Anharmonicity of the excited octupole band in actinides using supersymmetric quantum mechanics

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Background: Low-lying octupole collective excitations play an important role in the description of the structure of nuclei in the actinide region. Ground state alternating parity rotational bands combining both positive and negative parity states are known in several nuclei. However, only recently it has been discovered in ^{240}Pu an excited positive parity rotational band having an octupole nature and demonstrating strong anharmonicity of the octupole motion in the band head energies.
Purpose: To suggest a model describing both ground state and excited alternating parity bands, which includes a description of the anharmonic effects in the bandhead excitation energies and can be used to predict the energies of the excited rotational bands of octupole nature and the $E1$ transition probabilities.
Methods: The mathematical technique of the supersymmetric quantum mechanics with a collective Hamiltonian depending only on the octupole collective variable which keeps axial symmetry is used to describe the ground state and excited alternating parity rotational bands.
Results: The excitation energies of the states belonging to the lowest negative parity and the excited positive parity bands are calculated for ^{232}Th, ^{238}U, and ^{240}Pu. The $E1$ transition matrix elements are also calculated for ^{240}Pu.
Conclusions: It is shown that the suggested model describes the excitation energies of the states of the lowest negative parity band with the accuracy around 10 keV. The anharmonicity in the bandhead energy of the excited positive parity band is described also. The bandhead energy of the excited positive parity band is described with the accuracy around 100 keV.

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I. INTRODUCTION

Collective modes play a very important role in the structure of heavy nuclei. Among them the isoscalar collective quadrupole mode is widespread and well investigated. We know of the one- and two-phonon quadrupole states in vibrational nuclei, complicated spectra of the positive parity collective states in transitional nuclei, and the well-defined rotational bands in deformed axially symmetric nuclei. Low-lying collective octupole excitations have also been observed in many nuclei. However, they are known mainly as the one-phonon vibrational excitations. The more interesting information about octupole excitations is centered in the actinide nuclei. In these nuclei there are known long alternating parity bands which have been observed up to angular momentum $I = 30$. Beautiful new data have been obtained at the Argonne Tandem Linac Accelerator System (ATLAS) facility at the Argonne National Laboratory [1]. In addition to the long ground state alternating parity rotational band an excited band with positive parity has been observed in ^{240}Pu. The properties of the states of this band clearly indicate the octupole nature of the excited band: there are strong $E1$ transitions to the negative parity states of the ground state alternating parity band, however the $E2$ transitions from the states of this band to the positive parity states of the ground state band have not been observed. Among the recent results we mention measurements of the electric octupole transition strength at the Isotope Separator On Line Device (ISOLDE) at CERN, which give clear evidence for strong octupole correlations in the light Ra isotopes [2].

We present below a quantitative interpretation of these data based on the following physical picture. We assume that the main role in the description of the properties of these states is played by the octupole mode with $K = 0$ (β_{30}), which preserves axial symmetry. We present the method to calculate the excitation energies of the lowest negative parity states and of the excited positive parity band and the wave functions of these states. As shown below, the anharmonic effects are not important at low angular momentum for the lowest negative parity band. For the excited positive parity band the anharmonic effects are important, however, already at low I. With angular momentum increase the anharmonic effects become important for all states. It was shown in Ref. [3] that a second-order phase transition from octupole-nondeformed to octupole-deformed states takes place at I around 12 to 14 in the ground state alternating parity bands of ^{232}Th, ^{238}U, and ^{240}Pu.

The results obtained by the authors of Ref. [4] have shown that with the Gauss ansatz for the wave functions of the states of the ground state positive parity band the octupole potential at small I is very similar to the harmonic oscillator potential. As a consequence, the energy of the band head of the excited positive parity band (0_2^+ state) is approximately equal to twice the energy of the lowest 1^- state. This strongly contradicts the observed situation in ^{240}Pu. To improve this it is necessary to

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modify the Gauss ansatz in such a way as to keep the previous description of the lowest negative parity band, but improve a description of the excited positive parity band at low I. In other words, we should find a new ansatz for the positive parity states of the ground state band which produces a new octupole potential with the following properties. Up to the energies of the states of the first excited negative parity band it is close at every I to the previously used potential. However, around the energies of the second excited band the new potential should be wider than that obtained with the Gauss ansatz. As a result the energy of the 0_2^+ state will be shifted down in agreement with the experimental situation.

Octupole excitations have been a subject of a large number of theoretical studies. For instance, the model of Ref. [5] has been applied to some actinides in Ref. [6]. The collective model has been applied to the description of the octupole states in Refs. [7,8]. Alternating approaches to negative-parity bands in the actinides have been given in a cluster model in Refs. [9,10]. The algebraic approach to octupole correlations [11], fully classified in Refs. [12,13], has been applied in the actinides in Ref. [14]. The microscopic description of the evolution from spherical to octupole-deformed shapes has been considered in the framework of the relativistic mean-field theory in the actinides in Ref. [15].

There is, however, an effect observed in light Ra and Th isotopes [16,17], which are soft with respect to the quadrupole deformation: the parity splitting, i.e., a shift of the energy of the negative-parity states with respect to the positive-parity ones in the alternating parity bands, being positive at low and medium values of I but changing the sign at high I. This effect is produced by the band crossing with the aligned octupole phonon or the two-quasiparticle band [18–21] and cannot be described in the framework of our model. For this reason we consider the states of the alternating parity band only up to those values of I for which the parity splitting is positive.

In the consideration below, the mathematical technique of supersymmetric quantum mechanics is used.

II. HAMILTONIAN AND BASIC STATES

The Hamiltonian of the model used can be presented as

$$H_I = -\frac{\hbar^2}{2B}\frac{d^2}{d\beta_{30}^2} + V_I^{(1)}(\beta_{30}), \tag{1}$$

where the subscript I indicates that the shape of the potential depends on the angular momentum I because the potential energy term includes also the rotational energy. We do not take into account a possible dependence of the inertia coefficient B on octupole deformation to restrict the number of the parameters used. It is assumed here that the quadrupole deformation is rigid, i.e., well-deformed axially symmetric nuclei with the ratio $E(4_1^+)/E(2_1^+)$ close to 3.3 are considered. This model has common features with the algebraic model of Ref. [11] and the dinuclear system model [10].

We did not suggest any parametrization of $V_I^{(1)}$ but follow the procedure given by supersymmetric quantum mechanics [22,23] to obtain the potential. The experimental data on the excitation energies of the negative parity states of the ground state alternating parity bands are used to determine the parameters of the potential $V_I^{(1)}$ completely. In determining $V_I^{(1)}$ we have taken into account the experimental fact that the excitation energy of the bandhead of the excited positive parity octupole band demonstrates strong anharmonicity, i.e., a strong deviation of the excitation energy of this state from twice the energy of the first excited 1^- state.

In our approach we do not calculate the energies of the positive parity states of the ground state band. We calculate the energy differences between the energies of the excited states and some reference energies. For even values of the angular momentum I the reference energies are the experimental energies of the positive parity states of the ground state band. For odd I the reference energies are determined by interpolation using the energies of the neighboring positive parity states of the ground state band. The interpolation formula is the following [16]

$$E^*_{\text{inter},I+1} = \frac{(I+1)E^*_{\text{gs},I+2} + (I+2)E^*_{\text{gs},I}}{2I+3}, \tag{2}$$

where I is even.

To take into account anharmonic effects in the bandhead energies we use the following ansatz for the positive parity wave function of the states belonging to the ground state band. Since being the wave function of the lowest states for a given I this wave function has no nodes, we assume that it can be presented by the following expression

$$\Psi_I^{(+)} \sim (\cosh^{-l}\lambda[\beta_{30} - \beta_m(I)] + \cosh^{-l}\lambda[\beta_{30} + \beta_m(I)]), \tag{3}$$

which is the sum of the two associated Legendre polynomials $P_l^l\{\tanh\lambda[\beta_{30} \pm \beta_m(I)]\}$ [24]. This wave function is symmetric with respect to a $\beta_{30} \to -\beta_{30}$ transformation.

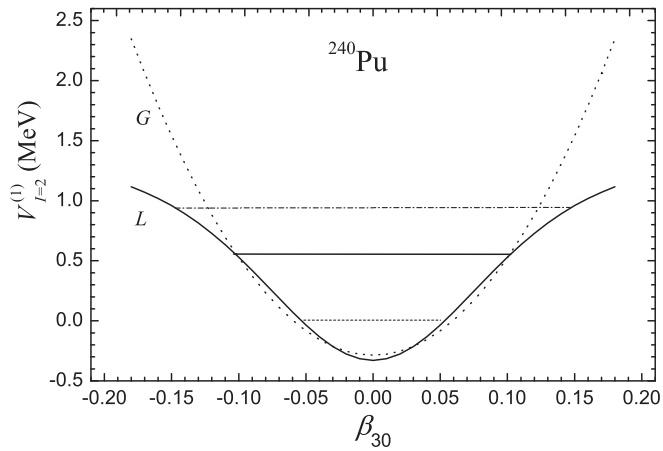

FIG. 1. The potentials $V_I^{(1)}$ as a function of β_{30} calculated with $I = 2$ using the Legendre ansatz (L) and the Gauss ansatz (G) for the wave functions of the ground state band. The straight solid line indicates the position of the first excited negative parity state. Dashed line indicates position of the lowest positive parity states. Dot-dashed line indicates position of the second excited positive parity states. The level energies are obtained using the Legendre ansatz.

In contrast to our previous publication [4], we use here the associated Legendre polynomials instead of the Gauss function. The reason is the following. The use of the Gauss function corresponds at low I to harmonic octupole vibrations. The corresponding potential coincides with a good accuracy at $I = 0$ with the harmonic oscillator potential. The use of the associated Legendre polynomials give us a possibility to include the consideration of anharmonic effects even at $I = 0$. The reason is a different behavior of these functions at large β_{30} although they generate very similar potentials at small β_{30} as shown in Fig. 1. Indeed, it is seen in Fig. 1 that around the energy of the first excited state, indicated by the solid line, the two potentials are quite close to each other. It happens because the parameters $\hbar\omega$, c_0, and c_1 are fitted separately for both potentials so as to describe the excitation energies of the lowest negative parity states at every I. These states are just the first excited states in the potential. However, around the energy of the second excited state, indicated in Fig. 1 by the dashed line, the potentials start to deviate one from the other. We mention that, instead of the associated Legendre polynomials, other functions with a similar behavior around the energies of the first and second excited states but different asymptotic behavior and therefore with a larger number of the parameters can be used. However, the choice of the associated Legendre polynomials is very convenient.

Following the procedure of supersymmetric quantum mechanics we substitute the wave function (3) into the Schrödinger equation with the Hamiltonian (1) and obtain the potential $V_I^{(1)}$. The potential produced in this way is a finite depth potential. This is a consequence of the use of the associated Legendre polynomials as the ansatz. As noted, this ansatz has been suggested instead of the Gauss ansatz to have a wider potential at the energy of the second excited state (two-phonon state in the harmonic approximation) compared to the harmonic oscillator potential. At the same time both potentials, namely, those produced by the Legendre polynomial ansatz and Gauss ansatz are similar at the energies of the lowest and the first excited states.

The use of the finite depth potential means that the number of the rotational bands in the model is restricted. This situation is not unique. It is well known that in the interacting boson model the anharmonicity of the quadrupole mode is included into consideration by the use of the SU(6) dynamical symmetry group whose generators are applied to construct the Hamiltonian of the model. The eigenstates of this Hamiltonian are characterized by the maximal possible number of the quadrupole bosons (d bosons) in the basis wave functions. In other words, the total number of the eigenstates of the model Hamiltonian is finite. If we use the collective octupole Hamiltonian (1) with the potential $V_I^{(1)}$ as a finite depth potential, then for every value of the angular momentum I there is only a finite number of excited states. These excitations belong to different rotational bands and can be considered as the intrinsic excitations of a deformed nucleus. At the same time it does not mean that the quadrupole and the octupole deformation parameters are restricted from above. This would be in contradiction to, for instance, the asymmetric fission process. This method of the reduction of the basis is used only to describe the low-lying collective excitations which include mainly the quasiparticle components belonging to the valence shell. We mention that finite depth potentials have been considered in the algebraic theory of nuclear reactions [25].

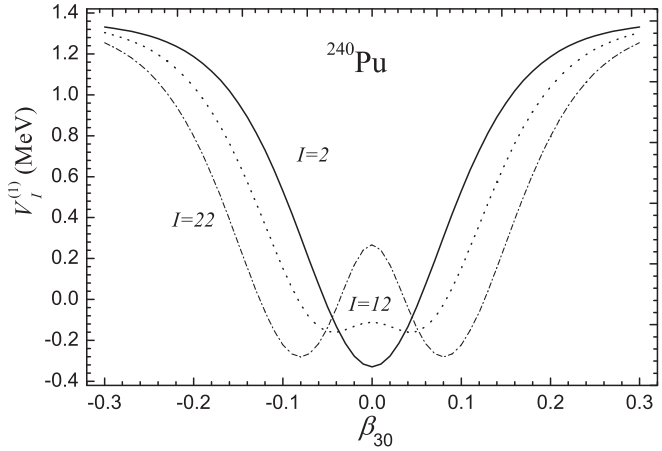

FIG. 2. The potentials $V_I^{(1)}$ as a function of β_{30} calculated with $I = 2$, 12, 22 using the Legendre ansatz for the ground state wave functions.

In Fig. 2 the collective octupole potentials calculated for different values of I using the ansatz (3) are shown.

Following Ref. [3], we rewrite the expression (3) as

$$d\Psi_I^{(+)} \sim (\cosh^{-l} \lambda\beta_m(I)[\beta_{30}/\beta_m(I) - 1] + \cosh^{-l} \lambda\beta_m(I)[\beta_{30}/\beta_m(I) + 1]). \quad (4)$$

It is convenient to introduce instead of λ the new parameter ω

$$\lambda = \sqrt{\frac{B\omega}{\hbar}} \quad (5)$$

and the parameter

$$s_3(I) = \sqrt{\frac{B\omega}{\hbar}}\beta_m(I) \quad (6)$$

as in Ref. [3]. Then

$$\Psi_I^{(+)} \sim (\cosh^{-l} s_3(I)[\beta_{30}/\beta_m(I) - 1] + \cosh^{-l} s_3(I)[\beta_{30}/\beta_m(I) + 1]). \quad (7)$$

We see that the wave function (7) is a function of the variable $\beta_{30}/\beta_m(I)$ and depends only on the parameter $s_3(I)$.

Following the procedure of supersymmetric quantum mechanics we first determine the supersymmetric potential $W(\beta_{30})$

$$W(\beta_{30}) = -\frac{\hbar}{\sqrt{2B}}\frac{\Psi_I'^{(+)}}{\Psi_I^{(+)}}, \quad (8)$$

which can be used to determine the potential $V_I^{(1)}$ for the Schrödinger equation, whose lowest state wave function is (7),

$$V_I^{(1)} = W^2 - \frac{\hbar}{\sqrt{2B}}W' + E_I^* = \frac{\hbar^2}{2B}\frac{\Psi_I''^{(+)}}{\Psi_I^{(+)}} + E_I^* \quad (9)$$

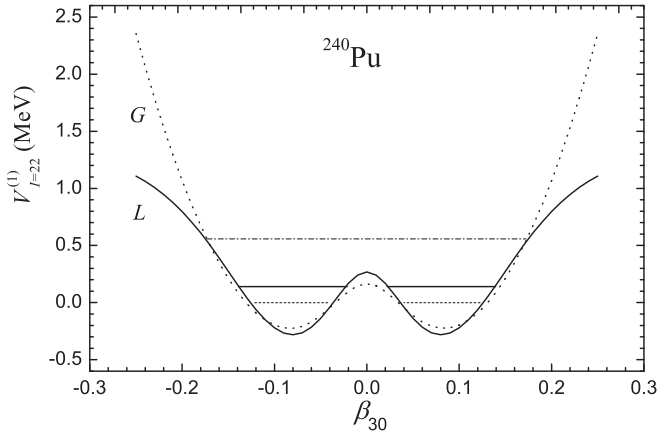

FIG. 3. The potentials $V_I^{(1)}$ as a function of β_{30} calculated with $I = 22$ using the Legendre ansatz (L) and the Gauss ansatz (G) for the wave functions of the ground state band. The straight solid line indicates the position of the first excited negative parity state. Dashed line indicates position of the lowest positive parity states. Dot-dashed line indicates position of the second excited positive parity states. The level energies are obtained using the Legendre ansatz.

and its supersymmetric partner potential $V_I^{(2)}$

$$V_I^{(2)} = W^2 + \frac{\hbar}{\sqrt{2B}} W' + E_I^* = \frac{\hbar^2}{2B}\left[2\left(\frac{\Psi_I'^{(+)}}{\Psi_I^{(+)}}\right)^2 - \frac{\Psi_I''^{(+)}}{\Psi_I^{(+)}}\right] + E_I^*. \quad (10)$$

Here for even I the quantity E_I^* is the experimental excitation energy of the lowest state with spin I and positive parity. For odd I this energy is determined by interpolation as was described above [see Eq. (2)].

Varying the value of the parameter s_3 we have found that the potential $V_I^{(1)}$ evolves from that having one minimum at $\beta_{30} = 0$ when s_3 is small to the two-center potential for large values of s_3 (see Fig. 3). However, the partner potential $V_I^{(2)}$ is the one-center potential for physically interesting values of s_3. This fact makes a solution of the Schrödinger equation with the potential $V_I^{(2)}$ simpler than a solution of the Schrödinger equation with the potential $V_I^{(1)}$ since in this case some set of orthogonal well-known special functions can be used as a basis. At the same time the eigenvalues of the Hamiltonian with the potential $V_I^{(2)}$ coincide with those obtained using the potential $V_I^{(1)}$, excluding the lowest eigenvalue of this Hamiltonian. This is an important advantage related to the application of supersymmetric quantum mechanics to this problem.

The lowest state of the Hamiltonian with the potential $V_I^{(2)}$ can be constructed using as the basis the associated Legendre polynomials $P^{l-1}_{l-3+2n}[\tanh(s_3 \frac{\beta_{30}}{\beta_m(I)})]$ with $n = 1, 2, 3, \ldots$. These functions are orthogonal

$$\int_{-\infty}^{+\infty} P_l^m(\tanh x) P_k^m(\tanh x) \frac{dx}{\cosh^2 x} = \frac{2(l+m)!}{(2l+1)(l-m)!}\delta_{kl}. \quad (11)$$

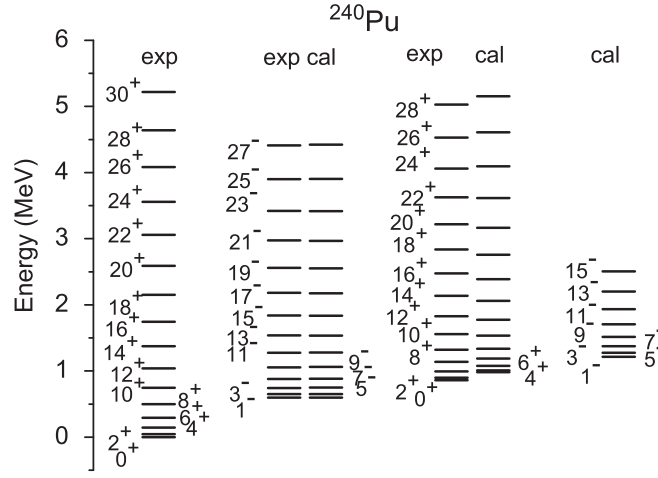

FIG. 4. The calculated and experimental spectra of the ground and excited bands of octupole nature of both parities in ^{240}Pu. The experimental data are taken from Refs. [1,27]. The values of the energies are given in the Tables I to IV.

The first excited state of the Hamiltonian with the potential $V_I^{(2)}$ has negative parity. The wave function of this state can be constructed using the following basis: $P^{l-2}_{l-3+2n}[\tanh(s_3 \frac{\beta_{30}}{\beta_m(I)})]$. For the second excited state of this Hamiltonian the basis is formed by the functions $P^{l-3}_{l-3+2n}[\tanh(s_3 \frac{\beta_{30}}{\beta_m(I)})]$, and so on.

III. RESULTS OF CALCULATIONS AND DISCUSSIONS

It was shown in Ref. [4] based on the results of the calculations of the parity splitting for the Gauss ansatz of the ground state wave functions that the angular momentum dependence of $s_3(I)$ can be parameterized by the linear function

$$s_3(I) = c_0 + c_1 \cdot I, \quad (12)$$

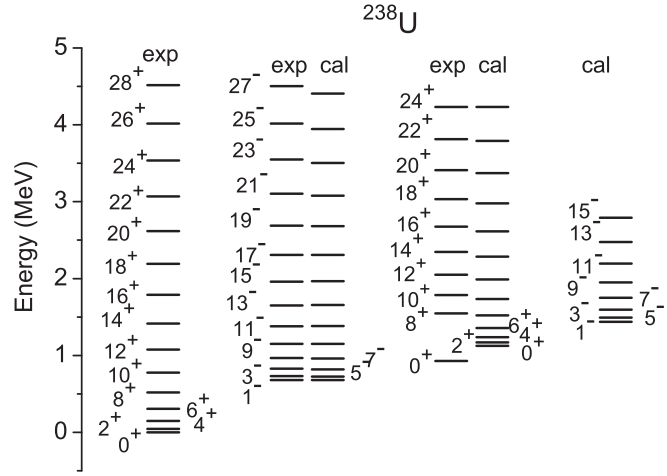

FIG. 5. The calculated and experimental spectra of the ground and excited bands of octupole nature of both parities in ^{238}U. The experimental data are taken from Refs. [26,27]. The values of the energies are given in the Tables I to IV.

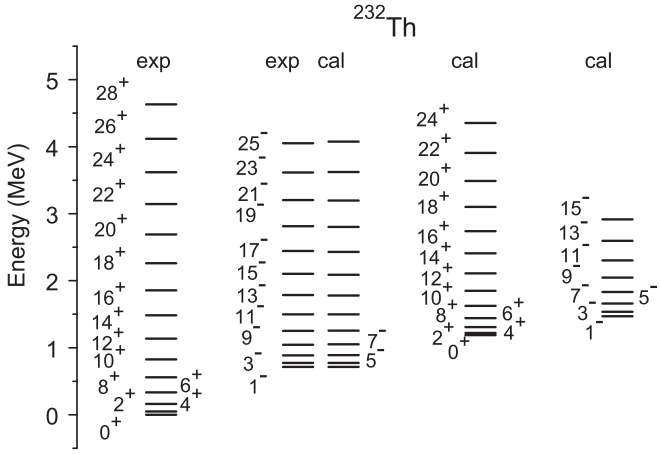

FIG. 6. The calculated and experimental spectra of the ground and excited bands of octupole nature of both parities in ^{232}Th. The experimental data are taken from Ref. [27]. The values of the energies are given in the Tables I to IV.

where c_0 and c_1 are the fit parameters. They are determined to get the best description of the energies of the negative parity members of the ground state alternating parity band. It is shown by our present calculations that this assumption is also valid for the Legendre ansatz suggested. The parameter c_0 does not play an important role and, as shown in Ref. [4], can be put equal to zero without the essential loss of accuracy of the description of the experimental data. For high values of the angular momentum the use of this parameter reduces a deviation of the calculated values from the experimental ones by about 10 keV.

TABLE I. Experimental and calculated excitation energies of the states of the lowest negative parity band for ^{240}Pu, ^{238}U, and ^{232}Th given in keV. The experimental data are taken from Refs. [1,26,27]. The values of the parameters are the following: for ^{240}Pu $\hbar\omega = 170$ keV, $c_0 = 0.0187$, and $c_1 = 0.0305$; for ^{238}U $\hbar\omega = 204$ keV, $c_0 = 0.0920$, and $c_1 = 0.0218$; for ^{232}Th $\hbar\omega = 206$ keV, $c_0 = 0.0378$, and $c_1 = 0.0222$.

I	^{240}Pu		^{238}U		^{232}Th	
	exp	cal	exp	cal	exp	cal
1	598	596	680	680	714	712
3	649	652	732	724	774	774
5	742	748	827	819	884	888
7	878	884	966	962	1043	1050
9	1056	1061	1151	1152	1250	1254
11	1277	1278	1379	1384	1499	1498
13	1540	1536	1649	1656	1785	1778
15	1841	1835	1959	1965	2102	2089
17	2182	2174	2307	2308	2445	2431
19	2560	2552	2689	2682	2813	2801
21	2973	2967	3104	3080	3204	3198
23	3420	3418	3548	3503	3616	3623
25	3900	3904	4017	3945	4050	4075
27	4410	4422	4504	4407	4506	4558

TABLE II. Experimental and calculated excitation energies of the excited positive parity band of octupole nature for ^{240}Pu, ^{238}U, and ^{232}Th given in keV. The experimental data are taken from Refs. [1,26,27]. The values of the parameters $\hbar\omega$, c_0, and c_1 are the same as have been used in Table I.

I	^{240}Pu		^{238}U		^{232}Th	
	exp	cal	exp	cal	exp	cal
0	861	982	927	1154	–	1187
2	900	1010	–	1171	–	1220
4	992	1077	–	1239	–	1306
6	1138	1186	–	1356	–	1442
8	1323	1337	1546	1522	–	1624
10	1556	1533	1787	1734	–	1848
12	1829	1775	2049	1989	–	2111
14	2136	2060	2347	2284	–	2408
16	2474	2388	2676	2615	–	2739
18	2836	2758	3032	2979	–	3100
20	3217	3166	3412	3373	–	3489
22	3626	3613	3812	3792	–	3907
24	4062	4095	4233	4233	–	4352

The next step is to fix the integer number l, which characterizes the wave function (3) and determines the number of eigenstate of the Hamiltonian with the potential $V_I^{(1)}$. Since in ^{240}Pu there are found experimentally three rotational bands: the ground state positive parity band, the negative parity band based on the 1_1^- state, and the excited positive parity band based on the 0_2^+ state the parameter l cannot be smaller than $l = 3$. If we assume that a second excited $K = 0$ band of negative parity states of an octupole nature can exist we must take l equal to 4. The results of the calculations of the spectra of the positive and the negative parity states, whose origin is related to the octupole mode, are presented in Fig. 4 for ^{240}Pu and in Figs. 5 and 6 for ^{238}U and ^{232}Th, respectively. The values of

TABLE III. Calculated excitation energies of the states of the excited negative parity collective band of octupole nature for ^{240}Pu, ^{238}U, and ^{232}Th given in keV. The values of the parameters $\hbar\omega$, c_0, and c_1 are the same as given in the caption of Table I.

I	^{240}Pu	^{238}U	^{232}Th
1	1216	1438	1470
3	1274	1491	1538
5	1373	1596	1660
7	1517	1750	1832
9	1705	1951	2049
11	1933	2194	2304
13	2201	2475	2594
15	2506	2791	2916
17	2848	3139	3265
19	3224	3514	3640
21	3635	3912	4040
23	4077	4332	4463
25	4552	4769	4909
27	5057	5223	5377

TABLE IV. Calculated excitation energies of the states of the lowest negative parity and of the first excited positive parity bands for ^{240}Pu given in keV. The values of the parameters are the same as presented in Table I for ^{240}Pu. The results obtained using the associated Legendre polynomials as the ansatz for the positive parity wave functions of the states of the ground state band are marked by "Legendre." The results obtained using the Gauss ansatz are marked by "Gauss."

I	$E_1(I^-)$		I	$E_2(I^+)$	
	Legendre	Gauss		Legendre	Gauss
1	596	594	0	982	1166
3	652	646	2	1010	1187
5	748	741	4	1077	1238
7	884	878	6	1186	1325
9	1061	1057	8	1337	1453
11	1278	1278	10	1533	1628
13	1536	1538	12	1775	1848
15	1835	1840	14	2060	2112
17	2174	2181	16	2388	2427
19	2552	2561	18	2758	2786
21	2967	2978	20	3166	3190
23	3418	3432	22	3613	3637
25	3904	3920	24	4095	4129

the parameters $\hbar\omega$, c_0, and c_1 are given in the caption of the Table I. We have accepted $l = 4$.

The results presented in the tables and the figures show that the description of the excitation energies of the lowest negative parity states is quite good for ^{240}Pu up to $I = 23$ using both ansatzs: based on the associated Legendre polynomials and based on the Gaussian function (see Table IV). For ^{238}U and ^{232}Th only the results obtained using the ansatz based on the associated Legendre polynomials are presented. They are good up to I equal to 19 and 23, respectively. The calculated excitation energy of the excited octupole 0^+ state in ^{240}Pu obtained using the Legendre ansatz is higher than the experimental one by 120 keV. However, a significant part of the anharmonic effect in the bandhead energy is taken into account in the calculations with this ansatz. We mention that agreement would be better for $l = 3$. In other words, the experimental fact that the excitation energy of the bandhead of the excited positive parity octupole band strongly deviates from twice the energy of the first excited 1^- state is mainly taken into account by the new potential $V_I^{(1)}$. The results presented in Table IV also show that if the ansatz based on the Gaussian function is used the excitation energy of the 0_2^+ state is equal to approximately twice the energy of the 1_1^- state. It is also seen from the results presented in the Table IV that the differences between the energies of the positive parity states of the excited band obtained with the Legendre ansatz and the Gauss ansatz decrease with increasing I. The reason for this is illustrated in Figs. 1 and 3. In the case of $I = 2$ (see Fig. 1) the excited positive parity state is in the region where the difference between the potentials obtained using the Legendre and Gauss ansatzs is much larger than for $I = 22$ (see Fig. 3).

The excited positive parity states of octupole nature are observed also in ^{238}U for $I \geqslant 8$ [26]. Their description in our calculations is quite satisfactory even for the highest observed values of the angular momentum. There is no experimental data on these states in ^{232}Th. Thus the calculated result for ^{232}Th is a prediction. Comparing the calculated results with the experimental data for ^{240}Pu and ^{238}U we see that the experimental energies of the 0_2^+ state are lower than the calculated ones by 100 keV or more. We mention that the parameters $\hbar\omega$, c_0, and c_1 are fitted so as to get a better description of the energies of the negative parity states of the ground state alternating parity band.

In Figs. 4 to 6 are shown also the states of the excited negative parity bands which decay by strong $E1$ transitions to the excited positive parity states discussed above. However, there is no experimental information on these states and their experimental detection or nondetection is of great interest. The calculated excitation energies of the second excited 1^- states in ^{240}Pu, ^{238}U, and ^{232}Th are equal to 1.2–1.4 MeV.

The results of calculations with the Legendre ansatz of the $E1$ transition matrix elements

$$M(E1)_{1\mu} = (1 + \chi)C_{LD}AZeD^1_{\mu 0}\beta_{20}\beta_{30} \qquad (13)$$

between the lowest excited negative parity states and the positive parity states of the ground state band and of the excited double octupole phonon band are presented in Tables V and VI

TABLE V. Calculated $E1$ transition matrix elements between the ground and the lowest excited negative parity states $\langle I_1^+||E1||(I+1)_1^-\rangle$ and between the second excited positive parity states and the lowest negative parity states $\langle I_2^+||E1||(I+1)_1^-\rangle$ given in the units e·fm. The calculations are done for ^{240}Pu. The values of the parameters $\hbar\omega$, c_0, and c_1 are the same as presented in Table I. The values of $\hbar^2/B$ are taken to be equal to 1/172.5 MeV.

I	$\langle I_1^+\|\|M(E1)\|\|(I+1)_1^-\rangle$	$\langle I_2^+\|\|M(E1)\|\|(I+1)_1^-\rangle$	I	$\langle I_1^+\|\|M(E1)\|\|(I+1)_1^-\rangle$	$\langle I_2^+\|\|M(E1)\|\|(I+1)_1^-\rangle$
0	0.08	0.05	14	0.54	0.58
2	0.19	0.17	16	0.61	0.63
4	0.24	0.27	18	0.70	0.66
6	0.29	0.36	20	0.80	0.71
8	0.34	0.41	22	0.90	0.75
10	0.41	0.48	24	1.00	0.78
12	0.46	0.53	26	1.08	0.81

TABLE VI. Comparison of the $E1$ transition matrix elements calculated with the Legendre ansatz and the Gauss ansatz. Calculations are done for ^{240}Pu. The calculated matrix elements are those connecting the states of the ground state band and the lowest excited negative parity band $\langle I_1^+||M(E1)||(I+1)_1^-\rangle$ and those connecting the states of the excited positive parity band and of the lowest excited negative parity band $\langle I_2^+||M(E1)||(I+1)_1^-\rangle$. The values of the parameters are the same as in Table V.

I	$\frac{\langle I_1^+\|\|M(E1)\|\|(I+1)_1^-\rangle_{\text{Gauss}}}{\langle I_1^+\|\|M(E1)\|\|(I+1)_1^-\rangle_{\text{Legendre}}}$	$\frac{\langle I_2^+\|\|M(E1)\|\|(I+1)_1^-\rangle_{\text{Gauss}}}{\langle I_2^+\|\|M(E1)\|\|(I+1)_1^-\rangle_{\text{Legendre}}}$
2	1.0	1.60
4	1.0	1.75
6	1.0	1.76
8	1.0	1.96
10	0.96	1.96
12	1.0	1.94
14	0.96	1.88
16	0.97	1.86
18	0.98	1.85
20	0.98	1.76
22	0.96	1.68
24	0.95	1.57
26	0.94	1.29

for ^{240}Pu. Only the results obtained for the $\langle I^+||E1||(I+1)^-\rangle$ matrix elements are shown. In Eq. (13) we used $C_{LD} = 0.0007$ fm and $\chi = -0.7$. It is seen from these results that in agreement with the experimental data [1]: the transition matrix elements $\langle I_1^+||E1||(I+1)_1^-\rangle$ and $\langle I_2^+||E1||(I+1)_1^-\rangle$ are of the same magnitude. It is interesting to mention that the ratio $\langle I_2^+||E1||(I+1)_1^-\rangle / \langle I_1^+||E1||(I+1)_1^-\rangle$ is approximately equal to 2 if the Gauss ansatz for the positive parity states of the ground state band is used. Qualitatively, it is explained by the fact that with the Gauss ansatz we are close to the harmonic approximation and in the case of harmonic vibrations transition matrix elements increase with increase of the number of phonons in the states connected by the transition operator. Thus, the experimental data on the $E1$ transitions also indicate the presence of anharmonic effects in the properties of the states of the double octupole phonon band. It is seen from the Table V that the $E1$ transition matrix elements increase with I approximately linearly. The reason is the linear increase of $s_3(I)$.

In Table VII are given the calculated values of the $E2$ transition matrix elements. It is seen that the $E2$ transitions between the excited and the ground positive parity bands are much weaker than the intraband transitions. The nonzero values of the intraband $E2$ transition matrix elements are obtained because the wave functions depend on the angular momentum. Would they be independent of I these $E2$ transition matrix elements would be equal to zero because β_{20} is considered as a constant in our approach.

IV. SUMMARY

In summary, we have shown that the experimental data on the excitation spectra of the states of octupole nature belonging to the ground and the excited alternating parity bands are described quite satisfactorily by our approach. The mathematical technique of supersymmetric quantum mechanics is used for the simplification of the calculations. It is shown also that the results obtained using the Legendre ansatz explain the experimental fact that the transition matrix elements $\langle I_1^+||E1||(I+1)_1^-\rangle$ and $\langle I_2^+||E1||(I+1)_1^-\rangle$ are of the same magnitude, in contrast to those obtained with the Gauss ansatz. The results of the calculations of the $E2$ transition matrix elements demonstrate that the $E2$ transitions between the excited and the ground positive parity bands are much weaker than the intraband transitions. This approach is based on the assumption that the main role in the description of the properties of the ground and excited alternating parity bands of actinides is played by the octupole mode preserving axial symmetry. We mention, however, that in ^{240}Pu there is known a 0^+ state located 229 keV higher than the 0^+ state of octupole nature. In contrast to the 0_2^+ state this state decays by the $E0$ transition to the ground state [27].

It is shown also that the use of the associated Legendre polynomials as the ansatz for the wave functions of the positive parity states of the ground state band instead of the Gauss ansatz used in our previous paper [4] gives us a possibility to include into consideration anharmonicity in the bandhead energies.

TABLE VII. Calculated $E2$ transition matrix elements between the ground band and the excited positive parity band of octupole nature $\langle I_1^+||E2||(I+2)_2^+\rangle$ and the intraband transition matrix elements given in eb. The calculations are done for ^{240}Pu.

I	$\langle I_1^+\|\|E2\|\|(I+2)_2^+\rangle$	$\langle I_1^+\|\|E2\|\|(I+2)_1^+\rangle$	$\langle I_2^+\|\|E2\|\|(I+2)_2^+\rangle$
2	0.8	4.0	3.1
4	0.7	5.2	4.7
6	0.6	6.1	5.8
8	0.5	6.9	6.6
10	0.5	7.6	7.4
12	0.4	8.2	8.1
14	0.3	8.8	8.7
16	0.3	9.3	9.2
18	0.2	9.8	9.8
20	0.2	10.3	10.3
22	0.1	10.8	10.7

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