$E_n^C = -\frac{(Z+2)^2e^4\mu_{\pi A}}{2n^2}. \tag{14}$$

From Eq. (13) it follows that for a given total pion-nucleus angular momentum (l) there is an infinite number of resonances with increasing density as $E \to E_{\rm thr}$. It is easy to see that the width of the resonance region is determined by the energy of the first Coulomb level. For reaction (7) the energy of the lowest pionic atom level is given by

$$E_1^C = -\frac{\mu_{\pi N}(Z+2)^2e^4}{2}. \tag{15}$$

As shown in the Appendix, in the vicinity of each resonance energy $E = E_{nl}^{\rm res}$ the S matrix can be approximated by the Breit-Wigner formula as

$$S_l \approx e^{2i\delta_{\pi A,l}}\left(1 - \sum_{n=1}^{\infty}\frac{i\Gamma_{nl}^e}{E-E_{nl}^{\rm res}-i\Gamma_{nl}^e/2}\right), \tag{16}$$

where Γ_{nl}^e is the elastic width of each resonance. In applications to real processes the upper limit should be replaced by some finite number N which is determined by experimental energy resolution.

A simple procedure for generalization of this formalism to include the important case where one of the particles created in the opening reaction channel is unstable was proposed in Ref. [18]. In this case Eq. (16) is replaced by

$$S_l \approx e^{2i\delta_{\pi A,l}}\left(1 - \sum_{n=1}^{N}\frac{i\Gamma_{nl}^e}{E-E_{n,l}^{\rm res}-i(\Gamma_{nl}^e+\Gamma)/2}\right), \tag{17}$$

where Γ represents the particle's energy width. In the considered case this particle is the nucleus created in the DCX reaction.

Formula (16) can be simplified if the width of the created nucleus is much larger than the elastic widths of corresponding subthreshold resonances. Indeed, if $\Gamma \gg \Gamma_{nl}^e$, one can neglect the quantities Γ_{nl}^e in the denominators and present Eq. (16) in the form

$$S_l \approx e^{2i\delta_{\pi A,l}}\left(1 - \frac{i\Gamma_{{\rm tot},l}^e}{E-E_l^{\rm res}-i\Gamma/2}\right), \tag{18}$$

where

$$\Gamma_{{\rm tot},l}^e = \sum_{n=1}^{\infty}\Gamma_{nl}^e, \tag{19}$$

and $E_l^{\rm res}$ is some average value of the subthreshold resonance energy. This formula can be rewritten as

$$S_l \approx e^{2i\delta_{\pi A,l}}\left(1 - \gamma_l\frac{i\Gamma}{E-E_l^{\rm res}-i\Gamma/2}\right), \tag{20}$$

where $\gamma_l \equiv \Gamma_{{\rm tot},l}^e/\Gamma$. The derived formula presents a single-term Breit-Wigner approximation for the infinite series of the subthreshold resonances contributing to the scattering process. It is important to note that $\Gamma_{{\rm tot},l}^e$ is an effective elastic width representing the contribution from all resonances at a given orbital momentum l.

IV. CALCULATIONS

In scattering of positive pions from ^{12}C the DCX reaction creates two oppositely charged particles (π^-,^{12}O) which can form a pionic atom below the threshold of this reaction. This reaction has a positive Q value (32.06 MeV). In addition, a positive pion needs to overcome the nucleus Coulomb barrier. Therefore, the threshold energy of the DCX reaction channel will be determined by the sum of the reaction Q value and the magnitude of the Coulomb repulsion barrier δV_C, i.e., $E_{\rm thr} \approx Q + \delta V_C$, where $\delta V_C = {\rm Ze}^2/R$, $R = r_0A^{1/3}$, and $r_0 = 1.1$ fm. For (π^+,^{12}C), $\delta V_C \approx 3.43$ MeV. Therefore, the threshold energy $E^{\rm thr} \approx 35.45$ MeV. The resonance energies are shifted down from the threshold energy by the amount of the corresponding binding energy in accordance with Eq. (13). The lowest resonance energy corresponds to the $1s$ state in the (π^-,^{12}O) atom. Using Eq. (15) one can obtain that $E_1^C \approx -0.23$ MeV, and the value of the resonance energy $E_{1s}^{\rm res} \approx 35.26$ MeV. In the calculations below, this value will be

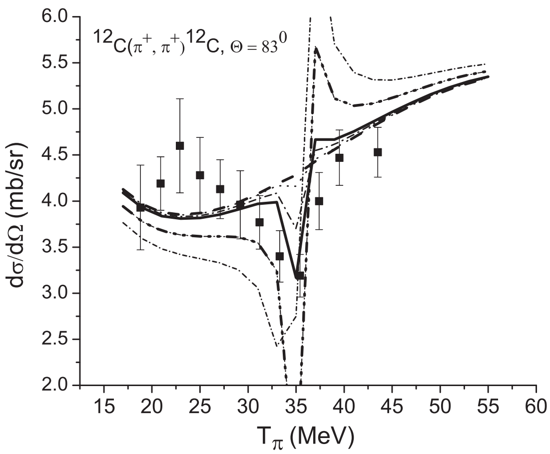

FIG. 2. Excitation function at 83°. Experimental data are taken from Ref. [3]. The dashed line represents the UST calculations without taking into account the effect of subthreshold resonances ($\gamma_0 = 0$); the dotted line corresponds to $\gamma_0 = 0.01$; the dash-dotted line, $\gamma_0 = 0.05$; the solid line, $\gamma_0 = 0.1$; the dash-dot-dotted line, $\gamma_0 = 0.5$; and the short-dash-dotted line, $\gamma_0 = 1.0$.

used as the average subthreshold resonance energy in Eq. (20) as well.

The ^{12}O nucleus is unstable. The width of the ground state of unbound ^{12}O is known with a large uncertainty: $\Gamma = 0.40 \pm 0.25$ MeV [20]. The main decay mode is two-proton emission to the ground state of ^{10}C. The "elastic" strong interaction width of the 1s state of (π^-,^{12}O) is about 10^{-3} MeV (see, e.g., [27]). Because the elastic width is much smaller than the width ^{12}O one can use the derived one-term Breit-Wigner approximation, (20).

The spin and parity of the ^{12}O ground state is 0^+. Therefore, the subthreshold s resonances can be generated by the pion s wave only. The s-wave S matrix is given by

$$S_0 \approx e^{2i\delta_{\pi A,0}}\left(1 - \gamma_0 \frac{i\Gamma}{E - E_0^{\text{res}} - i\Gamma/2}\right), \qquad (21)$$

where, in accordance with (20), $\gamma_0 = \Gamma^e_{\text{tot},0}/\Gamma$ is the ratio of the total elastic width of the 1s level to the decay width of ^{12}O, and $E_0^{\text{res}} = E^{\text{thr}} + E_1^C$.

The 2s energy level in the (π^-,^{12}O) atom is separated by 0.18 MeV, and the distance between the higher energy levels is rapidly decreasing as $\sim 1/n^3$. Therefore, one can expect that only several low-lying levels will make a noticeable contribution to Γ^e_{tot}. In our calculations we consider this quantity as a free parameter to be determined from the data.

In Fig. 2 we present calculations at 83° for different values of $\gamma_0 = \Gamma^e_{\text{tot},0}/\Gamma$. This parameter determines the magnitude of the effective elastic width $\Gamma^e_{\text{tot},0}$. In our calculations $\Gamma = 0.4$ MeV. It is seen that the best description of the data is obtained with $\gamma_0 = 0.1$, which corresponds to $\Gamma^e_{\text{tot},0} = 0.04$ MeV. The threshold energy $E^{\text{res}} = 35.26$ MeV.

The results of calculations of the excitation function with and without taking into account the subthreshold resonance effect at all scattering angles measured in Ref. [3] are presented in Fig. 3. These calculations were performed with $\gamma_0 = 0.1$

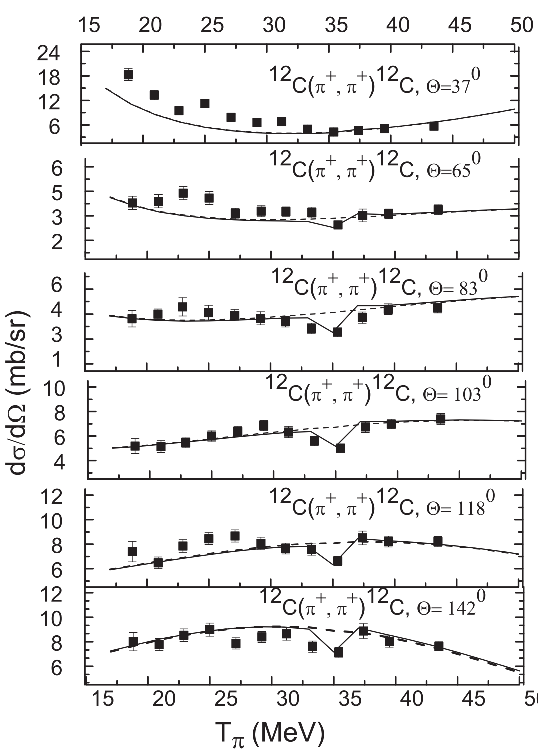

FIG. 3. Excitation functions at fixed scattering angles. Filled squares are the data from [3], the lines represent UST calculations without ($\gamma_0 = 0$) and with ($\gamma_0 = 0.1$) inclusion of the subthreshold resonance effect from formation of the (π^-,^{12}O) atom.

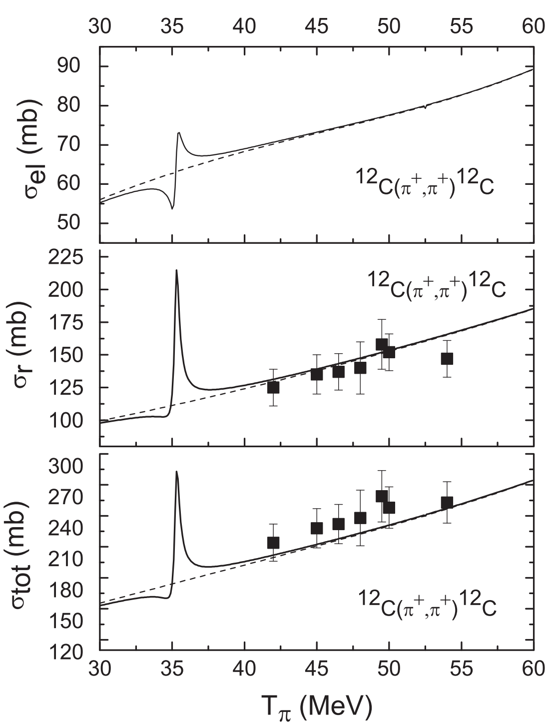

FIG. 4. Total cross sections: σ_{el}, σ_{tot}, and $\sigma_r = \sigma_{\text{tot}} - \sigma_{\text{el}}$. Lines represent the results of UST calculations without ($\gamma_0 = 0$) and with ($\gamma_0 = 0.1$) consideration of the subthreshold resonance effect. Experimental data are taken from Ref. [28]

($\Gamma^{e}_{\text{tot},0} \approx 0.04$ MeV), which was found to provide the best description of the data at 83°.

Figure 4 shows the effect of the S-wave subthreshold resonance on the total cross sections. One can see that the reaction cross section can reach the magnitude of ∼300 mb.

At each resonance energy the partial cross section reaches its "kinematic" maximum $\sigma_{l,\text{res}} = \frac{4\pi}{k^2}(2l+1)$. For example, for $T_\pi = 35$ MeV with $l = 0$, $\sigma_{0,\text{res}} = \frac{4\pi}{k^2} \approx 500$ mb. It means that even at low energies the reaction cross section can be quite large, comparable to the cross pion-nucleus sections at the Δ_{33} resonance region. There are no direct systematic experimental data on total cross sections at the subthreshold resonance region. The data from Ref. [28] cover the energy region from 45 to 65 MeV and are in agreement with the UST calculations.

V. CONCLUSION

In this paper we have presented an explanation for the energy dependence of the excitation function in scattering of positive pions from ^{12}C at pion energies in the 30- to 35-MeV region which were observed in Ref. [3]. It is shown that these oscillations can be explained by formation of the quasistationary pionic atom states in the vicinity of the threshold of the DCX reaction channel. The threshold energy is determined by the reaction Q value and the positive pion's kinetic energy required to overcome the Coulomb barrier. In the considered case the threshold energy is about 35 MeV.

The width of the resonance region is determined by the magnitude of the first Coulomb level of the pionic atom, which is about 0.23 MeV for the $(\pi^-, ^{12}\text{O})$ atom. The narrowness of this region may explain why other experimental groups (a detailed comparison of existing experimental data is given in Ref. [3]) did not see these oscillations. Fortunately, in Ref. [3] the data sets at different pion energies include the pion's energy at 35.4 MeV.

In the presented analysis there is one free parameter: the elastic width of the subthreshold resonances. The best description of the oscillations was obtained with $\Gamma^{e}_{\text{tot},0} \approx 0.04$ MeV. It is important to note that this value refers to the integrated elastic width, which accounts for the contribution of an infinite series of resonances at a given orbital momentum. In Sec. II it is shown that one can approximate the sum over all resonances by a single Breit-Wigner formula, (20), if the particle's decay width is much larger than the corresponding elastic widths. In the considered reaction this condition is satisfied, as the ^{12}O decay width is ∼0.4 MeV.

As the pion energy approaches the subthreshold resonance energies the reaction cross section varies significantly as shown in Fig. 4. This means that, despite the fact that the DCX reaction cross section is quite small by itself at low energies, the subthreshold resonance effect amplifies the DCX role in pion-nucleus dynamics. In addition, if the decay width of the nucleus created owing to the DCX is much larger than the corresponding elastic width, the final state of the quasibound system is determined by the nuclear decay. In other words, one can say that positive pions become effective "burners" of the nuclei when their energy matches the energy of subthreshold resonances caused by the formation of pionic atom states below the threshold of the DCX reaction.

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APPENDIX: *R*-MATRIX FORMALISM AND SUBTHRESHOLD RESONANCES

In this Appendix we present a brief derivation of formulas (8) and (9) for the S matrix using the R-matrix method [18,19], which allows us to separate strong-interaction short-range effects from the long-range effects caused by Coulomb interaction. It is assumed that the forces are central and the reaction region is confined by a sphere of radius R. A general expression for the S matrix in terms of the corresponding R matrix is given by

$$\hat{S} = (\psi^{(+)} - \hat{R}\psi^{(+)\prime})^{-1}(\psi^{(-)} - \hat{R}\psi^{(-)\prime}), \tag{A1}$$

where $\psi^{(\pm)}$ and $\psi^{(\pm)\prime}$ are the radial scattering wave functions and their derivatives in the region outside the reaction region and are calculated at $r = R$. Below, we omit the partial wave index l in all quantities, for simplicity.

For a two-channel reaction, one can derive the following expression for the elastic scattering S matrix using (A1):

$$S_{11} = \frac{\psi_1^-}{\psi_1^+}\frac{(1 - R_{11}\tau_1^*)(1 - R_{22}\tau_2) - R_{12}^2\tau_1^*\tau_2}{(1 - R_{11}\tau_1)(1 - R_{22}\tau_2) - R_{12}^2\tau_1\tau_2}, \tag{A2}$$

where $\tau_1 \equiv \psi_1^{(+)\prime}/\psi_1^{(+)}$ and $\tau_2 \equiv \psi_2^{(+)\prime}/\psi_2^{(+)}$ are the logarithmic derivatives of the wave functions calculated at $r = R$ in channels 1 and 2. The wave function $\psi_2^{(+)} \sim G_l + iF_l$, where G_l and F_l are the standard Coulomb functions. Channel 1 corresponds to the elastic scattering process; channel 2, to the DCX reaction.

From (A2) it follows that the S matrix can be presented as the product of two quantities,

$$S_{11} = S^{(1)}S^{(2)}, \tag{A3}$$

where

$$S^{(1)} = \frac{\psi_1^-}{\psi_1^+}\frac{(1 - R_{11}\tau_1^*)}{(1 - R_{11}\tau_1)} \tag{A4}$$

and

$$S^{(2)} = \frac{1 - \Omega_{12}^*\tau_2}{1 - \Omega_{12}\tau_2}, \quad \Omega_{12} \equiv R_{22} - \frac{R_{12}^2\tau_1}{1 - R_{11}\tau_1}. \tag{A5}$$

In Eq. (8), $S^{(1)} \equiv \exp(2i\delta_{\pi A})$, $S^{(2)} \equiv S^{\text{res}} = \exp(2i\delta^{\text{res}})$.

The logarithmic derivative τ_2 in (A5) can be calculated analytically [18,19]. In the vicinity of the threshold of channel 2 one can present $S^{(2)}$ as

$$S^{(2)} = \frac{\alpha + \beta\zeta t}{\alpha^* + \beta^*\zeta t}, \tag{A6}$$

where

$$t = \begin{cases} i & \text{for} \quad E > E_{\text{thr}}, \\ (-1)^{l+1}\cot(\pi\eta) & \text{for} \quad E < E_{\text{thr}}. \end{cases} \tag{A7}$$

Here, η is defined by Eq. (11), and the complex parameters α and β are given by

$$\alpha \equiv 1 - \Omega_{12}{}^*(p_l - q_l), \quad \beta \equiv 1 - \Omega_{12}{}^* p_l, \\ p_l \equiv \frac{l+1}{R}, \quad q_l = \frac{2l+1}{l+1}\frac{l+1}{R}. \tag{A8}$$

It is important to note that for $l = 0$, $p_l - q_l = 0$ and $\alpha = 1$.

Below the threshold, using (A7), we obtain the following expression for $S^{\text{res}} \equiv S^{(2)}$:

$$S^{\text{res}} = \frac{\alpha^* + \beta^*(-1)^{l+1}\zeta\cot(\pi\eta)}{\alpha + \beta(-1)^{l+1}\zeta\cot(\pi\eta)}, \tag{A9}$$

which represents Eq. (9) for corresponding phase shifts.

The subthreshold resonance energies are determined by Eq. (12), and in the vicinity of the resonance energy $E = E_{nl}^{\text{res}}$, the S matrix can be approximated by the Breit-Wigner formula, (16). To derive it we start from formula (A9), presenting the pion energy as $E = E_{nl}^{\text{res}} + \epsilon$ and expanding $\cot(\pi\eta)$ in powers of ϵ. The first two terms of this expansion are

$$\cot(\pi\eta) = (-1)^l \frac{\alpha_1}{\beta_1\zeta} + \left(1 + \frac{\alpha_1^2}{\beta_1^2\zeta^2}\right)\pi\sqrt{\frac{E_n^C}{E_{nl}}}\frac{\epsilon}{2E_{nl}}, \tag{A10}$$

where E_{nl} and E_n^C are determined by Eqs. (13) and (14), respectively.

Substituting (A10) into (A9) one can obtain

$$S^{\text{res}} = \frac{\beta}{\beta^*}\frac{c+\epsilon}{c^*+\epsilon}, \tag{A11}$$

where

$$c = (-1)^{l+1}\frac{\alpha - \beta\zeta\omega}{\beta\zeta\nu}, \quad \omega = \frac{\alpha_1}{\beta_1\zeta}, \\ \nu = \frac{1}{2}\pi n(1+\omega^2)\xi^3\frac{1}{E_n^C}, \tag{A12}$$

and the parameters ζ and ξ are defined by Eqs. (10) and (13), respectively. Finally, this formula can be represented in the Breit-Wigner form as

$$S^{\text{res}} = \frac{\beta}{\beta^*}\left(1 - \frac{i\Gamma^e}{E - E^{\text{res}} + i\Gamma^e/2}\right), \tag{A13}$$

where $\Gamma^e = 2c_2$ and $E^{\text{res}} = E^{\text{thr}} + E_{nl} - c_1$, $c_1 = \text{Re}c$, and $c_2 = -\text{Im}c$. $\Gamma^e \sim R_{12}^2$ can be interpreted as the decay width from channel 2 back to channel 1. Using Eqs. (A5) and (A8) it can be shown that the phase factor $\frac{\beta}{\beta^*} \approx 1$ and $c_2 << E_{nl}$ if $|R_{12}|$ is much less than $|R_{11}|$ and $|R_{22}|$. This is justified in the considered case of the DCX channel, as the DCX cross section is less than the corresponding elastic scattering cross section by an order of magnitude. Therefore, finally, we obtain Eq. (16),

$$S^{\text{res}} \approx \left(1 - \frac{i\Gamma^e}{E - E^{\text{res}} + i\Gamma^e/2}\right), \tag{A14}$$

where $E^{\text{res}} = E^{\text{thr}} + E_{nl}$.

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