# Spin-$J$ nucleon-pair approximation with a $J$-pairing interaction for a single-$j$ shell

G. J. Fu,[1] Y. M. Zhao,[1,*] and A. Arima[1,2]

[1] *INPAC, Department of Physics and Shanghai Key Lab for Particle Physics and Cosmology, Shanghai Jiao Tong University, Shanghai 200240, China*

[2] *Musashi Gakuen, 1-26-1 Toyotamakami Nerima-ku, Tokyo 176-8533, Japan*

(Received 24 July 2013; revised manuscript received 30 August 2013; published 4 November 2013)

In this paper we study the spin-$J$ nucleon-pair approximation for a single-$j$ shell, with a simple $J$-pairing interaction. For a four-nucleon system, the $J$-pair truncation works remarkably well, in particular for isospin $T = 0$ states. The lowest $I = 0$ states of four nucleons are exactly described by the $J$-pair approximation. If both $J = 0$ and $J = 2j$ pairing interactions are considered, the $J = 2j$ pair truncation works very well in *one* of the two lowest $I = 2j - 1$ states. For six nucleons, $J$-pair approximations are very good for $J \sim 2j$, and the seniority scheme provides us with exact solutions to the $J = 0$ pairing interaction.

DOI: 10.1103/PhysRevC.88.054303 PACS number(s): 21.10.Hw, 21.60.Cs

## I. INTRODUCTION

The pairing correlation is one of the most prominent correlations in low-energy phenomena of atomic nuclei. Theoretically, the seniority scheme of Racah [1] provides us with solutions of the isovector monopole pairing Hamiltonian. The generalization of the monopole pairing correlations are considered in the broken pair model [2,3] and the nucleon-pair approximation of the shell model [4,5].

The so-called $J$-pairing correlations have been discussed for a single-$j$ shell in Ref. [6], where spin-$J$ pair approximations under the spin-$J$ pairing interaction have been examined for four identical particles. Recently, proton-neutron pairs with spin $J = 9$ and isospin $T = 0$ were suggested to be the key components in low-lying states of $^{96}$Cd, $^{94}$Ag, and $^{92}$Pd [7]. In Ref. [8] states of spin two and spin $2j - 1$ for four-particle systems with isospin $T = 1$ were studied for the single $j = 7/2$ shell and the $j = 9/2$ shell. In Refs. [9,10] isoscalar pairing interactions, especially the spin-$2j$ pairing interaction, were studied for two valence protons and two valence neutrons in a single-$j$ shell.

In this paper we study spin-$J$ nucleon-pair ($J$-pair for short) approximation under the $J$-pairing interaction in a single-$j$ shell space for four and six nucleons. Our $J$-pairing interaction $H_J$ is defined as follows:

$$H_J = -\sum_{M=-J}^{J}\sum_{\tau=-\mathbb{T}}^{\mathbb{T}} A_{M\tau}^{(J\mathbb{T})^\dagger} A_{M\tau}^{(J\mathbb{T})},$$

$$A_{M\tau}^{(J\mathbb{T})^\dagger} = \frac{1}{\sqrt{2}}(a_j{}^\dagger \times a_j{}^\dagger)_{M\tau}^{J\mathbb{T}},$$

where $J$ and $M$ are the spin of a pair and its projection, respectively, and $\mathbb{T}$ and $\tau$ are the isospin of a nucleon pair and its projection, respectively. For a single-$j$ shell, $\mathbb{T} = 1$ if $J$ is even, and $\mathbb{T} = 0$ if $J$ is odd. For simplicity the index $\mathbb{T}$ can be suppressed without confusion. We use $I$ and $T$ to denote the total spin and isospin of a given system, respectively.

This paper is organized as follows. In Sec. II we study the validity of $J$-pair truncations with the $J$-pairing interaction for four nucleons, and in Sec. III we come to the cases with six nucleons. Our summary and conclusion are given in Sec. IV.

## II. FOUR NUCLEONS

In this section we study four nucleons. For convenience, we use the following notation for a state with spin $I$ and isospin $T$, constructed by one pair with spin $J_1$ and another with spin $J_2$:

$$|j^4[J_1J_2]IT\rangle = \frac{1}{\sqrt{N_{J_1J_2;J_1J_2}^{(IT)}}}(A^{(J_1)^\dagger} \times A^{(J_2)^\dagger})^{(IT)}|0\rangle,$$

where $N_{J_1J_2;J_1J_2}^{(IT)}$ is the norm given as below

$$\begin{aligned} N_{J_1'J_2';J_1J_2}^{(IT)} &\equiv \langle 0|(A^{(J_1')} \times A^{(J_2')})^{(IT)}(A^{(J_1)^\dagger} \times A^{(J_2)^\dagger})^{(IT)}|0\rangle \\ &= \delta_{J_1'J_1}\delta_{J_2'J_2} + (-)^{I+T}\delta_{J_1'J_2}\delta_{J_2'J_1} \\ &\quad -4\begin{bmatrix} j & j & J_1 \\ j & j & J_2 \\ J_1' & J_2' & I \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1+(-)^{J_1}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1+(-)^{J_2}}{2} \\ \frac{1+(-)^{J_1'}}{2} & \frac{1+(-)^{J_2'}}{2} & T \end{bmatrix}, \end{aligned} \tag{1}$$

where $\delta_{J_1'J_1}$ is 1 if $J_1' = J_1$ and is 0 otherwise; $[\cdots]$ is a nine-$j$ coefficient. The matrix element of $H_J$ between two arbitrary basis vectors is as follows:

$$\langle j^4[J_1'J_2']IT|H_J|j^4[J_1J_2]IT\rangle = -\frac{\sum_{J'} N_{J_1'J_2';J'J}^{(IT)} N_{J'J;J_1J_2}^{(IT)}}{\sqrt{N_{J_1'J_2';J_1'J_2'}^{(IT)} N_{J_1J_2;J_1J_2}^{(IT)}}}.$$

Some special cases of this formula are

$$\langle j^4[00]0T|H_0|j^4[00]0T\rangle = -\left[2 + \frac{2 - T(T+1)}{2j+1}\right],$$

$$\langle j^4[0I]IT|H_0|j^4[0I]IT\rangle = -\left[1 + \frac{(1+(-)^I) - T(T+1)}{2j+1}\right],$$

for $I \neq 0$. The first formula presents us with the eigenenergy of the seniority-zero state under the monopole pairing interaction and the second gives eigenenergies of seniority-two states.

*Corresponding author: ymzhao@sjtu.edu.cn

 

For the $J$-pairing interaction, we assume the wave functions of the lowest states are given by two $J$ pairs, i.e., $|j^4[JJ]IT\rangle$, and that the corresponding eigenvalue is

$$E_J^{\rm P}(I,T) \equiv \langle j^4[JJ]IT|H_J|j^4[JJ]IT\rangle.$$

We use the superscript "P" to represent the abbreviation "pair approximation," and "SM" to represent "shell model."

### A. Validity of the $J$-pair approximation

We use two quantities as fingerprints for the validity of the nucleon-pair approximation. The first is the overlap between the wave function based on the spin-$J$ nucleon-pair approximation and that given by the exact shell model, and the second is the relative deviation of the eigenvalue in the nucleon-pair approximation from the exact result, i.e., $\epsilon_{IT}^J = |\frac{E_J^{\rm P}(I,T)-E_J^{\rm SM}(I,T)}{E_J^{\rm SM}(I,T)}|$. If the pair approximation is good for a given state, the overlap between the two sets of wave functions is close to 1, and the value of $\epsilon_{IT}^J$ is close to zero.

In Fig. 1 we exemplify our results by $j=21/2$, thus spin $J$ of our nucleon pair ranges from 0 to 21. One sees that the $J$-pair approximation is remarkably good for almost all yrast states with $T=0$ and reasonably good for $T=1$ and 2 for $I$ below 5.

There are a few features worth pointing out. First, we note that the lowest $I=0$ states with $T=0,2$ under the $J$-pairing Hamiltonian are precisely described by two $J$ pairs. An explanation for this has been already discussed in Ref. [6], by using the fact that there is only one nonzero eigenvalue in $I=0$ and $n=4$ states for identical particles. Below we show that this explanation is equally applicable for $\mathbb{T}=0$ and $J$ odd pairs. For the $J$-pairing Hamiltonian $H_J$ we define

$$|j^4J\rangle = |j^4[JJ]I=0T\rangle,$$

where $T$ equals 0 and 2 if $J$ is even, and equals 0 if $J$ is odd. We define

$$|j^4J'\rangle = |j^4[J'J']I=0T\rangle - \frac{N^{(0T)}_{JJ;J'J'}}{\sqrt{N^{(0T)}_{JJ;JJ}N^{(0T)}_{J'J';J'J'}}}|j^4J\rangle,$$

where $J'\neq J$. $|j^4J'\rangle$ is orthogonal to $|j^4J\rangle$. Using Eq. (1), we have

$$\langle j^4J_1|H_J|j^4J_2\rangle = -\delta_{J_1J}\delta_{J_2J}N^{(0T)}_{JJ;JJ}.$$

This means that the $J$-pairing Hamiltonian has only one nonzero matrix element corresponding to the wave function of $|j^4[JJ]I=0T\rangle$, i.e., the $J$-pair truncated wave function for $I=0$, $T=0$ or 2.

By using this feature, we readily explain Fig. 5 of Ref. [10], where Neergård presented the squared overlap between the wave function with the monopole-pairing interaction and that with the $J=2j$ pairing interaction, for the lowest state of $I=0$ and $T=0$ in a single-$j$ shell. Neergård found that the squared overlap is equal to 1 for $j=1/2$ and gradually decreases with $j$. According to our work, the wave function with the monopole-pairing interaction is $|j^4[00]00\rangle$, and that with the $J=2j$ pairing interaction is $|j^4[2j2j]00\rangle$. The squared overlap is given by

$$\frac{3(4j+1)}{(2j+1)(2j+2)\left(1+\frac{(2j)!^2}{(4j)!}\right)} \to 3/j$$

for the large $j$ limit. Here Eq. (1) and some special values of nine-$j$ coefficients are used.

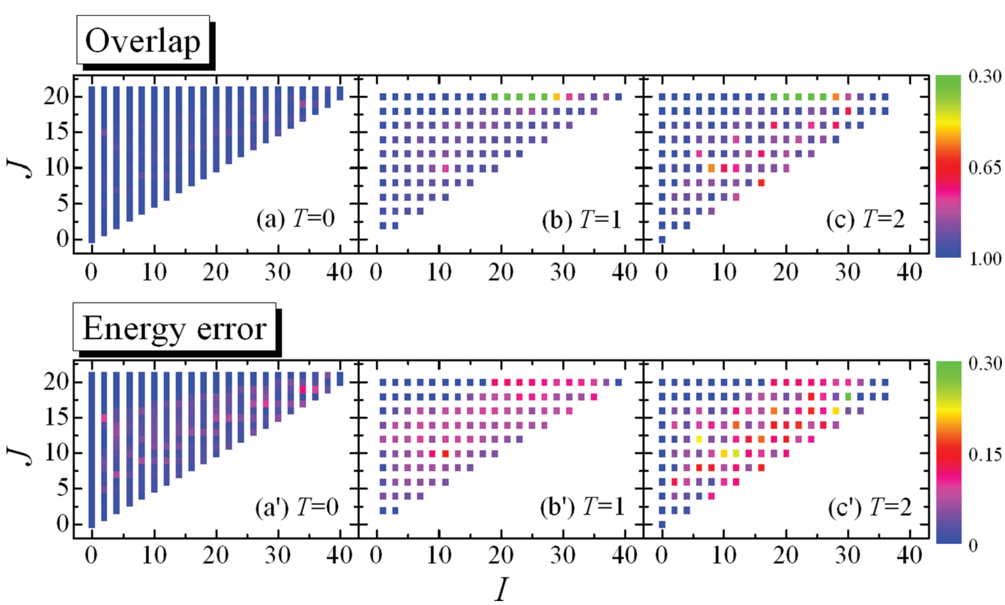


FIG. 1. (Color online) Comparison of $J$-pair approximation calculations with shell model calculations with the $J$-pairing interaction $H_J$ for four nucleons in the single $j=21/2$ shell. $J$ equals $0, 1, 2, \ldots, 20, 21$ $(=2j)$. Here we consider the lowest states of spin $I$ for $T=0, 1, 2$. The upper panels present the overlaps between wave functions of the shell model calculations and those of $J$-pair truncated calculations, and the lower present the relative deviations of the calculated energies. Blue color means $J$-pair approximation are very good and green color means the $J$-pair truncation is not good. The seniority-zero state for $T=0$ and 2 are precisely represented by two $S$ pairs with $\mathbb{T}=1$, as shown in panels (a), (a′), (c), and (c′).

The second feature is that the $D$-pair (one isovector nucleon pair with spin two) approximation is very good under the quadrupole-pairing interaction $H_{J=2}$. Using two $D$ pairs, one can construct states with $T = 0$ and $I = 0, 2, 4$, $T = 1$ and $I = 1, 3$, and $T = 2$ and $I = 0, 2, 4$. According to our numerical experiments, for all these states, the overlaps between the wave functions of the $D$-pair approximation and exact wave functions are from 0.9883 to 1, and the relative energy deviations are from 0.0164 to 0. In addition, we note three sets of asymptotic degenerate states: (1) $T = 0$ and $I = 2, 4$; (2) $T = 1$ and $I = 1, 3$; (3) $T = 2$ and $I = 2, 4$.

The third point we would like to make is that the $J = 2j$ pair approximation works remarkably well for the yrast states with the $J = 2j$ pairing interaction. In these cases the overlaps are from 0.9523 to 1, and the relative energy deviations range from 0.0446 to 0. A similar result is for $\mathbb{T} = 1$, $J \sim 2j - 1$, and $T = 1, 2$. For the $J = 2j - 1$ pairing interaction, spin-$(2j - 1)$ nucleon-pair approximation is very good for $T = 2$ when $I$ is not large or $I \approx I_{\max}$, as pointed out and explained in Ref. [6]. Here we note that the $J$-pair ($J \sim 2j - 1$) approximation is asymptotically a good truncation for $T = 1$ states.

### B. Lowest two states of $I = 2j - 1$

In Ref. [9] Zamick and Escuderos performed shell model calculations for low-lying states of $^{96}$Cd with two valence proton holes and two valence neutron holes in the single $j = 9/2$ shell, with various sets of interactions. They found that the wave functions of the yrast states of $T = 0$ under the $J = 9$ pairing interaction $H_{J=9}$ are very close to those obtained under the realistic interaction, except for the states of $I \sim 8$. The $8_1^+$ state cannot be described by $|j^4[99]IT\rangle$ [11]. Because the $J = 0$ and $J = 2j$ pairing interactions are the most strongly attractive in atomic nuclei, Zamick and Escuderos calculated the $8_1^+$ state wave function under a simple Hamiltonian ($H_{J=0} + H_{J=9}$), and found the overlap between the $8_1^+$ state wave function based on such a simple Hamiltonian and that obtained under the realistic interaction will be larger than 0.98 [9]. In Refs. [11,12] it was also found that the $8_1^+$ state of $^{96}$Cd is well described by $|j^4[08]IT\rangle$, a seniority-two configuration.

Here we are interested in the competition between the $|j^4[2j2j]IT\rangle$ configuration and the seniority-two configuration $|j^4[02j - 1]IT\rangle$ for low-lying states of $I = 2j - 1$ and $T = 0$, with a Hamiltonian with both the $J = 0$ and $J = 2j$ pairing interactions. We adopt the following Hamiltonian

$$H(\delta) = (1 - \delta)H_{J=0} + \delta H_{J=2j},$$

where $\delta$ is an adjustable parameter ranging between 0 to 1. We calculate the squared overlaps of the shell model wave function with pair truncated wave functions $|j^4[2j2j]IT\rangle$ and $|j^4[02j - 1]IT\rangle$, for the lowest $I = 8$, $T = 0$ state (denoted by $x_1^2$) and the second lowest $I = 8$, $T = 0$ state (denoted by $x_2^2$).

In Fig. 2 we plot the $x_1^2$, $x_2^2$, and $x_1^2 + x_2^2$ versus our adjustable parameter $\delta$. One sees that $x_1^2$ of seniority-two configuration $|j^4[02j - 1]IT\rangle$ is close to one as $\delta$ is small, and decreases suddenly as $\delta$ increases from 0.50 to 0.55; meanwhile $x_1^2$ of the spin-$J = 2j$ pair approximation $|j^4[2j2j]IT\rangle$ is close to zero for small $\delta$ and increases sharply at $\delta = 0.50 \sim 0.55$. The situation for the second lowest $T = 0$, $I = 8$ state is different. According to our calculated results [see Fig. 2(b)] the $J = 2j$-pair approximation is a good approximation for small $\delta$ and the seniority scheme is more relevant (but not good) for large $\delta$. Summing up $x_1^2$ and $x_2^2$, one sees in Fig. 2(c) that $x_1^2 + x_2^2 \sim 1.0$ for the $J = 2j$-pair approximation. For the seniority scheme, $x_1^2 + x_2^2 \sim 1.0$ for small $\delta$ and decreases as $\delta$ increases. We also exemplify our study for the lowest two $20^+$ states in the single $j = 21/2$ shell [see in Figs. 2(a′) to 2(c′)], where one sees similar features (except that the evolution versus $\delta$ is now abrupt). We also note that the $J = 2j$-pair truncation and the seniority scheme are asymptotically orthogonal to each other, as $|j^4[2j2j]IT\rangle$ and $|j^4[02j - 1]IT\rangle$ is very small (the overlap is 0.0996 for $j = 9/2$ and 0.0007 for $j = 21/2$).

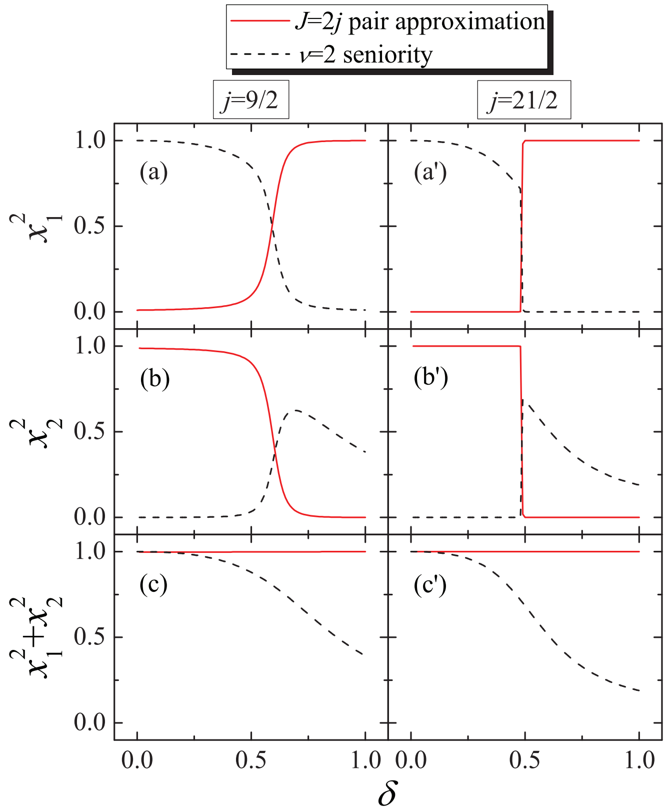


FIG. 2. (Color online) Squared overlaps $(x_1^2, x_2^2)$ between nucleon-pair truncated wave functions and shell model wave functions for the lowest two states of $I = 2j - 1$ and $T = 0$, with a Hamiltonian $H(\delta) = (1 - \delta)H_{J=0} + \delta H_{J=2j}$, where $\delta$ is an adjustable parameter between 0 to 1. $x_1^2$ corresponds to the lowest state, and $x_2^2$ the second lowest. Plots in red correspond to $J = 2j$ nucleon pair approximation (i.e., $|j^4[2j2j]IT\rangle$) and dashed plots in black correspond to seniority-two wave function (i.e., $|j^4[02j - 1]IT\rangle$). In the left panels they are obtained for $j = 9/2$, and in the right panels $j = 21/2$.

## III. SIX-NUCLEON SYSTEMS

We exemplify our results of six nucleons by a $j = 11/2$ shell, for which the spin of each nucleon pair $J$ runs from

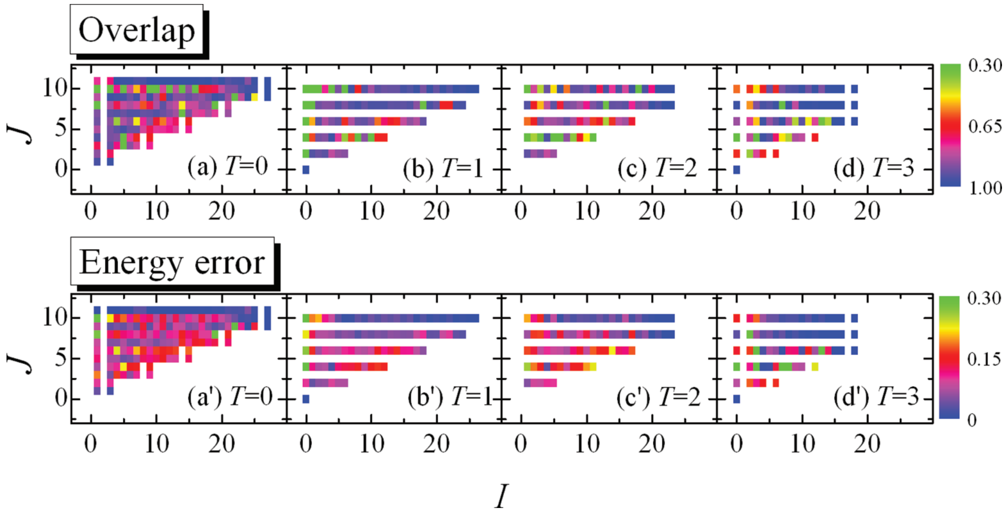


FIG. 3. (Color online) Same as Fig. 1, except that nucleon numbers $n = 6$ and $j = 11/2$. The seniority-zero state is precisely represented by $J$-pair truncation for $J = I = 0$, with $T = 1$ or 3, as shown in panels (b), (b′), (d), and (d′).

0 to 11. For each $J$, our nucleon-pair basis is given by

$$\frac{1}{\sqrt{\mathcal{N}}}((A^{(J)\dagger} \times A^{(J)\dagger})^{(I'T')} \times A^{(J)\dagger})^{(IT)}|0\rangle,$$

where $1/\sqrt{\mathcal{N}}$ is the normalization factor, and $I'$ ($T'$) is an intermediate spin (isospin). In the $J$-pair subspace, we calculate eigenenergies and wave functions of the yrast states with spin $I$ and isospin $T$ by using our nucleon-pair approximation model [13]. The construction of a complete set of basis states by using nucleon pairs has been discussed for $SD$ pairs in Ref. [14] and for arbitrary nucleon pairs in Ref. [15]. In short, the number of states is enumerated in terms of an $m$ scheme, and their linear independence is demonstrated by numerical experiments based on the eigenvalues of the norm matrix. There is freedom in choosing the intermediate spins, and different sets of the basis are equivalent.

In Figs. 3(a) through 3(d) we present our calculated overlaps between wave functions by using the $J$-pair approximation and those by the shell model, and in Figs. 3(a′) through 3(d′) relative deviations of the calculated energies in these two spaces. One sees that the $J$-pair approximation for nucleon number $n = 6$ is not as good as for $n = 4$. Yet $J$-pair truncation provides us with a reasonable approach for most states with $T = 0$, in particular for the case of $J \sim 2j$. This might be the reason why isoscalar spin-aligned pairs are dominant building blocks in low-lying states of $^{94}$Ag.

For $J = 0$, the seniority scheme provides us with the solution, as studied in the quasispin formalism with isospin [16], according to which the eigenvalue of the monopole-pairing interaction is

$$-\frac{1}{2\Omega}\left[\frac{(n-\nu)(4\Omega+6-n-\nu)}{4} + t(t+1) - T(T+1)\right]$$
$$= -\frac{1}{12}\left[\frac{(6-\nu)(24-\nu)}{4} + t(t+1) - T(T+1)\right], \quad (2)$$

where $\Omega$ is the allowed number of pairs, i.e., $\Omega = j + 1/2 = 6$; $n$ is the particle number, 6; $\nu$ is the seniority; $t$ is called the reduced isospin; and $T$ is the total isospin of the state.

According to Eq. (2), the lowest state has seniority zero (i.e., $S$) and reduced isospin $t = 0$ for a given $T = 1, 3$. The corresponding eigenvalues are $-17/6$ and $-2$, respectively. For odd $I$, $T = 0$, and $t = 0$, we have the seniority-two states

$$\frac{1}{\sqrt{\mathcal{N}}}((A^{(0)\dagger} \times A^{(0)\dagger})^{(00)} \times A^{(I)\dagger})^{(IT=0)}|0\rangle.$$

For even $I$, $T = 1$, and $t = 1$, the seniority-two states are

$$\frac{1}{\sqrt{\mathcal{N}}}((A^{(0)\dagger} \times A^{(0)\dagger})^{(00)} \times A^{(I)\dagger})^{(IT=1)}|0\rangle,$$
$$\frac{1}{\sqrt{\mathcal{N}}}((A^{(0)\dagger} \times A^{(0)\dagger})^{(02)} \times A^{(I)\dagger})^{(IT=1)}|0\rangle.$$

According to Eq. (2), all eigenvalues of these states are equal to $-11/6$, and are reproduced in our numerical calculations.

## IV. SUMMARY

In this paper, we study $J$-pair approximations under the $J$-pairing interaction in a single-$j$ shell. For each $J$-pairing interaction, we calculate the lowest states with spin $I$ and isospin $T$, as well as the overlaps between the shell model wave functions and those by using the $J$-pair approximation.

For four nucleons in a single-$j$ shell, $J$-pair approximations work well for most yrast states (especially for those of $T = 0$), under the $J$-pairing interaction. The lowest states of $I = 0$ and $T = 0$ (or 2) are precisely represented by two spin-$J$ pairs. The $D$-pair approximation is found to be very good for the quadrupole pairing interaction, and $J$-pair approximations are very good for all $T = 0$ states, and for $T = 1$ and 2 states with $I$ not large or $I \approx I_{\max}$, when $J \sim 2j$. A remarkable example is that the $J = 2j$-pair approximation works very well in the lowest states of $T = 0$ under the $J = 2j$ pairing interaction, with calculated overlaps between the shell model wave functions and $J$-pair truncated wave functions larger than 0.9523, and a relative energy deviation lower than 0.0446.

We study the competition between the $J = 2j$ pair approximation (i.e., $|j^4[2j2j]IT\rangle$) the seniority-two picture (i.e., $|j^4[02j-1]IT\rangle$) for the lowest two states with $I = 2j-1$ and $T = 0$, in the presence of both the monopole and $J = 2j$ pairing interactions, $H(\delta) = (1-\delta)H_{J=0} + \delta H_{J=2j}$. For the lowest state, the seniority picture is dominant when the monopole pairing interaction is relatively strong, and the picture of the $J = 2j$-pair approximation is dominant when the $J = 2j$ pairing interaction is relatively strong. The transition from seniority picture to the $J = 2j$ pair picture arises abruptly at $\delta \sim 0.5$.

Our study of six nucleons is performed by using a $j = 11/2$ shell. Although the $J$-pair approximation for six nucleons is not as remarkable as for four nucleons, it provides us with a reasonable approximation for many yrast states, in particular, the $J$-pair approximation is very good for yrast states of $T = 0$ when $J$ is close to $2j$. For the monopole pairing interaction the seniority scheme presents exact solutions.

## ACKNOWLEDGMENTS

We thank the National Natural Science Foundation of China (Grants No. 11225524 and No. 11145005), the 973 Program of China (Grant No. 2013CB834401), and the Science & Technology Committee of Shanghai city (Grant No. 11DZ2260700) for financial support.

[1] G. Racah, Phys. Rev. **62**, 438 (1942); **63**, 367 (1943).
[2] Y. K. Gambhir, S. Haq, and J. K. Suri, Ann. Phys. (NY) **133**, 154 (1981).
[3] K. Allaart, E. Boeker, G. Bonsignori, M. Savoia, and Y. K. Gambhir, Phys. Rep. **169**, 209 (1988).
[4] J. Q. Chen, B. Q. Chen, and A. Klein, Nucl. Phys. A **554**, 61 (1993); J. Q. Chen, *ibid.* **562**, 218 (1993); **626**, 686 (1997).
[5] Y. M. Zhao, N. Yoshinaga, S. Yamaji, J. Q. Chen, and A. Arima, Phys. Rev. C **62**, 014304 (2000).
[6] Y. M. Zhao, A. Arima, J. N. Ginocchio, and N. Yoshinaga, Phys. Rev. C **68**, 044320 (2003); Y. M. Zhao and A. Arima, *ibid.* **70**, 034306 (2004).
[7] B. Cederwall *et al.*, Nature (London) **469**, 68 (2011).
[8] L. Zamick and A. Escuderos, Nucl. Phys. A **889**, 8 (2012).
[9] L. Zamick and A. Escuderos, Phys. Rev. C **87**, 044302 (2013).
[10] K. Neergård, Phys. Rev. C **88**, 034329 (2013).
[11] C. Qi, J. Blomqvist, T. Bäck, B. Cederwall, A. Johnson, R. J. Liotta, and R. Wyss, Phys. Rev. C **84**, 021301 (2011); Z. X. Xu, C. Qi, J. Blomqvist, R. J. Liotta, and R. Wyss, Nucl. Phys. A **877**, 51 (2012).
[12] G. J. Fu, J. J. Shen, Y. M. Zhao, and A. Arima, Phys. Rev. C **87**, 044312 (2013).
[13] G. J. Fu, Y. Lei, Y. M. Zhao, S. Pittel, and A. Arima, Phys. Rev. C **87**, 044310 (2013).
[14] J. Q. Chen and Y. A. Luo, Nucl. Phys. A **639**, 615 (1998).
[15] Y. Lei, Z. Y. Xu, Y. M. Zhao, and A. Arima, Phys. Rev. C **80**, 064316 (2009).
[16] M. Ichimura, *On the Quasispin Formalism* (Pergamon, Oxford, 1968).