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Heine-Stieltjes correspondence and a new angular momentum projection for many-particle systems

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A new angular momentum projection for systems of particles with arbitrary spins is formulated based on the Heine-Stieltjes correspondence, which can be regarded as the solutions of the mean-field-plus -pairing model in the strong-pairing interaction $G \to \infty$ limit. Properties of the Stieltjes zeros of the extended Heine-Stieltjes polynomials, whose roots determine the projected states, and the related Van Vleck zeros are discussed. An electrostatic interpretation of these zeros is presented. As examples, applications to n nonidentical particles of spin $1/2$ and to identical bosons or fermions are made to elucidate the procedure and properties of the Stieltjes zeros and the related Van Vleck zeros. It is shown that the new angular momentum projection for n identical bosons or fermions can be simplified with the branching multiplicity formula of $\mathrm{U}(N) \downarrow \mathrm{O}(3)$ and the special choices of the parameters used in the projection. Especially, it is shown that the solutions for identical bosons can always be expressed in terms of zeros of Jacobi polynomials. However, unlike nonidentical particle systems, the n-coupled states of identical particles are nonorthogonal with respect to the multiplicity label after the projection.

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I. INTRODUCTION

The angular momentum projection or construction of many-body wave functions with a definite total angular momentum from a set of single-particle product states has practical value in quantum many-body physics [1–3]. For a few particle systems, the Clebsch-Gordan coefficients, $3j$ symbols or Wigner coefficients, can be used straightforwardly for this purpose. However, with increasing particle numbers, the Clebsch-Gordan couplings become tedious and cumbersome because, with increasing particle numbers, the number of intermediate angular momentum quantum numbers that is required to label different states with the same total angular momentum grows combinatorially. In practical applications, the projection technique of Löwdin has been one of the most popular [1]. This method uses the angular momentum projection operator to project a set of single-particle product states into states with a definite total angular momentum, which requires solution of the eigenvalue problem of the projection operator matrix constructed from the relevant single-particle product states. In [3], Biedenharn and Louck proposed the Wigner operator method, which combines Clebsch-Gordan couplings with results from the theory of the symmetric groups. However, their method can only be worked out explicitly for n nonidentical particles of spin $1/2$. In the case of the nuclear shell model, other procedures are used to construct states with a definite total angular momentum quantum number J. One, called the M scheme, starts with the total quantum number of the angular momentum projection onto the third axis $M = J$ and utilizes a simple subtraction procedure to extract states with a good total angular momentum [4], and another uses direct angular momentum couplings and is usually referred to as the J-coupled scheme for identical particles or the JT-coupled scheme when applied to a proton-neutron system [5]. Alternatively, the projection operator constructed in terms of an integration of the product of the rotational group element and its matrix element (Wigner's D function) of a given angular momentum over the Euler angles can also be used [2], as, for example, in the construction of the Elliott basis [6] of $\mathrm{SU}(3) \supset \mathrm{SO}(3)$ and in projected shell-model calculations [7]. These methods can all be relatively easily implemented in computer codes designed for their respective purposes. Their drawbacks lie in the fact that much CPU time is needed when the dimension of the subspace spanned by the relevant single-particle product states is really large, especially when the projection operator is constructed in terms of an integration of the product of the rotational-group element and its matrix element of a given angular momentum over the Euler angles is used, because the Wheeler-Hill integral involved is difficult to treat accurately in the code.

Recently, it has been shown that the angular momentum projection may be realized by solving a set of Bethe ansatz equations (BAEs) [8,9]. The purpose of this work is to show that the BAEs can be solved relatively easily from zeros of the associated extended Heine-Stieltjes polynomials from the Heine-Stieltjes correspondence [9–14]. In Sec. II, we revisit the Bethe ansatz method for the angular momentum projection. The Heine-Stieltjes correspondence related to the problem, together with properties of the Heine-Stieltjes polynomials and their electrostatic interpretation, is studied in Sec. III. As an example, the application to n nonidentical particles with spin $1/2$ is shown in Sec. IV, which is also related to the eigenvalue problem of the pure pairing interactions among valence nucleon pairs over a set of deformed Nilsson orbits, while applications to systems of identical bosons and fermions are discussed in Sec. V. A brief summary is given in Sec. VI.

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II. THE BETHE ANSATZ METHOD FOR ANGULAR MOMENTUM PROJECTION

Let $\{J^{\alpha}_{\mu};\mu=+,-,0\}$, where $\alpha=1,2,\ldots,n$, be generators of the αth copy of the SU(2) algebra which satisfy the commutation relations

$$[J^{\alpha}_{+},J^{\beta}_{-}]=2\delta_{\alpha\beta}J^{\alpha}_{0},\quad \left[J^{\alpha}_{0},J^{\beta}_{\pm}\right]=\pm\delta_{\alpha\beta}J^{\alpha}_{\pm}, \tag{1}$$

and $|j_\alpha,m_\alpha\rangle$ be the corresponding orthonormal basis vectors with angular momentum quantum number j_α and quantum number m_α of its third component. According to the Bethe ansatz method, one can write an n-coupled state with total angular momentum $J=\sum_\alpha j_\alpha-k$ and $M=J$ as

$$|\zeta;J,M=J)=J_-\left(x^{(\zeta)}_1\right)J_-\left(x^{(\zeta)}_2\right)\ldots J_-\left(x^{(\zeta)}_k\right)|\mathrm{h.w.}\rangle, \tag{2}$$

where $|\mathrm{h.w.}\rangle=\prod_{\alpha=1}^{n}|j_\alpha,m_\alpha=j_\alpha\rangle$ is the SU(2) highest weight state satisfying $J^{\alpha}_{+}|\mathrm{h.w.}\rangle=0$ for any α,

$$J_-\left(x^{(\zeta)}_i\right)=\sum_{\alpha=1}^{n}\frac{1}{x^{(\zeta)}_i-\epsilon_\alpha}J^{\alpha}_{-}, \tag{3}$$

in which the parameters $\{\epsilon_\alpha\}$ can be any set of unequal numbers, and ζ is used to distinguish different n-coupled states with the same angular momentum J. Because (2) is the highest weight state of the angular momentum J, it should satisfy the condition

$$J_+|\zeta;J,M=J)=J_+J_-\left(x^{(\zeta)}_1\right)J_-\left(x^{(\zeta)}_2\right)\ldots J_-\left(x^{(\zeta)}_k\right)|\mathrm{h.w.}\rangle=0, \tag{4}$$

where $J_+=\sum_\alpha J^{\alpha}_{+}$. Using the commutation relations, (1), (4) requires that the BAEs

$$\sum_{\alpha=1}^{n}\frac{2j_\alpha}{x^{(\zeta)}_i-\epsilon_\alpha}-\sum_{t=1(\neq i)}^{k}\frac{2}{x^{(\zeta)}_i-x^{(\zeta)}_t}=0 \tag{5}$$

must be satisfied for $i=1,2,\ldots,k$. It is clear that the multiplicity label $\zeta=1,2,\ldots,d(n,k)$ in (2) is taken to be the label of different solutions $\{x^{(\zeta)}\}$ of Eq. (5). It can be verified [8,9] that the number of solutions $d(n,k)$ of Eq. (5) equals exactly the multiplicity in the reduction $j_1\otimes j_2\otimes\cdots\otimes j_n\downarrow J$, which can be calculated by

$$d(n,k)=\eta(n,k)-\sum_{\mu=0}^{k-1}d(n,\mu), \tag{6}$$

where

$$\eta(n,k)=\sum_{\mu_1=0}^{2j_1}\cdots\sum_{\mu_n=0}^{2j_n}\delta_{q,k}, \tag{7}$$

in which $q=\sum_{i=1}^{n}\mu_i$. From Eqs. (6) and (7), the multiplicity $d(n,k)$ can be calculated recursively from $d(n,0)=1$.

Once the solutions $\{x^{(\zeta)}_1,\ldots,x^{(\zeta)}_k\}$ are obtained from Eq. (5), the n-coupled state with any M can be expressed in the standard way as

$$|\zeta;J,M)=\sqrt{\frac{(J+M)!}{(2J)!(J-M)!}}J_-^{J-M}|\zeta;J,J), \tag{8}$$

where $|\zeta;J,J)$ is given by Eq. (2).

Because the uncoupled basis vectors $\{|j_\alpha,m_\alpha\rangle\}$ are orthonormal, substituting (3) into (2), one can find that the non-normalized angular momentum multicoupling coefficient is given by

$$(j_1,j_1-\mu_1;\ldots;j_n,j_n-\mu_n|\zeta;J,J)=S^{(k)}\left(\beta_1^{\mu_1},\ldots,\beta_n^{\mu_n}\right)\prod_{i=1}^{n}\sqrt{\frac{(2j_i)!\mu_i!}{(2j_i-\mu_n)!}}, \tag{9}$$

where the condition $\sum_\alpha^n\mu_\alpha=k$ must be satisfied, $S^{(k)}(\beta_1^{\mu_1},\ldots,\beta_n^{\mu_n})$ is the $k\times n$-variable symmetric function, in which $\beta_\alpha^{\mu_\alpha}$ is the shorthand notation of taken μ_α variables $\{\beta_{i_1,\alpha},\ldots,\beta_{i_{\mu_\alpha},\alpha}\}$ with $i_1\neq i_2\neq\cdots\neq i_{\mu_\alpha}$ from $\{\beta_{1,\alpha},\ldots,\beta_{k,\alpha}\}$, and

$$\beta_{i,\alpha}=\frac{1}{x^{(\zeta)}_i-\epsilon_\alpha}. \tag{10}$$

When $n=2$ and $k=2$, for example, we have $S^{(2)}(\beta_1^2)=\beta_{11}\beta_{12}$, $S^{(2)}(\beta_1,\beta_2)=\beta_{11}\beta_{22}+\beta_{21}\beta_{12}$, and $S^{(2)}(\beta_2^2)=\beta_{12}\beta_{22}$. The normalized angular momentum multicoupling coefficient is

$$\langle j_1,j_1-\mu_1,\ldots,j_n,j_n-\mu_n|\zeta;J,J\rangle=(j_1,j_1-\mu_1,\ldots,j_n,j_n-\mu_n|\zeta;J,J)/\mathcal{N}, \tag{11}$$

where

$$\mathcal{N}=\left(\sum_{\mu_1\ldots\mu_n}(j_1,j_1-\mu_1,\ldots,j_n,j_n-\mu_n|\zeta;J,J)^2\right)^{\frac{1}{2}}, \tag{12}$$

in which the summation should be restricted by $\sum_\alpha^n\mu_\alpha=k$.

In comparison to the traditional projection methods [1,2,4,5], the Bethe ansatz method for angular momentum projection is more efficient, as it only needs to solve k-coupled algebraic BAEs. However, one must solve the $d(n,k)$-dimensional matrix eigenvalue problem using the traditional projection methods. The dimension $d(n,k)$ increases with increasing n and k in a nonpolynomial way as shown in (6). Therefore, the Bethe ansatz method for angular momentum projection is advantageous if there is a simple way to solve the BAEs, (5).

III. THE HEINE-STIELTJES CORRESPONDENCE

It has been shown that BAEs similar to those shown in (5) may be solved from zeros of the corresponding extended Heine-Stieltjes polynomials [9–14]. Through the Heine-Stieltjes correspondence [9,10], for the BAEs, (5), one may consider the second-order Fuchsian equation

$$A_n(x)y''_k(x)+B_{n-1}(x)y'_k(x)-V_{n-2}(x)y_k(x)=0, \tag{13}$$

where $A_n(x)=\prod_{\alpha=1}^{n}(x-\epsilon_\alpha)$ is a polynomial of degree n, the polynomial $B_{n-1}(x)$ is given as

$$B_{n-1}(x)/A_n(x)=-\sum_{\alpha=1}^{n}\frac{2j_\alpha}{x-\epsilon_\alpha}, \tag{14}$$

and $V_{n-2}(x)$ is called the Van Vleck polynomial [15] of degree $n-2$, which is determined according to Eq. (13). Let

$\{x_i, i = 1, 2, \ldots, k\}$ be zeros of the extended Heine-Stieltjes polynomial $y_k(x)$, which are often called Stieltjes zeros. We may write $y_k(x) = \prod_{i=1}^{k}(x - x_i)$. At any zero x_i of $y_k(x)$, there is the identity

$$\frac{y_k''(x_i)}{y_k'(x_i)} = \sum_{t=1(t\neq i)}^{k} \frac{2}{x_i - x_t}. \tag{15}$$

It is obvious that, at any zero x_i of $y_k(x)$, (13) results in the BAEs, (5). Generally, we also have

$$\frac{y_k''(x)}{y_k(x)} = \sum_{1\leqslant i<t\leqslant k} \frac{2}{(x - x_i)(x - x_t)} = \sum_{1\leqslant i\neq t\leqslant k} \frac{2}{x - x_i}\frac{1}{(x_i - x_t)}, \tag{16}$$

$$\frac{y_k'(x)}{y_k(x)} = \sum_{i=1}^{k} \frac{1}{x - x_i}. \tag{17}$$

Substituting (16) and (17) into (13), we have

$$V_{n-2}^{(\zeta)}(x) = A_n(x)\sum_{i=1}^{k}\frac{1}{x - x_i^{(\zeta)}}\left(\sum_{t\neq i}\frac{2}{x_i^{(\zeta)} - x_t^{(\zeta)}} - \sum_{\alpha}\frac{2j_\alpha}{x - \epsilon_\alpha}\right). \tag{18}$$

By using the BAEs, (5), (18) becomes

$$V_{n-2}^{(\zeta)}(x) = A_n(x)\sum_{\alpha=1}^{n}\frac{1}{x - \epsilon_\alpha}\left(\sum_{i=1}^{k}\frac{2j_\alpha}{x_i^{(\zeta)} - \epsilon_\alpha}\right). \tag{19}$$

Equation (19) shows that zeros $\{\bar{x}_l^{(\zeta)}; l = 1, 2, \ldots, n-2\}$ of the Van Vleck polynomial $V_{n-2}^{(\zeta)}(x)$ related to the ζth extended Heine-Stieltjes polynomial $y_k^{(\zeta)}(x) = \prod_{i=1}^{k}(x - x_i^{(\zeta)})$ are determined by

$$\sum_{\alpha=1}^{n}\frac{1}{\bar{x}_l^{(\zeta)} - \epsilon_\alpha}\left(\sum_{i=1}^{k}\frac{2j_\alpha}{\epsilon_\alpha - x_i^{(\zeta)}}\right) = 0. \tag{20}$$

$\{\bar{x}_l^{(\zeta)}; l = 1, 2, \ldots, n-2\}$ are called Ven Vleck zeros related to the ζth extended Heine-Stieltjes polynomial $y_k^{(\zeta)}(x)$. Once the Van Vleck zeros are obtained from Eq. (20), $V_{n-2}^{(\zeta)}(x)$ can be expressed explicitly as

$$V_{n-2}^{(\zeta)}(x) = c_{n,k}\prod_{l=1}^{n-2}\left(x - \bar{x}_l^{(\zeta)}\right), \tag{21}$$

where $c_{n,k}$ is a constant depending on n, k, and the parameters ϵ_α ($\alpha = 1, \ldots, n$).

If ϵ_α ($\alpha = 1, \ldots, n$) are chosen to be real, according to the Stieltjes results [15], the electrostatic interpretation of the location of zeros of the extended Heine-Stieltjes polynomial $y_k(x)$ may be stated as follows. Put n negative fixed charges $-j_\alpha$ at ϵ_α for $\alpha = 1, \ldots, n$ along a real line, respectively, and allow k positive unit charges to move freely on the two-dimensional complex plane. Therefore, up to a constant, the total energy functional may be written as

$$U(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k}\sum_{\alpha}^{n} j_\alpha \ln|x_i - \epsilon_\alpha| - \sum_{1\leqslant i\neq t\leqslant k}\ln|x_i - x_t|. \tag{22}$$

The BAEs given in Eq. (5) imply that there are $d(n,k)$ different configurations for the position of the k positive charges $\{x_1^{(\zeta)}, \ldots, x_k^{(\zeta)}\}$ with $\zeta = 1, 2, \ldots, d(n,k)$, corresponding to global minima of the total energy.

Similarly, let

$$\rho_\alpha^{(\zeta)}(k) = 2j_\alpha\sum_{i=1}^{k}\frac{1}{\epsilon_\alpha - x_i^{(\zeta)}}, \tag{23}$$

which is now called Van Vleck charges related to the zeros of the ζth extended Heine-Stieltjes polynomial $y_k^{(\zeta)}(x)$. Put n Van Vleck charges $\rho_\alpha^{(\zeta)}(k)$ at positions ϵ_α for $\alpha = 1, \ldots, n$ along a real line, respectively, and allow one unit charge to move freely on the two-dimensional complex plane. Equation (20) provides $n-2$ possible equilibrium positions $\{\bar{x}_l^{(\zeta)}; l = 1, 2, \ldots, n-2\}$ of the unit moving charge for the electrostatic system.

Let

$$\Lambda(x) = \frac{y_k'(x)}{y_k(x)} = \sum_{i=1}^{k}\frac{1}{x - x_i}. \tag{24}$$

As shown in [16], $\Lambda(x)$ satisfies the Riccati-type equation

$$\Lambda'(x) + \Lambda^2(x) + \sum_{i=1}^{k}\sum_{\alpha}^{n}\frac{2j_\alpha}{(x - x_i)(\epsilon_\alpha - x_i)} = 0 \tag{25}$$

in this case. The Van Vleck charges ρ_α can be expressed as

$$\rho_\alpha = 2j_\alpha\Lambda(\epsilon_\alpha). \tag{26}$$

There is a series of high-order differential equations [16] for $\Lambda(\epsilon_\alpha)$. For example, the lowest order one is

$$(1 - 2j_\beta)\Lambda'(\epsilon_\beta) + \Lambda^2(\epsilon_\beta) + \sum_{\alpha\neq\beta}2j_\alpha\frac{\Lambda(\epsilon_\beta) - \Lambda(\epsilon_\alpha)}{\epsilon_\alpha - \epsilon_\beta} = 0. \tag{27}$$

It seems that the solutions of $\Lambda(\epsilon_\alpha)$ of Eq. (27) or from a series of high-order differential equations can be used to determine the Van Vleck zeros and, eventually, solve the BAEs, (5), as shown in [16]. It should be noted that the solutions of $\{\Lambda(\epsilon_\alpha)\}$ from these Riccati-type equations depend only on the parameters $\{\epsilon_\alpha\}$ and $\{j_\alpha\}$, and do not explicitly depend on k and ζ. Therefore, the solutions of $\{\Lambda(\epsilon_\alpha)\}$ from these Riccati-type equations are numerous. One should try to search for a set of solutions $\{\Lambda(\epsilon_\alpha)\}$ corresponding to specific k and ζ from solutions with all possible k and ζ obtained from Eq. (27), which explains why the other expressions of $\{\Lambda(\epsilon_\alpha)\}$ in terms of symmetric functions of $\{x_1, \ldots, x_k\}$ should also be used to solve the problem [16].

In order to avoid the previously mentioned ambiguity, in the following we insist on using the method outlined in [10].

We write

$$y_k(x) = \sum_{j=0}^{k} a_j x^j, \quad V_{n-2}(x) = \sum_{j=0}^{n-2} b_j x^j, \tag{28}$$

where $\{a_j\}$ and $\{b_j\}$ are the expansion coefficients to be determined. Substitution of (28) into (13) yields two matrix equations. By solving these two matrix equations, we can obtain the solutions of $\{a_j\}$ and $\{b_j\}$ for given k. Because there is freedom to choose the parameters $\{\epsilon_\alpha; \alpha = 1, \ldots, n\}$, we find the following parameter settings to be a simple and convenient choice owing to the fact that there is an additional reflection symmetry in (5):

$$\begin{aligned} \epsilon_\alpha &= -(p+1-\alpha) \quad \text{for} \quad \alpha \leqslant p, \\ \epsilon_{\alpha+p} &= \alpha \qquad\qquad\quad\; \text{for} \quad \alpha \geqslant 1 \end{aligned} \tag{29}$$

when $n = 2p$ and

$$\begin{aligned} \epsilon_\alpha &= -(p+1-\alpha) \quad \text{for} \quad \alpha \leqslant p, \\ \epsilon_{p+1} &= 0, \\ \epsilon_{\alpha+p+1} &= \alpha \qquad\qquad\quad\; \text{for} \quad \alpha \geqslant 1 \end{aligned} \tag{30}$$

when $n = 2p+1$. With such a choice, in addition to the S_k permutation symmetry among indices $i = 1, \ldots, k$ of $\{x_1, \ldots, x_k\}$, the Stieltjes zeros $\{x_i\}$ have the following additional reflection symmetries: (i) If $\{x_1, \ldots, x_k\}$ is a set of Stieltjes zeros, $\{-x_1, \ldots, -x_k\}$ is another set. (ii)When n is even, there are many sets of solutions with $\{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-1} = -x_k\}$ when k is even and $\{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-2} = -x_{k-1}, x_k = 0\}$ when k is odd. When n is odd, there are many sets of solutions with $\{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-1} = -x_k\}$ when k is even. Solutions satisfying property (ii) are self-reflectional. Property (i) is strong, namely, such pairs of solutions always exist, which is obvious with the substitutions of x_i with $-x_i$ in Eq. (5) for $i = 1, \ldots, k$. However, property (ii) only applies to a subset of solutions, namely, there are other sets of solutions which may not follow property (ii). One can verify that the substitution of $\{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-1} = -x_k\}$ for k even or $\{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-2} = -x_{k-1}, x_k = 0\}$ for k odd into Eq. (5) indeed yields k consistent equations when n is even, which implies that $\{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-1} = -x_k\}$ for k even and $\{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-2} = -x_{k-1}, x_k = 0\}$ for k odd are possible solutions when n is even. For odd-n cases, self-reflectional solutions only exist when k is even. Because the parameters chosen satisfy the interlacing condition $\epsilon_1 < \cdots < \epsilon_n$, zeros of $y_k(x)$ may be arranged to satisfy the interlacing condition, $\mathbf{Re}(x_1) \leqslant \mathbf{Re}(x_2) \leqslant \cdots \leqslant \mathbf{Re}(x_k)$, where $\mathbf{Re}(x_i)$ lies in one of the $n-1$ intervals $(\epsilon_1, \epsilon_2), \ldots, (\epsilon_{n-1}, \epsilon_n)$, in which the equality is only possible when the adjacent zeros are complex conjugate with each other. When two zeros are conjugate with each other with $x_i = x_{i+1}^*$, it is obvious that $\mathbf{Re}(x_i)$ and $\mathbf{Re}(x_{i+1})$ are in the same interval $(\epsilon_\alpha, \epsilon_{\alpha+1})$. The number of different such allowed configurations gives the possible solutions of $y_k(x)$ and the corresponding $V_{n-2}(x)$. Therefore, these properties are very helpful to simplify the problem and to search for solutions of (5).

IV. APPLICATION TO SYSTEMS WITH NONIDENTICAL SPIN-1/2 PARTICLES

Generally, the Bethe ansatz method for angular momentum projection with the Heine-Stieltjes correspondence shown in previous sections can be applied to construct a state with a definite angular momentum from a set of uncoupled single-particle states of both nonidentical- and identical-particle systems. Because identical-particle systems have additional permutation symmetries, namely, symmetry among identical bosons or antisymmetry among identical fermions with respect to the single-particle coordinate permutations, the procedure outlined in previous sections can be simplified. Such simplifications and applications are reported in the next section. In this section, we focus only on a nonidentical-particle case, in which we strictly follow the method described previously because no further simplification can be made for nonidentical-particle systems.

As the simplest but nontrivial example, we consider n nonidentical particles of spin 1/2, which was previously studied by Louck and Biedenharn using the pattern calculus with the Yamanocchi symbol of an irrep of S_n as the upper pattern used to label the outer multiplicity of SU(2) $\times \cdots \times$ SU(2) $\downarrow$ SU(2), and the SU(2) basis of the same irrep as the lower pattern [3,17]. This is the only case that can be solved analytically using the Wigner operator method. However, as shown in [3], the construction of a coupled state with a definite angular momentum for nonidentical particles of arbitrary spin can never be expressed analytically using the pattern calculus, principally because of unsolved problems relating to the upper patterns. Specifically, using the pattern calculus, the n-coupled state with total angular momentum J of a spin-1/2 system may be written as [3,17]

$$|(i_1 \ldots i_n); JM\rangle = \sum_{k_1 \ldots k_n} \left\langle \begin{matrix} 2J\,0 \\ J+M \end{matrix} \middle| \left\langle \begin{matrix} i_1 \\ 1\,0 \\ k_n \end{matrix} \right\rangle \cdots \left\langle \begin{matrix} i_n \\ 1\,0 \\ k_1 \end{matrix} \right\rangle \middle| \begin{matrix} 0\,0 \\ 0 \end{matrix} \right\rangle \times \prod_{i=1}^{n} \left| \frac{1}{2}, k_i - \frac{1}{2} \right\rangle, \tag{31}$$

where $(i_1 \ldots i_n)$ with $i_s = 0$ or 1 for $s = 1, \ldots, n$ is used as the multiplicity label, the sum should be restricted with $k_i = 0$ or 1 for $i = 1, \ldots, n$, and the expansion coefficient

$$\left\langle \begin{matrix} 2J\,0 \\ J+M \end{matrix} \middle| \left\langle \begin{matrix} i_1 \\ 1\,0 \\ k_n \end{matrix} \right\rangle \cdots \left\langle \begin{matrix} i_n \\ 1\,0 \\ k_1 \end{matrix} \right\rangle \middle| \begin{matrix} 0\,0 \\ 0 \end{matrix} \right\rangle \tag{32}$$

should be calculated consecutively with

$$\left\langle \begin{matrix} 1 \\ 1\,0 \\ 1 \end{matrix} \right\rangle \left| \begin{matrix} 2j\;0 \\ j+m \end{matrix} \right\rangle = \left(\frac{j+m+1}{2j+1} \right)^{1/2} \left| \begin{matrix} 2j+1\,0 \\ j+m+1 \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} 1 \\ 1\,0 \\ 0 \end{matrix} \right\rangle \left| \begin{matrix} 2j\;0 \\ j+m \end{matrix} \right\rangle = \left(\frac{j-m+1}{2j+1} \right)^{1/2} \left| \begin{matrix} 2j+1\,0 \\ j+m \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} 0 \\ 1\,0 \\ 1 \end{matrix} \right| \left. \begin{matrix} 2j\,0 \\ j+m \end{matrix} \right\rangle = -\left(\frac{j-m}{2j+1}\right)^{1/2} \left| \begin{matrix} 2j-1\,0 \\ j+m \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} 0 \\ 1\,0 \\ 0 \end{matrix} \right| \left. \begin{matrix} 2j\,0 \\ j+m \end{matrix} \right\rangle = \left(\frac{j+m}{2j+1}\right)^{1/2} \left| \begin{matrix} 2j-1\,0 \\ j+m-1 \end{matrix} \right\rangle. \quad (33)$$

Though the expression of the expansion coefficients shown by (32) is analytic, evaluation of (32) according to the rules shown in (33) is still cumbersome; especially, the coefficients for many permissible upper patterns $(i_1 \ldots i_n)$ that may lie in the null space are zero, which, however, cannot be ruled out beforehand. This is the main drawback in using the upper pattern to resolve the outer multiplicity problem of unitary groups [18]. In contrast, roots of the BAEs, (5), provide all possible coupled states with the same angular momentum J as shown by (2) and (3), which are mutually orthogonal with respect to the multiplicity label ζ. Solutions of (5) can be obtained from Eq. (13) with the explicit expressions shown in (28). Moreover, the new angular momentum projection method outlined in Sec. II is not restricted to systems consisting of particles with the same spin but can be applied to systems consisting of particles with arbitrary spins.

The above example is closely related to the construction of eigenstates of the pure pairing Hamiltonian in the deformed Nilsson basis with

$$\hat{H}_{\rm S} = -GS^+S^-, \quad (34)$$

where $S^+ = \sum_\mu S_\mu^+ = \sum_\mu a_{\mu\uparrow}^\dagger a_{\mu\downarrow}^\dagger$ and $S^- = (S^+)^\dagger$, in which $S_\mu^+ = a_{\mu\uparrow}^\dagger a_{\mu\downarrow}^\dagger$ ($S_\mu^- = a_{\mu\downarrow} a_{\mu\uparrow}$) are pair creation (annihilation) operators. The up and down arrows in these expressions refer to time-reversed states. For simplicity, we only consider the seniority zero cases. The eigenstates of (34) can be constructed in the following way [19]: Because each Nilsson level can be occupied by at most one pair owing to the Pauli principle, the local states can be regarded as quasi-spin-1/2 states. $|\frac{1}{2}, \frac{1}{2}\rangle$ stands for a one-pair state, while $|\frac{1}{2}, -\frac{1}{2}\rangle$ stands for a no-pair state. Then, similarly to (2), any allowed total quasi-spin S and $M_S = S$ state of p pairs over n Nilsson levels can be written as

$$|\zeta; S, M_S = S\rangle = \mathcal{N} S^-\big(x_1^{(\zeta)}\big) \ldots S^-\big(x_t^{(\zeta)}\big)|\text{h.w.}\rangle \quad (35)$$

with $p = n - t$ pairs, where $\mathcal{N}$ is the normalization constant defined by (12), $S = n/2 - t$, $|\text{h.w.}\rangle \equiv |\frac{1}{2}, \frac{1}{2}; \ldots; \frac{1}{2}, \frac{1}{2}\rangle$ is the product of n copies of the local state with the highest weight of quasispin 1/2, and

$$S^-\big(x_i^{(\zeta)}\big) = \sum_{\mu=1}^{n} \frac{1}{x_i^{(\zeta)} - \epsilon_\mu} S_\mu^-, \quad (36)$$

in which the parameters $\{\epsilon_\mu\}$ can be any set of unequal numbers, and ζ is used to distinguish different n-coupled states with the same quasispin S. The variables $\{x_1^{(\zeta)}, \ldots, x_t^{(\zeta)}\}$ should satisfy

$$\sum_{\mu=1}^{n} \frac{1}{x_i^{(\zeta)} - \epsilon_\mu} - \sum_{l=1(\neq i)}^{t} \frac{2}{x_i^{(\zeta)} - x_l^{(\zeta)}} = 0 \quad (37)$$

for $i = 1, 2, \ldots, t$. It is clear that the multiplicity label $\zeta = 1, 2, \ldots, d(n,t)$ in (37) is taken to be the label of different solutions $\{x^{(\zeta)}\}$ of Eq. (37). It can be verified that the number of solutions $d(n,t)$ of Eq. (37) equals exactly the multiplicity in the reduction $j_1 \otimes j_2 \otimes \cdots \otimes j_n \downarrow J$ with $j_l = \frac{1}{2}$ for $1 \leqslant l \leqslant n$, which can be calculated from Eq. (6) with

$$\eta(n,t) = \sum_{\mu_1=0}^{1} \cdots \sum_{\mu_n=0}^{1} \delta_{q,t} \quad (38)$$

for this case, in which $q = \sum_{i=1}^{n} \mu_i$. From Eqs. (6) and (38), the multiplicity $d(n,t)$ can be calculated recursively with $d(n,0) = 1$, which indicates that there are $d(n,t)$ different states with the same quasispin $S = n/2 - t$. For this case, there is a closed form of $d(n,t)$ with

$$d(n,t) = \frac{(1+n-2t)n!}{(1+n-t)(n-t)!t!}, \quad (39)$$

which equals exactly the dimension of the irrep $[n-t, t]$ of the permutation group S_n [3,19] and is consistent with the result obtained from Eqs. (6) and (38). In this case, $c_{n,k}$ in the Van Vleck polynomials, (21), can be obtained in solving the corresponding Fuchsian equation (13) with

$$c_{n,k} = -(n-k+1)k. \quad (40)$$

Finally, the state with quasispin S and any M_S can be expressed as

$$|\zeta; S, M_S\rangle = \sqrt{\frac{(S+M_S)!}{(2S)!(S-M_S)!}}(S^-)^{S-M_S}|\zeta; S, S\rangle \quad (41)$$

for a system with $p = n/2 + M_S$ pairs.

In order to demonstrate the method and properties of the zeros outlined in previous sections, in the following we display results of the method for relatively simple cases with $n = 8$, $t = 1, 4$ and $n = 7$, $t = 2, 3$ as examples of even- and odd-n cases, respectively. The multiplicities $d(8,t)$ with $0 \leqslant t \leqslant 4$ and $d(7,t)$ with $0 \leqslant t \leqslant 3$ are listed in Table I. With parameters $\{\epsilon_\alpha\}$ chosen according to (29) and (30), we find that there are exactly $d(n,t)$ different solutions for given n and t as listed in Tables II–V. For any case, it can be verified that any zero $x_i^{(\zeta)}$ of $y_t^{(\zeta)}(x)$ indeed lies in one of the $n-1$ intervals $(\epsilon_1, \epsilon_2) \ldots (\epsilon_{n-1}, \epsilon_n)$. It is obvious that $y_1^{(1)}(x)$ in Table II, $y_4^{(1)}(x), \ldots, y_4^{(6)}(x)$ in Table III, and $y_2^{(1)}(x)$, $y_2^{(2)}(x)$ in Table IV are self-reflectional, while the solutions in most cases satisfy reflection symmetry property (i). For example, $y_4^{(7)}(x) = y_4^{(8)}(-x)$, $y_4^{(9)}(x) = y_4^{(10)}(-x)$, $y_4^{(11)}(x) = y_4^{(12)}(-x)$,

TABLE I. Multiplicity $d(8,t)$ for $S = 4 - t$ and $d(7,t)$ for $S = \frac{7}{2} - t$.

t	$S = 4 - t$	$d(8,t)$	$S = \frac{7}{2} - t$	$d(7,t)$
0	4	1	$\frac{7}{2}$	1
1	3	7	$\frac{5}{2}$	6
2	2	20	$\frac{3}{2}$	14
3	1	28	$\frac{1}{2}$	14
4	0	14		

TABLE II. Extended Heine-Stieltjes polynomials $y_1^{(\zeta)}(x)$ for constructing $S=3$ states with $n=8$ and $t=1$ according to (35) and corresponding Van Vleck polynomials $V_6^{(\zeta)}(x)$.

	Extended Heine-Stieltjes polynomial $y_1^{(\zeta)}(x)$	Van Vleck polynomial $V_6^{(\zeta)}(x)$
$\zeta=1$	x	$-8(x-3.679)(x-2.59)(x-1.502)(x+1.502)(x+2.59)(x+3.679)$
$\zeta=2$	$x+3.679$	$-8(x-3.679)(x-1.502)(x+1.502)(x+2.59)(x-2.59)x$
$\zeta=3$	$x-3.679$	$-8(x+3.679)(x+1.502)(x-1.502)(x-2.59)(x+2.59)x$
$\zeta=4$	$x-2.59$	$-8(x-3.679)(x+3.679)(x-1.502)(x+1.502)(x+2.59)x$
$\zeta=5$	$x+2.59$	$-8(x+3.679)(x-3.679)(x+1.502)(x-1.502)(x-2.59)x$
$\zeta=6$	$x+1.502$	$-8(x-3.679)(x-2.59)(x-1.502)(x+2.59)(x+3.679)x$
$\zeta=7$	$x-1.502$	$-8(x+3.679)(x+2.59)(x+1.502)(x-2.59)(x-3.679)x$

and $y_4^{(13)}(x)=y_4^{(14)}(-x)$ when $n=8$ and $t=4$. The Van Vleck polynomial satisfies the same reflection property as that of the corresponding extended Heine-Stieltjes polynomial. In addition, one can verify that the Van Vleck zeros of $V_{n-2}^{(\zeta)}(x)$ indeed satisfy Eq. (20). With Stieltjes zeros $\{x_i\}$ of $y_t^{(\zeta)}(x)$ obtained from Tables II–V, one can verify that the eigenstates, (41), are mutually orthogonal with respect to the multiplicity label ζ:

$$\langle\zeta;S,M_S|\zeta';S',M_S'\rangle=\delta_{\zeta,\zeta'}\delta_{S,S'}\delta_{M_S,M_S'}. \tag{42}$$

Once the eigenstates, (41), of (34) are obtained, the results can be used for constructing eigenstates and calculating eigenvalues of any mean-field-plus-pairing model by using the progressive diagonalization scheme as shown [19]. Furthermore, Eqs. (5) and (37) can be regarded as the same BAEs [20] in determining solutions of the mean-field plus-pairing model in the strong-pairing interaction $G\to\infty$ limit by replacing the parameters $\{\epsilon_\alpha\}$ with $\{2\varepsilon_\alpha\}$, where $\{\varepsilon_\alpha\}$ are single-particle energies in the corresponding orbits of the mean field [9].

V. APPLICATION TO IDENTICAL-PARTICLE SYSTEMS

Classification and construction of identical-particle states for a given angular momentum quantum number are fundamental, especially in nuclear structure theory. n-coupled states of l bosons can be constructed as the basis vectors of symmetric irreducible representations of $U(2l+1)\supset O(2l+1)\supset O(3)$ as shown in [21–23], while those of j fermions can be constructed as the basis vectors of antisymmetric irreducible

TABLE III. Extended Heine-Stieltjes polynomials $y_4^{(\zeta)}(x)$ for constructing $S=0$ states with $n=8$ and $t=4$ according to (35) and corresponding Van Vleck polynomials $V_6^{(\zeta)}(x)$.

	Extended Heine-Stieltjes polynomial $y_4^{(\zeta)}(x)$	Van Vleck polynomial $V_6^{(\zeta)}(x)$
$\zeta=1$	$(x^2+0.379415)(x^2+10.53874)$	$-20(x^2-13.0977)(x^2-6.2593)(x^2-1.9184)$
$\zeta=2$	$(x^2-12.56879)(x^2+0.57829)$	$-20(x^2-2.3151)(x^2-4.9901x+6.4066)(x^2+4.9901x+6.4066)$
$\zeta=3$	$(x^2+0.8145)(x^2-5.7144)$	$-20(x^2-12.8761)(x^2-2.8274x+2.1700)(x^2+2.8274x+2.1700)$
$\zeta=4$	$(x^2-2.23204)(x^2-12.3149)$	$-20(x^2+0.5558)(x^2-5.3854x+7.4056)(x^2+5.3854x+7.4056)$
$\zeta=5$	$(x^2+4.82433)(x^2-1.80406)$	$-20(x^2-13.0201)(x^2-6.1361)(x^2+0.2502)$
$\zeta=6$	$(x^2-5.07233x+6.61367)(x^2+5.07233x+6.61367)$	$-20(x^2-12.5294)(x^2-2.2932)(x^2+0.5729)$
$\zeta=7$	$(x^2-1.2051x-8.3519)(x^2+1.2051x+0.8179)$	$-20(x^2+2.0270x-5.5609)(x^2-4.8900x+6.1511)$ $\times(x^2+2.8630x+2.2150)$
$\zeta=8$	$(x^2+1.2051x-8.3519)(x^2-1.2051x+0.8179)$	$-20(x^2-2.0270x-5.5609)(x^2+4.8900x+6.1511)$ $\times(x^2-2.8630x+2.2150)$
$\zeta=9$	$(x^2-5.01525x+6.47201)(x^2+5.01525x+5.28038)$	$-20(x^2+5.0460x+5.3283)(x^2+0.3380x+0.5727)$ $\times(x^2-5.3840x+7.4014)$
$\zeta=10$	$(x^2+5.01525x+6.47201)(x^2-5.01525x+5.28038)$	$-20(x^2-5.0460x+5.3283)(x^2-0.3380x+0.5727)$ $\times(x^2+5.3840x+7.4014)$
$\zeta=11$	$(x^2+2.162707x+3.56055)(x^2-2.162707x-5.00227)$	$-20(x^2+6.051x+8.8268)(x^2-1.3451x-0.3943)$ $\times(x^2-4.7057x+5.6852)$
$\zeta=12$	$(x^2-2.162707x+3.56055)(x^2+2.162707x-5.00227)$	$-20(x^2-6.051x+8.8268)(x^2+1.3451x-0.3943)$ $\times(x^2+4.7057x+5.6852)$
$\zeta=13$	$(x^2+1.0751x+2.9229)(x^2-1.0751x-3.2612)$	$-20(x^2-3.3149x-1.0537)(x^2-2.7178x+2.1494)$ $\times(x^2+6.0327x+8.7374)$
$\zeta=14$	$(x^2-1.0751x+2.9229)(x^2+1.0751x-3.2612)$	$-20(x^2+3.3149x-1.0537)(x^2+2.7178x+2.1494)$ $\times(x^2-6.0327x+8.7374)$

TABLE IV. Extended Heine-Stieltjes polynomials $y_2^{(\zeta)}(x)$ for constructing $S = 3/2$ states with $n = 7$ and $t = 2$ according to (35) and corresponding Van Vleck polynomials $V_5^{(\zeta)}(x)$.

	Extended Heine-Stieltjes polynomial $y_2^{(\zeta)}(x)$	Van Vleck polynomial $V_5^{(\zeta)}(x)$
$\zeta = 1$	$(x - 2.646)(x + 2.646)$	$-12(x - 1.5275)(x + 1.5275)x^3$
$\zeta = 2$	$(x - 1.5275)(x + 1.5275)$	$-12(x - 2.646)(x + 2.646)x^3$
$\zeta = 3$	$x^2 - 1.270x + 0.5564$	$-12(x - 2.63447)(x - 1.4683x)(x + 0.430964)(x + 1.54863)(x + 2.65236x)$
$\zeta = 4$	$x^2 + 1.270x + 0.5564$	$-12(x + 2.63447)(x + 1.4683x)(x - 0.430964)(x - 1.54863)(x - 2.65236x)$
$\zeta = 5$	$(x - 2.582)(x - 0.6325)$	$-12(x + 0.4792)(x + 1.5636)(x + 2.6585)(x^2 - 3.3621x + 2.9644)$
$\zeta = 6$	$(x + 2.582)(x + 0.6325)$	$-12(x - 0.4792)(x - 1.5636)(x - 2.6585)(x^2 + 3.3621x + 2.9644)$
$\zeta = 7$	$(x - 2.620)(x + 0.4611)$	$-12(x - 0.779279)(x + 1.5569)(x + 2.6556)(x^2 - 2.5336x + 1.66393)$
$\zeta = 8$	$(x + 2.620)(x - 0.4611)$	$-12(x + 0.779279)(x - 1.5569)(x - 2.6556)(x^2 + 2.5336x + 1.66393)$
$\zeta = 9$	$(x - 1.484)(x + 0.4196)$	$-12(x - 2.63681)(x + 1.5462)(x + 2.6515)(x^2 - 1.1171x + 0.4745)$
$\zeta = 10$	$(x + 1.484)(x - 0.4196)$	$-12(x + 2.63681)(x - 1.5462)(x - 2.6515)(x^2 + 1.1171x + 0.4745)$
$\zeta = 11$	$x^2 - 3.831x + 3.719$	$-12(x - 2.5078)(x - 0.6085)(x + 0.4859)(x + 1.5666)(x + 2.6598)$
$\zeta = 12$	$x^2 + 3.831x + 3.719$	$-12(x + 2.5078)(x + 0.6085)(x - 0.4859)(x - 1.5666)(x - 2.6598)$
$\zeta = 13$	$(x - 2.637)(x + 1.546)$	$-12(x - 1.4827)(x + 0.4211)(x + 2.6516)(x^2 - 1.1358x + 0.4844)$
$\zeta = 14$	$(x + 2.637)(x - 1.546)$	$-12(x + 1.4827)(x - 0.4211)(x - 2.6516)(x^2 + 1.1358x + 0.4844)$

representations of $U(2j+1) \supset Sp(2j+1) \supset O(3)$ as shown in [24] and [25]. The Bethe ansatz method for angular momentum projection with the Heine-Stieltjes correspondence shown in previous sections can also be applied to construct states with definite angular momentum from a set of uncoupled single-particle product states for identical-particle systems, which can be done as follows: First, we solve the BAEs, (5), for nonidentical particle systems with the same spin and then construct the coupled state, (2). Once the coupled state, (2), is expanded in terms of single-particle product states, we take all particles to be identical, which is called assimilation. For identical-fermion systems, the Pauli principle forbidden single-particle product states will be automatically ruled out after the assimilation. Because of the additional permutation symmetry with respect to the single-particle coordinate permutations, the procedure outlined in previous sections can be simplified. In this section, we show how the the procedure is carried out.

A. Identical bosons

Let the single-particle states of boson with angular momentum l be $|l, m\rangle \equiv |m\rangle$ with $m = -l, -l+1, \ldots, l$. According to (2), the n-coupled state with total angular momentum $L = nl - k$ and $M_L = L$

$$|\zeta; L, M_L = L) = L_-\big(x_1^{(\zeta)}\big) \ldots L_-\big(x_k^{(\zeta)}\big)|\text{h.w.}\rangle, \tag{43}$$

where $|\text{h.w.}\rangle = \prod_{\alpha=1}^{n} |m_\alpha = l\rangle$ is the highest weight state,

$$L_-\big(x_i^{(\zeta)}\big) = \sum_{\alpha=1}^{n} \frac{1}{x_i^{(\zeta)} - \epsilon_\alpha} L_-^\alpha, \tag{44}$$

TABLE V. Extended Heine-Stieltjes polynomials $y_3^{(\zeta)}(x)$ for constructing $S = 1/2$ states with $n = 7$ and $t = 3$ according to (35) and corresponding Van Vleck polynomials $V_5^{(\zeta)}(x)$.

	Extended Heine-Stieltjes polynomial $y_3^{(\zeta)}(x)$	Van Vleck polynomial $V_5^{(\zeta)}(x)$
$\zeta = 1$	$(x - 0.6354)(x^2 - 2.020x + 2.296)$	$-15(x - 2.581)(x - 1.432)(x + 0.4002)(x + 1.514)(x + 2.630)$
$\zeta = 2$	$(x + 0.6354)(x^2 + 2.020x + 2.296)$	$-15(x + 2.581)(x + 1.432)(x - 0.4002)(x - 1.514)(x - 2.630)$
$\zeta = 3$	$(x - 2.542)(x^2 + 0.5857x + 0.2296)$	$-15(x - 0.4874)(x + 1.473)(x + 2.619)(x^2 - 3.213x + 2.754)$
$\zeta = 4$	$(x + 2.542)(x^2 - 0.5857x + 0.2296)$	$-15(x + 0.4874)(x - 1.473)(x - 2.619)(x^2 + 3.213x + 2.754)$
$\zeta = 5$	$(x - 2.582)(x + 0.5705)(x + 2.540)$	$-15(x - 0.6329)(x^2 - 2.867x + 2.204)(x^2 + 3.394x + 3.026)$
$\zeta = 6$	$(x + 2.582)(x - 0.5705)(x - 2.540)$	$-15(x + 0.6329)(x^2 + 2.867x + 2.204)(x^2 - 3.394x + 3.026)$
$\zeta = 7$	$(x - 0.3106)(x^2 + 2.634x + 1.912)$	$-15(x - 2.625)(x - 1.498)(x + 2.562)(x^2 + 1.097x + 0.4387)$
$\zeta = 8$	$(x + 0.3106)(x^2 - 2.634x + 1.912)$	$-15(x + 2.625)(x + 1.498)(x - 2.562)(x^2 - 1.097x + 0.4387)$
$\zeta = 9$	$(x - 1.434)(x + 0.5286)(x + 2.529)$	$-15(x - 2.612)(x^2 - 1.116x + 0.4798)(x^2 + 3.403x + 3.044)$
$\zeta = 10$	$(x + 1.434)(x - 0.5286)(x - 2.529)$	$-15(x + 2.612)(x^2 + 1.116x + 0.4798)(x^2 - 3.403x + 3.044)$
$\zeta = 11$	$(x - 2.593)(x^2 + 3.468x + 3.144)$	$-15(x - 0.7007)(x + 0.5593)(x + 2.528)(x^2 - 2.562x + 1.738)$
$\zeta = 12$	$(x + 2.593)(x^2 - 3.468x + 3.144)$	$-15(x + 0.7007)(x - 0.5593)(x - 2.528)(x^2 + 2.562x + 1.738)$
$\zeta = 13$	$(x - 1.461)(x^2 + 3.299x + 2.884)$	$-15(x - 2.616)(x + 0.5086)(x + 2.537)(x^2 - 0.7961x + 0.3266)$
$\zeta = 14$	$(x + 1.461)(x^2 - 3.299x + 2.884)$	$-15(x + 2.616)(x - 0.5086)(x - 2.537)(x^2 + 0.7961x + 0.3266)$

in which the parameters $\{\epsilon_\alpha\}$ can usually be any set of unequal numbers, and L_-^α is the angular momentum lowering operator acting only on the αth copy of the single-particle state, and $L_+ = \sum_\alpha L_+^\alpha$, similar to the nonidentical particle case. The corresponding BAE is

$$\sum_{\alpha=1}^{n} \frac{2l}{x_i^{(\zeta)} - \epsilon_\alpha} - \sum_{t=1(\neq i)}^{k} \frac{2}{x_i^{(\zeta)} - x_t^{(\zeta)}} = 0 \tag{45}$$

for $i = 1, 2, \ldots, k$. Upon substituting the solutions $\{x_i\}$ of (45) into (43), (43) gives the final result after assimilation.

It can be easily proven that the n-coupled state with $L = ln - 1$ is 0. Because

$$\begin{aligned} L_-^\alpha|\text{h.w.}\rangle &= \sqrt{2l} \prod_{\beta=1(\neq\alpha)}^{n} |m_\beta = l\rangle|m_\alpha = l-1\rangle \\ &= \sqrt{2l} \prod_{\beta=1}^{n-1} |m_\beta = l\rangle|m_n = l-1\rangle, \end{aligned} \tag{46}$$

owing to the fact that these bosons are identical, (43) becomes

$$\begin{aligned} |\zeta; L = M_L = nl - 1) &= L_-(x^{(\zeta)})|\text{h.w.}\rangle \\ &= \sqrt{2l} \sum_{\alpha=1}^{n} \frac{2l}{x^{(\zeta)} - \epsilon_\alpha} \prod_{\beta=1}^{n-1} |m_\beta = l\rangle|m_n = l-1\rangle, \end{aligned} \tag{47}$$

which is 0 because

$$\sum_{\alpha=1}^{n} \frac{2l}{x^{(\zeta)} - \epsilon_\alpha} = 0 \tag{48}$$

according to Eq. (45) when $k = 1$.

When $k \geqslant 2$, the number of states, (43), with $L = 2l - k$ may be calculated in the following way: Let $P_n(k)$ be the number of different n partitions of the integer k with $k = \sum_{i=1}^{n} \xi_i$, where $2l \geqslant \xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_n \geqslant 0$. Then the number of linearly independent states shown in (43) for l bosons $D_\text{B}(n,k) = P_n(k) - P_n(k-1)$, which gives the multiplicity of a given $L = nl - k$ in the reduction $\text{U}(2l+1) \downarrow \text{O}(3)$ for the symmetric irreducible representation $[n, \dot{0}]$ of $\text{U}(2l+1)$. Generally, $D_\text{B}(n,k)$ is far less than $d(n,k)$ shown in (6) for nonidentical particles. Therefore, for given L, n-coupled states, (43), obtained from solutions of (45) are overcomplete for identical-particle systems when $k \geqslant 2$. Actually, states, (43), obtained from different solutions of (45), up to a normalization constant, are all the same when $D_\text{B}(n,k) = 1$. When $D_\text{B}(n,k) \geqslant 2$, the solutions, (43), are not orthogonal with respect to the multiplicity label, and many solutions of (43) can be expressed by a linear combination of other solutions of (43).

Simplification can be made to overcome such complexity, mainly because there is a freedom to choose the parameters $\{\epsilon_\alpha\}$ in (44). When $D_\text{B}(n,k) = 1$, we set

$$\epsilon_\alpha = -1 \quad \text{for} \quad \alpha \leqslant p, \quad \epsilon_{\alpha+p} = 1 \quad \text{for} \quad \alpha \geqslant 1 \tag{49}$$

when $n = 2p$ and

$$\epsilon_\alpha = -1 \quad \text{for} \quad \alpha \leqslant p+1, \quad \epsilon_{\alpha+p+1} = 1 \quad \text{for} \quad \alpha \geqslant 1 \tag{50}$$

when $n = 2p+1$. With this choice, Eq. (45) becomes

$$\frac{2l(p+r)}{x_i + 1} + \frac{2lp}{x_i - 1} - \sum_{t=1(\neq i)}^{k} \frac{2}{x_i - x_t} = 0 \tag{51}$$

for $i = 1, 2, \ldots, k$, where $r = 0$ when $n = 2p$ and $r = 1$ when $n = 2p+1$, which are exactly the Niven equations for zeros of the Jacobi polynomial $P_k^{[-2lp-1,-2l(p+r)-1]}(x)$. There is only one set of zeros of (51) which is sufficient for (43) when $D_\text{B}(n,k) = 1$. Therefore, (43) with zeros of the Jacobi polynomial $P_k^{[-2lp-1,-2l(p+r)-1]}(x)$ are n-coupled states with $L = nl - k$ when the parameters $\{\epsilon_\alpha\}$ are chosen according to (49) or (50) when $D_\text{B}(n,k) = 1$.

For example, there is only one state with $L = 6$ for $n = 4$ d bosons ($l = 2$). According to (49), we set $\{\epsilon_1 = \epsilon_2 = -1, \epsilon_3 = \epsilon_4 = 1\}$. Substituting two zeros $\{x_1 = -0.2582\imath, x_2 = 0.2582\imath\}$ of the Jacobi polynomial $P_2^{[-7,-7]}(x)$ into (43), we get

$$|L = M_L = 6\rangle = -0.5222|2,2,1,1\rangle + 0.8528|2,2,2,0\rangle \tag{52}$$

after assimilation and normalization.

When $D_\text{B}(n,k) \geqslant 2$, we have many ways to set the parameters $\{\epsilon_\alpha\}$. The simplest way is to choose the two-value parametrization with $\epsilon_{\alpha_1} = \epsilon_{\alpha_2} = \cdots = \epsilon_{\alpha_r} = -1$ and the rest parameters $\epsilon_\beta = 1$ when $\beta \neq \alpha_i$ for $i = 1, 2, \ldots, r$. Obviously, there are $2^n - 2$ different ways to do such a parametrization, from which one can choose $D_\text{B}(n,k)$ of them. Zeros of the corresponding Jacobi polynomial can be used to obtain the final results from (43). It seems that $D_\text{B}(n,k) \leqslant 2^n - 2$ is always satisfied for $n \geqslant 2$, though we are unable to prove this inequality in general. Therefore, the above two-value parametrization seems sufficient to resolve the multiplicity.

For example, there are two coupled states of $n = 4$ d bosons with $L = 4$. One can set $\{\epsilon_1 = \epsilon_2 = -1, \epsilon_3 = \epsilon_4 = 1\}$ for one solution with

$$\begin{aligned} |\zeta = 1, L = M_L = 4\rangle &= 0.2208|1,1,1,1\rangle \\ &- 0.7211|2,1,1,0\rangle + 0.6403|2,2,0,0\rangle \\ &- 0.1030|2,2,1,-1\rangle + 0.1030|2,2,2,-2\rangle \end{aligned} \tag{53}$$

and set $\{\epsilon_1 = -1, \epsilon_2 = \epsilon_3 = \epsilon_4 = 1\}$ for another solution with

$$\begin{aligned} |\zeta = 2, L = M_L = 4\rangle &= 0.0827|1,1,1,1\rangle \\ &- 0.2702|2,1,1,0\rangle - 0.1161|2,2,0,0\rangle \\ &+ 0.6733|2,2,1,-1\rangle - 0.6733|2,2,2,-2\rangle. \end{aligned} \tag{54}$$

In this case, the final coupled states (53) and (54) are not orthogonal with respect to the multiplicity label ζ, namely, $\langle\zeta = 1|\zeta = 2\rangle \neq 0$. In order to orthonormalize them, the Gram-Schimidt process may be adopted.

More complicated parametrizations are always possible. For example, we can also set

$$\begin{aligned} &\epsilon_\alpha = -1 \quad \text{for} \quad \alpha \leqslant p, \\ &\epsilon_{p+1} = 0, \quad \epsilon_{\beta+p+1} = 1 \quad \text{for} \quad \beta \geqslant 1, \end{aligned} \tag{55}$$

TABLE VI. Extended Heine-Stieltjes polynomials $y_4^{(\zeta)}(x)$ for $n=4$ d bosons coupled to $L=M_L=4$ with $\{\epsilon_1=-1, \epsilon_2=0, \epsilon_3=\epsilon_4=1\}$.

	Extended Heine-Stieltjes polynomial $y_4^{(\zeta)}(x)$
$\zeta=1$	$(0.1382-0.7053x+x^2)(0.2800-0.6172x+x^2)$
$\zeta=2$	$(0.6106+1.5493x+x^2)(0.9545+1.8343x+x^2)$
$\zeta=3$	$(-0.2918+x)(0.4284+x)(0.1284-0.4857x+x^2)$
$\zeta=4$	$(0.0876-0.5298x+x^2)(0.3138+1.0483x+x^2)$
$\zeta=5$	$(-0.2874+x)(0.6446+x)(0.6007+1.4124x+x^2)$

where the integer p can be chosen arbitrarily. Thus, the BAEs, (45), become

$$\frac{2lp}{x_i^{(\zeta)}+1}+\frac{2l}{x_i^{(\zeta)}}+\frac{2l(n-p-1)}{x_i^{(\zeta)}-1}-\sum_{t=1(\neq i)}^{k}\frac{2}{x_i^{(\zeta)}-x_t^{(\zeta)}}=0 \tag{56}$$

for $i=1,2,\ldots,k$. Then one can choose $D_{\rm B}(n,k)$ solutions of (56) to get the results. When we set $\{\epsilon_1=-1, \epsilon_2=0, \epsilon_3=\epsilon_4=1\}$ for the previous $L=4$ example of four d bosons, there are five solutions of (56) with the extended Heine-Stieltjes polynomials listed in Table VI.

The corresponding coupled states after normalization are

$$\begin{aligned}|\zeta=1, L=M_L=4\rangle &= 0.1974|1,1,1,1\rangle\\ &-0.6448|2,1,1,0\rangle+0.6502|2,2,0,0\rangle\\ &-0.2475|2,2,1,-1\rangle+0.2475|2,2,2,-2\rangle;\end{aligned}$$

$$\begin{aligned}|\zeta=2, L=M_L=4\rangle &= -0.0432|1,1,1,1\rangle\\ &+0.1412|2,1,1,0\rangle+0.2252|2,2,0,0\rangle\\ &-0.6810|2,2,1,-1\rangle+0.6810|2,2,2,-2\rangle;\end{aligned}$$

$$\begin{aligned}|\zeta=3, L=M_L=4\rangle &= 0.1926|1,1,1,1\rangle\\ &-0.6290|2,1,1,0\rangle+0.2642|2,2,0,0\rangle\\ &+0.4987|2,2,1,-1\rangle-0.4987|2,2,2,-2\rangle;\end{aligned}$$

$$\begin{aligned}|\zeta=4, L=M_L=4\rangle &= 0.0829|1,1,1,1\rangle\\ &-0.2707|2,1,1,0\rangle+0.5085|2,2,0,0\rangle\\ &-0.5750|2,2,1,-1\rangle+0.5750|2,2,2,-2\rangle;\end{aligned}$$

$$\begin{aligned}|\zeta=5, L=M_L=4\rangle &= -0.2355|1,1,1,1\rangle\\ &+0.7692|2,1,1,0\rangle-0.5740|2,2,0,0\rangle\\ &-0.1081|2,2,1,-1\rangle+0.1081|2,2,2,-2\rangle.\end{aligned}$$

Because $D_{\rm B}(4,4)=2$ in this case, we may choose

$$\begin{aligned}|\chi=1\rangle &= |\zeta=1\rangle,\\ |\chi=2\rangle &= c_1|\zeta=1\rangle+c_2|\zeta=2\rangle,\end{aligned}$$

where $c_1=1/\mathcal{N}$ and $c_2=-\frac{1}{\mathcal{N}\langle\zeta=1|\zeta=2\rangle}$ with the normalization constant

$$\mathcal{N}=\left(\langle\zeta=1|\zeta=1\rangle+\frac{\langle\zeta=2|\zeta=2\rangle}{\langle\zeta=1|\zeta=2\rangle^2}-2\right)^{1/2}$$

according to the Gram-Schmidt process. Then one finds

$$\begin{aligned}|\zeta=3\rangle &= 0.3686|\chi=1\rangle+0.9296|\chi=2\rangle,\\ |\zeta=4\rangle &= 0.8062|\chi=1\rangle-0.59168|\chi=2\rangle,\\ |\zeta=5\rangle &= -0.8622|\chi=1\rangle-0.5066|\chi=2\rangle.\end{aligned}$$

This example shows that coupled states with zeros of other polynomials can indeed be expressed as linear combinations of the chosen two owing to the overcompleteness.

B. Identical fermions

Let the single-particle states of fermions with spin j be $|j,m\rangle\equiv|m\rangle$ with $m=-j,-j+1,\ldots,j$. Unlike identical bosons, we have verified that the parameters $\{\epsilon_\alpha\}$ must be a set of unequal numbers for identical fermions. Initially, we need to solve the BAEs, (5), for nonidentical particles with the same spin $j_\alpha=j$ for $\alpha=1,2,\ldots,n$. After (2) is expanded in terms of the single-particle product states, we take all particles to be identical. The Pauli exclusion will automatically rule out any forbidden single-particle product states after such assimilation. The result of (2) gives the final coupled state with total angular momentum $J=nj-k$ and $M_J=J$. However, the two-value parametrization schemes for identical bosons shown previously cannot be used for identical fermions, mainly because the single-particle product states are totally antisymmetric with respect to permutations among different single-particle states. As a consequence, the coupled state is 0 if one chooses any two-value parametrization scheme in $\{\epsilon_\alpha\}$ for identical fermions.

Similarly to identical bosons, the number of linearly independent states obtained from (2), $D_{\rm F}(n,k)$, can be calculated as follows: Let $Q_n(k)$ be the number of different n partitions of the integer k with $k=\sum_{i=1}^n\xi_i$, where $2j+1-n\geqslant\xi_1\geqslant\xi_2\geqslant\cdots\geqslant\xi_n\geqslant 0$. Then the number of linearly independent states obtained from (2) for j fermions, $D_{\rm F}(n,k)=Q_n(k)-Q_n(k-1)$, which gives the multiplicity of a given $J=nj-k$ in the reduction $\mathrm{U}(2j+1)\downarrow\mathrm{O}(3)$ for the antisymmetric irreducible representation $[1^n,\dot{0}]$ of $\mathrm{U}(2l+1)$. Generally, $D_{\rm F}(n,k)$ is far less than $d(n,k)$ shown in (6) for nonidentical particles. Therefore, (2) obtained from solutions of (5) are also overcomplete. Similarly to identical bosons, one only needs to choose $D_{\rm F}(n,k)$ solutions of (5). When the parameters $\{\epsilon_\alpha\}$ are chosen according to (29) or (30), the coupled state, (2), satisfies the symmetry

$$\begin{aligned}&J_-\big(x_1^{(\zeta)}\big)J_-\big(x_2^{(\zeta)}\big)\ldots J_-\big(x_k^{(\zeta)}\big)|\mathrm{h.w.}\rangle\\ &\quad=J_-\big(-x_1^{(\zeta)}\big)J_-\big(-x_2^{(\zeta)}\big)\ldots J_-\big(-x_k^{(\zeta)}\big)|\mathrm{h.w.}\rangle.\end{aligned} \tag{57}$$

Therefore, only one reflectional symmetry pair of Stieltjes zeros should be considered.

In the following, we take $n=3$ $j=9/2$ identical fermions as examples. In this case, there is only one coupled state with $J=M_J=17/2$ $(k=5)$, for which there are *six* extended Heine-Stieltjes polynomials as listed in Table VII, which clearly shows that $y_5^{(2)}(x)=y_5^{(1)}(-x)$, $y_5^{(4)}(x)=y_5^{(3)}(-x)$, and $y_5^{(6)}(x)=y_5^{(5)}(-x)$. But three solutions $y_5^{(1)}(x)$, $y_5^{(3)}(x)$, and

TABLE VII. Extended Heine-Stieltjes polynomials $y_5^{(\zeta)}(x)$ for $J = M_J = 17/2$ ($k = 5$) coupled states of $n = 3$, $j = 9/2$ identical fermions with $\{\epsilon_1 = -1, \epsilon_2 = 0, \epsilon_3 = 1\}$.

	Extended Heine-Stieltjes polynomial $y_5^{(\zeta)}(x)$
$\zeta = 1$	$(-0.6191 + x)(0.60249 - 1.3616x + x^2)(0.4249 - 1.2622x + x^2)$
$\zeta = 2$	$(0.6191 + x)(0.60249 + 1.3616x + x^2)(0.4249 + 1.2622x + x^2)$
$\zeta = 3$	$(0.4849 + x)(0.4559 - 1.2046x + x^2)(0.34201 - 1.1556x + x^2)$
$\zeta = 4$	$(-0.4849 + x)(0.4559 + 1.2046x + x^2)(0.34201 + 1.1556x + x^2)$
$\zeta = 5$	$(0.5371 + x)(0.2723 - 1.0146x + x^2)(0.3451 - 1.0889x + x^2)$
$\zeta = 6$	$(-0.5371 + x)(0.2723 + 1.0146x + x^2)(0.3451 + 1.0889x + x^2)$

$y_5^{(5)}(x)$ all result in one coupled state,

$$|J = M_J = 17/2\rangle = 0.7746|9/2, 5/2, 3/2\rangle - 0.6325|9/2, 7/2, 1/2\rangle, \quad (58)$$

up to a normalization constant after assimilation.

Because $D_{\rm F}(3, 9) = 2$ for $j = 9/2$ identical fermions, $J = 9/2$ should occur twice. While there are 10 extended Heine-Stieltjes polynomials as listed in Table VIII, which shows that $y_9^{(2)}(x) = y_9^{(1)}(-x)$, $y_9^{(4)}(x) = y_9^{(3)}(-x)$, $y_9^{(6)}(x) = y_9^{(5)}(-x)$, $y_9^{(8)}(x) = y_9^{(7)}(-x)$, and $y_9^{(10)}(x) = y_9^{(9)}(-x)$, we need to choose only 2 of them to get the coupled states according to (2). The coupled state with $y_9^{(1)}(x)$ is

$$|\zeta = 1, J = M_J = 9/2\rangle = 0.2105|5/2, 3/2, 1/2\rangle - 0.1684|7/2, 3/2, -1/2\rangle + 0.1575|7/2, 5/2, -3/2\rangle - 0.3384|9/2, 1/2, -1/2\rangle + 0.4415|9/2, 3/2, -3/2\rangle - 0.5446|9/2, 5/2, -5/2\rangle + 0.5446|9/2, 7/2, -7/2\rangle,$$

and that with $y_9^{(3)}(x)$ is

$$|\zeta = 2, J = M_J = 9/2\rangle = 0.1506|5/2, 3/2, 1/2\rangle - 0.1205|7/2, 3/2, -1/2\rangle + 0.1127|7/2, 5/2, -3/2\rangle - 0.3913|9/2, 1/2, -1/2\rangle + 0.4651|9/2, 3/2, -3/2\rangle - 0.5388|9/2, 5/2, -5/2\rangle + 0.5388|9/2, 7/2, -7/2\rangle.$$

After the Gram-Schmidt orthonormalization, we have

$$|\chi = 1, J = M_J = 9/2\rangle = |\zeta = 1, J = M = 9/2\rangle,$$
$$|\chi = 2, J = M_J = 9/2\rangle = 0.5526|5/2, 3/2, 1/2\rangle - 0.4421|7/2, 3/2, -1/2\rangle + 0.4135|7/2, 5/2, -3/2\rangle + 0.5164|9/2, 1/2, -1/2\rangle - 0.2457|9/2, 3/2, -3/2\rangle - 0.0250|9/2, 5/2, -5/2\rangle + 0.0250|9/2, 7/2, -7/2\rangle.$$

Then the other three coupled states, corresponding to $y_9^{(5)}(x)$, $y_9^{(7)}(x)$, and $y_9^{(9)}(x)$, respectively, can be expressed as linear combinations of $|\chi = 1\rangle$ and $|\chi = 2\rangle$.

VI. SUMMARY

In summary, a new angular momentum projection for many-particle systems is formulated based on the Heine-Stieltjes correspondence, which can be regarded as the solutions of the mean-field-plus-pairing model in the strong-pairing interaction $G \to \infty$ limit [9]. With the special choice of the parameters $\{\epsilon_\alpha\}$, the solutions of the associated BAEs are simplified because of the additional reflection symmetries. Properties of the Stieltjes zeros and the related Van Vleck zeros are discussed. The electrostatic interpretation of these zeros are presented. As an example, the application to n nonidentical particles with spin 1/2 is made to elucidate the procedure and properties of the Stieltjes zeros and the related Van Vleck zeros. It is clear that the new angular momentum projection can be used for nonidentical particles with arbitrary spins. It is shown that the new angular momentum projection for identical bosons or fermions can be simplified with the branching multiplicity formula of $\mathrm{U}(N) \downarrow \mathrm{O}(3)$ and the special choices of the parameters used in the projection. Especially, it is shown that the coupled states of identical bosons can always be expressed in terms of zeros of Jacobi

TABLE VIII. Extended Heine-Stieltjes polynomials $y_9^{(\zeta)}(x)$ for $J = 9/2$ coupled states of $n = 3$, $j = 9/2$ identical fermions with $\{\epsilon_1 = -1, \epsilon_2 = 0, \epsilon_3 = 1\}$.

	Extended Heine-Stieltjes polynomial $y_9^{(\zeta)}(x)$
$\zeta = 1$	$(1.7186 - 2.3241x + x^2)(0.9532 - 1.7796x + x^2)(0.6707 - 1.5615x + x^2)(0.5516 - 1.4664x + x^2)(-0.7196 + x)$
$\zeta = 2$	$(1.7186 + 2.3241x + x^2)(0.9532 + 1.7796x + x^2)(0.6707 + 1.5615x + x^2)(0.5516 + 1.4664x + x^2)(0.7196 + x)$
$\zeta = 3$	$(1.3423 - 1.8543x + x^2)(0.7134 - 1.4807x + x^2)(0.4992 - 1.3433x + x^2)(0.4228 - 1.2929x + x^2)(0.4142 + x)$
$\zeta = 4$	$(1.3423 + 1.8543x + x^2)(0.7134 + 1.4807x + x^2)(0.4992 + 1.3433x + x^2)(0.4228 + 1.2929x + x^2)(-0.4142 + x)$
$\zeta = 5$	$(0.9728 - 1.4201x + x^2)(0.5034 - 1.2196x + x^2)(0.3611 - 1.1571x + x^2)(0.1818 + 0.8182x + x^2)(-0.5708 + x)$
$\zeta = 6$	$(0.9728 + 1.4201x + x^2)(0.5034 + 1.2196x + x^2)(0.3611 + 1.1571x + x^2)(0.1818 - 0.8182x + x^2)(0.5708 + x)$
$\zeta = 7$	$(0.6395 - 1.0463x + x^2)(0.3362 - 1.0087x + x^2)(0.2589 - 1.0027x + x^2)(0.2069 + 0.7932x + x^2)(0.4142 + x)$
$\zeta = 8$	$(0.6395 + 1.0463x + x^2)(0.3362 + 1.0087x + x^2)(0.2589 + 1.0027x + x^2)(0.2069 - 0.7932x + x^2)(-0.4142 + x)$
$\zeta = 9$	$(0.1801 - 0.8267x + x^2)(0.2527 - 0.7578x + x^2)(0.3775 + 0.7973x + x^2)(0.2253 + 0.8689x + x^2)(0.4423 + x)$
$\zeta = 10$	$(0.1801 + 0.8267x + x^2)(0.2527 + 0.7578x + x^2)(0.3775 - 0.7973x + x^2)(0.2253 - 0.8689x + x^2)(-0.4423 + x)$

polynomials. However, unlike nonidentical particle systems, the coupled states of identical particles are nonorthogonal with respect to the multiplicity label after the projection. In order to establish orthonormalized coupled states for identical particles, the Gram-Schimidt process may be adopted. It will be advantageous in the application, for example, to shell-model calculations if matrix elements of one- and two-body operators under the angular momentum projected basis can be calculated easily, which seems possible as shown in [26], where explicit expressions for the expectation values of one- and two-body operators in the mean-field-plus-pairing model were obtained, on which relevant research is in progress.

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