Normalization of strongly intensive quantities

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Recently, two strongly intensive quantities, $\Delta[A, B]$ and $\Sigma[A, B]$, designed for the study of event-by-event fluctuations in high-energy collisions were introduced. They are defined in terms of two extensive event observables A and B. In this paper a special normalization of the $\Delta[A, B]$ and $\Sigma[A, B]$ fluctuation measures is proposed. It ensures that they are dimensionless and yields a common scale required for a quantitative comparison of fluctuations of different, in general dimensional, extensive quantities. Namely, the properly normalized strongly intensive measures assume the value one for fluctuations given by the independent particle model, and they are equal to zero when the A and B observables have constant values in all collision events.

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I. INTRODUCTION

Intensive quantities are defined within the grand canonical ensemble of statistical mechanics. They depend on temperature and chemical potential(s), but they are independent of the system volume. Strongly intensive quantities [1] are, in addition, independent of volume fluctuations. They were suggested for and are used in studies of event-by-event fluctuations of hadron production in nucleus-nucleus collisions at high energies. This is because, in these collisions, the volume of created states varies from collision to collision, and moreover it is difficult or even impossible to measure.

Strongly intensive quantities are defined using two arbitrary, *extensive state quantities* A and B. Here, we call A and B *extensive* when the first moments of their distributions for the ensemble of possible states is proportional to volume. They are referred to as *state quantities* as they characterize the states of the considered system, e.g., final states (or equivalently events) of nucleus-nucleus collisions or microstates of the grand canonical ensemble. For example, A and B may stand for multiplicities of pions and kaons in a particular state, respectively.

The simplest family of strongly intensive quantities is given by the ratio of the first moments (i.e., average values) of A and B:

$$R[A, B] = \frac{\langle A\rangle}{\langle B\rangle}, \tag{1}$$

where averaging $\langle\cdots\rangle$ is performed over the ensemble of considered states.

There are two families of strongly intensive quantities which depend on the second and first moments of A and B and thus allow the study of state-by-state fluctuations [1]. These are

$$\Delta[A, B] = \frac{1}{C_\Delta}[\langle B\rangle\,\omega[A] - \langle A\rangle\,\omega[B]], \tag{2}$$

$$\Sigma[A, B] = \frac{1}{C_\Sigma}[\langle B\rangle\,\omega[A] + \langle A\rangle\,\omega[B] - 2\,(\langle AB\rangle - \langle A\rangle\langle B\rangle)], \tag{3}$$

where

$$\omega[A] \equiv \frac{\langle A^2\rangle - \langle A\rangle^2}{\langle A\rangle}, \quad \omega[B] \equiv \frac{\langle B^2\rangle - \langle B\rangle^2}{\langle B\rangle} \tag{4}$$

are scaled variances of A and B. The normalization factors C_Δ and C_Σ are required to be proportional to the first moment of any extensive quantity.

It is important to stress that $\Delta[A, B]$ and $\Sigma[A, B]$ are independent of system size fluctuations, not only for the grand canonical ensemble of states. They are also independent of the average number of sources and source number fluctuations in the model of independent particle sources, for example, in the wounded nucleon model [2].

Usage of strongly intensive quantities for fluctuations has a long history. The first quantity of this type, introduced in 1992, was the so-called Φ measure of fluctuations [3]. According to the current classification the Φ measure belongs to the Σ family [1]. It is defined as the difference of the quantity calculated for a studied ensemble (e.g., central Pb + Pb collisions) and its value obtained within an independent particle model (IPM) which preserves basic features of the ensemble. Thus, by construction, $\Phi = 0$ if the studied ensemble satisfies the assumptions of the IPM. In general, Φ is a dimensional quantity and it does not assume a characteristic value for the case of nonfluctuating A and B (variances of A and B distributions equal to 0). The latter properties were clearly disturbing in numerous applications of Φ when attempting to characterize fluctuations in experimental data [4] and models [5].

In this paper we propose a specific choice of the C_Δ and C_Σ normalization factors which makes the quantities $\Delta[A, B]$ and $\Sigma[A, B]$ dimensionless and leads to $\Delta[A, B] = \Sigma[A, B] = 1$ in the IPM. Moreover, from the definition of $\Delta[A, B]$ and $\Sigma[A, B]$ it follows that $\Delta[A, B] = \Sigma[A, B] = 0$ in the case of absence of fluctuations of A and B, i.e., for $\omega[A] = \omega[B] = \langle AB\rangle - \langle A\rangle\langle B\rangle = 0$. Thus the proposed normalization of $\Delta[A, B]$ and $\Sigma[A, B]$ leads to a common scale for which the values of the fluctuations measures calculated for different state quantities A and B can be compared.

The paper is organized as follows. In Sec. II we introduce an independent particle model within which we calculate the $\Delta[A, B]$ and $\Sigma[A, B]$ quantities. The calculation details are given in Appendix A. Appendix B gives explicit expressions of $\Delta[A, B]$ and $\Sigma[A, B]$ for three choices of the quantities A and B. Specific models which share the properties of the IPM are discussed in Sec. III. Section IV presents the proposal for the normalization of $\Delta[A, B]$ and $\Sigma[A, B]$ discusses the procedure to calculate them for a given ensemble of states and provides numerical examples. A summary in Sec. V closes the article.

II. $\Sigma[A, B]$ AND $\Delta[A, B]$ IN AN INDEPENDENT PARTICLE MODEL

In the independent particle model one assumes the following:

(i) The state[1] quantities A and B can be expressed as

$$A = \alpha_1 + \alpha_2 + \ldots + \alpha_N, \quad B = \beta_1 + \beta_2 + \ldots + \beta_N, \quad (5)$$

where α_j and β_j denote single-particle contributions to A and B, respectively, and N is the number of particles.

(ii) Interparticle correlations are absent, i.e. the probability of any multiparticle state is the product of probability distributions $P(\alpha_j, \beta_j)$ of single-particle states, and these probability distributions are the same for all $j = 1, \ldots, N$ and independent of N:

$$P_N(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_N, \beta_N) = \mathcal{P}(N) \times P(\alpha_1, \beta_1) \times P(\alpha_2, \beta_2) \times \cdots \times P(\alpha_N, \beta_N), \quad (6)$$

where $\mathcal{P}(N)$ is an arbitrary multiplicity distribution of particles.

It is easy to show (see Appendix A) that within the IPM the first and second moments of A and B are equal to

$$\langle A\rangle = \overline{\alpha}\langle N\rangle, \quad \langle A^2\rangle = \overline{\alpha^2}\langle N\rangle + \overline{\alpha}^2[\langle N^2\rangle - \langle N\rangle], \quad (7)$$

$$\langle B\rangle = \overline{\beta}\langle N\rangle, \quad \langle B^2\rangle = \overline{\beta^2}\langle N\rangle + \overline{\beta}^2[\langle N^2\rangle - \langle N\rangle], \quad (8)$$

$$\langle AB\rangle = \overline{\alpha\,\beta}\langle N\rangle + \overline{\alpha}\cdot\overline{\beta}[\langle N^2\rangle - \langle N\rangle]. \quad (9)$$

The values of $\langle A\rangle$ and $\langle B\rangle$ are proportional to the average number of particles, $\langle N\rangle$, and, thus, to the average size of the system. These quantities are extensive. The quantities $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\alpha^2}$, $\overline{\beta^2}$, $\overline{\alpha\,\beta}$ are the first and second moments of the single-particle distribution $P(\alpha, \beta)$. Within the IPM they are independent of $\langle N\rangle$ and play the role of intensive quantities. By using Eq. (7) the scaled variance $\omega[A]$ which describes the state-by-state fluctuations of A can be expressed as

$$\omega[A] \equiv \frac{\langle A^2\rangle - \langle A\rangle^2}{\langle A\rangle} = \frac{\overline{\alpha^2} - \overline{\alpha}^2}{\overline{\alpha}} + \overline{\alpha}\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle} \equiv \omega[\alpha] + \overline{\alpha}\omega[N], \quad (10)$$

where $\omega[\alpha]$ is the scaled variance of the single-particle quantity α, and $\omega[N]$ is the scaled variance of N. A similar expression follows from Eq. (8) for the scaled variance $\omega[B]$. The scaled variances $\omega[A]$ and $\omega[B]$ depend on the fluctuations of the particle number via $\omega[N]$. Therefore, $\omega[A]$ and $\omega[B]$ are not strongly intensive quantities.

From Eqs. (7)–(9) one obtains expressions for $\Delta[A, B]$ and $\Sigma[A, B]$, namely,

$$\Delta[A, B] = \frac{\langle N\rangle}{C_\Delta}[\overline{\beta}\omega[\alpha] - \overline{\alpha}\omega[\beta]], \quad (11)$$

$$\Sigma[A, B] = \frac{\langle N\rangle}{C_\Sigma}[\overline{\beta}\omega[\alpha] + \overline{\alpha}\omega[\beta] - 2(\overline{\alpha\,\beta} - \overline{\alpha}\cdot\overline{\beta})]. \quad (12)$$

Thus, the requirement that

$$\Delta[A, B] = \Sigma[A, B] = 1, \quad (13)$$

within the IPM, leads to

$$C_\Delta = \langle N\rangle[\overline{\beta}\omega[\alpha] - \overline{\alpha}\omega[\beta]], \quad (14)$$

$$C_\Sigma = \langle N\rangle[\overline{\beta}\omega[\alpha] + \overline{\alpha}\omega[\beta] - 2(\overline{\alpha\,\beta} - \overline{\alpha}\cdot\overline{\beta})]. \quad (15)$$

Two comments are in order here. First, Eqs. (7)–(9) have the same structure as Eqs. (2)–(4) of Ref. [1] obtained within the model of independent sources. The only difference is that the number of sources, N_S, in the model of independent sources is replaced by the number of particles, N, in the IPM. Each source can produce many particles, and the number of these particles varies from source to source and from event to event. Besides, properties of particles emitted from the same source may be correlated. Therefore, in general, the model of independent sources does not satisfy the assumptions of the IPM. Nevertheless, the formal similarity between the two models can be exploited and it gives the following rule of one-to-one correspondence: all results obtained within the IPM can be found from the expressions obtained within the model of independent sources, by assuming artificially that each source always produces one particle. Second, only the first and second moments of two extensive quantities A and B are required in order to define the strongly intensive quantities Δ and Σ. However, in order to calculate the proposed normalization factors C_Σ and C_Δ additional information is needed, namely, the first and second moments of single-particle contributions to A and B as well as the mean number of particles. Note that in special cases the factors C_Σ and C_Δ may assume the value zero and thus the proposed normalization is not possible (see Sec. IV B for details).

Explicit expressions for Eqs. (14) and (15) for three choices of A and B are given in Appendix B. The first two cases correspond to the study of "transverse momentum" and "chemical" fluctuations. The third choice is the most general. The examples presented in Appendix B cover a broad spectrum of cases. In Appendix B1 each particle contributes by one unit to an event multiplicity N and by its p_T to the event

[1] By "state," we mean, e.g., a microstate of the grand canonical ensemble or a final state of a nucleus-nucleus collision.

transverse momentum P_T. On the other hand, in the case of fluctuations of kaon and pion multiplicities, K and π, considered in Appendix B1, each particle is either a pion or a kaon; i.e., it contributes either to K or to π. Nevertheless, introducing the single-particle identity variables w_K and w_π we have succeeded in treating these different cases as well as the most general situation of partially overlapping A and B quantities within the same mathematical formalism.

III. EXAMPLES OF INDEPENDENT PARTICLE MODELS

In this section two specific models which satisfy the IPM assumptions, i.e., Eqs. (5) and (6), are presented and discussed.

A. Grand canonical ensemble

The most popular model which satisfies the IPM assumptions is the ideal Boltzmann multicomponent gas in the grand canonical ensemble (GCE) formulation. Here we refer to it as the IB-GCE. In the IB-GCE the probability of any microscopic state is equal to the product of probabilities of single-particle states. These probabilities are independent of particle multiplicity. Thus, the IB-GCE satisfies assumption (6) of the IPM.

The IB-GCE predicts a specific form of the multiplicity distribution $\mathcal{P}(N)$, namely, the Poisson distribution and, thus, $\omega[N] = 1$. Moreover, it also predicts the specific form of the single-particle probability in momentum space, namely, the Boltzmann distribution,

$$f_B(\mathbf{p}) = C\,\exp\left(-\frac{\sqrt{\mathbf{p}^2 + m^2}}{T}\right), \qquad (16)$$

where $\mathbf{p}$ and m are particle momentum and mass, respectively, T is the system temperature, and C is the normalization constant.

Note that by introducing quantum statistics one destroys the correspondence between the GCE and the IPM. This is because of (anti)correlation between particles in the same quantum state for the (Fermi) Bose ideal gas. Moreover, correlations between particles are introduced if instead of resonances their decay products are considered. Note that it is necessary to include strong decays of resonances in order to compare the GCE predictions to experimental results.

The correspondence between the IB-GCE and the IPM remains valid even if the volume varies from microstate to microstate[2] but local properties of the system, i.e., temperature and chemical potentials, are independent of the system volume. Let volume fluctuations be given by the probability density function $F(V)$. The averaging over all microstates includes the averaging over the microstates with fixed volume and the averaging over the volume fluctuations. The volume fluctuations broaden the $\mathcal{P}(N)$ distribution and increase its scaled variance:

$$\omega[N] \equiv \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle} = 1 + \frac{\langle N\rangle}{\langle V\rangle}\cdot\frac{\overline{V^2} - \overline{V}^2}{\overline{V}}, \qquad (17)$$

where $\overline{V^k} \equiv \int dV\,F(V)V^k$ for $k = 1, 2$. The first term on the right-hand side of Eq. (17) corresponds to the particle number fluctuations in the IB-GCE at a fixed volume V (i.e., this is the scaled variance of the Poisson distribution), and the second term is the contribution due to the volume fluctuations. Equation (6) remains valid in this example; therefore, the IB-GCE with arbitrary volume fluctuations satisfies the IPM assumptions.

B. Mixed event model

The mixed event model is defined by the Monte Carlo procedure frequently used by experimentalists in order to create a sample of artificial events in which correlations and fluctuations present in the original ensemble of events are partly removed. Then the original and mixed events are analyzed in the same way and the corresponding results are compared in order to extract the magnitude of a signal of interest, which by construction should be present in the original events and absent in the mixed events. The mixed event procedure is in particular popular in studies of resonance production, particle correlations due to quantum statistics, and event-by-event fluctuations (see Ref. [7] for examples).

There are many variations of the mixed event model. Here we describe the one which in the limit of an infinite number of the original and mixed events gives results identical to the IPM.

The procedure to create a mixed event which corresponds to the given ensemble of original events consists of two steps, namely,

(i) a mixed event multiplicity, N, is drawn from the set of multiplicities of all original events;

(ii) N particles for the mixed event are drawn randomly with replacement from the set of all particles from all original events.

Then these are repeated to create the next mixed event and the procedure is stopped when the desired number of mixed events is reached. In the limit of an infinite number of original events, the probability of having two particles from the same original event in a single mixed event is zero and thus particles in the mixed events are uncorrelated. Therefore, in this limit the mixed event model satisfies the IPM assumptions. Note that, for an infinite number of mixed events, the first moments of all extensive quantities and all single-particle distributions of the original and mixed events are identical.

IV. DETERMINATION OF $\Delta[A, B]$ AND $\Sigma[A, B]$

The strongly intensive quantities $\Delta[A, B]$ and $\Sigma[A, B]$ were introduced for the study of state-by-state fluctuations of any extensive quantities A and B in a given ensemble of states. For example, states may refer to data for nucleus-nucleus collisions recorded by an experiment or generated within a Monte Carlo model. In this section, we first explicitly present

[2]Statistical ensembles with volume fluctuations were discussed in Ref. [6].

how $\Delta[A, B]$ and $\Sigma[A, B]$ with their normalization factors can be calculated for a given ensemble of states. Then, we comment on selected properties of $\Delta[A, B]$ and $\Sigma[A, B]$ and illustrate the procedure of their determination by numerical examples.

A. Normalization procedure

Let the ensemble of states Ω and the extensive state quantities A and B be given. We propose to define the normalization factors C_Σ and C_Δ in Eqs. (2) and (3) such that $\Delta[A, B] = \Sigma[A, B] = 1$ in the IPM with the multiplicity distribution $\mathcal{P}(N)$ and the single-particle distribution $P(\alpha, \beta)$ identical to those of the ensemble Ω. The IPM which corresponds to the ensemble Ω will be denoted as IPM-Ω. The normalization factors C_Δ and C_Σ calculated within the IPM-Ω are then given by Eqs. (14) and (15), where all entering quantities should be calculated from the ensemble Ω.

The procedure of calculating $\Delta[A, B]$ and $\Sigma[A, B]$ given by Eqs. (2) and (3) with the normalization factors defined by Eqs. (14) and (15) consists of the following steps:

(i) Calculate the Ω-ensemble state averages of the first and second moments of extensive quantities A and B.
(ii) Calculate the first and second moments of single-particle quantities α and β, as well as the average number of particles, $\langle N\rangle$, entering Eqs. (14) and (15); the averaging is performed over the Ω-ensemble.
(iii) Calculate $\Delta[A, B]$ and $\Sigma[A, B]$ according to Eqs. (2) and (3).

Note that the results obtained within the IPM-Ω can be approximated by the corresponding ones obtained using mixed events constructed from the Ω ensemble (see Sec. III B for details). As has been noted there, the mixed event construction satisfies the IPM assumptions if the number of mixed events approaches infinity.

B. Comments

There is an important difference between the $\Sigma[A, B]$ and $\Delta[A, B]$ quantities. Namely, in order to calculate $\Delta[A, B]$ one needs to measure only the first two moments: $\langle A\rangle$, $\langle B\rangle$ and $\langle A^2\rangle$, $\langle B^2\rangle$. This can be done by independent measurements of the distributions $P_A(A)$ and $P_B(B)$. The quantity $\Sigma[A, B]$ includes the correlation term $\langle AB\rangle - \langle A\rangle\langle B\rangle$, and thus it requires, in addition, simultaneous measurements of A and B in order to obtain the joint distribution $P_{AB}(A, B)$. The quantities $\Sigma[A, B]$ and $\Delta[A, B]$ also have properties under exchange of A and B, namely, $\Sigma[A, B] = \Sigma[B, A]$ and $\Delta[A, B] = -\Delta[B, A]$. Using the last relation one can always define $\Delta \geqslant 0$ by exchanging the A and B quantities.

In the IPM the A and B quantities are expressed in terms of sums of the single-particle variables α and β. Thus in order to calculate the normalization C_Δ and C_Σ factors one has to measure the single-particle quantities α and β. However, this may not always be possible within a given experimental setup. For example, A and B may be energies of particles measured by two calorimeters. Then one can study fluctuations in terms of $\Delta[A, B]$ and $\Sigma[A, B]$ but cannot calculate the normalization factors proposed here.

A special discussion is needed when C_Δ and/or C_Σ equal zero, and the normalization procedure may lead to the singular behavior of the Δ and/or Σ quantities. First, let us look at the case of transverse momentum fluctuations ($A = P_T$ and $B = N$). According to Eq. (B5) one obtains $C_\Delta = C_\Sigma = \langle N\rangle \cdot \omega[p_T]$. Thus, for $\omega[p_T] = 0$, $C_\Delta = C_\Sigma = 0$. This is possible only for an unphysical case when all particles in all events have the same transverse momentum, $p_T = \overline{p_T}$. The total transverse momentum $P_T = \overline{p_T}\, N$ still may fluctuate because of the particle number fluctuations. This corresponds to the IPM with $P(p_T) = \delta(p_T - \overline{p_T})$ and from Eq. (6) one easily calculates

$$\langle P_T\rangle = \overline{p_T}\cdot\langle N\rangle, \left\langle P_T^2\right\rangle = \overline{p_T}^2\cdot\langle N^2\rangle,\ \langle P_T N\rangle = \overline{p_T}\cdot\langle N^2\rangle, \tag{18}$$

and $\Delta[P_T, N] = \Sigma[P_T, N] = 1$, as it should be in the IPM. Thus, in the above example, in spite of having $C_\Delta = C_\Sigma = 0$, no singularity of Δ and Σ appears.

As a second example let us consider A and B as being particle multiplicities. According to Eqs. (B11) and (B12) one obtains

$$C_\Delta = \langle B\rangle - \langle A\rangle, \quad C_\Sigma = \langle B\rangle + \langle A\rangle. \tag{19}$$

It is clear that $C_\Sigma > 0$, but it may happen that $C_\Delta = 0$. Therefore, normalization of the Δ quantity may not be possible. For example, the average multiplicities of protons and kaons can be equal to each other: $\langle p\rangle = \langle K\rangle$. As discussed above, the normalization is given by the requirement $\Delta = 1$ within the IPM. For example, for the Boltzmann ideal gas in the GCE one has $\omega[A] = \omega[B] = 1$, and thus $\Delta[A, B]$ would be equal to 1 even for $\langle A\rangle = \langle B\rangle$. However, in general, $\langle A\rangle = \langle B\rangle$ does not imply $\omega[A] = \omega[B] = 1$, e.g., $\omega[p] \neq \omega[K]$ if quantum statistics is assumed. Therefore, one should have in mind that the proposed normalization makes the Δ quantity of particle number fluctuations sensitive to small deviations from the IPM results for $\langle A\rangle \approx \langle B\rangle$.

C. Numerical examples

The proposed procedure is illustrated by numerical results obtained within the ultra-relativistic quantum molecular dynamics (UrQMD) model [8] and an ideal gas of quantum particles.

Figure 1 shows the collision energy dependence of different fluctuation measures discussed in this paper in the CERN Super Proton Synchrotron (SPS) energy range. In this example, $A = P_T$ is the total transverse momentum of negatively charged hadrons and $B = H^-$ is their multiplicity (see Appendix B1). The UrQMD simulations were performed for inelastic $p + p$ interactions and for the 7% most central Xe + La collisions. This choice of reactions is motivated by the experimental program of the NA61/SHINE Collaboration [9] at the CERN SPS. NA61/SHINE already reported the first results on event-by-event fluctuations in $p + p$ interactions [10], and results for nucleus-nucleus (Be + Be, Ar + Ca, and Xe + La) collisions will become available within the next couple of years. A comparison between experimental data and models is beyond the scope of this paper. The top plots show

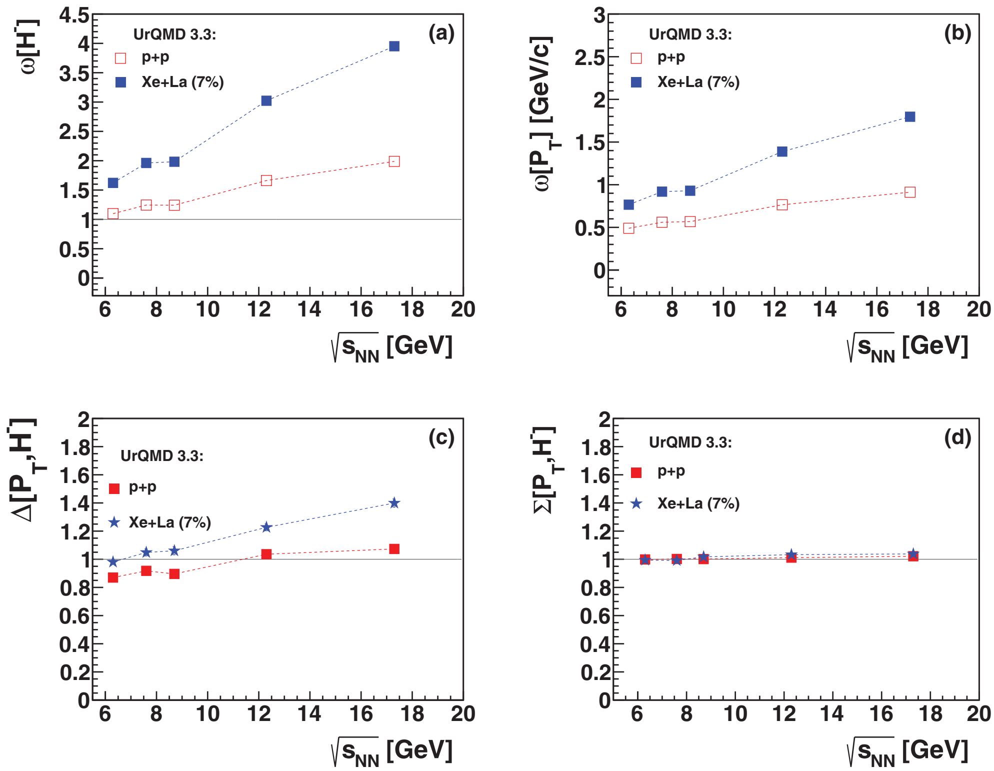

FIG. 1. (Color online) Fluctuation measures calculated within the UrQMD model for negatively charged hadrons produced in inelastic $p + p$ interactions and the 7% most central Xe + La collisions as functions of collision energy in the CERN SPS energy range. The top plots show intensive measures of fluctuations, namely, the scaled variances of (a) the negatively charged hadron multiplicity, $\omega[H^-]$, and (b) the sum of magnitudes of their transverse momenta, $\omega[P_T]$. The bottom plots show the corresponding strongly intensive measures (c) $\Delta[P_T, H^-]$ and (d) $\Sigma[P_T, H^-]$. Statistical uncertainties are smaller than the symbol size and were calculated using the subsample method.

intensive fluctuation measures, namely, the scaled variance of the negatively charged particle multiplicity distribution, $\omega[H^-]$, and of the distribution of the sum of the magnitudes of their transverse momenta, $\omega[P_T]$. The bottom plots show the corresponding strongly intensive measures $\Delta[P_T, H^-]$ and $\Sigma[P_T, H^-]$ normalized as proposed in this paper according to Eqs. (14) and (15) with their explicit form given in Eq. (B5).

The scaled variance of H^- and P_T is significantly larger in central Xe + La collisions than in $p + p$ interactions. To a large extent this is due to fluctuations of the number of nucleons which interacted (wounded nucleons) (see Ref. [11] for a detailed discussion of this issue). The advantages of the $\Delta[P_T, H^-]$ and $\Sigma[P_T, H^-]$ quantities are obvious from the results presented in the bottom plots. First, they are not directly sensitive to fluctuations of the collision geometry (the number of wounded nucleons), in contrast to the scaled variance. Thus, the remaining small differences between results for central Xe + La collisions and $p + p$ interactions are entirely due to deviations of the UrQMD model from the independent source model. Second, they are dimensionless and expressed in units common for all energies and reactions as well as for different choices of state quantities A and B. Due to the particular normalization proposed in this article, they assume the value one for the independent particle model and zero in the absence of event-by-event fluctuations.

The strongly intensive fluctuation measures $\Delta[P_T, N]$ and $\Sigma[P_T, N]$ have been recently studied in Ref. [12] for the ideal Bose and Fermi gases within the GCE. As was already noted the GCE for the Boltzmann approximation satisfies the conditions of the IPM; i.e., Eq. (13) is valid in the IB-GCE. The following general relations have been found [12]:

$$\Delta^{\text{Bose}}[P_T, N] < \Delta^{\text{Boltzmann}} = 1 < \Delta^{\text{Fermi}}[P_T, N], \quad (20)$$

$$\Sigma^{\text{Fermi}}[P_T, N] < \Sigma^{\text{Boltzmann}} = 1 < \Sigma^{\text{Bose}}[P_T, N]; \quad (21)$$

i.e., the Bose statistics makes $\Delta[P_T, N]$ smaller than unity and $\Sigma[P_T, N]$ larger than unity, whereas the Fermi statistics works in the opposite way. The Bose statistics of pions appears to be the main source of quantum statistics effects in a hadron gas with a temperature typical for the hadron system created in $A + A$ collisions. It gives about a 20% decrease of $\Delta[P_T, N]$ and a 10% increase of $\Sigma[P_T, N]$ at $T \cong 150$ MeV in comparison to the IPM results (13). The Fermi statistics of protons modifies insignificantly $\Delta[P_T, N]$ and $\Sigma[P_T, N]$ for the typical T and μ_B. Note that UrQMD takes into account several sources of fluctuations and correlations, e.g., the exact conservation laws and resonance decays. On the other hand, it does not include the effects of Bose and Fermi statistics.

V. SUMMARY

The strongly intensive quantities $\Delta[A, B]$ and $\Sigma[A, B]$ are fluctuation measures which are independent of the system volume and its fluctuations within the grand canonical ensemble of statistical mechanics. Moreover, they are independent of the number of wounded nucleons and its fluctuations within the wounded nucleon model. Strongly intensive quantities are expected to be useful in studies of fluctuations in hadron production in nucleus-nucleus collisions at high energies. In this paper a special normalization of strongly intensive quantities is proposed. It ensures that they are dimensionless and yields a common scale enabling a quantitative comparison of fluctuations of different extensive state quantities. With the proposed normalization $\Delta[A, B]$ and $\Sigma[A, B]$ assume the value one for fluctuations given by the independent particle model and zero in the absence of state-by-state fluctuations.

The paper includes details of calculations and explicit formulas for "transverse momentum" and "chemical" fluctuations as well as for the most general case of fluctuations of two extensive motional quantities for partly overlapping sets of particles. Moreover, numerical examples are given using final states of high-energy collisions generated by the UrQMD model.

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APPENDIX A: CALCULATION DETAILS OF THE IPM

In this Appendix details of the derivation of Eqs. (7)–(9) within the independent particle model defined by Eqs. (5) and (6) are given.

The functions entering Eq. (6) satisfy the normalization conditions

$$\sum_N \mathcal{P}(N) = 1, \int d\alpha\, d\beta\, P(\alpha, \beta) = 1. \tag{A1}$$

The average values of α_j and α_j^2 are

$$\langle\alpha_1\rangle = \langle\alpha_2\rangle = \cdots = \langle\alpha_N\rangle = \int d\alpha d\beta\, \alpha\, P(\alpha, \beta) \equiv \overline{\alpha}, \tag{A2}$$

$$\langle\alpha_1^2\rangle = \langle\alpha_2^2\rangle = \cdots = \langle\alpha_N^2\rangle = \int d\alpha d\beta\, \alpha^2\, P(\alpha, \beta) \equiv \overline{\alpha^2}. \tag{A3}$$

Similarly,

$$\langle\beta_1\rangle = \langle\beta_2\rangle = \cdots = \langle\beta_N\rangle = \int d\alpha\, d\beta \beta\, P(\alpha, \beta) \equiv \overline{\beta}, \tag{A4}$$

$$\langle\beta_1^2\rangle = \langle\beta_2^2\rangle = \cdots = \langle\beta_N^2\rangle = \int d\alpha\, d\beta \beta^2 P(\alpha, \beta) \equiv \overline{\beta^2}. \tag{A5}$$

At $i \neq j$ one finds

$$\begin{aligned} \langle\alpha_i\alpha_j\rangle &= \langle\alpha_i\rangle\langle\alpha_j\rangle = \overline{\alpha}\cdot\overline{\alpha} = \overline{\alpha}^2, \\ \langle\beta_i\beta_j\rangle &= \langle\beta_i\rangle\langle\beta_j\rangle = \overline{\beta}\cdot\overline{\beta} = \overline{\beta}^2. \end{aligned} \tag{A6}$$

The state averages of A and B are equal to

$$\begin{aligned} \langle A\rangle &= \left\langle \sum_{j=1}^N \alpha_j \right\rangle = \sum_N \mathcal{P}(N) \sum_{j=1}^N \langle\alpha_j\rangle \\ &= \sum_N \mathcal{P}(N)\overline{\alpha}\cdot N = \overline{\alpha}\,\langle N\rangle, \end{aligned} \tag{A7}$$

$$\begin{aligned} \langle B\rangle &= \left\langle \sum_{j=1}^N \beta_j \right\rangle = \sum_N \mathcal{P}(N) \sum_{j=1}^N \langle\beta_j\rangle \\ &= \sum_N \mathcal{P}(N)\overline{\beta}\cdot N = \overline{\beta}\,\langle N\rangle. \end{aligned} \tag{A8}$$

For the second moments of A and B one obtains

$$\begin{aligned} \langle A^2\rangle &= \langle(\alpha_1+\alpha_2+\ldots+\alpha_N)^2\rangle = \left\langle \sum_{j=1}^N \alpha_j^2 + \sum_{1\neq i<j\leqslant N} \alpha_i\alpha_j \right\rangle \\ &= \sum_N \mathcal{P}(N)\left[\sum_{j=1}^N \langle\alpha_j^2\rangle + \sum_{1\leqslant i\neq j\leqslant N}\langle\alpha_i\alpha_j\rangle\right] \\ &= \overline{\alpha^2}\,\langle N\rangle + \overline{\alpha}^2\,[\langle N^2\rangle - \langle N\rangle], \end{aligned} \tag{A9}$$

and similarly

$$\langle B^2\rangle = \overline{\beta^2}\,\langle N\rangle + \overline{\beta}^2\,[\langle N^2\rangle - \langle N\rangle]. \tag{A10}$$

Finally, for $\langle AB\rangle$ one finds

$$\begin{aligned} \langle AB\rangle &= \langle(\alpha_1+\alpha_2+\ldots+\alpha_N)\times(\beta_1+\beta_2+\ldots+\beta_N)\rangle \\ &= \sum_N \mathcal{P}(N)\left[\sum_{j=1}^N \langle\alpha_j\beta_j\rangle + \sum_{1\leqslant i\neq j\leqslant N}\langle\alpha_i\beta_j\rangle\right] \\ &= \overline{\alpha\beta}\,\langle N\rangle + \overline{\alpha}\cdot\overline{\beta}\,[\langle N^2\rangle - \langle N\rangle]. \end{aligned} \tag{A11}$$

APPENDIX B: EXAMPLES FOR THREE CHOICES OF A AND B

In this Appendix explicit expressions for $\Delta[A, B]$ and $\Sigma[A, B]$ and their normalization factors C_Σ and C_Δ calculated within the IPM are given for two popular choices of the extensive state quantities A and B which correspond to the study of "transverse momentum" and "chemical" fluctuations. Finally, the most general case is considered, which corresponds to the selection of two extensive motional quantities for partly overlapping sets of particles.

1. "Transverse momentum" fluctuations

The first [3] and the most popular [4,5] application of the Φ measure was the study of transverse momentum fluctuations. In the formalism, introduced in Ref. [1], this corresponds to the following choice of the extensive state quantities A and B:

$$A \equiv P_T = p_T^{(1)} + p_T^{(2)} + \ldots + p_T^{(N)}, \tag{B1}$$

$$B \equiv N = w^{(1)} + w^{(2)} + \ldots + w^{(N)}, \tag{B2}$$

where $p_T^{(j)}$ is the absolute value of the transverse momentum of the jth particle[3] and $w^{(j)}$ is the particle identity [13] which equals one for all particles: $w^{(j)} = 1$.

Thus, for the single-particle quantities

$$\alpha = p_T, \quad \beta = w = 1, \tag{B3}$$

one gets

$$\begin{aligned} &\overline{\alpha} = \overline{p_T}, \quad \overline{\alpha^2} = \overline{p_T^2}, \\ &\overline{\beta} = \overline{\beta^2} = \overline{w} = \overline{w^2} = 1, \quad \overline{\alpha\beta} = \overline{p_T}, \end{aligned} \tag{B4}$$

where $\overline{p_T}$ and $\overline{p_T^2}$ are the average values of p_T and p_T^2 calculated from the properly normalized single-particle transverse momentum distribution.[4] Consequently, Eqs. (14) and (15) give

$$C_\Delta = C_\Sigma = \langle N\rangle \cdot \frac{\overline{p_T^2} - \overline{p_T}^2}{\overline{p_T}} \equiv \langle N\rangle \cdot \omega[p_T]. \tag{B5}$$

As was already mentioned in Sec. II, only the first and second moments of two extensive quantities P_T and N are required to calculate the strongly intensive measures $\Delta[P_T, N]$ and $\Sigma[P_T, N]$. However, in order to calculate the proposed normalization factors C_Δ and C_Σ additional information may be necessary. In the considered example, one also needs the second moment of the single-particle p_T distribution, $\overline{p_T^2}$.

Let us recall here that $\Sigma(P_T, N)$ is directly related to the Φ_{p_T} measure of transverse momentum fluctuations (for the explicit expression see Ref. [1]). The only difference is in the scale used to quantify fluctuations measured by both quantities. Namely, the Φ_{p_T} measure is defined as the difference of the event quantity calculated for the studied ensemble (e.g., central Pb + Pb collisions) and its value obtained within the independent particle model. Consequently, $\Phi_{p_T} = 0$ if the studied ensemble satisfies the assumptions of the IPM. Moreover, Φ_{p_T} is a dimensional quantity and does not assume a characteristic value for the case of nonfluctuating A and B. These undesired properties of Φ_{p_T} are removed when fluctuations are measured using $\Sigma(P_T, N)$ normalized as proposed in this article.

2. "Chemical" fluctuations

In the jargon of high-energy nuclear physics "chemical" fluctuations refer to fluctuations of particle-type composition of the system. In order to be specific let us consider relative fluctuations of the number of charged pions $\pi \equiv \pi^+ + \pi^-$ and kaons $K \equiv K^+ + K^-$:

$$\begin{aligned} A \equiv K &= w_K^{(1)} + w_K^{(2)} + \ldots + w_K^{(N)}, \\ B \equiv \pi &= w_\pi^{(1)} + w_\pi^{(2)} + \ldots + w_\pi^{(N)}, \end{aligned} \tag{B6}$$

where $w_\pi^{(j)}$ and $w_K^{(j)}$ are the pion and kaon identities of the jth particle.[5] Particle identities were introduced first in Ref. [13] and used in the study of "chemical" fluctuations in terms of the Φ measure [4,5,13].

In this example one defines the kaon $w_K^{(j)}$ and pion $w_\pi^{(j)}$ identities as $w_K^{(j)} = 1$ and $w_\pi^{(j)} = 0$ if the jth particle is a kaon and as $w_K^{(j)} = 0$ and $w_\pi^{(j)} = 1$ if the jth particle is a pion.

For the single-particle quantities

$$\alpha = w_K, \quad \beta = w_\pi, \tag{B7}$$

one obtains

$$\begin{aligned} &\overline{w_K} = \overline{w_K^2} = \frac{\langle K\rangle}{\langle N\rangle} \equiv k, \\ &\overline{w_\pi} = \overline{w_\pi^2} = \frac{\langle \pi\rangle}{\langle N\rangle} = 1 - k, \quad \overline{w_K\, w_\pi} = 0, \end{aligned} \tag{B8}$$

where $N = K + \pi$. Then from Eq. (B8) it follows that

$$\omega[w_K] \equiv \frac{\overline{w_K^2} - \overline{w_K}^2}{\overline{w_K}} = 1 - k, \quad \omega[w_\pi] \equiv \frac{\overline{w_\pi^2} - \overline{w_\pi}^2}{\overline{w_\pi}} = k, \tag{B9}$$

$$\overline{w_K w_\pi} - \overline{w_K} \cdot \overline{w_\pi} = -k \cdot (1 - k). \tag{B10}$$

Therefore, Eqs. (14) and (15) give

$$C_\Delta = \langle N\rangle \cdot (1 - 2k) = \langle \pi\rangle - \langle K\rangle, \tag{B11}$$

$$C_\Sigma = \langle N\rangle = \langle \pi\rangle + \langle K\rangle. \tag{B12}$$

As seen from Eqs. (B11) and (B12) the normalization factors C_Δ and C_Σ depend only on the first moments of the extensive state quantities K and π.

However, in general, more information is needed to calculate C_Δ and C_Σ. As an illustration let us consider partly overlapping sets of particles, e.g., the number of charged kaons $K = K^+ + K^-$ and all negatively charged particles H^-. The extensive state quantities A and B are

$$A \equiv K = w_K^{(1)} + w_K^{(2)} + \cdots + w_K^{(N)}, \tag{B13}$$

$$B \equiv H^- = w_-^{(1)} + w_-^{(2)} + \cdots + w_-^{(N)}, \tag{B14}$$

where $w_K^{(j)}$ and $w_-^{(j)}$ are the kaon and negatively charged hadron identities of the jth particle. The kaon $w_K^{(j)}$ and negatively charged hadron $w_-^{(j)}$ identities are defined as $w_K^{(j)} = 1$ and $w_-^{(j)} = 0$ if the jth particle is a K^+, $w_K^{(j)} = 1$ and $w_-^{(j)} = 1$ if the jth particle is a K^-, and $w_K^{(j)} = 0$ and $w_-^{(j)} = 1$ if the jth particle is a negative hadron but not a K^-.

For the single-particle quantities

$$\alpha = w_K, \quad \beta = w_-, \tag{B15}$$

one obtains

$$\overline{w_K} = \overline{w_K^2} = \frac{\langle K^+\rangle + \langle K^-\rangle}{\langle N\rangle} \equiv k_+ + k_- \equiv k, \tag{B16}$$

$$\overline{w_-} = \overline{w_-^2} = \frac{\langle H^-\rangle}{\langle N\rangle} \equiv h_-, \quad \overline{w_K\, w_-} = \frac{\langle K^-\rangle}{\langle N\rangle} = k_-, \tag{B17}$$

[3] Similarly, one can consider sums of any other motional variables, e.g., particle energies, rapidities, etc.

[4] In high-energy physics single-particle distributions are called *inclusive distributions*.

[5] Similarly, one can consider sums of any other particle identities, e.g., negatively charged particles, baryons, etc.

where $N = K^+ + H^-$. Then from Eqs. (B16) and (B17) it follows that

$$\omega[w_K] \equiv \frac{\overline{w_K^2} - \overline{w_K}^2}{\overline{w_K}} = 1 - k,$$
$$\omega[w_-] \equiv \frac{\overline{w_-^2} - \overline{w_-}^2}{\overline{w_-}} = 1 - h_-, \tag{B18}$$

$$\overline{w_K w_-} - \overline{w_K} \cdot \overline{w_-} = k_- - k \cdot h_-. \tag{B19}$$

Therefore, for Eqs. (14) and (15) one finds

$$C_\Delta = \langle N\rangle \cdot (h_- - k) = \langle H^-\rangle - \langle K\rangle, \tag{B20}$$
$$C_\Sigma = \langle N\rangle \cdot (h_- + k - 2k_-) = \langle H^-\rangle + \langle K\rangle - 2\,\langle K^-\rangle. \tag{B21}$$

Thus in this case the normalization factors depend on $\langle K\rangle$ and $\langle H^-\rangle$ and, in addition, on $\langle K^-\rangle$.

3. The most general case

The most general case, which up to now was not considered in the literature, concerns relative fluctuations of two motional extensive quantities, e.g., energy of charged kaons, E_K, and transverse momentum of all negatively charged hadrons, P_T^-. These two sets of particles are partly overlapping. This example corresponds to the following choice of the extensive state quantities A and B:

$$A \equiv E_K = w_K^{(1)}\epsilon^{(1)} + w_K^{(2)}\epsilon^{(2)} + \cdots + w_K^{(N)}\epsilon^{(N)}, \tag{B22}$$
$$B \equiv P_T^- = w_-^{(1)}p_t^{(1)} + w_-^{(2)}p_t^{(2)} + \cdots + w_-^{(N)}p_t^{(N)}, \tag{B23}$$

where $w_K^{(j)}$ and $w_-^{(j)}$ are the kaon and negatively charged hadron identities of the jth particle, and $\epsilon^{(j)}$ and $p_t^{(j)}$ are its energy and transverse momentum. Note that for $\epsilon = p_t = 1$ Eqs. (B22) and (B23) are reduced to Eqs. (B13) and (B14), respectively.

For the single-particle quantities

$$\alpha = w_K\,\epsilon, \quad \beta = w_-\,p_t, \tag{B24}$$

one obtains

$$\overline{\alpha} = k \cdot \overline{\epsilon}, \quad \overline{\alpha^2} = k \cdot \overline{\epsilon^2}, \quad \overline{\beta} = h_- \cdot \overline{p_t},$$
$$\overline{\beta^2} = h_- \cdot \overline{p_t^2}, \quad \overline{\alpha\beta} = k_- \cdot \overline{\epsilon\, p_t}, \tag{B25}$$

where $(n = 1,\ 2)$

$$\overline{\epsilon^n} = \int d\epsilon\, \epsilon^n\, f_K(\epsilon),\ \overline{p_t^n} = \int dp_t\, p_t^n\, f(p_t), \tag{B26}$$

$$\overline{\epsilon\, p_t} = \int d^3p\,\sqrt{\mathbf{p}^2 + m_K^2}\; p_t\, f_{K^-}(\mathbf{p}). \tag{B27}$$

In order to calculate these averages [(B26) and (B27)] one needs to know the single-particle ϵ distribution for kaons, $f_K(\epsilon)$, the p_t distribution for negatively charged hadrons, $f(p_t)$, and the $\mathbf{p}$ distribution for K^-. Then from Eq. (B25) it follows that

$$\omega[\alpha] = \omega[\epsilon] + (1 - k)\cdot\overline{\epsilon},$$
$$\omega[\beta] = \omega[p_t] + (1 - h_-)\cdot\overline{p_t}, \tag{B28}$$
$$\overline{\alpha\beta} - \overline{\alpha}\cdot\overline{\beta} = k_- \cdot \overline{\epsilon\, p_t} - k\, h_-\,\overline{\epsilon}\cdot\overline{p_t}. \tag{B29}$$

Finally, one finds for the normalization factors

$$C_\Delta = \langle P_T^-\rangle \cdot [\,\omega[\epsilon] + \overline{p_t}\,] - \langle E_K\rangle \cdot [\,\omega[p_t] + \overline{\epsilon}\,], \tag{B30}$$
$$C_\Sigma = \langle P_T^-\rangle \cdot [\,\omega[\epsilon] + \overline{p_t}\,] + \langle E_K\rangle \cdot [\,\omega[p_t] + \overline{\epsilon}\,] - 2\,\langle K^-\rangle \cdot \overline{\epsilon\, p_t}. \tag{B31}$$

For the special case $\epsilon = p_t = 1$ one gets

$$\overline{\epsilon} = \overline{\epsilon^2} = \overline{p_t} = \overline{p_t^2} = \overline{\epsilon\, p_t} = 1, \tag{B32}$$

leading to

$$\omega[\epsilon] = \omega[p_t] = 0, \quad \langle E_K\rangle \to \langle K\rangle, \quad \langle P_T\rangle \to \langle H^-\rangle, \tag{B33}$$

and Eqs. (B30) and (B31) reduce to Eqs. (B20) and (B21), respectively.

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