New type of nuclear collective motion: The spin scissors mode

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The coupled dynamics of low-lying modes and various giant resonances are studied with the help of the Wigner function moments method on the basis of time-dependent Hartree-Fock equations in the harmonic oscillator model including spin-orbit potential plus quadrupole-quadrupole and spin-spin residual interactions. New low-lying spin-dependent modes are analyzed. Special attention is paid to the spin scissors mode.

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I. INTRODUCTION

The idea of the possible existence of the collective motion in deformed nuclei similar to the scissors motion continues to attract the attention of physicists who extend it to various kinds of objects, not necessarily nuclei (e.g., magnetic traps, see the review by Heyde *et al.* [1]) and invent new sorts of scissors, for example, the rotational oscillations of neutron skin against a proton-neutron core [2].

The nuclear scissors mode was predicted [3–6] as a counter-rotation of protons against neutrons in deformed nuclei. However, its collectivity turned out to be small. From random-phase approximation (RPA) results which were in qualitative agreement with experiment, it was even questioned whether this mode is collective at all [7,8]. Purely phenomenological models (such as, e.g., the two rotors model [9]) and the sum rule approach [10] did not clear up the situation in this respect. Finally in a very recent review [1] it is concluded that the scissors mode is "weakly collective, but strong on the single-particle scale" and, further [1], "The weakly collective scissors mode excitation has become an ideal test of models—especially microscopic models—of nuclear vibrations. Most models are usually calibrated to reproduce properties of strongly collective excitations (e.g., of $J^\pi = 2^+$ or 3^- states, giant resonances, ...). Weakly-collective phenomena, however, force the models to make genuine predictions and the fact that the transitions in question are strong on the single-particle scale makes it impossible to dismiss failures as a mere detail, especially in the light of the overwhelming experimental evidence for them in many nuclei [11,12]."

The Wigner function moments (WFM) or phase space moments method turns out to be very useful in this situation. On the one hand it is a purely microscopic method, because it is based on the time-dependent Hartree-Fock (TDHF) equation. On the other hand the method works with average values (moments) of operators which have a direct relation to the considered phenomenon and, thus, make a natural bridge with the macroscopic description. This makes it an ideal instrument to describe the basic characteristics (energies and excitation probabilities) of collective excitations such as, in particular, the scissors mode. Our investigations have shown that already the minimal set of collective variables, i.e., phase space moments up to quadratic order, is sufficient to reproduce the most important property of the scissors mode: its inevitable coexistence with the isovector giant quadrupole resonance implying a deformation of the Fermi surface.

Further developments of the WFM method, namely, the switch from TDHF to time-dependent Hartree-Fock Bogoliubov equations, i.e., taking into account pair correlations, allowed us to improve considerably the quantitative description of the scissors mode [13,14]: for rare-earth nuclei the energies are reproduced with ~10% accuracy and $B(M1)$ values are reduced by about a factor of 2 with respect to their non-superfluid values. However, they remain about two times too high with respect to experiment. We suspect that the reason for this last discrepancy is hidden in the spin degrees of freedom, which have so far been ignored by the WFM method. One cannot exclude, that due to spin-dependent interactions some part of the force of $M1$ transitions is shifted to the energy region of 5–10 MeV, where a 1^+ resonance of spin nature is observed [7].

In a recent paper [15] the WFM method was applied for the first time to solve the TDHF equations including spin dynamics. As a first step, only the spin-orbit interaction was included in the consideration, being the most important one among all possible spin-dependent interactions because it enters into the mean field. This allows one to understand the structure of necessary modifications of the method avoiding cumbersome calculations. The most remarkable result was the discovery of a new type of nuclear collective motion: rotational oscillations of "spin-up" nucleons with respect to "spin-down" nucleons (the spin scissors mode). It turns out that the experimentally observed group of peaks in the energy interval 2–4 MeV corresponds very likely to two different types of motion: the conventional (orbital) scissors mode and this new kind of mode, i.e., the spin scissors mode.

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Three low-lying excitations of a new nature were found: isovector and isoscalar spin scissors and the excitation generated by the relative motion of the orbital angular momentum and the spin of the nucleus (they can change their absolute values and directions keeping the total spin unchanged). In the frame of the same approach ten high-lying excitations were also obtained: well-known isoscalar and isovector giant quadrupole resonances (GQR), two resonances of a new nature describing isoscalar and isovector quadrupole vibrations of "spin-up" nucleons with respect to "spin-down" nucleons, and six resonances which can be interpreted as spin-flip modes of various kinds and multipolarity.

The obtained results are very interesting; however, they are only intermediate in our investigation of $M1$ modes. Our finite goal is to get reasonable agreement with experimental data for the conventional scissors mode, especially for its $B(M1)$ factors which remain about two times too strong. We should keep in mind that only the standard spin-orbit potential was taken into account in the paper [15]; spin-dependent residual interactions were completely neglected.

The aim of this work is to get a qualitative understanding of the influence of the spin-spin force on the new states analyzed in Ref. [15], for instance, the spin scissors mode. As a matter of fact we find that the spin-spin interaction does not change the general picture of the positions of excitations described in Ref. [15]. It pushes all levels up proportionally to its strength without changing their order. The most interesting result concerns the $B(M1)$ values of both scissors modes—the spin-spin interaction strongly redistributes $M1$ strength in favor of the spin scissors mode. This is a very promising fact, because it shows that after taking into account in addition pairing [16] one may achieve agreement with experiment.

One of the main points of the present work is, indeed, be that we are able to give a tentative explanation of a recent experimental finding [17] where the $B(M1)$ values in ^{232}Th of the two low-lying magnetic states are inverted in strength in favor of the lowest, i.e., the spin scissors mode, when cranking up the spin-spin interaction. Indeed, the explanation with respect to a triaxial deformation given in Ref. [17] yields a stronger $B(M1)$ value for the higher-lying state, contrary to observation, as remarked by the authors themselves.

The paper is organized as follows. In Sec. II the TDHF equations for the 2×2 density matrix are formulated and their Wigner transform is found. In Sec. III the model Hamiltonian is analyzed and the mean field generated by the spin-spin interaction is derived. In Sec. IV the collective variables are defined and the respective dynamical equations are given. In Sec. V the results of our calculations of energies and $B(M1)$ and $B(E2)$ values are discussed. Last, remarks and the outlook are presented in Sec. VI. The mathematical details are concentrated in Appendices A and B.

II. WIGNER TRANSFORMATION OF TDHF EQUATION WITH SPIN

The TDHF equation in operator form reads [16]

$$i\hbar\dot{\hat{\rho}} = [\hat{h}, \hat{\rho}]. \tag{1}$$

Let us consider its matrix form in coordinate space keeping all spin indices:

$$\begin{aligned} i\hbar\langle \mathbf{r}, s|\dot{\hat{\rho}}|\mathbf{r}'', s''\rangle = \sum_{s'}\int d^3r'[\langle \mathbf{r}, s|\hat{h}|\mathbf{r}', s'\rangle\langle \mathbf{r}', s'|\hat{\rho}|\mathbf{r}'', s''\rangle \\ -\langle \mathbf{r}, s|\hat{\rho}|\mathbf{r}', s'\rangle\langle \mathbf{r}', s'|\hat{h}|\mathbf{r}'', s''\rangle]. \end{aligned} \tag{2}$$

We do not specify the isospin indices to make the formulas more transparent. They are reintroduced at the end.

These equations are solved by the method of moments. The idea of the method is illustrated in the section "Conservation of expectation values of symmetry operators" in Ref [16]. For an arbitrary, time-independent, single-particle operator $\hat{G} = \sum g_{lm}\hat{c}_l^\dagger\hat{c}_m$, from Eq. (1) we obtain for the expectation values in the state $|\Phi(t)\rangle$ (Slater determinant) the following equation:

$$\begin{aligned} i\hbar\frac{d}{dt}\langle\Phi|\hat{G}|\Phi\rangle = i\hbar\,\mathrm{Tr}\,(g\dot{\rho}) = \mathrm{Tr}\,(g[h, \rho]) = \mathrm{Tr}([g, h]\rho) \\ = \langle\Phi|[\hat{G}, \hat{H}]|\Phi\rangle. \end{aligned} \tag{3}$$

The last equality is fulfilled only in the case of full self-consistency of the Hartree-Fock Hamiltonian $\hat{h}$. Equation (3) is the essence of the WFM method. It is seen, that in the case when $\hat{G}$ is the symmetry operator of the original many-body Hamiltonian $\hat{H}$, i.e., $[\hat{G}, \hat{H}] = 0$, its expectation value is conserved also in the HF approach (due to self-consistency, even if $[g, h] \neq 0$) generating the zero energy spurious mode. From the point of view of the WFM method such expectation values are just the integrals of motion. On the other hand, when $[\hat{G}, \hat{H}] \neq 0$, Eq. (3) represents the dynamical equation for the average value $\langle\hat{G}\rangle$. It is natural to expect that it will be coupled with dynamical equations for average values of some other operators [18,19].

From the technical point of view it is more convenient to work with the Wigner function $f(\mathbf{r}, \mathbf{p})$ instead of the density matrix $\langle\mathbf{r}|\hat{\rho}|\mathbf{r}'\rangle$, which allows us to consider its moments in the phase space $(\mathbf{r}, \mathbf{p})$. To this end we rewrite Eq. (2) with the help of the Wigner transformation [16]. To make the formulas more readable we do not write out the coordinate dependence $(\mathbf{r}, \mathbf{p})$ of the functions. With the conventional notation

$$\uparrow \text{ for } \quad s = \tfrac{1}{2} \quad \text{and} \quad \downarrow \text{ for } \quad s = -\tfrac{1}{2}$$

the Wigner transform of Eq. (2) can be written as

$$\begin{aligned} i\hbar\dot{f}^{\uparrow\uparrow} &= i\hbar\{h^{\uparrow\uparrow}, f^{\uparrow\uparrow}\} + h^{\uparrow\downarrow}f^{\downarrow\uparrow} - f^{\uparrow\downarrow}h^{\downarrow\uparrow} + \frac{i\hbar}{2}\{h^{\uparrow\downarrow}, f^{\downarrow\uparrow}\} - \frac{i\hbar}{2}\{f^{\uparrow\downarrow}, h^{\downarrow\uparrow}\} - \frac{\hbar^2}{8}\{\{h^{\uparrow\downarrow}, f^{\downarrow\uparrow}\}\} + \frac{\hbar^2}{8}\{\{f^{\uparrow\downarrow}, h^{\downarrow\uparrow}\}\} + \cdots, \\ i\hbar\dot{f}^{\uparrow\downarrow} &= f^{\uparrow\downarrow}(h^{\uparrow\uparrow} - h^{\downarrow\downarrow}) + \frac{i\hbar}{2}\{(h^{\uparrow\uparrow} + h^{\downarrow\downarrow}), f^{\uparrow\downarrow}\} - \frac{\hbar^2}{8}\{\{(h^{\uparrow\uparrow} - h^{\downarrow\downarrow}), f^{\uparrow\downarrow}\}\} - h^{\uparrow\downarrow}(f^{\uparrow\uparrow} - f^{\downarrow\downarrow}) \\ &\quad + \frac{i\hbar}{2}\{h^{\uparrow\downarrow}, (f^{\uparrow\uparrow} + f^{\downarrow\downarrow})\} + \frac{\hbar^2}{8}\{\{h^{\uparrow\downarrow}, (f^{\uparrow\uparrow} - f^{\downarrow\downarrow})\}\} + \cdots, \end{aligned} \tag{4}$$

where the functions h and f are the Wigner transforms of $\hat{h}$ and $\hat{\rho}$ respectively, $\{f,h\}$ is the Poisson bracket of the functions f and h, and $\{\{f,h\}\}$ is their double Poisson bracket; the dots stand for terms proportional to higher powers of $\hbar$. The remaining two equations are obtained by the obvious change of arrows $\uparrow\leftrightarrow\downarrow$.

It is useful to rewrite the above equations in terms of functions $f^+ = f^{\uparrow\uparrow} + f^{\downarrow\downarrow}$ and $f^- = f^{\uparrow\uparrow} - f^{\downarrow\downarrow}$. By analogy with isoscalar $f^n + f^p$ and isovector $f^n - f^p$ functions one can name the functions f^+ and f^- as spin-scalar and spin-vector ones, respectively. We have

$$\begin{aligned}
i\hbar\dot{f}^+ &= \frac{i\hbar}{2}\{h^+,f^+\} + \frac{i\hbar}{2}\{h^-,f^-\} + i\hbar\{h^{\uparrow\downarrow},f^{\downarrow\uparrow}\} + i\hbar\{h^{\downarrow\uparrow},f^{\uparrow\downarrow}\} + \cdots,\\
i\hbar\dot{f}^- &= \frac{i\hbar}{2}\{h^+,f^-\} + \frac{i\hbar}{2}\{h^-,f^+\} - 2h^{\downarrow\uparrow}f^{\uparrow\downarrow} + 2h^{\uparrow\downarrow}f^{\downarrow\uparrow} + \frac{\hbar^2}{4}\{\{h^{\downarrow\uparrow},f^{\uparrow\downarrow}\}\} - \frac{\hbar^2}{4}\{\{h^{\uparrow\downarrow},f^{\downarrow\uparrow}\}\} + \cdots,\\
i\hbar\dot{f}^{\uparrow\downarrow} &= -h^{\uparrow\downarrow}f^- + h^-f^{\uparrow\downarrow} + \frac{i\hbar}{2}\{h^{\uparrow\downarrow},f^+\} + \frac{i\hbar}{2}\{h^+,f^{\uparrow\downarrow}\} + \frac{\hbar^2}{8}\{\{h^{\uparrow\downarrow},f^-\}\} - \frac{\hbar^2}{8}\{\{h^-,f^{\uparrow\downarrow}\}\} + \cdots,\\
i\hbar\dot{f}^{\downarrow\uparrow} &= h^{\downarrow\uparrow}f^- - h^-f^{\downarrow\uparrow} + \frac{i\hbar}{2}\{h^{\downarrow\uparrow},f^+\} + \frac{i\hbar}{2}\{h^+,f^{\downarrow\uparrow}\} - \frac{\hbar^2}{8}\{\{h^{\downarrow\uparrow},f^-\}\} + \frac{\hbar^2}{8}\{\{h^-,f^{\downarrow\uparrow}\}\} + \cdots,
\end{aligned} \tag{5}$$

where $h^\pm = h^{\uparrow\uparrow} \pm h^{\downarrow\downarrow}$.

III. MODEL HAMILTONIAN

The microscopic Hamiltonian of the model, harmonic oscillator with spin-orbit potential plus separable quadrupole-quadrupole and spin-spin residual interactions is given by

$$\hat{H} = \sum_{i=1}^{A}\left[\frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}_i^2 - \eta\hat{\mathbf{l}}_i\hat{\mathbf{S}}_i\right] + \hat{H}_{\rm qq} + \hat{H}_{\rm ss}, \tag{6}$$

with

$$\hat{H}_{\rm qq} = \sum_{\mu=-2}^{2}(-1)^\mu\left\{\bar{\kappa}\sum_i^Z\sum_j^N + \frac{\kappa}{2}\left[\sum_{i,j(i\neq j)}^Z + \sum_{i,j(i\neq j)}^N\right]\right\} \times q_{2-\mu}(\mathbf{r}_i)q_{2\mu}(\mathbf{r}_j), \tag{7}$$

$$\hat{H}_{\rm ss} = \sum_{\mu=-1}^{1}(-1)^\mu\left\{\bar{\chi}\sum_i^Z\sum_j^N + \frac{\chi}{2}\left[\sum_{i,j(i\neq j)}^Z + \sum_{i,j(i\neq j)}^N\right]\right\} \times \hat{S}_{-\mu}(i)\hat{S}_\mu(j)\,\delta(\mathbf{r}_i - \mathbf{r}_j), \tag{8}$$

where N and Z are the numbers of neutrons and protons and $\hat{S}_\mu$ are spin matrices [20]:

$$\hat{S}_1 = -\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad \hat{S}_0 = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \hat{S}_{-1} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{9}$$

The quadrupole operator $q_{2\mu}(\mathbf{r}) = \sqrt{16\pi/5}\,r^2Y_{2\mu}(\theta,\phi)$ can be written as the tensor product:

$$q_{2\mu}(\mathbf{r}) = \sqrt{6}\{r\otimes r\}_{2\mu},$$

where $\{r\otimes r\}_{\lambda\mu} = \sum_{\sigma,\nu} C_{1\sigma,1\nu}^{\lambda\mu} r_\sigma r_\nu$; cyclic coordinates r_{-1}, r_0, and r_1 are defined in Ref. [20]; and $C_{1\sigma,1\nu}^{\lambda\mu}$ is a Clebsch-Gordan coefficient.

A. Mean field

Let us analyze the mean field generated by this Hamiltonian.

1. Spin-orbit potential

Written in cyclic coordinates, the spin-orbit part of the Hamiltonian reads

$$\hat{h}_{ls} = -\eta\sum_{\mu=-1}^{1}(-)^\mu\hat{l}_\mu\hat{S}_{-\mu} = -\eta\begin{pmatrix}\hat{l}_0\frac{\hbar}{2} & \hat{l}_{-1}\frac{\hbar}{\sqrt{2}}\\ -\hat{l}_1\frac{\hbar}{\sqrt{2}} & -\hat{l}_0\frac{\hbar}{2}\end{pmatrix},$$

where [20]

$$\hat{l}_\mu = -\hbar\sqrt{2}\sum_{\nu,\alpha} C_{1\nu,1\alpha}^{1\mu} r_\nu\nabla_\alpha \tag{10}$$

and

$$\begin{aligned}
\hat{l}_1 &= \hbar(r_0\nabla_1 - r_1\nabla_0) = -\frac{1}{\sqrt{2}}(\hat{l}_x + i\hat{l}_y),\\
\hat{l}_0 &= \hbar(r_{-1}\nabla_1 - r_1\nabla_{-1}) = \hat{l}_z,\\
\hat{l}_{-1} &= \hbar(r_{-1}\nabla_0 - r_0\nabla_{-1}) = \frac{1}{\sqrt{2}}(\hat{l}_x - i\hat{l}_y),\\
\hat{l}_x &= -i\hbar(y\nabla_z - z\nabla_y),\quad \hat{l}_y = -i\hbar(z\nabla_x - x\nabla_z),\\
\hat{l}_z &= -i\hbar(x\nabla_y - y\nabla_x).
\end{aligned} \tag{11}$$

Matrix elements of $\hat{h}_{ls}$ in coordinate space can be obviously written [15] as

$$\begin{aligned}
&\langle\mathbf{r}_1,s_1|\hat{h}_{ls}|\mathbf{r}_2,s_2\rangle\\
&= -\frac{\hbar}{\sqrt{2}}\eta\left[\frac{\hat{l}_0(\mathbf{r}_1)}{\sqrt{2}}(\delta_{s_1\uparrow}\delta_{s_2\uparrow} - \delta_{s_1\downarrow}\delta_{s_2\downarrow})\right.\\
&\quad\left. + \hat{l}_{-1}(\mathbf{r}_1)\delta_{s_1\uparrow}\delta_{s_2\downarrow} - \hat{l}_1(\mathbf{r}_1)\delta_{s_1\downarrow}\delta_{s_2\uparrow}\right]\delta(\mathbf{r}_1 - \mathbf{r}_2).
\end{aligned} \tag{12}$$

The Wigner transform of Eq. (12) reads [15]

$$\begin{aligned}
h_{ls}^{s_1s_2}(\mathbf{r},\mathbf{p}) = -\frac{\hbar}{2}\eta[&l_0(\mathbf{r},\mathbf{p})(\delta_{s_1\uparrow}\delta_{s_2\uparrow} - \delta_{s_1\downarrow}\delta_{s_2\downarrow})\\
&+ \sqrt{2}l_{-1}(\mathbf{r},\mathbf{p})\delta_{s_1\uparrow}\delta_{s_2\downarrow} - \sqrt{2}l_1(\mathbf{r},\mathbf{p})\delta_{s_1\downarrow}\delta_{s_2\uparrow}],
\end{aligned} \tag{13}$$

where $l_\mu = -i\sqrt{2}\sum_{\nu,\alpha} C_{1\nu,1\alpha}^{1\mu} r_\nu p_\alpha$.

2. Quadrupole-quadrupole interaction

The contribution of $\hat{H}_{\rm qq}$ to the mean-field potential is easily found by replacing one of the $q_{2\mu}$ operators by the average value. We have

$$V^{\tau}_{\rm qq} = 6\sum_{\mu}(-1)^{\mu} Z^{\tau+}_{2-\mu}\{r\otimes r\}_{2\mu}. \tag{14}$$

Here

$$\begin{aligned} Z^{n+}_{2\mu} &= \kappa R^{n+}_{2\mu} + \bar{\kappa} R^{p+}_{2\mu}, \quad Z^{p+}_{2\mu} = \kappa R^{p+}_{2\mu} + \bar{\kappa} R^{n+}_{2\mu},\\ R^{\tau+}_{\lambda\mu}(t) &= \int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu} f^{\tau+}(\mathbf{r},\mathbf{p},t), \end{aligned} \tag{15}$$

with $\int d(\mathbf{p},\mathbf{r}) \equiv (2\pi\hbar)^{-3}\int d^3p\int d^3r$ and τ being the isospin index.

3. Spin-spin interaction

The analogous expression for $\hat{H}_{\rm ss}$ is found in the standard way, with the Hartree-Fock contribution given [16] by

$$\Gamma_{kk'}(t) = \sum_{ll'} \bar{v}_{kl'k'l}\rho_{ll'}(t), \tag{16}$$

where $\bar{v}_{kl'k'l}$ is the antisymmetrized matrix element of the two-body interaction $v(1,2)$. Identifying the indices k, k', l, and l' with the set of coordinates $(\mathbf{r}, s, \tau)$, i.e. (position, spin, isospin), one rewrites Eq. (16) as

$$V^{\rm HF}(\mathbf{r}_1,s_1,\tau_1;\mathbf{r}'_1,s'_1,\tau'_1;t) = \int d^3r_2\int d^3r'_2\sum_{s_2,s'_2}\sum_{\tau_2,\tau'_2}\langle\mathbf{r}_1,s_1,\tau_1;\mathbf{r}_2,s_2,\tau_2|\hat{v}|\mathbf{r}'_1,s'_1,\tau'_1;\mathbf{r}'_2,s'_2,\tau'_2\rangle_{a.s.}\rho(\mathbf{r}'_2,s'_2,\tau'_2;\mathbf{r}_2,s_2,\tau_2;t).$$

Let us consider the neutron-proton part of the spin-spin interaction. In this case

$$\hat{v} = v(\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2)\sum_{\mu=-1}^{1}(-1)^{\mu}\hat{S}_{-\mu}(1)\hat{S}_{\mu}(2)\delta_{\tau_1 p}\delta_{\tau_2 n},$$

where $\hat{\mathbf{r}}_1$ is the position operator: $\hat{\mathbf{r}}_1|\mathbf{r}_1\rangle = \mathbf{r}_1|\mathbf{r}_1\rangle$, $\langle\mathbf{r}_1|\hat{\mathbf{r}}_1|\mathbf{r}'_1\rangle = \langle\mathbf{r}_1|\mathbf{r}'_1\rangle\mathbf{r}'_1 = \delta(\mathbf{r}_1-\mathbf{r}'_1)\mathbf{r}'_1$.

For the Hartree term one finds

$$\begin{aligned} &\langle\mathbf{r}_1,s_1,\tau_1;\mathbf{r}_2,s_2,\tau_2|\hat{v}|\mathbf{r}'_1,s'_1,\tau'_1;\mathbf{r}'_2,s'_2,\tau'_2\rangle\\ &\quad = \delta(\mathbf{r}_1-\mathbf{r}'_1)\delta(\mathbf{r}_2-\mathbf{r}'_2)v(\mathbf{r}'_1-\mathbf{r}'_2)\sum_{\mu=-1}^{1}(-1)^{\mu}\langle s_1,\tau_1;s_2,\tau_2|\hat{S}_{-\mu}(1)\hat{S}_{\mu}(2)\delta_{\tau_1 p}\delta_{\tau_2 n}|s'_1,\tau'_1;s'_2,\tau'_2\rangle, \end{aligned}$$

$$\begin{aligned} V^{H}(\mathbf{r}_1,s_1,\tau_1;\mathbf{r}'_1,s'_1,\tau'_1;t) &= \int d^3r_2\int d^3r'_2\sum_{s_2,s'_2}\sum_{\tau_2,\tau'_2}\langle\mathbf{r}_1,s_1,\tau_1;\mathbf{r}_2,s_2,\tau_2|\hat{v}|\mathbf{r}'_1,s'_1,\tau'_1;\mathbf{r}'_2,s'_2,\tau'_2\rangle\rho(\mathbf{r}'_2,s'_2,\tau'_2;\mathbf{r}_2,s_2,\tau_2;t)\\ &= \delta_{\tau_1 p}\delta_{\tau'_1 p}\sum_{s_2,s'_2}\sum_{\mu=-1}^{1}(-1)^{\mu}\langle s_1|\hat{S}_{-\mu}(1)|s'_1\rangle\langle s_2|\hat{S}_{\mu}(2)|s'_2\rangle\delta(\mathbf{r}_1-\mathbf{r}'_1)\int d^3r_2 v(\mathbf{r}_1-\mathbf{r}_2)\rho(\mathbf{r}_2,s'_2,n;\mathbf{r}_2,s_2,n;t). \end{aligned}$$

The Fock term reads

$$\begin{aligned} &\langle\mathbf{r}_1,s_1,\tau_1;\mathbf{r}_2,s_2,\tau_2|\hat{v}|\mathbf{r}'_2,s'_2,\tau'_2;\mathbf{r}'_1,s'_1,\tau'_1\rangle\\ &\quad = \delta(\mathbf{r}_1-\mathbf{r}'_2)\delta(\mathbf{r}_2-\mathbf{r}'_1)v(\mathbf{r}'_2-\mathbf{r}'_1)\sum_{\mu=-1}^{1}(-1)^{\mu}\langle s_1,\tau_1;s_2,\tau_2|\hat{S}_{-\mu}(1)\hat{S}_{\mu}(2)\delta_{\tau_1 p}\delta_{\tau_2 n}|s'_2,\tau'_2;s'_1,\tau'_1\rangle, \end{aligned}$$

$$\begin{aligned} V^{F}(\mathbf{r}_1,s_1,\tau_1;\mathbf{r}'_1,s'_1,\tau'_1;t) &= -\int d^3r_2\int d^3r'_2\sum_{s_2,s'_2}\sum_{\tau_2,\tau'_2}\langle\mathbf{r}_1,s_1,\tau_1;\mathbf{r}_2,s_2,\tau_2|\hat{v}|\mathbf{r}'_2,s'_2,\tau'_2;\mathbf{r}'_1,s'_1,\tau'_1\rangle\rho(\mathbf{r}'_2,s'_2,\tau'_2;\mathbf{r}_2,s_2,\tau_2;t)\\ &= -\delta_{\tau_1 p}\delta_{\tau'_1 n}\sum_{s_2,s'_2}\sum_{\mu=-1}^{1}(-1)^{\mu}\langle s_1|\hat{S}_{-\mu}(1)|s'_2\rangle\langle s_2|\hat{S}_{\mu}(2)|s'_1\rangle v(\mathbf{r}_1-\mathbf{r}'_1)\rho(\mathbf{r}_1,s'_2,p;\mathbf{r}'_1,s_2,n;t). \end{aligned}$$

Taking into account the relations

$$\langle s|\hat{S}_{-1}|s'\rangle = \frac{\hbar}{\sqrt{2}}\delta_{s\downarrow}\delta_{s'\uparrow}, \quad \langle s|\hat{S}_{0}|s'\rangle = \frac{\hbar}{2}\delta_{s,s'}(\delta_{s\uparrow}-\delta_{s\downarrow}), \quad \langle s|\hat{S}_{1}|s'\rangle = -\frac{\hbar}{\sqrt{2}}\delta_{s\uparrow}\delta_{s'\downarrow},$$

and $v(\mathbf{r}-\mathbf{r}')=\bar{\chi}\delta(\mathbf{r}-\mathbf{r}')$, one finds the following for the mean field generated by the proton-neutron part of $\hat{H}_{\rm ss}$:

$$\begin{aligned}\Gamma_{pn}(\mathbf{r},s,\tau;\mathbf{r}',s',\tau';t)=\bar{\chi}\frac{\hbar^2}{4}\{&\delta_{\tau p}\delta_{\tau' p}[\delta_{s\downarrow}\delta_{s'\uparrow}\rho(\mathbf{r},\downarrow,n;\mathbf{r}',\uparrow,n;t)+\delta_{s\uparrow}\delta_{s'\downarrow}\rho(\mathbf{r},\uparrow,n;\mathbf{r}',\downarrow,n;t)]\\&-\delta_{\tau p}\delta_{\tau' n}[\delta_{s\downarrow}\delta_{s'\downarrow}\rho(\mathbf{r},\uparrow,p;\mathbf{r}',\uparrow,n;t)+\delta_{s\uparrow}\delta_{s'\uparrow}\rho(\mathbf{r},\downarrow,p;\mathbf{r}',\downarrow,n;t)]\\&+\frac{1}{2}\delta_{\tau p}\delta_{\tau' p}(\delta_{s\uparrow}\delta_{s'\uparrow}-\delta_{s\downarrow}\delta_{s'\downarrow})[\rho(\mathbf{r},\uparrow,n;\mathbf{r}',\uparrow,n;t)-\rho(\mathbf{r},\downarrow,n;\mathbf{r}',\downarrow,n;t)]\\&+\frac{1}{2}\delta_{\tau p}\delta_{\tau' n}[\delta_{s\uparrow}\delta_{s'\downarrow}\rho(\mathbf{r},\uparrow,p;\mathbf{r}',\downarrow,n;t)+\delta_{s\downarrow}\delta_{s'\uparrow}\rho(\mathbf{r},\downarrow,p;\mathbf{r}',\uparrow,n;t)\\&-\delta_{s\uparrow}\delta_{s'\uparrow}\rho(\mathbf{r},\uparrow,p;\mathbf{r}',\uparrow,n;t)-\delta_{s\downarrow}\delta_{s'\downarrow}\rho(\mathbf{r},\downarrow,p;\mathbf{r}',\downarrow,n;t)]\}\delta(\mathbf{r}-\mathbf{r}')+\bar{\chi}\frac{\hbar^2}{4}\{p\leftrightarrow n\}\delta(\mathbf{r}-\mathbf{r}').\end{aligned}\tag{17}$$

The expression for the mean field $\Gamma_{pp}(\mathbf{r},s,\tau;\mathbf{r}',s',\tau';t)$ generated by the proton-proton part of $\hat{H}_{\rm ss}$ can be obtained from Eq. (17) by replacing index n by p and the strength constant $\bar{\chi}$ by χ. The proton mean field is defined as the sum of these two terms $\Gamma_{pp}(\mathbf{r},s,p;\mathbf{r}',s',p;t)+\Gamma_{pn}(\mathbf{r},s,p;\mathbf{r}',s',p;t)$. Its Wigner transform can be written as

$$\begin{aligned}V_p^{ss'}(\mathbf{r},t)=&\,3\chi\frac{\hbar^2}{8}\{\delta_{s\downarrow}\delta_{s'\uparrow}n_p^{\downarrow\uparrow}+\delta_{s\uparrow}\delta_{s'\downarrow}n_p^{\uparrow\downarrow}-\delta_{s\downarrow}\delta_{s'\downarrow}n_p^{\uparrow\uparrow}-\delta_{s\uparrow}\delta_{s'\uparrow}n_p^{\downarrow\downarrow}\}\\&+\bar{\chi}\frac{\hbar^2}{8}\{2\delta_{s\downarrow}\delta_{s'\uparrow}n_n^{\downarrow\uparrow}+2\delta_{s\uparrow}\delta_{s'\downarrow}n_n^{\uparrow\downarrow}+(\delta_{s\uparrow}\delta_{s'\uparrow}-\delta_{s\downarrow}\delta_{s'\downarrow})(n_n^{\uparrow\uparrow}-n_n^{\downarrow\downarrow})\},\end{aligned}\tag{18}$$

where $n_\tau^{ss'}(\mathbf{r},t)=\int\frac{d^3p}{(2\pi\hbar)^3}f_\tau^{ss'}(\mathbf{r},\mathbf{p},t)$. The Wigner transform of the neutron mean field $V_n^{ss'}$ is obtained from Eq. (18) by the obvious change of indices $p\leftrightarrow n$. The Wigner function f and the density matrix ρ are connected by the relation $f_{\tau\tau'}^{ss'}(\mathbf{r},\mathbf{p},t)=\int d^3q\, e^{-i\mathbf{pq}/\hbar}\rho(\mathbf{r}_1,s,\tau;\mathbf{r}_2,s',\tau';t)$, with $\mathbf{q}=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{r}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$. Integrating this relation over $\mathbf{p}$ with $\tau'=\tau$ one finds

$$n_\tau^{ss'}(\mathbf{r},t)=\rho(\mathbf{r},s,\tau;\mathbf{r},s',\tau;t).$$

By definition the diagonal elements of the density matrix describe the proper densities. Therefore $n_\tau^{ss}(\mathbf{r},t)$ is the density of spin-up nucleons (if $s=\uparrow$) or spin-down nucleons (if $s=\downarrow$). Off diagonal in spin elements of the density matrix $n_\tau^{ss'}(\mathbf{r},t)$ are spin-flip characteristics and can be called spin-flip densities.

IV. EQUATIONS OF MOTION

Integrating the set of Eqs. (5) over phase space with the weights

$$W=\{r\otimes p\}_{\lambda\mu},\quad \{r\otimes r\}_{\lambda\mu},\quad \{p\otimes p\}_{\lambda\mu},\ \text{and } 1,\tag{19}$$

one gets dynamic equations for the following collective variables:

$$\begin{aligned}L_{\lambda\mu}^{\tau\varsigma}(t)&=\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu}f^{\tau\varsigma}(\mathbf{r},\mathbf{p},t),\quad R_{\lambda\mu}^{\tau\varsigma}(t)=\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu}f^{\tau\varsigma}(\mathbf{r},\mathbf{p},t),\\P_{\lambda\mu}^{\tau\varsigma}(t)&=\int d(\mathbf{p},\mathbf{r})\{p\otimes p\}_{\lambda\mu}f^{\tau\varsigma}(\mathbf{r},\mathbf{p},t),\quad F^{\tau\varsigma}(t)=\int d(\mathbf{p},\mathbf{r})f^{\tau\varsigma}(\mathbf{r},\mathbf{p},t),\end{aligned}\tag{20}$$

where $\varsigma=+,\,-,\,\uparrow\downarrow,\,\downarrow\uparrow$. We already called the functions $f^+=f^{\uparrow\uparrow}+f^{\downarrow\downarrow}$ and $f^-=f^{\uparrow\uparrow}-f^{\downarrow\downarrow}$ spin-scalar and spin-vector ones, respectively. It is, therefore, natural to call the corresponding collective variables $X_{\lambda\mu}^+(t)$ and $X_{\lambda\mu}^-(t)$ spin-scalar and spin-vector variables. The required expressions for $h^\pm$, $h^{\uparrow\downarrow}$, and $h^{\downarrow\uparrow}$ are

$$h_\tau^+=\frac{p^2}{m}+m\,\omega^2r^2+12\sum_\mu(-1)^\mu Z_{2\mu}^{\tau+}(t)\{r\otimes r\}_{2-\mu}+V_\tau^+(\mathbf{r},t),$$

$$h_\tau^-=-\hbar\eta l_0+V_\tau^-(\mathbf{r},t),\quad h_\tau^{\uparrow\downarrow}=-\frac{\hbar}{\sqrt{2}}\eta l_{-1}+V_\tau^{\uparrow\downarrow}(\mathbf{r},t),\quad h_\tau^{\downarrow\uparrow}=\frac{\hbar}{\sqrt{2}}\eta l_1+V_\tau^{\downarrow\uparrow}(\mathbf{r},t),$$

where according to Eq. (18)

$$\begin{aligned}&V_p^+(\mathbf{r},t)=-3\frac{\hbar^2}{8}\chi n_p^+(\mathbf{r},t),\quad V_p^-(\mathbf{r},t)=3\frac{\hbar^2}{8}\chi n_p^-(\mathbf{r},t)+\frac{\hbar^2}{4}\bar{\chi}n_n^-(\mathbf{r},t),\\&V_p^{\uparrow\downarrow}(\mathbf{r},t)=3\frac{\hbar^2}{8}\chi n_p^{\uparrow\downarrow}(\mathbf{r},t)+\frac{\hbar^2}{4}\bar{\chi}n_n^{\uparrow\downarrow}(\mathbf{r},t),\quad V_p^{\downarrow\uparrow}(\mathbf{r},t)=3\frac{\hbar^2}{8}\chi n_p^{\downarrow\uparrow}(\mathbf{r},t)+\frac{\hbar^2}{4}\bar{\chi}n_n^{\downarrow\uparrow}(\mathbf{r},t),\end{aligned}\tag{21}$$

and the neutron potentials V_n^ς are obtained by the obvious change of indices $p\leftrightarrow n$.

The integration yields

$$\dot{L}^{+}_{\lambda\mu} = \frac{1}{m}P^{+}_{\lambda\mu} - m\,\omega^2 R^{+}_{\lambda\mu} + 12\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{}^{11j}_{2\lambda 1}\right\}\{Z^{+}_{2}\otimes R^{+}_{j}\}_{\lambda\mu}$$
$$- i\hbar\frac{\eta}{2}\left[\mu L^{-}_{\lambda\mu} + \sqrt{(\lambda-\mu)(\lambda+\mu+1)}L^{\uparrow\downarrow}_{\lambda\mu+1} + \sqrt{(\lambda+\mu)(\lambda-\mu+1)}L^{\downarrow\uparrow}_{\lambda\mu-1}\right]$$
$$- \int d^3r\left[\frac{1}{2}n^{+}\{r\otimes\nabla\}_{\lambda\mu}V^{+} + \frac{1}{2}n^{-}\{r\otimes\nabla\}_{\lambda\mu}V^{-} + n^{\downarrow\uparrow}\{r\otimes\nabla\}_{\lambda\mu}V^{\uparrow\downarrow} + n^{\uparrow\downarrow}\{r\otimes\nabla\}_{\lambda\mu}V^{\downarrow\uparrow}\right],$$

$$\dot{L}^{-}_{\lambda\mu} = \frac{1}{m}P^{-}_{\lambda\mu} - m\,\omega^2 R^{-}_{\lambda\mu} + 12\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{}^{11j}_{2\lambda 1}\right\}\{Z^{+}_{2}\otimes R^{-}_{j}\}_{\lambda\mu} - i\hbar\frac{\eta}{2}\mu L^{+}_{\lambda\mu} - \frac{\hbar^2}{2}\eta\delta_{\lambda,1}[\delta_{\mu,-1}F^{\uparrow\downarrow} + \delta_{\mu,1}F^{\downarrow\uparrow}]$$
$$- \frac{1}{2}\int d^3r[n^{-}\{r\otimes\nabla\}_{\lambda\mu}V^{+} + n^{+}\{r\otimes\nabla\}_{\lambda\mu}V^{-}] - 2\frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu}[h^{\uparrow\downarrow}f^{\downarrow\uparrow} - h^{\downarrow\uparrow}f^{\uparrow\downarrow}],$$

$$\dot{L}^{\uparrow\downarrow}_{\lambda\mu+1} = \frac{1}{m}P^{\uparrow\downarrow}_{\lambda\mu+1} - m\,\omega^2 R^{\uparrow\downarrow}_{\lambda\mu+1} + 12\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{}^{11j}_{2\lambda 1}\right\}\{Z^{+}_{2}\otimes R^{\uparrow\downarrow}_{j}\}_{\lambda\mu+1}$$
$$- i\hbar\frac{\eta}{4}\sqrt{(\lambda-\mu)(\lambda+\mu+1)}L^{+}_{\lambda\mu} + \frac{\hbar^2}{4}\eta\delta_{\lambda,1}[\delta_{\mu,0}F^{-} + \sqrt{2}\delta_{\mu,-1}F^{\uparrow\downarrow}]$$
$$- \frac{1}{2}\int d^3r[n^{\uparrow\downarrow}\{r\otimes\nabla\}_{\lambda\mu+1}V^{+} + n^{+}\{r\otimes\nabla\}_{\lambda\mu+1}V^{\uparrow\downarrow}] - \frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu+1}[h^{-}f^{\uparrow\downarrow} - h^{\uparrow\downarrow}f^{-}],$$

$$\dot{L}^{\downarrow\uparrow}_{\lambda\mu-1} = \frac{1}{m}P^{\downarrow\uparrow}_{\lambda\mu-1} - m\,\omega^2 R^{\downarrow\uparrow}_{\lambda\mu-1} + 12\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{}^{11j}_{2\lambda 1}\right\}\{Z^{+}_{2}\otimes R^{\downarrow\uparrow}_{j}\}_{\lambda\mu-1}$$
$$- i\hbar\frac{\eta}{4}\sqrt{(\lambda+\mu)(\lambda-\mu+1)}L^{+}_{\lambda\mu} + \frac{\hbar^2}{4}\eta\delta_{\lambda,1}[\delta_{\mu,0}F^{-} - \sqrt{2}\delta_{\mu,1}F^{\downarrow\uparrow}]$$
$$- \frac{1}{2}\int d^3r[n^{\downarrow\uparrow}\{r\otimes\nabla\}_{\lambda\mu-1}V^{+} + n^{+}\{r\otimes\nabla\}_{\lambda\mu-1}V^{\downarrow\uparrow}] - \frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu-1}[h^{\downarrow\uparrow}f^{-} - h^{-}f^{\downarrow\uparrow}],$$

$$\dot{F}^{-} = 2\eta[L^{\downarrow\uparrow}_{1-1} + L^{\uparrow\downarrow}_{11}],\quad \dot{F}^{\uparrow\downarrow} = -\eta[L^{-}_{1-1} - \sqrt{2}L^{\uparrow\downarrow}_{10}],\quad \dot{F}^{\downarrow\uparrow} = -\eta[L^{-}_{11} + \sqrt{2}L^{\downarrow\uparrow}_{10}],$$

$$\dot{R}^{+}_{\lambda\mu} = \frac{2}{m}L^{+}_{\lambda\mu} - i\hbar\frac{\eta}{2}\left[\mu R^{-}_{\lambda\mu} + \sqrt{(\lambda-\mu)(\lambda+\mu+1)}R^{\uparrow\downarrow}_{\lambda\mu+1} + \sqrt{(\lambda+\mu)(\lambda-\mu+1)}R^{\downarrow\uparrow}_{\lambda\mu-1}\right],$$

$$\dot{R}^{-}_{\lambda\mu} = \frac{2}{m}L^{-}_{\lambda\mu} - i\hbar\frac{\eta}{2}\mu R^{+}_{\lambda\mu} - 2\frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu}[h^{\uparrow\downarrow}f^{\downarrow\uparrow} - h^{\downarrow\uparrow}f^{\uparrow\downarrow}],$$

$$\dot{R}^{\uparrow\downarrow}_{\lambda\mu+1} = \frac{2}{m}L^{\uparrow\downarrow}_{\lambda\mu+1} - i\hbar\frac{\eta}{4}\sqrt{(\lambda-\mu)(\lambda+\mu+1)}R^{+}_{\lambda\mu} - \frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu+1}[h^{-}f^{\uparrow\downarrow} - h^{\uparrow\downarrow}f^{-}],$$

$$\dot{R}^{\downarrow\uparrow}_{\lambda\mu-1} = \frac{2}{m}L^{\downarrow\uparrow}_{\lambda\mu-1} - i\hbar\frac{\eta}{4}\sqrt{(\lambda+\mu)(\lambda-\mu+1)}R^{+}_{\lambda\mu} - \frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu-1}[h^{\downarrow\uparrow}f^{-} - h^{-}f^{\downarrow\uparrow}],$$

$$\dot{P}^{+}_{\lambda\mu} = -2m\,\omega^2 L^{+}_{\lambda\mu} + 24\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{}^{11j}_{2\lambda 1}\right\}\{Z^{+}_{2}\otimes L^{+}_{j}\}_{\lambda\mu}$$
$$- i\hbar\frac{\eta}{2}\left[\mu P^{-}_{\lambda\mu} + \sqrt{(\lambda-\mu)(\lambda+\mu+1)}P^{\uparrow\downarrow}_{\lambda\mu+1} + \sqrt{(\lambda+\mu)(\lambda-\mu+1)}P^{\downarrow\uparrow}_{\lambda\mu-1}\right]$$
$$- \int d^3r[\{J^{+}\otimes\nabla\}_{\lambda\mu}V^{+} + \{J^{-}\otimes\nabla\}_{\lambda\mu}V^{-} + 2\{J^{\downarrow\uparrow}\otimes\nabla\}_{\lambda\mu}V^{\uparrow\downarrow} + 2\{J^{\uparrow\downarrow}\otimes\nabla\}_{\lambda\mu}V^{\downarrow\uparrow}],$$

$$\dot{P}^{-}_{\lambda\mu} = -2m\,\omega^2 L^{-}_{\lambda\mu} + 24\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{}^{11j}_{2\lambda 1}\right\}\{Z^{+}_{2}\otimes L^{-}_{j}\}_{\lambda\mu} - i\hbar\frac{\eta}{2}\mu P^{+}_{\lambda\mu}$$
$$- \int d^3r[\{J^{-}\otimes\nabla\}_{\lambda\mu}V^{+} + \{J^{+}\otimes\nabla\}_{\lambda\mu}V^{-}] - 2\frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{p\otimes p\}_{\lambda\mu}[h^{\uparrow\downarrow}f^{\downarrow\uparrow} - h^{\downarrow\uparrow}f^{\uparrow\downarrow}],$$

$$\dot{P}^{\uparrow\downarrow}_{\lambda\mu+1} = -2m\,\omega^2 L^{\uparrow\downarrow}_{\lambda\mu+1} + 24\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{{}^{11j}_{2\lambda 1}\right\}\{Z^{+}_{2}\otimes L^{\uparrow\downarrow}_{j}\}_{\lambda\mu+1} - i\hbar\frac{\eta}{4}\sqrt{(\lambda-\mu)(\lambda+\mu+1)}P^{+}_{\lambda\mu}$$
$$- \int d^3r[\{J^{\uparrow\downarrow}\otimes\nabla\}_{\lambda\mu+1}V^{+} + \{J^{+}\otimes\nabla\}_{\lambda\mu+1}V^{\uparrow\downarrow}] - \frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{p\otimes p\}_{\lambda\mu+1}[h^{-}f^{\uparrow\downarrow} - h^{\uparrow\downarrow}f^{-}],$$

$$\dot{P}^{\downarrow\uparrow}_{\lambda\mu-1} = -2m\,\omega^2 L^{\downarrow\uparrow}_{\lambda\mu-1} + 24\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\left\{\begin{smallmatrix}1 & 1 & j\\ 2 & \lambda & 1\end{smallmatrix}\right\}\{Z^+_2\otimes L^{\downarrow\uparrow}_j\}_{\lambda\mu-1} - i\hbar\frac{\eta}{4}\sqrt{(\lambda+\mu)(\lambda-\mu+1)}P^+_{\lambda\mu}$$
$$-\int d^3r[\{J^{\downarrow\uparrow}\otimes\nabla\}_{\lambda\mu-1}V^+ + \{J^+\otimes\nabla\}_{\lambda\mu-1}V^{\downarrow\uparrow}] - \frac{i}{\hbar}\int d(\mathbf{p},\mathbf{r})\{p\otimes p\}_{\lambda\mu-1}[h^{\downarrow\uparrow}f^- - h^- f^{\downarrow\uparrow}], \quad (22)$$

where $\left\{\begin{smallmatrix}1 & 1 & j\\ 2 & \lambda & 1\end{smallmatrix}\right\}$ is the Wigner $6j$-symbol and $J^\varsigma_\nu(\mathbf{r},t) = \int \frac{d^3p}{(2\pi\hbar)^3}p_\nu f^\varsigma(\mathbf{r},\mathbf{p},t)$ is the current. For the sake of simplicity the isospin and the time dependence of tensors is not written out. It is easy to see that Eqs. (22) are nonlinear due to quadrupole-quadrupole and spin-spin interactions. We solve them in the small amplitude approximation, by linearizing the equations. This procedure helps also to solve another problem: to represent the integral terms in Eqs. (22) as the linear combination of collective variables (20), which allows one to close the whole set of Eqs. (22). The detailed analysis of the integral terms is given in Appendix A.

We are interested in the scissors mode with quantum number $K^\pi = 1^+$. Therefore, we only need the part of the dynamic equations with $\mu = 1$.

A. Linearized equations ($\mu = 1$): Isovector and isoscalar

Writing all variables as a sum of their equilibrium value plus a small deviation,

$$R_{\lambda\mu}(t) = R_{\lambda\mu}(\mathrm{eq}) + \mathcal{R}_{\lambda\mu}(t), \quad P_{\lambda\mu}(t) = P_{\lambda\mu}(\mathrm{eq}) + \mathcal{P}_{\lambda\mu}(t),$$
$$L_{\lambda\mu}(t) = L_{\lambda\mu}(\mathrm{eq}) + \mathcal{L}_{\lambda\mu}(t), \quad F(t) = F(\mathrm{eq}) + \mathcal{F}(t),$$

and neglecting quadratic deviations, one obtains the linearized equations. Naturally one needs to know the equilibrium values of all variables. Evident equilibrium conditions for an axially symmetric nucleus are

$$R^+_{2\pm1}(\mathrm{eq}) = R^+_{2\pm2}(\mathrm{eq}) = 0, \quad R^+_{20}(\mathrm{eq}) \neq 0. \quad (23)$$

It is obvious that all ground-state properties of the system of spin-up nucleons are identical to the ones of the system of nucleons with spin down. Therefore

$$R^-_{\lambda\mu}(\mathrm{eq}) = P^-_{\lambda\mu}(\mathrm{eq}) = L^-_{\lambda\mu}(\mathrm{eq}) = 0. \quad (24)$$

We also suppose

$$L^+_{\lambda\mu}(\mathrm{eq}) = L^{\uparrow\downarrow}_{\lambda\mu}(\mathrm{eq}) = L^{\downarrow\uparrow}_{\lambda\mu}(\mathrm{eq}) = 0 \quad \text{and}$$
$$R^{\uparrow\downarrow}_{\lambda\mu}(\mathrm{eq}) = R^{\downarrow\uparrow}_{\lambda\mu}(\mathrm{eq}) = 0. \quad (25)$$

Let us recall that all variables and equilibrium quantities $R^+_{\lambda 0}(\mathrm{eq})$ and $Z^+_{20}(\mathrm{eq})$ in Eqs. (22) have isospin indices $\tau = n, p$. All the difference between neutron and proton systems is contained in the mean-field quantities $Z^{\tau+}_{20}(\mathrm{eq})$ and V^ς_τ, which are different for neutrons and protons [see Eqs. (15) and (21)].

It is convenient to rewrite the dynamical equations in terms of isovector and isoscalar variables:

$$R_{\lambda\mu} = R^n_{\lambda\mu} + R^p_{\lambda\mu}, \quad P_{\lambda\mu} = P^n_{\lambda\mu} + P^p_{\lambda\mu}, \quad L_{\lambda\mu} = L^n_{\lambda\mu} + L^p_{\lambda\mu},$$
$$\bar{R}_{\lambda\mu} = R^n_{\lambda\mu} - R^p_{\lambda\mu}, \quad \bar{P}_{\lambda\mu} = P^n_{\lambda\mu} - P^p_{\lambda\mu}, \quad \bar{L}_{\lambda\mu} = L^n_{\lambda\mu} - L^p_{\lambda\mu}. \quad (26)$$

It also is natural to define isovector and isoscalar strength constants $\kappa_1 = \frac{1}{2}(\kappa - \bar{\kappa})$ and $\kappa_0 = \frac{1}{2}(\kappa + \bar{\kappa})$ connected by the relation $\kappa_1 = \alpha\kappa_0$ [18]. Then the equations for the neutron and proton systems are transformed into isovector and isoscalar ones. Supposing that all equilibrium characteristics of the proton system are equal to those of the neutron system, one decouples isovector and isoscalar equations. This approximations looks rather crude; nevertheless the possible corrections to it are very small, being of the order $(\frac{N-Z}{A})^2$. With the help of the above equilibrium relations one arrives at the following final set of equations for the isovector system:

$$\dot{\bar{\mathcal{L}}}^+_{21} = \frac{1}{m}\bar{\mathcal{P}}^+_{21} - \left[m\,\omega^2 - 4\sqrt{3}\alpha\kappa_0 R^{\mathrm{eq}}_{00} + \sqrt{6}(1+\alpha)\kappa_0 R^{\mathrm{eq}}_{20}\right]\bar{\mathcal{R}}^+_{21} - i\hbar\frac{\eta}{2}[\bar{\mathcal{L}}^-_{21} + 2\bar{\mathcal{L}}^{\uparrow\downarrow}_{22} + \sqrt{6}\bar{\mathcal{L}}^{\downarrow\uparrow}_{20}],$$

$$\dot{\bar{\mathcal{L}}}^-_{21} = \frac{1}{m}\bar{\mathcal{P}}^-_{21} - \left[m\,\omega^2 + \sqrt{6}\kappa_0 R^{\mathrm{eq}}_{20} - \frac{\sqrt{3}}{20}\hbar^2\left(\chi - \frac{\bar{\chi}}{3}\right)\left(\frac{I_1}{a_0^2} + \frac{I_1}{a_1^2}\right)\left(\frac{a_1^2}{\mathcal{A}_2} - \frac{a_0^2}{\mathcal{A}_1}\right)\right]\bar{\mathcal{R}}^-_{21} - i\hbar\frac{\eta}{2}\bar{\mathcal{L}}^+_{21},$$

$$\dot{\bar{\mathcal{L}}}^{\uparrow\downarrow}_{22} = \frac{1}{m}\bar{\mathcal{P}}^{\uparrow\downarrow}_{22} - \left[m\,\omega^2 - 2\sqrt{6}\kappa_0 R^{\mathrm{eq}}_{20} - \frac{\sqrt{3}}{5}\hbar^2\left(\chi - \frac{\bar{\chi}}{3}\right)\frac{I_1}{\mathcal{A}_2}\right]\bar{\mathcal{R}}^{\uparrow\downarrow}_{22} - i\hbar\frac{\eta}{2}\bar{\mathcal{L}}^+_{21},$$

$$\dot{\bar{\mathcal{L}}}^{\downarrow\uparrow}_{20} = \frac{1}{m}\bar{\mathcal{P}}^{\downarrow\uparrow}_{20} - \left[m\,\omega^2 + 2\sqrt{6}\kappa_0 R^{\mathrm{eq}}_{20}\right]\bar{\mathcal{R}}^{\downarrow\uparrow}_{20} + 4\sqrt{3}\kappa_0 R^{\mathrm{eq}}_{20}\bar{\mathcal{R}}^{\downarrow\uparrow}_{00} - i\hbar\frac{\eta}{2}\sqrt{\frac{3}{2}}\bar{\mathcal{L}}^+_{21}$$
$$+\frac{\sqrt{3}}{15}\hbar^2\left(\chi - \frac{\bar{\chi}}{3}\right)I_1\left[\left(\frac{1}{\mathcal{A}_2} - \frac{2}{\mathcal{A}_1}\right)\bar{\mathcal{R}}^{\downarrow\uparrow}_{20} + \sqrt{2}\left(\frac{1}{\mathcal{A}_2} + \frac{1}{\mathcal{A}_1}\right)\bar{\mathcal{R}}^{\downarrow\uparrow}_{00}\right],$$

$$\dot{\bar{\mathcal{L}}}^+_{11} = -3\sqrt{6}(1-\alpha)\kappa_0 R^{\mathrm{eq}}_{20}\bar{\mathcal{R}}^+_{21} - i\hbar\frac{\eta}{2}[\bar{\mathcal{L}}^-_{11} + \sqrt{2}\bar{\mathcal{L}}^{\downarrow\uparrow}_{10}],$$

$$\dot{\bar{\mathcal{L}}}^-_{11} = -\left[3\sqrt{6}\kappa_0 R^{\mathrm{eq}}_{20} - \frac{\sqrt{3}}{20}\hbar^2\left(\chi - \frac{\bar{\chi}}{3}\right)\left(\frac{I_1}{a_0^2} - \frac{I_1}{a_1^2}\right)\left(\frac{a_1^2}{\mathcal{A}_2} - \frac{a_0^2}{\mathcal{A}_1}\right)\right]\bar{\mathcal{R}}^-_{21} - \hbar\frac{\eta}{2}[i\bar{\mathcal{L}}^+_{11} + \hbar\bar{\mathcal{F}}^{\downarrow\uparrow}],$$

$$\dot{\bar{\mathcal{L}}}^{\downarrow\uparrow}_{10} = -\hbar\frac{\eta}{2\sqrt{2}}[i\bar{\mathcal{L}}^{+}_{11} + \hbar\bar{\mathcal{F}}^{\downarrow\uparrow}], \quad \dot{\bar{\mathcal{F}}}^{\downarrow\uparrow} = -\eta[\bar{\mathcal{L}}^{-}_{11} + \sqrt{2}\bar{\mathcal{L}}^{\downarrow\uparrow}_{10}], \quad \dot{\bar{\mathcal{R}}}^{+}_{21} = \frac{2}{m}\bar{\mathcal{L}}^{+}_{21} - i\hbar\frac{\eta}{2}[\bar{\mathcal{R}}^{-}_{21} + 2\bar{\mathcal{R}}^{\uparrow\downarrow}_{22} + \sqrt{6}\bar{\mathcal{R}}^{\downarrow\uparrow}_{20}],$$

$$\dot{\bar{\mathcal{R}}}^{-}_{21} = \frac{2}{m}\bar{\mathcal{L}}^{-}_{21} - i\hbar\frac{\eta}{2}\bar{\mathcal{R}}^{+}_{21}, \quad \dot{\bar{\mathcal{R}}}^{\uparrow\downarrow}_{22} = \frac{2}{m}\bar{\mathcal{L}}^{\uparrow\downarrow}_{22} - i\hbar\frac{\eta}{2}\bar{\mathcal{R}}^{+}_{21}, \quad \dot{\bar{\mathcal{R}}}^{\downarrow\uparrow}_{20} = \frac{2}{m}\bar{\mathcal{L}}^{\downarrow\uparrow}_{20} - i\hbar\frac{\eta}{2}\sqrt{\frac{3}{2}}\bar{\mathcal{R}}^{+}_{21},$$

$$\dot{\bar{\mathcal{P}}}^{+}_{21} = -2\big[m\,\omega^2 + \sqrt{6}\kappa_0 R^{\rm eq}_{20}\big]\bar{\mathcal{L}}^{+}_{21} + 6\sqrt{6}\kappa_0 R^{\rm eq}_{20}\bar{\mathcal{L}}^{+}_{11} - i\hbar\frac{\eta}{2}[\bar{\mathcal{P}}^{-}_{21} + 2\bar{\mathcal{P}}^{\uparrow\downarrow}_{22} + \sqrt{6}\bar{\mathcal{P}}^{\downarrow\uparrow}_{20}]$$
$$+\frac{3\sqrt{3}}{4}\hbar^2\chi\frac{I_2}{\mathcal{A}_1\mathcal{A}_2}[(\mathcal{A}_1 - \mathcal{A}_2)\bar{\mathcal{L}}^{+}_{21} + (\mathcal{A}_1 + \mathcal{A}_2)\bar{\mathcal{L}}^{+}_{11}],$$

$$\dot{\bar{\mathcal{P}}}^{-}_{21} = -2\big[m\,\omega^2 + \sqrt{6}\kappa_0 R^{\rm eq}_{20}\big]\bar{\mathcal{L}}^{-}_{21} + 6\sqrt{6}\kappa_0 R^{\rm eq}_{20}\bar{\mathcal{L}}^{-}_{11} - i\hbar\frac{\eta}{2}\bar{\mathcal{P}}^{+}_{21} + \frac{3\sqrt{3}}{4}\hbar^2\chi\frac{I_2}{\mathcal{A}_1\mathcal{A}_2}[(\mathcal{A}_1 - \mathcal{A}_2)\,\bar{\mathcal{L}}^{-}_{21} + (\mathcal{A}_1 + \mathcal{A}_2)\,\bar{\mathcal{L}}^{-}_{11}],$$

$$\dot{\bar{\mathcal{P}}}^{\uparrow\downarrow}_{22} = -\left[2m\,\omega^2 - 4\sqrt{6}\kappa_0 R^{\rm eq}_{20} - \frac{3\sqrt{3}}{2}\hbar^2\chi\frac{I_2}{\mathcal{A}_2}\right]\bar{\mathcal{L}}^{\uparrow\downarrow}_{22} - i\hbar\frac{\eta}{2}\bar{\mathcal{P}}^{+}_{21},$$

$$\dot{\bar{\mathcal{P}}}^{\downarrow\uparrow}_{20} = -\big[2m\,\omega^2 + 4\sqrt{6}\kappa_0 R^{\rm eq}_{20}\big]\bar{\mathcal{L}}^{\downarrow\uparrow}_{20} + 8\sqrt{3}\kappa_0 R^{\rm eq}_{20}\bar{\mathcal{L}}^{\downarrow\uparrow}_{00} - i\hbar\frac{\eta}{2}\sqrt{\frac{3}{2}}\bar{\mathcal{P}}^{+}_{21} + \frac{\sqrt{3}}{2}\hbar^2\chi\frac{I_2}{\mathcal{A}_1\mathcal{A}_2}[(\mathcal{A}_1 - 2\mathcal{A}_2)\bar{\mathcal{L}}^{\downarrow\uparrow}_{20} + \sqrt{2}(\mathcal{A}_1 + \mathcal{A}_2)\bar{\mathcal{L}}^{\downarrow\uparrow}_{00}],$$

$$\dot{\bar{\mathcal{L}}}^{\downarrow\uparrow}_{00} = \frac{1}{m}\bar{\mathcal{P}}^{\downarrow\uparrow}_{00} - m\omega^2\bar{\mathcal{R}}^{\downarrow\uparrow}_{00} + 4\sqrt{3}\kappa_0 R^{\rm eq}_{20}\,\bar{\mathcal{R}}^{\downarrow\uparrow}_{20} + \frac{1}{2\sqrt{3}}\hbar^2\left[\left(\chi - \frac{\bar{\chi}}{3}\right)I_1 - \frac{9}{4}\chi I_2\right]\left[\left(\frac{2}{\mathcal{A}_2} - \frac{1}{\mathcal{A}_1}\right)\bar{\mathcal{R}}^{\downarrow\uparrow}_{00} + \sqrt{2}\left(\frac{1}{\mathcal{A}_2} + \frac{1}{\mathcal{A}_1}\right)\bar{\mathcal{R}}^{\downarrow\uparrow}_{20}\right],$$

$$\dot{\bar{\mathcal{R}}}^{\downarrow\uparrow}_{00} = \frac{2}{m}\bar{\mathcal{L}}^{\downarrow\uparrow}_{00},$$

$$\dot{\bar{\mathcal{P}}}^{\downarrow\uparrow}_{00} = -2m\,\omega^2\bar{\mathcal{L}}^{\downarrow\uparrow}_{00} + 8\sqrt{3}\kappa_0 R^{\rm eq}_{20}\,\bar{\mathcal{L}}^{\downarrow\uparrow}_{20} + \frac{\sqrt{3}}{2}\hbar^2\chi I_2\left[\left(\frac{2}{\mathcal{A}_2} - \frac{1}{\mathcal{A}_1}\right)\bar{\mathcal{L}}^{\downarrow\uparrow}_{00} + \sqrt{2}\left(\frac{1}{\mathcal{A}_2} + \frac{1}{\mathcal{A}_1}\right)\bar{\mathcal{L}}^{\downarrow\uparrow}_{20}\right], \quad (27)$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined in Appendix B; $\kappa_0 = -m\bar{\omega}^2/(4Q_{00})$ [21] with $\bar{\omega}^2 = \omega^2/(1 + \frac{2}{3}\delta)$ and $Q_{00} = A\frac{3}{5}R_0^2$; and $a_{-1} = a_1 = R_0(\frac{1-(2/3)\delta}{1+(4/3)\delta})^{1/6}$ and $a_0 = R_0(\frac{1-(2/3)\delta}{1+(4/3)\delta})^{-1/3}$ are semiaxes of ellipsoid by which the shape of nucleus is approximated, where δ is the deformation parameter and $R_0 = 1.2A^{1/3}$ fm;

$$I_1 = \frac{\pi}{4}\int_0^\infty dr\, r^4\left(\frac{\partial n(r)}{\partial r}\right)^2, \quad I_2 = \frac{\pi}{4}\int_0^\infty dr\, r^2 n(r)^2, \quad n(r) = n^+_p + n^+_n = \frac{n_0}{1 + e^{(r-R_0)/a}}.$$

The isoscalar set of equations is easily obtained from Eqs. (27) by taking $\alpha = 1$, replacing $\bar{\chi} \to -\bar{\chi}$, and removing the overbar from all the variables.

B. Integrals of motion and the angular momentum conservation

Imposing the time evolution via $e^{iEt/\hbar}$ for all variables one transforms Eqs. (27) into a set of algebraic equations. It contains 19 equations. To find the eigenvalues we construct the 19×19 determinant and seek (numerically) for its zeros. We find three roots with exactly $E = 0$ and 16 roots that are nonzero: eight positive ones (shown in the tables) and eight negative ones (not shown, the situation is exactly same as with RPA; see Ref. [19] for connection of the WFM and the RPA).

The integrals of motion corresponding to Goldstone modes (zero roots) can be found analytically. In the isovector case we have

$$\hbar\frac{\eta}{2}(2i\bar{\mathcal{L}}^{+}_{11} + \hbar\bar{\mathcal{F}}^{\downarrow\uparrow}) + (1-\alpha)\left(2m\bar{\omega}^2\delta C^{\downarrow\uparrow} - \frac{9\sqrt{3}\,\chi I_2}{16\left(1 - \frac{2}{3}\delta\right)Q_{00}}\frac{\Xi}{m}\bar{\mathcal{P}}^{\downarrow\uparrow}_{00}\right) = \text{const}, \quad (28)$$

$$\sqrt{\frac{3}{2}}\bar{\mathcal{R}}^{\uparrow\downarrow}_{22} - \bar{\mathcal{R}}^{\downarrow\uparrow}_{20} - \sqrt{2}\bar{\mathcal{R}}^{\downarrow\uparrow}_{00} + \frac{\Xi}{m}\left(\sqrt{\frac{3}{2}}\bar{\mathcal{P}}^{\uparrow\downarrow}_{22} - \bar{\mathcal{P}}^{\downarrow\uparrow}_{20} - \sqrt{2}\bar{\mathcal{P}}^{\downarrow\uparrow}_{00}\right) = \text{const}, \quad (29)$$

$$i\hbar\frac{\eta}{2}\Xi\,\mathcal{L}^{+}_{21} + \frac{\Xi}{m}\bar{\mathcal{P}}^{\uparrow\downarrow}_{22} + \left(1 - \frac{\Xi}{4}m\hbar^2\eta^2\right)\bar{\mathcal{R}}^{\uparrow\downarrow}_{22} - \frac{\Xi}{8}m\hbar^2\eta^2(\bar{\mathcal{R}}^{-}_{21} + \sqrt{6}\bar{\mathcal{R}}^{\downarrow\uparrow}_{20})$$
$$-\frac{\sqrt{3}}{2\delta}\left\{m\hbar^2\Xi\left[\frac{3}{4}\eta^2 - (1-\alpha)\left(1 + \frac{\delta}{3}\right)\bar{\omega}^2\right] - 1\right\}C^{\downarrow\uparrow} = \text{const}, \quad (30)$$

where

$$C^{\downarrow\uparrow} \equiv \frac{2}{3}\sqrt{2}\,\delta\bar{\mathcal{R}}^{\downarrow\uparrow}_{20} + \left(1 + \frac{2}{3}\delta\right)\bar{\mathcal{R}}^{\downarrow\uparrow}_{00} + \frac{\Xi}{m}\left(1 + \frac{4}{3}\delta\right)\bar{\mathcal{P}}^{\downarrow\uparrow}_{00}, \quad \Xi \equiv \left[m\bar{\omega}^2\left(1 + \frac{4}{3}\delta\right) + \frac{9\chi I_2}{16\left(1 - \frac{2}{3}\delta\right)Q_{00}}\right]^{-1}.$$

Isoscalar integrals of motion are easily obtained from isovector ones by taking $\alpha = 1$ and removing bars above all variables. In the case of harmonic oscillations all constants are obviously equal to zero.

The physical sense of variables entering into the above integrals of motion can be understood with the help of their definitions (20). The variables (or matrix elements) $\mathcal{R}^{ss'}_{\lambda\mu}(t)$ describe the quadrupole ($\lambda = 2$) and monopole ($\lambda = 0$) deformation of the density of nucleons with spin s, if $s = s'$; otherwise they describe the simultaneous deformation and spin flip. The variables $\mathcal{P}^{ss'}_{\lambda\mu}(t)$ describe the analogous situation in the momentum space, i.e., the Fermi surface deformation, if $s = s'$, or the deformation accompanied by spin flip, if $s \neq s'$. The variables $\mathcal{L}^{ss'}_{\lambda\mu}(t)$ with $\lambda = 2, 0$ describe the same situation in the phase space $(\mathbf{r}, \mathbf{p})$. The variables $\mathcal{L}^{ss'}_{1\mu}(t)$ describe the dynamics of the orbital angular momentum of nucleons with spin s, if $s = s'$; otherwise they describe the dynamics of the orbital angular momentum together with spin flip. The variables $\mathcal{F}^{ss'}(t)$ describe the dynamics of the number of nucleons with spin s, if $s = s'$, or dynamics of spin, i.e., the spin flip, if $s \neq s'$.

Having this information we can give the physical interpretation of some integrals of motion. The first isoscalar integral is the most simple one,

$$2i\mathcal{L}^{+}_{11}(t) + \hbar\mathcal{F}^{\downarrow\uparrow}(t) = \text{const},$$

and has a clear physical interpretation—the conservation of the total angular momentum $\langle\hat{J}_1\rangle = \langle\hat{l}_1\rangle + \langle\hat{S}_1\rangle$. Really, by definition

$$\begin{aligned}\langle\hat{l}_1\rangle &= \mathrm{Tr}(\hat{l}_1\hat{\rho}) = \sum_s \int d^3r \int d^3r' \langle\mathbf{r}|\hat{l}_1|\mathbf{r}'\rangle\langle\mathbf{r}', s|\hat{\rho}|\mathbf{r}, s\rangle \\ &= \sum_s \int d^3r \int d^3r' \hat{l}_1(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')\langle\mathbf{r}', s|\hat{\rho}|\mathbf{r}, s\rangle \\ &= \int d^3r \hat{l}_1(\mathbf{r})[\langle\mathbf{r}|\hat{\rho}|\mathbf{r}\rangle^{\uparrow\uparrow} + \langle\mathbf{r}|\hat{\rho}|\mathbf{r}\rangle^{\downarrow\downarrow}] \\ &= \int d(\mathbf{p}, \mathbf{r}) l_1(\mathbf{r}, \mathbf{p}) f^{+}(\mathbf{r}, \mathbf{p}, t) \\ &= -i\sqrt{2}\int d(\mathbf{p}, \mathbf{r})\{r \otimes p\}_{11} f^{+}(\mathbf{r}, \mathbf{p}, t) = -i\sqrt{2}L^{+}_{11}(t).\end{aligned} \tag{31}$$

The average value of the spin operator $\hat{S}_1$ reads

$$\begin{aligned}\langle\hat{S}_1\rangle &= \mathrm{Tr}(\hat{S}_1\hat{\rho}) = \sum_{s,s'} \int d^3r \langle s|\hat{S}_1|s'\rangle\langle\mathbf{r}, s'|\hat{\rho}|\mathbf{r}, s\rangle \\ &= \sum_{s,s'} \langle s|\hat{S}_1|s'\rangle \int d(\mathbf{p}, \mathbf{r}) f^{s's}(\mathbf{r}, \mathbf{p}, t) \\ &= -\frac{\hbar}{\sqrt{2}}\sum_{s,s'} \delta_{s\uparrow}\delta_{s'\downarrow} F^{s's}(t) = -\frac{\hbar}{\sqrt{2}} F^{\downarrow\uparrow}(t).\end{aligned} \tag{32}$$

As a result $\langle\hat{J}_1\rangle = -\frac{1}{\sqrt{2}}(2iL^{+}_{11} + \hbar F^{\downarrow\uparrow})$. It is easy to see that such a combination of the respective equations of motion in Eqs. (27) is equal to zero in the isoscalar case ($\alpha = 1$); i.e., the total angular momentum is conserved. The isovector counterpart of this integral of motion implies that the relative (neutrons with respect of protons) total angular momentum oscillates in phase with the linear combination of three variables, $\bar{\mathcal{R}}^{\downarrow\uparrow}_{20}$, $\bar{\mathcal{P}}^{\downarrow\uparrow}_{00}$, and $\bar{\mathcal{R}}^{\downarrow\uparrow}_{00}$.

The second integral of motion can be interpreted saying that the definite combination of variables $(\sqrt{\frac{3}{2}}\bar{\mathcal{P}}^{\uparrow\downarrow}_{22} - \bar{\mathcal{P}}^{\downarrow\uparrow}_{20} - \sqrt{2}\bar{\mathcal{P}}^{\downarrow\uparrow}_{00})$, describing the quadrupole and monopole deformations of the Fermi surface together with the spin flip, oscillates out of phase with the exactly the same combination of variables $(\sqrt{\frac{3}{2}}\bar{\mathcal{R}}^{\uparrow\downarrow}_{22} - \bar{\mathcal{R}}^{\downarrow\uparrow}_{20} - \sqrt{2}\bar{\mathcal{R}}^{\downarrow\uparrow}_{00})$, describing the quadrupole and monopole deformations of the density distribution together with the spin flip. It is interesting to note that in the analogous problem without spin [18] there is the similar integral, saying that the nuclear density and the Fermi surface oscillate out of phase. The physical interpretation of the last integral seems not to be obvious.

Let us prove that the conservation of the total angular momentum follows from the set of Eqs. (22), which describe the motion without any restrictions on the values (small or large) of amplitudes. It is necessary to consider the first equation of Eqs. (22) in the isoscalar case for $\lambda = \mu = 1$. Having in mind that $R^{+}_{11} = P^{+}_{11} = 0$ we find

$$\begin{aligned}\dot{L}^{\tau+}_{11} &= 60\left\{{}^{112}_{211}\right\}\{Z^{\tau+}_2 \otimes R^{\tau+}_2\}_{11} - i\hbar\frac{\eta}{2}\left[L^{\tau-}_{11} + \sqrt{2}L^{\tau\downarrow\uparrow}_{10}\right] \\ &\quad - \int d^3r\left[\frac{1}{2}n^{\tau+}\{r \otimes \nabla\}_{11}V^{+}_{\tau} + \frac{1}{2}n^{\tau-}\{r \otimes \nabla\}_{11}V^{-}_{\tau}\right. \\ &\quad \left. + n^{\tau\downarrow\uparrow}\{r \otimes \nabla\}_{11}V^{\uparrow\downarrow}_{\tau} + n^{\tau\uparrow\downarrow}\{r \otimes \nabla\}_{11}V^{\downarrow\uparrow}_{\tau}\right].\end{aligned} \tag{33}$$

Let us analyze the first term. We have the following for protons:

$$\begin{aligned}\{Z^{p+}_2 \otimes R^{p+}_2\}_{11} &= \sum_{\nu\sigma} C^{11}_{2\nu,2\sigma} Z^{p+}_{2\nu} R^{p+}_{2\sigma} \\ &= \sum_{\nu\sigma} C^{11}_{2\nu,2\sigma}\left(\kappa R^{p+}_{2\nu} + \bar{\kappa} R^{n+}_{2\nu}\right) R^{p+}_{2\sigma} \\ &= \bar{\kappa}\sum_{\nu\sigma} C^{11}_{2\nu,2\sigma} R^{n+}_{2\nu} R^{p+}_{2\sigma}.\end{aligned} \tag{34}$$

We have used here the definition (15) of $Z^{\tau+}_{2\mu}$ and the equality $C^{11}_{2\nu,2\sigma} = -C^{11}_{2\sigma,2\nu}$. Analogously one finds the following for neutrons:

$$\begin{aligned}\{Z^{n+}_2 \otimes R^{n+}_2\}_{11} &= \sum_{\nu\sigma} C^{11}_{2\nu,2\sigma}\left(\kappa R^{n+}_{2\nu} + \bar{\kappa} R^{p+}_{2\nu}\right) R^{n+}_{2\sigma} \\ &= \bar{\kappa}\sum_{\nu\sigma} C^{11}_{2\nu,2\sigma} R^{p+}_{2\nu} R^{n+}_{2\sigma}.\end{aligned} \tag{35}$$

The sum of Eqs. (34) and (35) is obviously equal to zero.

The integral in Eq. (33) consists of four terms. The first one is [see the definition of V^{+}_{τ} in Eqs. (21)]

$$\begin{aligned}&-\frac{3}{16}\hbar^2\chi\int d^3r\, n^{+}_{\tau} C^{11}_{11,10}[r_1\nabla_0 - r_0\nabla_1] n^{+}_{\tau} \\ &\quad = -\frac{3}{32}\hbar^2\chi\int d^3r\, C^{11}_{11,10}[r_1\nabla_0 - r_0\nabla_1](n^{+}_{\tau})^2.\end{aligned} \tag{36}$$

Integrating by parts we find that this integral is equal to zero because $\nabla_1 r_0 = \nabla_0 r_1 = 0$. The second term of the integral in

Eq. (33) can be written (for protons) as

$$\frac{\hbar^2}{8}\int d^3r\, n_p^- C_{11,10}^{11}\,[r_1\nabla_0 - r_0\nabla_1]\left(\frac{3}{2}\chi n_p^- + \bar{\chi} n_n^-\right) = \frac{\hbar^2}{8}\bar{\chi}\int d^3r\, C_{11,10}^{11} n_p^-\,[r_1\nabla_0 - r_0\nabla_1]\, n_n^-. \quad (37)$$

Changing here the indices $p \leftrightarrow n$ we obtain the analogous integral for neutrons. Their sum is obviously equal to zero.

The third and fourth terms of the integral in Eq. (33) must be analyzed together. We have the following for protons:

$$\frac{\hbar^2}{4}C_{11,10}^{11}\int d^3r\left[n_p^{\downarrow\uparrow}(r_1\nabla_0 - r_0\nabla_1)\left(\frac{3}{2}\chi n_p^{\uparrow\downarrow} + \bar{\chi} n_n^{\uparrow\downarrow}\right) + n_p^{\uparrow\downarrow}(r_1\nabla_0 - r_0\nabla_1)\left(\frac{3}{2}\chi n_p^{\downarrow\uparrow} + \bar{\chi} n_n^{\downarrow\uparrow}\right)\right]$$
$$= \frac{\hbar^2}{4}C_{11,10}^{11}\,\bar{\chi}\int d^3r[n_p^{\downarrow\uparrow}(r_1\nabla_0 - r_0\nabla_1)\,n_n^{\uparrow\downarrow} + n_p^{\uparrow\downarrow}(r_1\nabla_0 - r_0\nabla_1)\,n_n^{\downarrow\uparrow}]. \quad (38)$$

The sum of this integral with the analogous one for neutrons (which is obtained by changing indices $p \leftrightarrow n$) is obviously equal to zero.

So, finally we have found that the isoscalar variant of Eq. (33) can be written [in variables defined in Eq. (26)] as

$$\dot{L}_{11}^{p+}(t) + \dot{L}_{11}^{n+}(t) \equiv \dot{L}_{11}^{+}(t) = -i\hbar\frac{\eta}{2}[L_{11}^{-}(t) + \sqrt{2}L_{10}^{\downarrow\uparrow}(t)]. \quad (39)$$

It is easy to see that the proper combination of this equation with the seventh equation in Eqs. (22) gives the required result:

$$-\frac{1}{\sqrt{2}}(2i\dot{L}_{11}^{+} + \hbar\dot{F}^{\downarrow\uparrow}) = \frac{d}{dt}\langle\hat{J}_1\rangle = 0;$$

i.e., the total angular momentum is conserved for arbitrary amplitudes, not only in a small amplitude approximation. One must note that this result is not influenced by the approximate treatment of integral terms in Eqs. (22).

V. RESULTS AND THEIR INTERPRETATION: DISCUSSIONS

The energies and excitation probabilities obtained by the solution of the *isovector* set of Eqs. (27) are given in Table I. The used spin-spin interaction is repulsive, the values of its strength constants being taken from Ref. [22], where the notation $\chi = K_s/A,\ \bar{\chi} = q\chi$ was introduced. The results without spin-spin interaction (variant I) are compared with those performed with two sets of constants: K_s and q (variants II and III). The first set of constants (variant II) was extracted by the authors of Ref. [22] from Skyrme forces following the standard procedure, the residual interaction being defined in terms of second derivatives of the Hamiltonian density $H(\rho)$ with respect to the one-body densities ρ. Different variants of Skyrme forces produce different strength constants of spin-spin interaction. The most consistent results are obtained with SG1, SG2 [23], and Sk3 [24] forces. We use here the spin-spin constants extracted from the Sk3 force. Another set of constants (variant III) was found by the authors of Ref. [22] phenomenologically in the calculations with a Woods-Saxon potential, when there is not any self-consistency between the mean field and the residual interaction. Our model is partially self-consistent; nevertheless we tentatively use this set just to have an idea about the dependence of the results on the values of strength constants. The strength of the spin-orbit interaction is taken from Ref. [25].

To avoid misunderstanding we want to recall here that quantum numbers of all levels are $K^\pi = 1^+$ (the projection of the total angular momentum and parity). The first column of Tables I and II demonstrates just the labels (λ, μ and spin projections $\uparrow, \downarrow$) of variables that are responsible (approximately, because all equations are coupled) for the generation of the corresponding eigenvalue.

One can see from Table I that the spin-spin interaction does not change the qualitative picture of the positions of the excitations described in Ref. [15]. It pushes all levels up proportionally to its strength (20–30% in the case II and 40–60% in the case III) without changing their order. The most interesting result concerns the relative $B(M1)$ values of the two low-lying scissors modes, namely, the spin scissors $(1,1)^-$ and the conventional (orbital) scissors $(1,1)^+$ mode. As can be noticed, the spin-spin interaction strongly redistributes $M1$ strength in favor of the spin scissors mode. We tentatively want to link this fact to the recent experimental finding in isotopes of Th and Pa [17]. Guttormsen *et al.* [17] have studied deuteron and ^{3}He-induced reactions on ^{232}Th and found in the residual nuclei ^{231,232,233}Th and ^{232,233}Pa "an unexpectedly strong integrated strength of $B(M1) = 11 - 15\mu_N^2$ in the $E_\gamma = 1.0$–3.5 MeV region." The $B(M1)$ force in most nuclei shows evident splitting into two Lorentzians. "Typically, the experimental splitting is $\Delta\omega_{M1} \sim 0.7$ MeV, and the ratio of the strengths between the lower and upper resonance components is $B_L/B_U \sim 2$." (Note a misprint in that paper: it is written erroneously $B_2/B_1 \sim 2$ whereas it should be $B_1/B_2 \sim 2$. To avoid misunderstanding, we write here B_L instead of B_1 and B_U instead of B_2.) The authors have tried to explain the splitting by a γ deformation. To describe the observed value of $\Delta\omega_{M1}$ the deformation $\gamma \sim 15^\circ$ is required, which leads to the ratio $B_L/B_U \sim 0.7$ in an obvious contradiction with experiment. The authors conclude that "the splitting may be due to other mechanisms." In this sense, we tentatively may argue as follows. On one side, theory [26] and experiment [27] give zero value of γ deformation for ^{232}Th. On the other side, it is easy to see that our theory suggests the required mechanism. The calculations performed for ^{232}Th give $\Delta\omega_{M1} \sim 0.32$ MeV and $B_L/B_U \sim 1.6$ for the first variant

TABLE I. Isovector energies and excitation probabilities of ^{164}Er. Deformation parameter $\delta = 0.25$; spin-orbit constant $\eta = 0.36$ MeV. Spin-spin interaction constants are as follows: I, $K_s = 0$ MeV; II, $K_s = 92$ MeV, $q = -0.8$; and III, $K_s = 200$ MeV, $q = -0.5$. Quantum numbers (including indices $\varsigma = +, -, \uparrow\downarrow, \downarrow\uparrow$) of variables responsible for the generation of the present level are shown in the first column. For example, $(1,1)^-$, spin scissors; $(1,1)^+$, conventional scissors; etc. The numbers in the last line are imaginary, so they are marked by the letter i.

$(\lambda,\mu)^\varsigma$	$E_{\rm iv}$ (MeV)			$B(M1)$ (μ_N^2)			$B(E2)$ (B_W)		
	I	II	III	I	II	III	I	II	III
$(1,1)^-$	1.61	2.02	2.34	3.54	5.44	7.91	0.12	0.36	0.82
$(1,1)^+$	2.18	2.45	2.76	5.33	4.48	2.98	1.02	1.23	1.26
$(0,0)^{\downarrow\uparrow}$	12.80	16.81	20.02	0.01	0.01	0.04	0.04	0.13	0.52
$(2,1)^-$	14.50	18.52	21.90	0.01	0.02	0.34	0.03	0.13	4.29
$(2,0)^{\downarrow\uparrow}$	16.18	20.61	24.56	0.02	0.23	0.03	0.18	3.09	0.44
$(2,2)^{\uparrow\downarrow}$	16.20	22.65	27.67	0	0.03	0	0	0.39	0.02
$(2,1)^+$	20.59	21.49	22.42	2.78	2.19	1.77	35.45	30.47	27.43
$(1,0)^{\downarrow\uparrow}$	$i0.26$	$i0.26$	$i0.26$	$-i5.4$	$-i5.4$	$-i5.4$	$i0$	$i0$	$i0$

of the spin-spin interaction and $\Delta\omega_{M1} \sim 0.28$ MeV and $B_L/B_U \sim 4.1$ for second one in reasonable agreement with experimental values. The inclusion of pair correlations will affect our results, but one may speculate that the agreement between the theory and experiment will be conserved at least qualitatively.

The energies and excitation probabilities obtained by the solution of the *isoscalar* set of Eqs. (27) are displayed in Table II. The general picture of the influence of the spin-spin interaction here is quite close to that observed in the isovector case. The only difference is the low-lying mode marked by $(1,1)^+$ which is practically insensitive to the spin-spin interaction. In Ref. [17] the assignment of the resonances to be of isovector type is only tentative based on the assumption that at such low energies there is no collective mode other than the isovector scissors mode. However, from Ref. [17] one cannot exclude that also an isoscalar spin scissors mode is mixed in. From our analysis we see that the isoscalar spin scissors where all nucleons with spin up counter-rotate with respect to the ones of spin down come more or less at the same energy as the isovector scissors. So it would be very important for the future to pin down precisely the quantum numbers of the resonances.

Let us discuss in more detail the nature of the predicted excitations. As one can see, the generalization of the WFM method by including spin dynamics allows one to reveal a variety of new types of nuclear collective motion involving spin degrees of freedom. Two isovector and two isoscalar low-lying eigenfrequencies and five isovector and five isoscalar high-lying eigenfrequencies have been found.

Three low-lying levels correspond to the excitation of new types of modes. For example, the isovector level marked by $(1,1)^-$ describes rotational oscillations of nucleons with the spin projection "up" with respect to nucleons with the spin projection "down"; i.e., one can talk of a nuclear spin scissors mode. Having in mind that this excitation is an isovector one, we can see that the resulting motion looks rather complex—proton spin scissors counter-rotate with respect to the neutron spin scissors. Thus the experimentally observed group of 1^+ peaks in the interval 2–4 MeV, associated usually with the nuclear scissors mode, in reality consists of the excitations of the "spin" scissors mode together with the conventional [1] scissors mode [the level $(1,1)^+$ in our case]. The isoscalar level $(1,1)^-$ describes the real spin scissors mode: all spin-up nucleons (protons together with neutrons) oscillate rotationally out of phase with all spin-down nucleons.

Such excitations were, undoubtedly, produced implicitly by other methods (e.g., RPA [1,2,22,28]), but they never were analyzed in such terms. It is interesting to note, for example, that in Ref. [2] the scissors mode was analyzed in so-called spin and orbital components. Roughly speaking there are two groups of states corresponding to these two types of

TABLE II. The same as in Table I, but for isoscalar excitations.

$(\lambda,\mu)^\varsigma$	$E_{\rm is}$ (MeV)			$B(M1)$ (μ_N^2)			$B(E2)$ (B_W)		
	I	II	III	I	II	III	I	II	III
$(1,1)^-$	1.71	2.04	2.40	0.07	0.05	0	1.12	0.65	0.39
$(1,1)^+$	0.37	0.37	0.37	−0.24	−0.24	−0.24	117.2	117.9	118.3
$(0,0)^{\downarrow\uparrow}$	12.83	15.59	18.72	0	0	0	0.66	0.31	0.15
$(2,1)^-$	14.51	17.40	20.65	0	0	0	0.12	0.06	0.03
$(2,0)^{\downarrow\uparrow}$	16.22	20.09	24.80	0	0	0	0.20	0.02	0.01
$(2,2)^{\uparrow\downarrow}$	16.20	19.43	23.09	0	0	0	0	0.07	0.04
$(2,1)^+$	10.28	11.92	13.60	0	0	0	66.50	57.78	50.87
$(1,0)^{\downarrow\uparrow}$	$i0.20$	$i0.20$	$i0.20$	$i0.12$	$i0.12$	$i0.12$	$i30.0$	$i29.8$	$i30.3$

components, not completely dissimilar to our finding. Whereas the nature of the orbital, i.e., conventional scissors, is quite clear, the authors did not analyze the character of their states which consist of the spin component. It can be speculated that those spin components just correspond to the isovector spin scissors mode discussed in our work here. It would be interesting to study whether our suggestion is correct or not. This could, for example, be done by analyzing the current patterns.

One more new low-lying mode [isoscalar at 0.37 MeV, marked by $(1,1)^+$] is generated by the relative motion of the orbital angular momentum and spin of the nucleus. They can change their absolute values and directions keeping the total spin unchanged. If there were not the spin-orbit coupling, orbital angular momentum and spin would be constants of motion separately, see dynamical equations for $\mathcal{L}^+_{11}$ (the orbital angular momentum variable) and $\mathcal{F}^{\downarrow\uparrow}$ (the spin variable) in the isoscalar variant of the set of Eqs. (27). Apparently spin-orbit force is too weak to lift up this zero mode strongly. Physically it is quite understandable that such a mode can exist. We want to call it the "collective spin-orbit mode." Another question is whether such a collective spin-orbit mode can be excited experimentally. In any case, to our knowledge a low-lying mode of this type with a strong $B(E2)$ has so far not been identified experimentally. On the other hand the negligibly small negative $B(M1)$ value probably has to do with the approximate treatment of integrals in the equations of motion (22) (especially the neglect by the terms generating fourth-order moments, see Appendix A). It would be very interesting to invent some method to search for such a collective state which is predicted by our model.

To complete the picture of the low-lying states, it is important to discuss the state which is slightly imaginary. Let us first state that the nature of this state has nothing to do with either spin scissors or conventional scissors. It can be seen from the structure of our equations that this state corresponds to a spin flip induced by the spin-orbit potential. Such a state is of purely quantal character and it cannot be hoped that we can accurately describe it with our WFM approach restricting the consideration by second-order moments only. For its correct treatment, we certainly should consider higher moments like fourth-order moments. The spin-orbit potential is the only term in our theory which couples the second-order moments to the fourth-order ones. As mentioned, we decoupled the system by neglecting the fourth-order moments. Therefore, it is no surprise that this particular spin-flip mode is not well described. Nevertheless, one may try to better understand the origin of this mode almost at zero energy. For this, we make the following approximation of our diagonalization procedure to get the eight eigenvalues listed in Table I. We neglect in Eqs. (27) all couplings between the set of variables $X^+_{\lambda\mu}, X^-_{\lambda\mu}$ and the set of variables $X^{\uparrow\downarrow}_{\lambda\mu}, X^{\downarrow\uparrow}_{\lambda\mu}$ $(X \equiv \mathcal{L}, \mathcal{R}, \mathcal{P}, \mathcal{F})$. To this end in the dynamical equations for $X^+_{\lambda\mu}, X^-_{\lambda\mu}$ we omit all terms containing $X^{\uparrow\downarrow}_{\lambda\mu}, X^{\downarrow\uparrow}_{\lambda\mu}$ and in the dynamical equations for $X^{\uparrow\downarrow}_{\lambda\mu}, X^{\downarrow\uparrow}_{\lambda\mu}$ we omit all terms containing $X^+_{\lambda\mu}, X^-_{\lambda\mu}$. In such a way we get two independent sets of dynamical equations. The first one (for $X^+_{\lambda\mu}, X^-_{\lambda\mu}$) was already studied in Ref. [15], where we have found that such approximation gives satisfactory (in comparison with the exact solution) results but must be used cautiously because of the problems with the angular momentum conservation. The second set of equations (for $X^{\uparrow\downarrow}_{\lambda\mu}, X^{\downarrow\uparrow}_{\lambda\mu}$) splits into three independent subsets. Two of them were already analyzed in Ref. [15] (it turns out that these subsets can be obtained also in the limit $\eta \to 0$, which was studied there), where it was shown that the results of approximate calculations are very close to those of exact calculations; i.e., the coupling between the respective variables $X^{\uparrow\downarrow}_{\lambda\mu}, X^{\downarrow\uparrow}_{\lambda\mu}$ and $X^+_{\lambda\mu}, X^-_{\lambda\mu}$ is very weak. The only new subset of equations reads

$$\dot{\bar{\mathcal{L}}}^{\downarrow\uparrow}_{10} = -\hbar^2 \frac{\eta}{2\sqrt{2}} \bar{\mathcal{F}}^{\downarrow\uparrow}, \quad \dot{\bar{\mathcal{F}}}^{\downarrow\uparrow} = -\eta\sqrt{2}\bar{\mathcal{L}}^{\downarrow\uparrow}_{10}. \tag{40}$$

The solution of these equations is $E = i\frac{\hbar}{\sqrt{2}}\eta = i0.255$, which practically coincides with the number of the full diagonalization. So the nonzero (purely imaginary) value of this root only comes from the fact that the z component of orbital angular momentum is not conserved (only total spin J is conserved). However, the violation of the conservation of the orbital angular momentum is very small as can be seen from the numbers. In any case, we see that this spin-flip state has nothing to do with either the spin scissors or the conventional scissors.

Two high-lying excitations of a new nature are found. They are marked by $(2,1)^-$ and following Ref. [28] can be called spin-vector giant quadrupole resonances. The isovector one corresponds to the following quadrupole motion: the proton system oscillates out of phase with the neutron system, whereas inside of each system spin-up nucleons oscillate out of phase with spin-down nucleons. The respective isoscalar resonance describes out of phase oscillations of all spin-up nucleons (protons together with neutrons) with respect to all spin-down nucleons.

Six high-lying modes can be interpreted as spin-flip giant monopole [marked by $(0,0)^{\downarrow\uparrow}$] and quadrupole [marked by $(2,0)^{\downarrow\uparrow}$ and $(2,2)^{\uparrow\downarrow}$] resonances.

This is a pertinent place to make the following citation from the review by Osterfeld [28]: "Similar oscillations to those in isospin space are also possible in spin space. Nucleons with spin up and spin down may move either in phase (spin-scalar $S = 0$ modes) or out of phase (spin-vector $S = 1$ modes). The latter class of states is also referred to as spin excitations or spin-flip transitions." On account of our results in this work, the latter statement that all spin excitations are of a spin-flip nature should be modified. We predict in this paper the existence of spin excitations of a non-spin-flip nature—the isovector and isoscalar spin scissors and the isovector and isoscalar spin-vector GQR.

VI. CONCLUDING REMARKS

In this work, we continued the investigation of spin modes [15] using the WFM to study the influence of spin-spin forces. The WFM, when pushed to high-order moments, is equivalent to the RPA [19]. For lower-rank moments, it yields a coarse-grained spectrum. It has the advantage that the moments allow for a direct physical interpretation and, thus, the spin or orbital structure of the found states comes directly to hand.

The inclusion of spin-spin interaction does not change qualitatively the picture concerning the spectrum of the spin modes found in Ref. [15]. It pushes all levels up without changing their order. However, it strongly redistributes $M1$ strength between the conventional and spin scissors mode in favor of the last one. Our calculations did not fully confirm the expectations mentioned in the Introduction, namely, that essentially only the low-lying part of the spectrum will be strongly influenced by the spin-spin force. Nevertheless our results turned out to be very useful, because they demonstrate that the spin-spin interaction together with pair correlations are able to push a substantial part of the $M1$ force out of the area of the conventional scissors mode, which is required for reasonable agreement with the experimental data.

In this respect, we should mention that we did not include pairing in this work. Inclusion of pairing would have complicated the formalism quite a bit. This shall be worked out in the future. We here wanted to study the features of spin dynamics in a most transparent way, staying, however, somewhat on the qualitative side. That is why we did not try to discuss in detail possible relations with experiment or to compare with the results of other theories. Nevertheless we mentioned the quite recent experimental work of Guttormsen *et al.* [17], where for the two low-lying magnetic states a stronger $B(M1)$ transition for the lower state with respect to the higher one was found. A tentative explanation in terms of a slight triaxial deformation in Ref. [17] failed. However, our theory can naturally predict such a scenario with a nonvanishing spin-spin force. It would indeed be very exciting, if the results of Ref. [17] had already revealed the isovector spin scissors mode. However, much deeper experimental and theoretical results must be obtained before a firm conclusion on this point is possible.

In light of the above results, the study of spin excitations with pairing included will be the natural continuation of this work. Pairing is important for a quantitative description of the conventional scissors mode. The same is expected for the novel spin scissors mode discussed here. The effect of pairing generally is to push up the spectrum in energy. Therefore, as just mentioned, it can be expected that the results will come into better agreement with experiment.

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APPENDIX A

All derivations of this section are done in the approximation of spherical symmetry. The inclusion of deformation makes the calculations more cumbersome without changing the final conclusions. Let us consider, as an example, the integral

$$I_h = \int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu}[h^{\uparrow\downarrow}f^{\downarrow\uparrow} - h^{\downarrow\uparrow}f^{\uparrow\downarrow}].$$

It can be divided into two parts corresponding to the contributions of spin-orbital and spin-spin potentials: $I_h = I_{so} + I_{ss}$, where

$$I_{so} = -\frac{\hbar}{\sqrt{2}}\eta\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu}[l_{-1}f^{\downarrow\uparrow} + l_1 f^{\uparrow\downarrow}],$$

$$I_{ss} = \int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu}[V_\tau^{\uparrow\downarrow}f^{\downarrow\uparrow} - V_\tau^{\downarrow\uparrow}f^{\uparrow\downarrow}],$$

$V_\tau^{ss'}$ being defined in Eqs. (21). It is easy to see that the integral I_{so} generates moments of the fourth order. According to the rules of the WFM method [29] this integral is neglected.

Let us analyze the integral I_{ss} (to be definite, for protons). In this case

$$V_p^{\uparrow\downarrow}(\mathbf{r}) = 3\frac{\hbar^2}{8}\chi n_p^{\uparrow\downarrow}(\mathbf{r}) + \frac{\hbar^2}{4}\bar{\chi} n_n^{\uparrow\downarrow}(\mathbf{r}),$$

$$V_p^{\downarrow\uparrow}(\mathbf{r}) = 3\frac{\hbar^2}{8}\chi n_p^{\downarrow\uparrow}(\mathbf{r}) + \frac{\hbar^2}{4}\bar{\chi} n_n^{\downarrow\uparrow}(\mathbf{r}).$$

It can be seen that I_{ss} is split into four terms of identical structure, so it is sufficient to analyze in detail only one part. For example

$$I_{ss4} = \int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu}n^{\downarrow\uparrow}f^{\uparrow\downarrow} = \int d^3r\{r\otimes J^{\uparrow\downarrow}\}_{\lambda\mu}n^{\downarrow\uparrow} = \sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\int d^3r r_\nu J_\alpha^{\uparrow\downarrow}n^{\downarrow\uparrow}, \quad \text{(A1)}$$

where $J_\alpha^{\uparrow\downarrow}(\mathbf{r},t) = \int\frac{d^3p}{(2\pi\hbar)^3}p_\alpha f^{\uparrow\downarrow}(\mathbf{r},\mathbf{p},t)$. The variation of this integral reads

$$\delta I_{ss4} = \sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\int d^3r r_\nu[n^{\downarrow\uparrow}(\text{eq})\delta J_\alpha^{\uparrow\downarrow} + J_\alpha^{\uparrow\downarrow}(\text{eq})\delta n^{\downarrow\uparrow}]. \quad \text{(A2)}$$

It is necessary to represent this integral in terms of the collective variables (20). This problem cannot be solved exactly, so we use the approximation suggested in Ref. [29] and expand the density and current variations as a series (see Appendix B).

Let us consider the second part of integral (A2). With the help of formula (B4) we find

$$I_2 \equiv \sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\int d^3r\, r_\nu J_\alpha^{\uparrow\downarrow}(\text{eq})\delta n^{\downarrow\uparrow} = -\sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\int d^3r\, r_\nu J_\alpha^{\uparrow\downarrow}(\text{eq})\sum_\beta(-1)^\beta \times\left\{N^{\downarrow\uparrow}_{\beta,-\beta}(t)n^+ + \sum_\gamma(-1)^\gamma N^{\downarrow\uparrow}_{\beta,\gamma}(t)\frac{1}{r}\frac{\partial n^+}{\partial r}r_{-\beta}r_{-\gamma}\right\}. \quad \text{(A3)}$$

Let us analyze at first the more simple part of this expression:

$$I_{2,1} \equiv -\sum_\beta(-1)^\beta N^{\downarrow\uparrow}_{\beta,-\beta}(t)\int d^3r\sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}r_\nu J_\alpha^{\uparrow\downarrow}(\text{eq})n^+ = -\sum_\beta(-1)^\beta N^{\downarrow\uparrow}_{\beta,-\beta}X_{\lambda\mu}. \quad \text{(A4)}$$

We are interested in the value of $\mu = 1$; therefore it is necessary to analyze two possibilities: $\lambda = 1$ and $\lambda = 2$.

In the case $\lambda = 1$ and $\mu = 1$ we have

$$X_{11} \equiv \int d^3r\, n^+ \sum_{\nu,\alpha} C^{11}_{1\nu,1\alpha} r_\nu J^{\uparrow\downarrow}_\alpha(\mathrm{eq}) = \int d^3r\, n^+ \frac{1}{\sqrt{2}}[r_1 J^{\uparrow\downarrow}_0(\mathrm{eq}) - r_0 J^{\uparrow\downarrow}_1(eq)]. \tag{A5}$$

By definition

$$J^{ss'}_\nu = \int \frac{d^3p}{(2\pi\hbar)^3} p_\nu f^{ss'}(\mathbf{r},\mathbf{p}) = -\frac{i\hbar}{2}[(\nabla_\nu - \nabla'_\nu)\rho(\mathbf{r},s;\mathbf{r}'s')]_{\mathbf{r}'=\mathbf{r}} = \frac{i\hbar}{2}\sum_k v_k^2[\phi_k(\mathbf{r},s)\nabla_\nu\phi^*_k(\mathbf{r},s') - \phi^*_k(\mathbf{r},s')\nabla_\nu\phi_k(\mathbf{r},s)], \tag{A6}$$

where $k \equiv n, l, j, m$ is a set of oscillator quantum numbers, v_k^2 are occupation numbers, and

$$\phi_{nljm}(\mathbf{r},s) = \mathcal{R}_{nl}(r)\sum_{\Lambda,\sigma} C^{jm}_{l\Lambda,\frac12\sigma} Y_{l\Lambda}(\theta,\phi)\chi_{\frac12\sigma}(s) = \mathcal{R}_{nlj}(r) C^{jm}_{lm-s,\frac12 s} Y_{lm-s}(\theta,\phi) \tag{A7}$$

are single-particle wave functions, $\chi_{\frac12\sigma}(s) = \delta_{\sigma,s}$ being spin functions. Inserting Eq. (A6) into Eq. (A5) one finds

$$X_{11} = \frac{i\hbar}{2}\frac{1}{\sqrt{2}}\sum_{nljm} v^2_{nljm}\int d^3r\, n^+(r)\mathcal{R}^2_{nlj}(r) C^{jm}_{l\Lambda,\frac12\frac12} C^{jm}_{l\Lambda',\frac12-\frac12}[Y_{l\Lambda}(r_1\nabla_0 - r_0\nabla_1)Y^*_{l\Lambda'} - Y^*_{l\Lambda'}(r_1\nabla_0 - r_0\nabla_1)Y_{l\Lambda}], \tag{A8}$$

with $\Lambda = m - \frac12$ and $\Lambda' = m + \frac12$. Remembering the definition (11) of the angular momentum $\hat{l}_1 = \hbar(r_0\nabla_1 - r_1\nabla_0)$ and using the relation [20] $\hat{l}_{\pm1}Y_{l\Lambda} = \mp\frac{1}{\sqrt{2}}\sqrt{(l\mp\Lambda)(l\pm\Lambda+1)}Y_{l\Lambda\pm1}$ one transforms Eq. (A8) into

$$\begin{aligned} X_{11} &= -\frac{i\hbar}{2}\frac{1}{\sqrt{2}}\sum_{nljm} v^2_{nljm}\int dr n^+(r) r^2 \mathcal{R}^2_{nlj}(r) C^{jm}_{l\Lambda,\frac12\frac12} C^{jm}_{l\Lambda',\frac12-\frac12}\frac{2}{\sqrt{2}}\sqrt{(l-\Lambda)(l+\Lambda+1)} \\ &= -i\hbar\sum_{nl}\sum_{m=\frac12}^{|l-\frac12|}\frac{\left[\left(l+\frac12\right)^2 - m^2\right]}{2l+1}\int dr n^+(r) r^2\left[v^2_{nll+\frac12 m}\mathcal{R}^2_{nll+\frac12}(r) - v^2_{nl|l-\frac12|m}\mathcal{R}^2_{nl|l-\frac12|}(r)\right]. \end{aligned} \tag{A9}$$

As can be seen, the value of this integral is determined by the difference of the wave functions of spin-orbital partners $(v\mathcal{R})^2_{nll+\frac12 m} - (v\mathcal{R})^2_{nl|l-\frac12|m}$, which is usually very small, so we neglect it. The only remarkable contribution can appear in the vicinity of the Fermi surface, where some spin-orbital partners with $j = l + \frac12$ and $j = |l - \frac12|$ can be disposed on different sides of the Fermi surface. In reality such a situation happens very frequently; nevertheless we do not take into account this effect, because the values of the corresponding integrals are considerably smaller than $R_{20}(\mathrm{eq})$, the typical input parameter of our model.

Let us consider now the integral $I_{2,1}$ [formula (A4)] for the case $\lambda = 2$ and $\mu = 1$. We have

$$X_{21} \equiv \int d^3r n^+ \sum_{\nu,\alpha} C^{21}_{1\nu,1\alpha} r_\nu J^{\uparrow\downarrow}_\alpha(\mathrm{eq}) = \int d^3r n^+ C^{21}_{11,10}[r_1 J^{\uparrow\downarrow}_0(\mathrm{eq}) + r_0 J^{\uparrow\downarrow}_1(\mathrm{eq})]. \tag{A10}$$

With the help of formulas (A6) and (A7) one can show by simple algebraic transformations that

$$\int d\Omega\, r_1 J^{\uparrow\downarrow}_0(\mathrm{eq}) = -\int d\Omega\, r_0 J^{\uparrow\downarrow}_1(\mathrm{eq}), \tag{A11}$$

where $\int d\Omega$ means the integration over angles. As a result $X_{21} = 0$.

Let us consider the second, more complicated, part of integral I_2:

$$I_{2,2} = -\sum_{\beta,\gamma}(-1)^{\beta+\gamma} N^{\downarrow\uparrow}_{-\beta,-\gamma}(t)\sum_{\nu,\alpha} C^{\lambda\mu}_{1\nu,1\alpha}\int d^3r r_\nu J^{\uparrow\downarrow}_\alpha(\mathrm{eq})\frac{1}{r}\frac{\partial n^+}{\partial r} r_\beta r_\gamma = -\sum_{\beta,\gamma}(-1)^{\beta+\gamma} N^{\downarrow\uparrow}_{-\beta,-\gamma}(t) X'_{\lambda\mu}(\beta,\gamma). \tag{A12}$$

For the case $\lambda = 1$ and $\mu = 1$,

$$\begin{aligned} X'_{11}(\beta,\gamma) &= \frac{1}{\sqrt{2}}\int d^3r\,\frac{1}{r}\frac{\partial n^+}{\partial r}[r_1 J^{\uparrow\downarrow}_0(\mathrm{eq}) - r_0 J^{\uparrow\downarrow}_1(\mathrm{eq})] r_\beta r_\gamma \\ &= y - \frac{i\hbar}{4}\sum_{nljm} v^2_{nljm}\int d^3r\,\frac{1}{r}\frac{\partial n^+}{\partial r}\mathcal{R}^2_{nlj}(r) C^{jm}_{l\Lambda,\frac12\frac12} C^{jm}_{l\Lambda',\frac12-\frac12}\sqrt{(l-\Lambda)(l+\Lambda+1)}[Y_{l\Lambda}Y^*_{l\Lambda} + Y^*_{l\Lambda'}Y_{l\Lambda'}] r_\beta r_\gamma. \end{aligned} \tag{A13}$$

The angular part of this integral is

$$\int d\Omega[Y_{l\Lambda}Y^*_{l\Lambda}+Y^*_{l\Lambda'}Y_{l\Lambda'}]r_\beta r_\gamma=\sum_{L,M}C^{LM}_{1\beta,1\gamma}\int d\Omega[Y_{l\Lambda}Y^*_{l\Lambda}+Y^*_{l\Lambda'}Y_{l\Lambda'}]\{r\otimes r\}_{LM}$$
$$=-\frac{2}{\sqrt{3}}r^2C^{00}_{1\beta,1\gamma}+\sqrt{\frac{8\pi}{15}}r^2\sum_M C^{2M}_{1\beta,1\gamma}\int d\Omega[Y_{l\Lambda}Y^*_{l\Lambda}+Y^*_{l\Lambda'}Y_{l\Lambda'}]Y_{2M}$$
$$=\frac{2}{3}r^2\delta_{\gamma,-\beta}\left\{1-\sqrt{\frac{5}{2}}C^{l0}_{l0,20}C^{1\beta}_{1\beta,20}\left[C^{l\Lambda}_{l\Lambda,2M}+C^{l\Lambda'}_{l\Lambda',2M}\right]\right\}. \quad \text{(A14)}$$

Therefore

$$X'_{11}(\beta,\gamma)=-\frac{i\hbar}{6}\delta_{\gamma,-\beta}\int dr\,\frac{\partial n^+(r)}{\partial r}r^3\sum_{nljm}\left\{1-\sqrt{\frac{5}{2}}C^{1\beta}_{1\beta,20}C^{l0}_{l0,20}\left[C^{l\Lambda}_{l\Lambda,20}+C^{l\Lambda'}_{l\Lambda',20}\right]\right\}$$
$$\times v^2_{nljm}\mathcal{R}^2_{nlj}(r)C^{jm}_{l\Lambda,\frac12\frac12}C^{jm}_{l\Lambda',\frac12-\frac12}\sqrt{(l-\Lambda)(l+\Lambda+1)}$$
$$=-\frac{i\hbar}{3}\delta_{\gamma,-\beta}\sum_{nl}\left\{1-\sqrt{\frac{5}{2}}C^{1\beta}_{1\beta,20}C^{l0}_{l0,20}\left[C^{l\Lambda}_{l\Lambda,20}+C^{l\Lambda'}_{l\Lambda',20}\right]\right\}$$
$$\times\sum_{m=\frac12}^{|l-\frac12|}\frac{\left[\left(l+\frac12\right)^2-m^2\right]}{2l+1}\int dr\frac{\partial n^+(r)}{\partial r}r^3\left[v^2_{nll+\frac12 m}\mathcal{R}^2_{nll+\frac12}(r)-v^2_{nl|l-\frac12|m}\mathcal{R}^2_{nl|l-\frac12|}(r)\right]. \quad \text{(A15)}$$

One sees that, exactly as in formula (A9), the value of this integral is determined by the difference of the wave functions of spin-orbital partners $(v\mathcal{R})^2_{nll+\frac12 m}-(v\mathcal{R})^2_{nl|l-\frac12|m}$ near the Fermi surface, so it can be omitted together with X_{11} following the same arguments.

The case $\lambda=2$ and $\mu=1$ can be analyzed in full analogy with formulas (A10) and (A11), which allows us to take $X'_{21}=0$.

So, we have shown that the integral I_2 can be approximated by zero. Let us consider now the first part of the integral (A2):

$$I_1=\sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\int d^3r\,r_\nu n^{\downarrow\uparrow}(\text{eq})\delta J^{\uparrow\downarrow}_\alpha=\sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\int d^3r\,r_\nu n^{\downarrow\uparrow}(\text{eq})n^+(r)\sum_\gamma(-1)^\gamma K^{\uparrow\downarrow}_{\alpha,-\gamma}(t)r_\gamma$$
$$=\sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\sum_\gamma(-1)^\gamma K^{\uparrow\downarrow}_{\alpha,-\gamma}(t)\int d^3r\,n^{\downarrow\uparrow}(\text{eq})n^+(r)\sum_{L,M}C^{LM}_{1\nu,1\gamma}\{r\otimes r\}_{LM}. \quad \text{(A16)}$$

This integral can be estimated in the approximation of constant density $n^+(r)=n_0$. Then

$$I_1=n_0\sum_{\nu,\alpha}C^{\lambda\mu}_{1\nu,1\alpha}\sum_\gamma(-1)^\gamma K^{\uparrow\downarrow}_{\alpha,-\gamma}(t)\sum_{L,M}C^{LM}_{1\nu,1\gamma}R^{\downarrow\uparrow}_{LM}(\text{eq})=0. \quad \text{(A17)}$$

It is easy to show that $R^{\downarrow\uparrow}_{LM}(\text{eq})=0$. Let us consider, for example, the case with $L=2$:

$$R^{\downarrow\uparrow}_{2M}=\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{2M}f^{\downarrow\uparrow}(\mathbf{r},\mathbf{p})=\int d^3r\{r\otimes r\}_{2M}n^{\downarrow\uparrow}(\mathbf{r})=\sqrt{\frac{8\pi}{15}}\int d^3r\,r^2Y_{2M}n^{\downarrow\uparrow}(\mathbf{r}). \quad \text{(A18)}$$

By definition

$$n^{ss'}(\mathbf{r})=\int\frac{d^3p}{(2\pi\hbar)^3}f^{ss'}(\mathbf{r},\mathbf{p})=\sum_k v_k^2\phi_k(\mathbf{r},s)\phi^*_k(\mathbf{r},s'), \quad \text{(A19)}$$

with ϕ_k defined in Eq. (A7). Therefore

$$R^{\downarrow\uparrow}_{2M}=\sqrt{\frac{8\pi}{15}}\int d^3r\,r^2Y_{2M}\sum_{nljm}v^2_{nljm}\mathcal{R}^2_{nlj}(r)C^{jm}_{l\Lambda',\frac12-\frac12}C^{jm}_{l\Lambda,\frac12\frac12}Y_{l\Lambda'}Y^*_{l\Lambda}$$
$$=\sqrt{\frac{2}{3}}\sum_{nljm}v^2_{nljm}\int dr\,r^4\mathcal{R}^2_{nlj}(r)C^{jm}_{l\Lambda,\frac12\frac12}C^{jm}_{l\Lambda',\frac12-\frac12}C^{l0}_{20,l0}C^{l\Lambda}_{2M,l\Lambda'}=0, \quad \text{(A20)}$$

where $\Lambda=m-\frac12$ and $\Lambda'=m+\frac12$. The zero is obtained due to summation over m. Really, the product $C^{jm}_{l\Lambda,\frac12\frac12}C^{jm}_{l\Lambda',\frac12-\frac12}=\pm\frac{\sqrt{(l+\frac12)^2-m^2}}{2l+1}$ (for $j=l\pm\frac12$) does not depend on the sign of m, whereas the Clebsh-Gordan coefficient $C^{l\Lambda}_{2M,l\Lambda'}$ changes its sign together with m.

Summarizing, we have demonstrated that $I_1 + I_2 \simeq 0$; hence one can neglect the contribution of the integrals I_h in the equations of motion.

It is necessary to analyze also the integrals with the weight $\{p \otimes p\}_{\lambda\mu}$:

$$I'_h = \int d(\mathbf{p},\mathbf{r})\{p \otimes p\}_{\lambda\mu}[h^{\uparrow\downarrow} f^{\downarrow\uparrow} - h^{\downarrow\uparrow} f^{\uparrow\downarrow}] = I'_{\rm so} + I'_{\rm ss}.$$

Again we neglect the contribution of the spin-orbital part $I'_{\rm so}$, which generates fourth-order moments. For the spin-spin contribution, we have

$$I'_{\rm ss4} = \int d(\mathbf{p},\mathbf{r})\{p \otimes p\}_{\lambda\mu} n^{\downarrow\uparrow}(\mathbf{r},t) f^{\uparrow\downarrow}(\mathbf{r},\mathbf{p},t) = \int d^3r\, \Pi^{\uparrow\downarrow}_{\lambda\mu}(\mathbf{r},t) n^{\downarrow\uparrow}(\mathbf{r},t), \tag{A21}$$

where $\Pi^{\uparrow\downarrow}_{\lambda\mu}(\mathbf{r},t) = \int \frac{d^3p}{(2\pi\hbar)^3}\{p \otimes p\}_{\lambda\mu} f^{\uparrow\downarrow}(\mathbf{r},\mathbf{p},t)$ is the pressure tensor. The variation of this integral reads

$$\delta I'_{\rm ss4} = \int d^3r[n^{\downarrow\uparrow}(\text{eq})\delta\Pi^{\uparrow\downarrow}_{\lambda\mu}(\mathbf{r},t) + \Pi^{\uparrow\downarrow}_{\lambda\mu}(\text{eq})\delta n^{\downarrow\uparrow}(\mathbf{r},t)]. \tag{A22}$$

The pressure tensor variation is defined in Appendix B. With formula (B6) one finds the following for the first part of Eq. (A22):

$$I'_1 = \int d^3r n^{\downarrow\uparrow}(\text{eq})\delta\Pi^{\uparrow\downarrow}_{\lambda\mu}(\mathbf{r},t) \simeq T^{\uparrow\downarrow}_{\lambda\mu}(t)\int d^3r n^{\downarrow\uparrow}(\text{eq}) n^+(r) \simeq T^{\uparrow\downarrow}_{\lambda\mu}(t) n_0 \int d^3r n^{\downarrow\uparrow}(\text{eq}) = 0. \tag{A23}$$

The last equality follows obviously from the definition of $n^{\downarrow\uparrow}$ (A19).

The second part of Eq. (A22) reads

$$I'_2 = \int d^3r\,\Pi^{\uparrow\downarrow}_{\lambda\mu}(\text{eq})\delta n^{\downarrow\uparrow}(r,t) = -\sum_\beta(-1)^\beta \int d^3r\,\Pi^{\uparrow\downarrow}_{\lambda\mu}(\text{eq})\left\{N^{\downarrow\uparrow}_{\beta,-\beta}(t) n^+ + \sum_\gamma (-1)^\gamma N^{\downarrow\uparrow}_{\beta,\gamma}(t)\frac{1}{r}\frac{\partial n^+}{\partial r} r_{-\beta} r_{-\gamma}\right\}. \tag{A24}$$

Let us consider at first the simpler part of this integral

$$-\sum_\beta (-1)^\beta N^{\downarrow\uparrow}_{\beta,-\beta}(t)\int d^3r\,\Pi^{\uparrow\downarrow}_{\lambda\mu}(\text{eq}) n^+(r). \tag{A25}$$

The value of the last integral is determined by the angular structure of the function $\Pi^{\uparrow\downarrow}_{\lambda\mu}(\mathbf{r})$. We are interested in $\lambda = 2$ and $\mu = 1$. By definition

$$\begin{aligned}
\Pi^{\uparrow\downarrow}_{21}(\mathbf{r}) &= \int \frac{d^3p}{(2\pi\hbar)^3}\{p \otimes p\}_{21} f^{\uparrow\downarrow}(\mathbf{r},\mathbf{p}) = \sum_{\nu,\sigma} C^{21}_{1\nu,1\sigma}\int \frac{d^3p}{(2\pi\hbar)^3} p_\nu p_\sigma f^{\uparrow\downarrow}(\mathbf{r},\mathbf{p}) \\
&= 2C^{21}_{11,10}\int \frac{d^3p}{(2\pi\hbar)^3} p_1 p_0 f^{\uparrow\downarrow}(\mathbf{r},\mathbf{p}) = -\frac{\hbar^2}{2\sqrt{2}}[(\nabla'_1 - \nabla_1)(\nabla'_0 - \nabla_0)\rho(\mathbf{r}'\uparrow,\mathbf{r}\downarrow)]_{\mathbf{r}'=\mathbf{r}} \\
&= -\frac{\hbar^2}{2\sqrt{2}}\sum_k v_k^2 \{[\nabla_1\nabla_0\phi_k(\mathbf{r},\uparrow)]\phi_k^*(\mathbf{r},\downarrow) - [\nabla_1\phi_k(\mathbf{r},\uparrow)][\nabla_0\phi_k^*(\mathbf{r},\downarrow)] \\
&\quad - [\nabla_0\phi_k(\mathbf{r},\uparrow)][\nabla_1\phi_k^*(\mathbf{r},\downarrow)] + \phi_k(\mathbf{r},\uparrow)[\nabla_1\nabla_0\phi_k^*(\mathbf{r},\downarrow)]\},
\end{aligned} \tag{A26}$$

with ϕ_k being defined by Eq. (A7). Taking into account formulas [20]

$$\nabla_{\pm1}Y_{l\lambda} = -\sqrt{\frac{(l\pm\Lambda+1)(l\pm\Lambda+2)}{2(2l+1)(2l+3)}}\frac{l}{r}Y_{l+1,\Lambda\pm1} - \sqrt{\frac{(l\mp\Lambda-1)(l\mp\Lambda)}{2(2l-1)(2l+1)}}\frac{l+1}{r}Y_{l-1,\Lambda\pm1},$$

$$\nabla_0 Y_{l\lambda} = -\sqrt{\frac{(l+1)^2-\Lambda^2}{(2l+1)(2l+3)}}\frac{l}{r}Y_{l+1,\Lambda} + \sqrt{\frac{l^2-\Lambda^2}{(2l-1)(2l+1)}}\frac{l+1}{r}Y_{l-1,\Lambda},$$

one finds that

$$\int d^3r\,\Pi^{\uparrow\downarrow}_{\lambda\mu}(\text{eq}) n^+(r) = \hbar^2\sum_{nljm} v^2_{nljm}\int dr\, n^+(r)\mathcal{R}^2_{nlj}(r)\big(\delta_{j,l+\frac12} - \delta_{j,l-\frac12}\big)\frac{l(l+1)\big[\big(l+\frac12\big)^2 - m^2\big]}{(2l+3)(2l+1)(2l-1)} m = 0 \tag{A27}$$

due to summation over m. The more complicated part of the integral (A24) is calculated in a similar way with the same result; hence $I'_2 = 0$.

So, we have shown that $I'_1 + I'_2 \simeq 0$; therefore one can neglect by the contribution of integrals I'_h (together with I_h) into equations of motion.

And finally, just a few words about the integrals with the weight $\{r \otimes r\}_{\lambda\mu}$:

$$I''_h = \int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu}[h^{\uparrow\downarrow}f^{\downarrow\uparrow} - h^{\downarrow\uparrow}f^{\uparrow\downarrow}] = I''_{\rm so} + I''_{\rm ss}.$$

The spin-orbital part $I''_{\rm so}$ is neglected and for the spin-spin part we have

$$I''_{ss4} = \int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu} n^{\downarrow\uparrow}(\mathbf{r},t) f^{\uparrow\downarrow}(\mathbf{r},\mathbf{p},t) = \int d^3r\{r\otimes r\}_{\lambda\mu} n^{\downarrow\uparrow}(\mathbf{r},t) n^{\uparrow\downarrow}(\mathbf{r},t). \tag{A28}$$

The variation of this integral reads

$$\delta I''_{\rm ss4} = \int d^3r\{r\otimes r\}_{\lambda\mu}[n^{\downarrow\uparrow}({\rm eq})\delta n^{\uparrow\downarrow}(\mathbf{r},t) + n^{\uparrow\downarrow}({\rm eq})\delta n^{\downarrow\uparrow}(\mathbf{r},t)]. \tag{A29}$$

With the help of formulas (A19) and (B4) the subsequent analysis becomes quite similar to that of the integral (A16) with the same result, i.e., $I''_h \simeq 0$.

The integrals $\int d(\mathbf{p},\mathbf{r})W_{\lambda\mu}[h^- f^{\downarrow\uparrow} - h^{\downarrow\uparrow}f^-]$ and $\int d(\mathbf{p},\mathbf{r})W_{\lambda\mu}[h^- f^{\uparrow\downarrow} - h^{\uparrow\downarrow}f^-]$, where $W_{\lambda\mu}$ is any of the abovementioned weights, can be analyzed in an analogous way with the same result.

APPENDIX B

According to the approximation suggested in Ref. [29], the variations of density, current, and pressure tensor are expanded as the following series:

$$\delta n^{\varsigma}(\mathbf{r},t) = -\sum_\beta (-1)^\beta \nabla_{-\beta}\left\{n^+(\mathbf{r})\left[N^{\varsigma}_\beta(t) + \sum_\gamma(-1)^\gamma N^{\varsigma}_{\beta,\gamma}(t) r_{-\gamma} + \sum_{\lambda',\mu'}(-1)^{\mu'}N^{\varsigma}_{\beta,\lambda'\mu'}(t)\{r\otimes r\}_{\lambda'-\mu'} + \cdots\right]\right\}, \tag{B1}$$

$$\delta J^{\varsigma}_\beta(\mathbf{r},t) = n^+(\mathbf{r})\left[K^{\varsigma}_\beta(t) + \sum_\gamma(-1)^\gamma K^{\varsigma}_{\beta,-\gamma}(t) r_\gamma + \sum_{\lambda',\mu'}(-1)^{\mu'}K^{\varsigma}_{\beta,\lambda'-\mu'}(t)\{r\otimes r\}_{\lambda'\mu'} + \cdots\right], \tag{B2}$$

$$\delta\Pi^{\varsigma}_{\lambda\mu}(\mathbf{r},t) = n^+(\mathbf{r})\left[T^{\varsigma}_{\lambda\mu}(t) + \sum_\gamma(-1)^\gamma T^{\varsigma}_{\lambda\mu,-\gamma}(t) r_\gamma + \sum_{\lambda',\mu'}(-1)^{\mu'}T^{\varsigma}_{\lambda\mu,\lambda'-\mu'}(t)\{r\otimes r\}_{\lambda'\mu'} + \cdots\right]. \tag{B3}$$

Putting these series into the integrals (A2) and (A22), one discovers immediately that all terms containing expansion coefficients N, K, and T with odd numbers of indices disappear due to axial symmetry. Furthermore, we truncate these series omitting all terms generating higher (than second) order moments. So, finally the following expressions are used:

$$\delta n^{\varsigma}(\mathbf{r},t) \simeq -\sum_\beta(-1)^\beta\nabla_{-\beta}\left\{n^+(\mathbf{r})\sum_\gamma(-1)^\gamma N^{\varsigma}_{\beta,\gamma}(t) r_{-\gamma}\right\} = -\sum_\beta(-1)^\beta\left\{N^{\varsigma}_{\beta,-\beta}(t)n^+ + \sum_\gamma(-1)^\gamma N^{\varsigma}_{\beta,\gamma}(t)\frac{1}{r}\frac{\partial n^+}{\partial r} r_{-\beta}r_{-\gamma}\right\}, \tag{B4}$$

$$\delta J^{\varsigma}_\beta(\mathbf{r},t) \simeq n^+(\mathbf{r})\sum_\gamma(-1)^\gamma K^{\varsigma}_{\beta,-\gamma}(t) r_\gamma, \tag{B5}$$

and

$$\delta\Pi^{\varsigma}_{\lambda\mu}(\mathbf{r},t) \simeq n^+(\mathbf{r})T^{\varsigma}_{\lambda\mu}(t). \tag{B6}$$

The coefficients $N^{\varsigma}_{\beta,\gamma}(t)$ and $K^{\varsigma}_{\beta,-\gamma}(t)$ are connected by the linear relations with the collective variables $\mathcal{R}^{\varsigma}_{\lambda\mu}(t)$ and $\mathcal{L}^{\varsigma}_{\lambda\mu}(t)$, respectively:

$$\mathcal{R}^{\varsigma}_{\lambda\mu} = \int d^3r\{r\otimes r\}_{\lambda\mu}\delta n^{\varsigma}(\mathbf{r}) = \frac{2}{\sqrt{3}}\left[\mathcal{A}_1 C^{\lambda\mu}_{1\mu,10}N^{\varsigma}_{\mu,0} - \mathcal{A}_2\left(C^{\lambda\mu}_{1\mu+1,1-1}N^{\varsigma}_{\mu+1,-1} + C^{\lambda\mu}_{1\mu-1,11}N^{\varsigma}_{\mu-1,1}\right)\right], \tag{B7}$$

where

$$\mathcal{A}_1 = \sqrt{2}\, R^{\rm eq}_{20} - R^{\rm eq}_{00} = \frac{Q_{00}}{\sqrt{3}}\left(1+\frac{4}{3}\delta\right), \quad \mathcal{A}_2 = R^{\rm eq}_{20}/\sqrt{2} + R^{\rm eq}_{00} = -\frac{Q_{00}}{\sqrt{3}}\left(1-\frac{2}{3}\delta\right),$$
$$R^{\rm eq}_{20} = Q_{20}/\sqrt{6}, \quad R^{\rm eq}_{00} = -Q_{00}/\sqrt{3}, \quad Q_{20} = \frac{4}{3}\delta Q_{00}, \quad Q_{00} = A\langle r^2\rangle = \frac{3}{5}AR_0^2; \tag{B8}$$

$$N^{\varsigma}_{-1,-1} = -\frac{\sqrt{3}\,\mathcal{R}^{\varsigma}_{2-2}}{2\mathcal{A}_2}, \quad N^{\varsigma}_{-1,0} = \frac{\sqrt{6}\,\mathcal{R}^{\varsigma}_{2-1}}{4\mathcal{A}_1}, \quad N^{\varsigma}_{-1,1} = -\frac{\mathcal{R}^{\varsigma}_{00} + \mathcal{R}^{\varsigma}_{20}/\sqrt{2}}{2\mathcal{A}_2},$$
$$N^{\varsigma}_{0,-1} = -\frac{\sqrt{6}\,\mathcal{R}^{\varsigma}_{2-1}}{4\mathcal{A}_2}, \quad N^{\varsigma}_{0,0} = \frac{\sqrt{2}\mathcal{R}^{\varsigma}_{2,0} - \mathcal{R}^{\varsigma}_{0,0}}{2\mathcal{A}_1}, \quad N^{\varsigma}_{0,1} = -\frac{\sqrt{6}\,\mathcal{R}^{\varsigma}_{21}}{4\mathcal{A}_2}, \tag{B9}$$
$$N^{\varsigma}_{1,-1} = N^{\varsigma}_{-1,1}, \quad N^{\varsigma}_{1,0} = \frac{\sqrt{6}\,\mathcal{R}^{\varsigma}_{21}}{4\mathcal{A}_1}, \quad N^{\varsigma}_{1,1} = -\frac{\sqrt{3}\,\mathcal{R}^{\varsigma}_{22}}{2\mathcal{A}_2};$$

$$\mathcal{L}^{\varsigma}_{\lambda,\mu} = \int d^3r\{r\otimes\delta J^{\varsigma}\}_{\lambda\mu} = \frac{1}{\sqrt{3}}(-1)^{\lambda}\left[\mathcal{A}_1 C^{\lambda\mu}_{1\mu,10}K^{\varsigma}_{\mu,0} - \mathcal{A}_2\left(C^{\lambda\mu}_{1\mu+1,1-1}K^{\varsigma}_{\mu+1,-1} + C^{\lambda\mu}_{1\mu-1,11}K^{\varsigma}_{\mu-1,1}\right)\right]; \tag{B10}$$

$$K^{\varsigma}_{-1,-1} = -\frac{\sqrt{3}\,\mathcal{L}^{\varsigma}_{2-2}}{\mathcal{A}_2}, \quad K^{\varsigma}_{-1,0} = \frac{\sqrt{3}\left(\mathcal{L}^{\varsigma}_{1-1} + \mathcal{L}^{\varsigma}_{2-1}\right)}{\sqrt{2}\,\mathcal{A}_1}, \quad K^{\varsigma}_{-1,1} = -\frac{\sqrt{3}\,\mathcal{L}^{\varsigma}_{10} + \mathcal{L}^{\varsigma}_{20} + \sqrt{2}\,\mathcal{L}^{\varsigma}_{00}}{\sqrt{2}\,\mathcal{A}_2},$$
$$K^{\varsigma}_{0,-1} = \frac{\sqrt{3}\left(\mathcal{L}^{\varsigma}_{1-1} - \mathcal{L}^{\varsigma}_{2-1}\right)}{\sqrt{2}\,\mathcal{A}_2}, \quad K^{\varsigma}_{0,0} = \frac{\sqrt{2}\mathcal{L}^{\varsigma}_{2,0} - \mathcal{L}^{\varsigma}_{0,0}}{\mathcal{A}_1}, \quad K^{\varsigma}_{0,1} = -\frac{\sqrt{3}\left(\mathcal{L}^{\varsigma}_{11} + \mathcal{L}^{\varsigma}_{21}\right)}{\sqrt{2}\,\mathcal{A}_2}, \tag{B11}$$
$$K^{\varsigma}_{1,-1} = \frac{\sqrt{3}\,\mathcal{L}^{\varsigma}_{10} - \mathcal{L}^{\varsigma}_{20} - \sqrt{2}\,\mathcal{L}^{\varsigma}_{00}}{\sqrt{2}\,\mathcal{A}_2}, \quad K^{\varsigma}_{1,0} = \frac{\sqrt{3}\left(\mathcal{L}^{\varsigma}_{21} - \mathcal{L}^{\varsigma}_{11}\right)}{\sqrt{2}\,\mathcal{A}_1}, \quad K^{\varsigma}_{1,1} = -\frac{\sqrt{3}\,\mathcal{L}^{\varsigma}_{22}}{\mathcal{A}_2}.$$

The coefficient $T^{\varsigma}_{\lambda\mu}(t)$ is connected with $\mathcal{P}^{\varsigma}_{\lambda\mu}(t)$ by the relation $\mathcal{P}^{\varsigma}_{\lambda\mu}(t) = AT^{\varsigma}_{\lambda\mu}(t)$, with A being the number of nucleons.

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