Time and space dependence of the electromagnetic field in relativistic heavy-ion collisions

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Exact analytical solution for the space-time evolution of electromagnetic field in electrically conducting nuclear matter produced in heavy-ion collisions is discussed. It is argued that the parameter that controls the strength of the matter effect on the field evolution is $\sigma\gamma b$, where σ is electrical conductivity, γ is the Lorentz boost-factor, and b is the characteristic transverse size of the matter. When this parameter is of the order 1 or larger, which is the case at the Relativistic Heavy Ion Collider and the Large Hadron Collider, the space-time dependence of the electromagnetic field completely differs from that in vacuum.

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In relativistic heavy-ion collisions, production of valence quarks in the central rapidity region (baryon stopping) is suppressed [1]. Hence, Z valence quarks of each nucleus continue to travel after heavy-ion collision along the straight lines in opposite directions. These valence quarks carry total electric charge $2Ze$ that creates electromagnetic field in the interaction region. Unlike the valence quarks, gluons and sea quarks are produced mostly in the central rapidity region, i.e., in a plane perpendicular to the collision axis. It has been argued in Refs. [2,3] that high-multiplicity events in heavy-ion collisions can be effectively described using relativistic hydrodynamics. In particular, matter produced in heavy-ion collisions can be characterized by a few transport coefficients. This approach has enjoyed remarkable phenomenological success (see, e.g., Ref. [4]). Since sea quarks carry an electric charge, the electromagnetic field created by valence quarks depends on the permittivity ϵ, permeability μ, and conductivity σ of the produced matter.

Consider the electromagnetic field created by a point charge e moving along the positive z axis with velocity v. It is governed by the following Maxwell equations:

$$\nabla \cdot \boldsymbol{B} = 0\,, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{1}$$

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= e\delta(z-vt)\delta(\boldsymbol{b})\,, \\ \nabla \times \boldsymbol{H} &= \frac{\partial \boldsymbol{D}}{\partial t} + \sigma \boldsymbol{E} + ev\hat{\boldsymbol{z}}\delta(z-vt)\delta(\boldsymbol{b})\,, \end{aligned} \tag{2}$$

where $\boldsymbol{r} = z\hat{\boldsymbol{z}} + \boldsymbol{b}$ (such that $\boldsymbol{b}\cdot\hat{\boldsymbol{z}} = 0$) is the position of the observation point. Performing the Fourier transform

$$\boldsymbol{E}(t,\boldsymbol{r}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \int \frac{d^2k_\perp}{(2\pi)^2} e^{-i\omega t + ik_z z + i\boldsymbol{k}_\perp\cdot\boldsymbol{b}} \times \boldsymbol{E}_{\omega\boldsymbol{k}}\,, \quad \text{etc.}\,, \tag{3}$$

we get

$$\boldsymbol{k}\cdot\boldsymbol{B}_{\omega k} = 0\,, \quad \boldsymbol{k}\times\boldsymbol{E}_{\omega k} = \mu\omega\boldsymbol{H}_{\omega\boldsymbol{k}}\,, \tag{4}$$

$$\begin{aligned} \epsilon\boldsymbol{k}\cdot\boldsymbol{E}_{\omega k} &= -2i\pi e\delta(\omega - k_z v)\,, \\ \boldsymbol{k}\times\boldsymbol{H}_{\omega k} &= -\omega\tilde{\epsilon}\boldsymbol{E}_{\omega\boldsymbol{k}} - 2\pi i e v\hat{\boldsymbol{z}}\delta(\omega - k_z v)\,, \end{aligned} \tag{5}$$

where $\tilde{\epsilon} = \epsilon + i\sigma/\omega$. The solution to these equations reads as follows (see, e.g., Ref. [5]):

$$\begin{aligned} \boldsymbol{H}_{\omega\boldsymbol{k}} &= -2\pi i e v \frac{\boldsymbol{k}\times\hat{\boldsymbol{z}}}{\omega^2\tilde{\epsilon}\mu - \boldsymbol{k}^2}\delta(\omega - k_z v)\,, \\ \boldsymbol{E}_{\omega\boldsymbol{k}} &= -2\pi i e \frac{\omega\mu v\hat{\boldsymbol{z}} - \boldsymbol{k}/\epsilon}{\omega^2\tilde{\epsilon}\mu - \boldsymbol{k}^2}\delta(\omega - k_z v)\,. \end{aligned} \tag{6}$$

Substituting (6) into (3), it is possible to take the integral over $\boldsymbol{k}$. However, integration over ω cannot be done in the general form, because the dispersion relations $\epsilon(\omega)$ and $\mu(\omega)$ depend on the matter properties.

The later time dependence of the electromagnetic field is determined by a singularity of (6) in the plane of complex ω that has the smallest imaginary part. We assume that the leading singularity is determined by electrical conductivity. (This gives a conservative estimate of the matter effect.) Therefore, we adopt a simple model $\epsilon = \mu = 1$, i.e., neglect the polarization and magnetization response of nuclear matter, but take into account its finite electrical conductivity. Plugging (6) into (3), we take, first, the trivial k_z integral. Integration over ω for positive values of $x_- = t - z/v$ is done by closing the integration contour over the pole in the lower half plane of complex ω. In the relativistic limit $\gamma = 1/\sqrt{1-v^2} \gg 1$, the result is [6]

$$\begin{aligned} \boldsymbol{H}(t,\boldsymbol{r}) &= H(t,\boldsymbol{r})\hat{\boldsymbol{\phi}} \\ &= \frac{e}{2\pi\sigma}\hat{\boldsymbol{\phi}} \int_0^\infty \frac{J_1(k_\perp b)k_\perp^2}{\sqrt{1+\frac{4k_\perp^2}{\gamma^2\sigma^2}}} \\ &\quad \times \exp\left\{\frac{1}{2}\sigma\gamma^2 x_-\left(1-\sqrt{1+\frac{4k_\perp^2}{\gamma^2\sigma^2}}\right)\right\} dk_\perp\,, \end{aligned} \tag{7}$$

$$\begin{aligned} E_z(t,\boldsymbol{r}) &= \frac{e}{4\pi}\int k_\perp J_0(k_\perp b)\frac{1-\sqrt{1+\frac{4k_\perp^2}{\gamma^2\sigma^2}}}{\sqrt{1+\frac{4k_\perp^2}{\gamma^2\sigma^2}}} \\ &\quad \times \exp\left\{\frac{1}{2}\sigma\gamma^2 x_-\left(1-\sqrt{1+\frac{4k_\perp^2}{\gamma^2\sigma^2}}\right)\right\} dk_\perp\,, \end{aligned} \tag{8}$$

$$\boldsymbol{E}_\perp(t,\boldsymbol{r}) = H(t,\boldsymbol{r})\hat{\boldsymbol{r}}\,, \tag{9}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors of polar coordinates in the transverse plane x, y. The electromagnetic field is a function

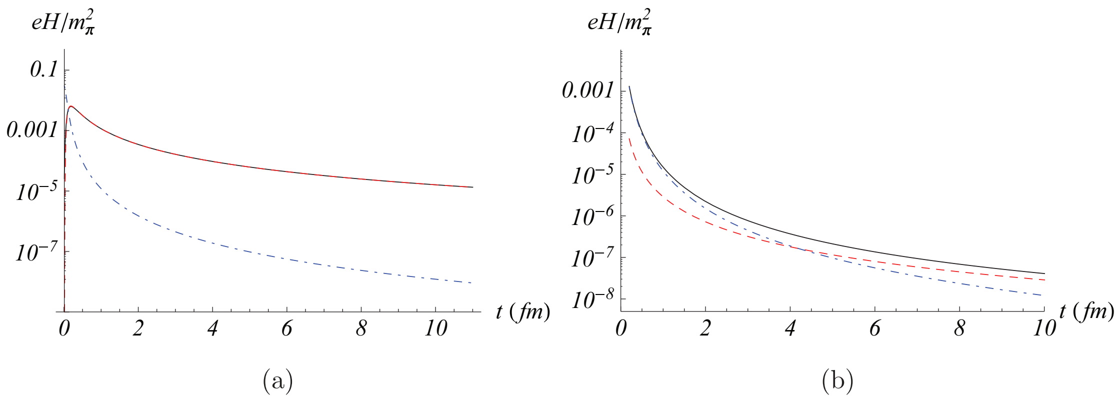

FIG. 1. (Color online) Time evolution of the magnetic field created by a point unit charge at $z = 0$, $b = 7.4$ fm, $\gamma = 100$ and (a) $\sigma = 5.8$ MeV and (b) $\sigma = 0.01$ MeV. The black solid line is a numerical computation of (7), the red dashed line is the "diffusion" approximation (11), and the blue dash-dotted line is a solution in free space.

of $\boldsymbol{r} - \boldsymbol{r}'$, where $\boldsymbol{r}$ and $\boldsymbol{r}' = vt\hat{\boldsymbol{z}}$ are the positions of the observation point and the moving charge correspondingly. In fact, it depends only on distances $z - vt = -vx_-$ and b.

Equations (7)–(9) have two instructive limits depending on the value of parameter $\gamma\sigma b$ that appears in the exponents once we notice that $k_\perp \sim 1/b$. If $\gamma\sigma b \ll 1$, then, after a simple integration, (7)–(9) reduce to the boosted Coulomb potential in free space as follows:

$$\boldsymbol{E} = \frac{e\gamma}{4\pi}\frac{\boldsymbol{b} - vx_-\hat{\boldsymbol{z}}}{(b^2 + \gamma^2 v^2 x_-^2)^{3/2}}, \qquad \boldsymbol{H} = \frac{e\gamma}{4\pi}\frac{v\hat{\boldsymbol{\phi}}}{(b^2 + \gamma^2 v^2 x_-^2)^{3/2}}. \tag{10}$$

This is the solution discussed in Ref. [7]. In the opposite limit $\gamma\sigma b \gg 1$, we expend the square root in (7) and (8) and derive

$$E_r = H_\phi = \frac{e}{2\pi}\frac{b\sigma}{4x_-^2}e^{-\frac{b^2\sigma}{4x_-}}, \qquad E_z = -\frac{e}{4\pi}\frac{x_- - b^2\sigma/4}{\gamma^2 x_-^3}e^{-\frac{b^2\sigma}{4x_-}}. \tag{11}$$

This is the solution pointed out in Ref. [8]. Notice that the electromagnetic field in (10) drops as $1/x_-^3$ at late times, whereas in conducting matter only as $1/x_-^2$. At the Relativistic Heavy Ion Collider $\gamma = 100$, $\sigma \approx 5.8$ MeV [9,10]. For $b = 7$ fm we estimate $\gamma\sigma b = 19$; hence, the field is given by the "diffusive" solution (11). This argument is augmented by numerical calculation presented in Fig. 1. In Fig. 1(a) we plot the result of numerical integration in (7) for $\sigma \approx 5.8$ MeV and compare it with the asymptotic solutions (10) and (11). It is seen that (11) completely overlaps with the exact solution at all times, except at $t < 0.1$ fm (not seen in the figure). To illustrate what happens at $\gamma\sigma b \ll 1$, we plotted in Fig. 1(b) the same formulas as in Fig. 1(a) calculated at artificially reduced conductivity $\sigma \approx 0.01$ MeV. One can clearly observe that at early time matter plays only a small role in the field time evolution which follows (10), whereas at later times the Foucault currents eventually slow down the magnetic field decline, which then follows (11). This conclusion supports our previous results [6,8] and disagrees with the recent claims made in Ref. [11].

The electromagnetic field of a charge moving at distance $\boldsymbol{b}'$ in the positive z direction with velocity v is given by (7)–(9) with $\boldsymbol{b}$ replaced by $\boldsymbol{b} - \boldsymbol{b}'$. It is denoted as $\boldsymbol{H}(x_-, |\boldsymbol{b} - \boldsymbol{b}'|)$, and so on. In the laboratory frame, all charges in a nucleus have approximately the same longitudinal coordinate $z' = vt$ and, hence, the same $x_- = -(z - z')/v$. Therefore, the electromagnetic field of the relativistic nucleus can be calculated as

$$\boldsymbol{H}_Z(x_-, \boldsymbol{b}) = \int \rho(\boldsymbol{r}')\boldsymbol{H}(x_-, |\boldsymbol{b} - \boldsymbol{b}'|)d^3r' = \int 2\sqrt{R_A^2 - b'^2}\rho\, \boldsymbol{H}(x_-, |\boldsymbol{b} - \boldsymbol{b}'|)d^2b', \quad \text{etc.}, \tag{12}$$

where $\rho = Z/(\frac{4}{3}\pi R_A^3)$ is the nuclear density and R_A is the nuclear radius and we used the fact that the $\rho dz'$ is boost invariant (in the z direction).[1] The electromagnetic field of a

[1]We neglect fluctuations of nucleon positions that can also give important contributions to electromagnetic field [12].

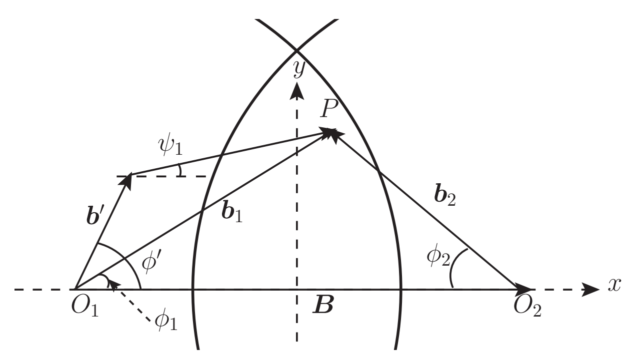

FIG. 2. The transverse plane geometry of heavy-ion collision. Thick lines depict nuclear boundaries. O_1 and O_2 are nuclear centers. $\boldsymbol{B}$ is the impact parameter and $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are positions of the observation point $P(x, y)$ with respect to the nuclear centers. $\boldsymbol{b}'$ is a position of an elementary charge in nucleus 1.

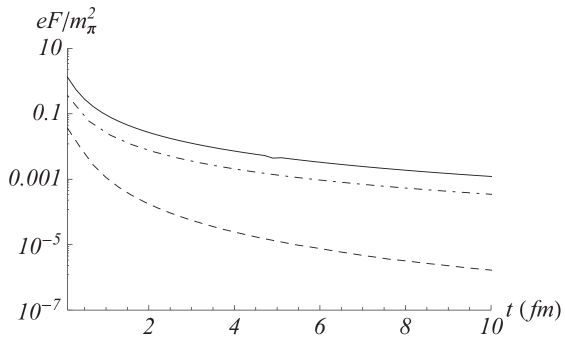

FIG. 3. The time dependence of the total electromagnetic magnetic field F at midrapidity $z = 0$, $\gamma = 100$, $B = 7$ fm, $t = 2$ fm. Solid line: $F = H_y$ at $x = y = 0$; dashed line $F = -H_x$ at $x = y = 1$ fm; dashed-dotted line $F = -E_y$ at $x = y = 1$ fm.

nucleus moving in the negative z direction is given by (12) with x_- replaced by $x_+ = t + z/v$.

Consider now the total electromagnetic field of two nuclei. The geometry of a heavy-ion collision in the transverse plane is depicted in Fig. 2. The magnetic field at point P with coordinates x, y is directed along the azimuthal angle direction $\hat{\boldsymbol{\phi}} = -\sin\psi_1\hat{\boldsymbol{x}} + \cos\psi_1\hat{\boldsymbol{y}}$, where ψ_1 is the angle between the vector $\boldsymbol{b}_1 - \boldsymbol{b}'$ and x axis, which can be related to vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ as follows:

$$\cos\psi_1 = \frac{(\boldsymbol{b}_1 - \boldsymbol{b}')\cdot\hat{\boldsymbol{x}}}{|\boldsymbol{b}_1 - \boldsymbol{b}'|} = \frac{b_1\cos\phi_1 - b'\cos\phi'}{\sqrt{b_1^2 + b'^2 - 2b_1b'\cos(\phi' - \phi_1)}}. \tag{13}$$

The transverse component of electric field has radial direction $\hat{\boldsymbol{r}} = \cos\psi_1\hat{\boldsymbol{x}} + \sin\psi_1\hat{\boldsymbol{y}}$. The final expression for the field of a nucleus is

$$\boldsymbol{H}_Z(x_-, \boldsymbol{b}_1) = \int 2\sqrt{R_A^2 - b'^2}\rho\, H(x_-, |\boldsymbol{b}_1 - \boldsymbol{b}'|) \times (-\sin\psi_1\hat{\boldsymbol{x}} + \cos\psi_1\hat{\boldsymbol{y}})d^2b', \tag{14}$$

$$\boldsymbol{E}_Z(x_-, \boldsymbol{b}_1) = \int 2\sqrt{R_A^2 - b'^2}\rho\, [H(x_-, |\boldsymbol{b}_1 - \boldsymbol{b}'|) \times (\cos\psi_1\hat{\boldsymbol{x}} + \sin\psi_1\hat{\boldsymbol{y}}) + E_z(x_-, |\boldsymbol{b}_1 - \boldsymbol{b}'|)\hat{\boldsymbol{z}}]d^2b', \tag{15}$$

with ψ_1 given by (13) and H, E_z by (7)–(9). Similar expressions hold for the other nucleus. The total electromagnetic field of two nuclei is given by

$$\begin{aligned}\boldsymbol{\mathcal{H}}(t, z, \boldsymbol{b}_1, \boldsymbol{b}_2) &= \boldsymbol{H}_Z(x_-, \boldsymbol{b}_1) + \boldsymbol{H}_Z(x_+, \boldsymbol{b}_2),\\ \boldsymbol{\mathcal{E}}(t, z, \boldsymbol{b}_1, \boldsymbol{b}_2) &= \boldsymbol{E}_Z(x_-, \boldsymbol{b}_1) + \boldsymbol{E}_Z(x_+, \boldsymbol{b}_2).\end{aligned} \tag{16}$$

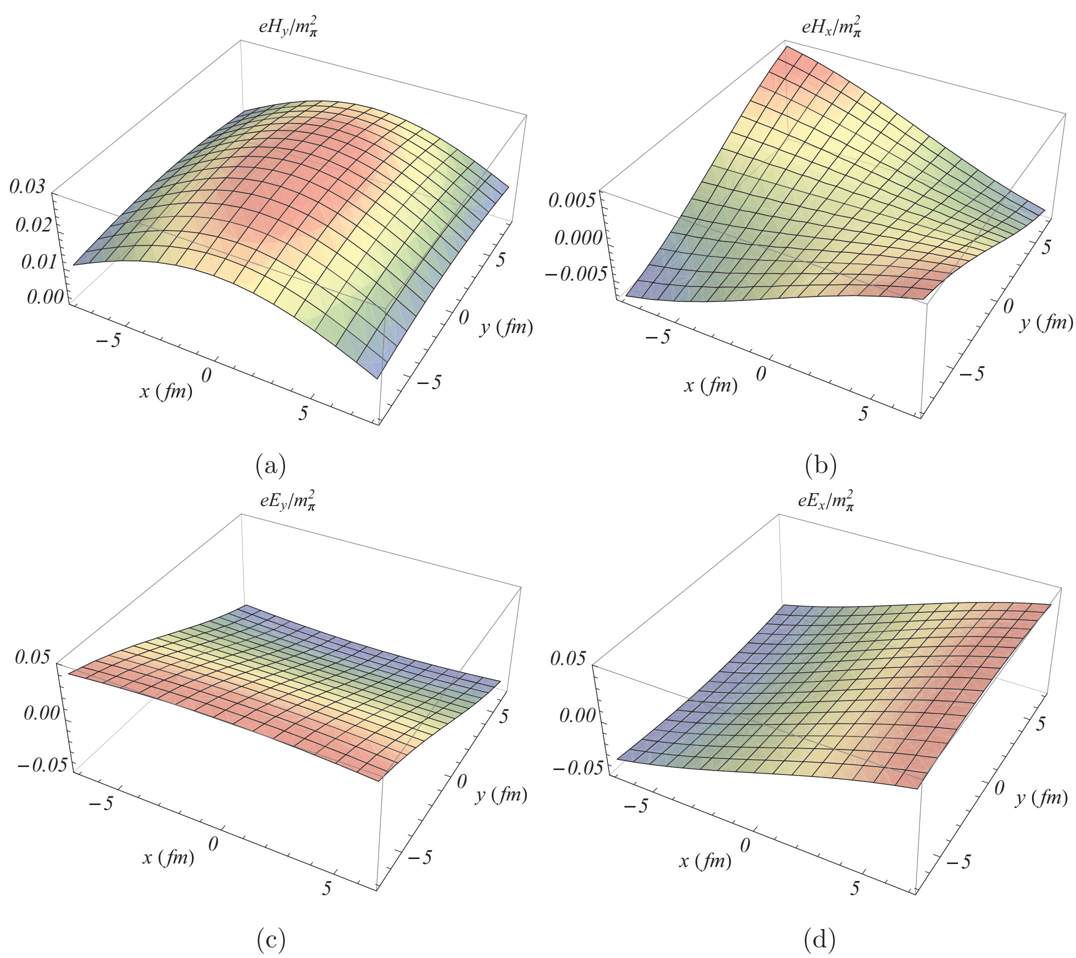

FIG. 4. (Color online) Structure of the field components in the transverse plane at midrapidity $z = 0$ and $\gamma = 100$, $B = 7$ fm.

In practice, one would like to know the electromagnetic field at a given impact parameter $\boldsymbol{B} = \boldsymbol{b}_1 - \boldsymbol{b}_2$ as a function of time t and coordinates x, y, z defined in a symmetric way shown in Fig. 2. This is accomplished using the following equations:

$$\tan\phi_{1,2} = \frac{y}{x \pm B/2}, \quad b_{1,2} = \sqrt{(x \pm B/2)^2 + y^2}. \tag{17}$$

The time dependence of the total magnetic field is shown in Fig. 3. As expected, the late time dependence of all components is the same and governed by (11).

Space dependence is exhibited in Fig. 4. We observe that the space variation of H_y is mild. Other transverse components vary more significantly as they are required to vanish at either $x = 0$ or $y = 0$ by symmetry. When averaged over the transverse plane, only the H_y component survives. However, one can think of observables sensitive to the field variations in the transverse plane.

In summary, we presented an exact analytical and numerical solution for the space and time dependencies of an electromagnetic field produced in heavy-ion collisions. We confirmed our previous result [8] that nuclear matter plays a crucial role in its time evolution.

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