Volume fluctuations and higher-order cumulants of the net baryon number

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We consider the effect of volume fluctuations on cumulants of the net baryon number. Based on a general formalism, we derive universal expressions for the net baryon number cumulants in the presence of volume fluctuations with an arbitrary probability distribution. The relevance of these fluctuations for the baryon-number cumulants and in particular for the ratios of cumulants is assessed in the Polyakov loop extended quark-meson model within the functional renormalization group. We show that the baryon number cumulants are generally enhanced by volume fluctuations and that the critical behavior of higher-order cumulants may be modified significantly.

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I. INTRODUCTION

One of the goals of the experiments with ultrarelativistic heavy ion collisions at Super Proton Synchrotron (SPS), Relativistic Heavy Ion Collider (RHIC), and Large Hadron Collider (LHC) energies is to probe the phase structure of strongly interacting matter and, in particular, to identify the deconfinement and chiral symmetry restoration transitions. In this context, the fluctuations of conserved charges may serve as a pertinent probe.

Fluctuations of the net baryon number and electric charge may provide an experimental signature for the hypothetical chiral critical endpoint [1,2]. Moreover, as recently noted [3–11], such fluctuations are also of interest at small baryon densities, since they reflect the critical dynamics of the underlying $O(4)$ transition, expected in QCD in the limit of massless light quarks [12,13]. Indeed, it was demonstrated that higher-order cumulants change sign in the crossover region of the QCD phase diagram [2,8,9]. Thus, the observation of a strong suppression the higher-order cumulants may be used to identify the chiral crossover transition in experiments.

The first measurements of fluctuations of the net baryon number, more precisely of the net proton number,[1] in heavy ion collisions at RHIC were obtained by the STAR Collaboration [15]. The analysis of cumulants of the fluctuations and of the probability distributions confirmed that the hadron resonance gas (HRG) model, which yields a quantitative description of particle yields in heavy ion collisions [16], provides a useful reference for the noncritical background contribution to the charge fluctuations [4]. Thus, critical fluctuations related to the dynamics of the chiral transition should be reflected in deviations of the measured net charge fluctuations from the HRG baseline. In this context, higher-order cumulants are of particular interest [8,10].

A detailed analysis of experimental data on moments of net proton number fluctuations and their probability distributions indeed exhibit deviations from the HRG. To verify the origin of these deviations, one must identify and assess effects, unrelated to the critical dynamics, which can influence the charge fluctuations. For instance, it was recently argued that constraints, owing to the conservation of the total baryon number in nucleus-nucleus collisions [17] or experimental acceptances in terms of kinematic variables [18], might modify the noncritical background contributions to higher-order cumulants of the net proton number fluctuations.

In this paper we study volume fluctuations as a further possible source of noncritical fluctuations, not accounted for in the HRG model results. We first present a transparent derivation of the cumulants of net baryon number, including the effect of volume fluctuations. The resulting cumulants are expressed in terms of cumulants of the net baryon number distribution at fixed volume and cumulants of the probability distribution for volume fluctuations.

We also provide a more formal derivation, making use of the cumulant generating functions. We stress that the final expressions are general, independent of the probability distributions for net baryon number and volume. The only assumption made is that the two sources of fluctuations are independent and that fluctuations of other thermodynamic parameters are negligible. This assumption is most likely justified for high-/energy heavy ion collisions, where the baryon chemical potential is close to zero. There, the thermalization[2] results in a freeze-out temperature independent of the initial conditions, while the volume fluctuations are determined by the collision geometry. At lower energies, fluctuations of the initial temperature and chemical potential may take the system to different freeze-out points, along the freeze-out curve. Therefore, at lower energies, the fluctuations of temperature, chemical potential, and volume are presumably correlated. Additional complications arise if the system passes close to a possible critical endpoint. In

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[1]As shown recently in Ref. [14], the net nucleon number cumulants can, to a good approximation, be deduced from the measured net proton cumulants.

[2]Thermalization is supported by the success of hydrodynamic and statistical models.

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this case, owing to the large correlation length, the volume fluctuations in the final state may be correlated with other thermodynamic variables.

With the limitations discussed above, we focus on the effect of volume fluctuations on the fluctuations of the net baryon number in the vicinity of the chiral crossover transition at vanishing chemical potential. We employ the functional renormalization group within the Polyakov loop extended quark-meson model, to properly account for the critical properties near the chiral phase transition.

The paper is organized as follows: In the next section we obtain the corrections due to volume fluctuations to the first four moments of the net baryon number fluctuations. In Sec. III we derive a general expression for the corrected cumulants, valid to any order, obtained using the cumulant generating functions. In Sec. IV we illustrate the role of volume fluctuations with a numerical study, and finally in Sec. V we state our conclusions.

II. HEURISTIC APPROACH

Consider a fixed volume V, where the net baryon number B fluctuates with the probability distribution $P(B,V)$. The nth order moments of the net baryon number are then defined by

$$\langle B^n\rangle_V = \sum_{B=-\infty}^{\infty} B^n P(B,V). \tag{1}$$

It is convenient to introduce reduced cumulants, corresponding to the net baryon number fluctuations per unit volume. The first four reduced cumulants are

$$\begin{aligned}
\kappa_1(T,\mu) &= \frac{1}{V}\langle B\rangle_V, \quad \kappa_2(T,\mu) = \frac{1}{V}\langle(\delta B)^2\rangle_V,\\
\kappa_3(T,\mu) &= \frac{1}{V}\langle(\delta B)^3\rangle_V,\\
\kappa_4(T,\mu) &= \frac{1}{V}\left[\langle(\delta B)^4\rangle_V - 3\langle(\delta B)^2\rangle_V^2\right],
\end{aligned} \tag{2}$$

where $\delta B = B - \bar{B}$ and $\bar{B} = \langle B\rangle_V$. The cumulants κ_i are, to leading order, independent of the volume V. In the following we neglect subleading surface effects, which could lead to a residual volume dependence of the cumulants.

The volume dependence of the moments follows from (2) and reads

$$\begin{aligned}
\langle B\rangle_V &= \kappa_1 V, \quad \langle B^2\rangle_V = \kappa_2 V + \kappa_1^2 V^2,\\
\langle B^3\rangle_V &= \kappa_3 V + 3\kappa_2\kappa_1 V^2 + \kappa_1^3 V^3,\\
\langle B^4\rangle_V &= \kappa_4 V + \left(4\kappa_3\kappa_1 + 3\kappa_2^2\right)V^2 + 6\kappa_2\kappa_1^2 V^3 + \kappa_1^4 V^4.
\end{aligned} \tag{3}$$

The coefficients in Eq. (3) are those of the Bell polynomials.

As an illustrative example, we consider the hadron resonance gas. In this model, the net baryon number fluctuations are given by the Skellam distribution [5,6] and the corresponding cumulants are particularly simple:[3]

$$\kappa_{2n+1}^{(HRG)} = \frac{1}{V}(\bar{B}_1 - \bar{B}_{-1}), \quad \kappa_{2n}^{(HRG)} = \frac{1}{V}(\bar{B}_1 + \bar{B}_{-1}), \tag{4}$$

where $\bar{B}_1 = \langle B_1\rangle$ is the mean number of baryons and $\bar{B}_{-1} = \langle B_{-1}\rangle$ that of antibaryons in V. The corresponding moments are obtained by inserting the cumulants (4) in (3).

We now allow for fluctuations of the volume. To this end, we introduce the volume probability distribution $\mathcal{P}(V)$, the corresponding moments

$$\langle V^n\rangle = \int V^n \mathcal{P}(V)dV, \tag{5}$$

and the reduced cumulants of the volume fluctuations, v_n. The latter are defined as in Eq. (2) with the replacements $V \to \langle V\rangle$ and $B \to V$. Thus, e.g., $v_1 = 1$ and $v_2 = (\langle V^2\rangle - \langle V\rangle^2)/\langle V\rangle$.

In the presence of volume fluctuations the moments of the net baryon number are given by

$$\begin{aligned}
\langle B^n\rangle &= \int dV\,\mathcal{P}(V)\sum_{B=-\infty}^{\infty} B^n P(B,V)\\
&= \int dV\,\mathcal{P}(V)\langle B^n\rangle_V.
\end{aligned} \tag{6}$$

It is now straightforward to compute the reduced cumulants, including the effect of volume fluctuations. Using Eqs. (2), (3), and (6), we find the general relations

$$\begin{aligned}
c_1 &= \kappa_1, \quad c_2 = \kappa_2 + \kappa_1^2 v_2,\\
c_3 &= \kappa_3 + 3\kappa_2\kappa_1 v_2 + \kappa_1^3 v_3,\\
c_4 &= \kappa_4 + \left(4\kappa_3\kappa_1 + 3\kappa_2^2\right)v_2 + 6\kappa_2\kappa_1^2 v_3 + \kappa_1^4 v_4,
\end{aligned} \tag{7}$$

which are valid for arbitrary probability distributions, provided the fluctuations in baryon number and volume are independent. We note that the form of (7) is determined by the volume dependence of the moments (3). Hence, the coefficients in (7) are also given by the Bell polynomials.

III. GENERAL DERIVATION

In the previous section, we explored the effect of volume fluctuations on the fluctuations of the net baryon number for the first few cumulants, where explicit calculations are tractable. In the following we derive a general expression for the cumulants, under the assumption that the fluctuations of baryon number and volume are independent.

A. Formalism

In general, the probability distributions introduced in Sec. II are characterized by the corresponding cumulant

[3]The normalization of the generalized susceptibilities given in [6] differs from the cumulants used here by a factor T^3.

generating functions[4]

$$\chi^B(t) = \ln \sum_{B=-\infty}^{\infty} P(B)\exp(B\,t), \tag{8}$$

$$\chi^V(s) = \ln \int_0^{\infty} dV\,\mathcal{P}(V)\exp(Vs). \tag{9}$$

The cumulants are obtained by expanding χ^B and χ^V in a series about the origin. The additivity of cumulants and thermodynamic principles imply that[5]

$$\chi^B(t) = V\zeta^B(t), \tag{10}$$

where ζ^B is a volume-independent function. In fact, ζ^B is the generating function for the reduced cumulants, defined in Eqs. (2):

$$\kappa_n = \frac{d^n}{dt^n}\zeta^B(t)\bigg|_{t=0}. \tag{11}$$

Similarly, we find for the reduced cumulants of volume fluctuations

$$v_n = \frac{1}{\langle V\rangle}\frac{d^n}{ds^n}\chi^V(s)\bigg|_{s=0}. \tag{12}$$

Our aim is to compute cumulants of the net baryon number *including* the effects of volume fluctuations. These cumulants are obtained from the cumulant generating function

$$\phi^B(t) = \ln \int dV\,\mathcal{P}(V)\sum_B P(B,V)e^{Bt}. \tag{13}$$

Using Eq. (10) we find

$$\sum_B P(B,V)e^{Bt} = e^{V\zeta^B(t)}, \tag{14}$$

and consequently

$$\phi^B(t) = \ln \int dV\mathcal{P}(V)e^{V\zeta^B(t)}. \tag{15}$$

A comparison with the definition of the cumulant generating function (9), yields

$$\phi^B(t) = \chi^V[\zeta^B(t)]. \tag{16}$$

This is the general form of the cumulant generating function for fluctuations of the net baryon number, including the effect of volume fluctuations. The corresponding reduced cumulants are given by a Taylor expansion of $\phi^B(t)$ about $t = 0$,

$$c_n = \frac{1}{\langle V\rangle}\frac{d^n}{dt^n}\phi^B(t)\bigg|_{t=0}. \tag{17}$$

[4]We assume that the integrals in (8) and (9) converge for t and s in an interval around the origin, so that the cumulant generating functions exist [19].

[5]The additivity of cumulants is valid only when there are no long range correlations in a system. This is the case near the chiral crossover transtion; however, it is not applicable in the vicinity of a possible critical point.

We note that since $\zeta^B(t=0) = 0$, no further normalization is needed in the calculation of the cumulants.

Using Faà di Bruno's formula [20], we obtain a closed form expression for the cumulants,

$$c_n = \sum_{i=1}^{n} v_n\, B_{n,i}(\kappa_1, \kappa_2, \ldots, \kappa_{n-i+1}), \tag{18}$$

where $B_{n,i}$ are Bell polynomials. This equation confirms and extends our previous results for the first four cumulants, given in Eq. (7). Thus, for an arbitrary probability distribution for the fluctuations of net baryon number as well as for the fluctuations of the volume, Eq. (18) yields cumulants that can be confronted with experiment. Conversely, given a model for the volume fluctuations, Eq. (18) can be used to extract cumulants of the net baryon number in a fixed volume.

B. Vanishing chemical potential and symmetric volume fluctuations

In the particular case of vanishing chemical potential, all odd cumulants of net baryon number fluctuations vanish, $\kappa_{2n+1} = 0$. For the sake of simplicity, we also assume that the fluctuations of the volume are symmetric, i.e., $v_{2n+1} = 0$ for $n > 1$. In this case the first three nonvanishing cumulants are given by

$$c_2^s = \kappa_2, \tag{19}$$

$$c_4^s = \kappa_4 + 3\kappa_2^2 v_2, \tag{20}$$

$$c_6^s = \kappa_6 + 15\kappa_2\kappa_4 v_2. \tag{21}$$

Thus, the cumulants c_n^s for $n < 8$ depend only on the second-order cumulant of the volume fluctuations, v_2. In other words, these cumulants are independent of the details of the probability distribution.

In the next section we use the above form to explore the effect of volume fluctuations on the cumulants of net baryon number.

IV. NUMERICAL RESULTS IN THE PQM MODEL

We illustrate the influence of volume fluctuations on net baryon number fluctuations, within a model calculation. Of particular interest is the modification of higher order moments near the chiral crossover transition. We adopt the Polyakov loop extended quark-meson model (PQM) and compute the cumulants in a nonperturbative scheme, the functional renormalisation group. Details on the calculations and on the derivation of the net baryon number fluctuations can be found in Ref. [9]. In this exploratory calculation, we consider only the case of *symmetric* volume fluctuations and vanishing baryon chemical potential.

In Ref. [8] it was shown that near the chiral crossover transition, higher cumulants of the net baryon number ($n > 4$) differ considerably from the predictions of the HRG model. In particular, it was suggested that negative values of κ_6 and κ_8 could be used to map out the chiral phase boundary.

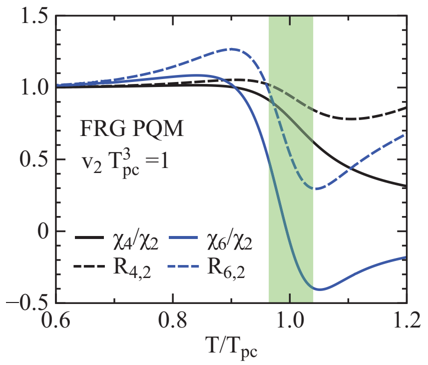

FIG. 1. (Color online) The ratios $R_{4,2}$ and $R_{6,2}$ [defined in Eqs. (24) and (25)] compared to χ_4/χ_2 and χ_6/χ_2 [see Eq. (23)] as functions of temperature, computed in the PQM model at vanishing chemical potential. The probability distribution for volume fluctuations is assumed to be symmetric, with the variance $v_2 T_{pc}^3 = 1$; see the text for details.

This potential signal for the QCD phase transition may be affected by volume fluctuations. Indeed, the second term in Eq. (21) yields a positive contribution to c_6. The strength of this contribution is directly proportional to the second cumulant v_2 of volume fluctuations and may thus change the sign of c_6.

To proceed with the calculations in the PQM model, we relate the cumulants κ_n to the generalized susceptibilities χ_n, defined by

$$\chi_n = \frac{\partial(p/T^4)}{\partial(\mu_B/T)^n} = \frac{\kappa_n}{T^3}. \tag{22}$$

It is useful to consider the ratios of cumulants,

$$R_{n,m} = \frac{c_n}{c_m}, \tag{23}$$

since many uncertainties cancel between the numerator and denominator. Using (20) and (21) we thus find

$$R_{4,2} = \frac{\chi_4}{\chi_2} + 3\chi_2 T^3 v_2, \tag{24}$$

$$R_{6,2} = \frac{\chi_6}{\chi_2} + 15\chi_4 T^3 v_2. \tag{25}$$

Figure 1 shows the effect of volume fluctuations on the $R_{4,2}$ and $R_{6,2}$ ratios, obtained in the PQM model at fixed $T^3 v_2 = 1$. The contribution of the volume fluctuations to both ratios are positive, and grows with temperature. This effect is also illustrated in Fig. 2, where the ratios $R_{n,m}$ at the crossover transition temperature are shown as functions of $v_2 T_{pc}^3$.

The above results indicate that volume fluctuations tend to suppress the signature of the chiral transition in the cumulants of net baryon number. Here, the ratio $R_{6,2}$ seems to be particularly sensitive. Consequently, the usefulness of fluctuations of conserved charges as a probe of criticality in heavy ion collisions, depends crucially on the possibility to control volume fluctuations.

In general, volume fluctuations are difficult to assess. In heavy ion collisions they depend on the centrality of the collision, on the definition used to fix the number of participants, and on the kinematic window, where the fluctuations are measured. Thus, v_2 is specific to a given experimental setup.

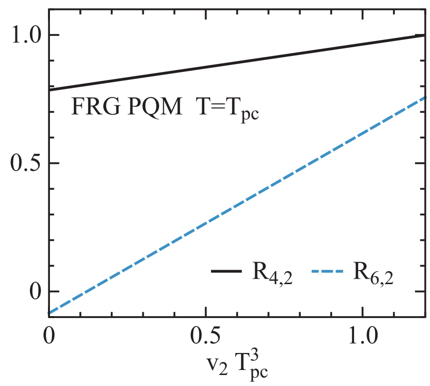

FIG. 2. (Color online) The ratios, $R_{4,2}$ and $R_{6,2}$ at the chiral crossover temperature T_{pc}, obtained in the PQM model, as functions of the dimensionless variance of the volume fluctuations, $v_2 T_{pc}^3$.

In order to explore the dependence of v_2 on the collision geometry, we performed a Glauber–Monte Carlo simulation, using the standard parameters for Au-Au collisions [21]. We assume that the volume is proportional to number of participants N_{part} times a volume factor V_0, which we fix to be equal to the volume of the proton, $V_0 = 2.83$ fm^3. We compute the fluctuations in N_{part} for a fixed number of charged particles N_{ch}. A similar procedure is adopted by the STAR Collaboration in their data analysis Ref. [15]. Since we adopted a small value for V_0, we expect that the resulting estimate of v_2 is effectively a lower limit. In Fig. 3, we show the dependence of v_2 on the number of charged particles, N_{ch}. We find that the reduced variance, v_2, is approximately constant except for very central collisions, where the volume fluctuations are strongly suppressed.

To assess the expected centrality dependence of the baryon number fluctuations, we use the Glauber result for v_2 and assume that the freeze-out temperature $T_{fr} = T_{pc}$ and that it depends only weakly on centrality. The resulting ratios $R_{4,2}$ and $R_{6,2}$ are shown in Fig. 4 as functions of N_{ch} and $\langle N_{part}\rangle$.

Recently, preliminary data on $R_{4,2}$ and $R_{6,2}$, obtained by the STAR Collaboration, were reported in Refs. [22,23]. It is found that both ratios are essentially independent of the number of participants in collisions ranging from $\langle N_{part}\rangle = 2$ to 350. As shown in Fig. 4, our model also yields a weak

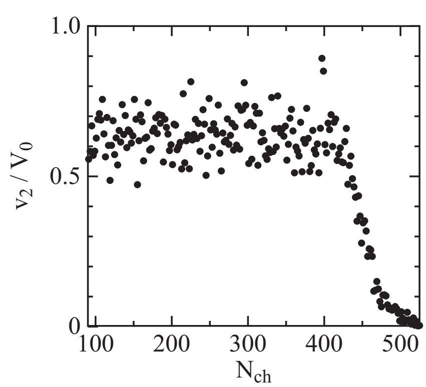

FIG. 3. The reduced variance of the volume fluctuations, v_2, as a function of the number of charged particles, N_{ch}.

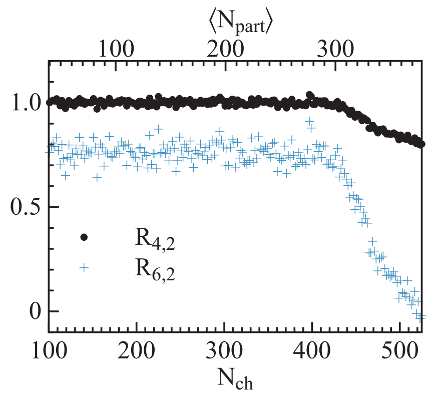

FIG. 4. (Color online) Ratios of cumulants, $R_{4,2}$ and $R_{6,2}$, as functions of the number of charges particles in Au-Au collisions, based on the PQM model (see text for details). The freeze-out temperature is assumed to be equal to $T_{\rm pc}$.

dependence of $R_{4,2}$ and $R_{6,2}$ on the number of participants, except for very central collisions where volume fluctuations are suppressed. We stress, however, that this schematic model is not expected to yield a quantitative description of the experimental data.

V. CONCLUSIONS

We have studied the influence of volume fluctuations on the properties of cumulants of net charge distributions in heavy ion collisions. In particular, we have computed the contribution of volume fluctuations to ratios of net-baryon-number cumulants.

In a heuristic approach we showed explicitly how the corrections due to volume fluctuations arise. The resulting expressions, which hold for arbitrary probability distributions, were confirmed and extended in a general formalism, where we employed cumulant generating functions to obtain a closed form for the cumulants, including the effect of volume fluctuations.

We assessed the effect of volume fluctuations on the kurtosis $R_{4,2}$ as well as on ratios involving higher-order cumulants, viz., $R_{6,2}$, in the Polyakov loop extended quark-meson model. A non-perturbative treatment of fluctuations was obtained by employing the functional renormalization group. We focused on the structure of ratios of cumulants near the chiral crossover transition, assuming that the probability distribution of volume fluctuations is approximately symmetric.

Finally, we showed that phenomenologically relevant ratios of cumulants of the net baryon number are enhanced by volume fluctuations. Consequently, the structure of these ratios, may be significantly modified by volume fluctuations. Therefore, we conclude that fluctuations of conserved charges in heavy ion collisions can provide robust probes of the chiral phase boundary if a good control of volume fluctuations can be achieved.

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