Scissors modes: The elusive breathing overtone

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The two-rotor model predicts two levels above the scissors modes with degenerate intrinsic energy. They have $J^\pi = 0^+, 2^+$ and are referred to as overtones. Their energy is below the threshold for nucleon emission, which should make them observable. The $J^\pi = 0^+$ overtone, which has the structure of an isovector breathing mode, has vanishing $E0$ amplitude so that it cannot be directly excited, but it could be reached in the decay of the $J = 2^+$ overtone. We discuss such a process and evaluate the $B(E2)$ strength, which turns out to be very small.

I. INTRODUCTION

The two-rotor model, which led to the prediction of the scissors modes [1], has approximately the spectrum of a planar harmonic oscillator, with some constraints on the states. There are two scissors modes with $J^\pi = 1^+, 2^+$. They differ by the rotational energy of the nucleus, but have the same intrinsic energy of the order of 3 MeV in the rare-earth region [2,3]. The first levels above the scissors modes, referred to as overtones, have $J^\pi = 0^+, 2^+$ and equal intrinsic energy of the order of 6 MeV. The nature of the collective motion of the 0^+ was known since the beginning; it is a kind of isovector breathing mode. The 2^+ was considered in Ref. [4] but its structure in the two-rotor model has been determined only recently [5]. It is a superposition of an isovector breathing mode and a scissors mode with $K = 2$.

In general one cannot expect that the higher lying excitations of semiclassical models are realized in nature, which is more so for the two-rotor model, because of the modest collectivity and substantial fragmentation of the scissors modes [2,3]. In this case, however, the energy of the overtones is below threshold for nucleon emission, so that their widths should remain of purely electromagnetic nature and small enough for these states to be observable. Even so there was not much hope to excite these states. Indeed the $E0$ amplitude for the 0^+ overtone vanishes, while the $E2$ amplitude for the 2^+ overtone was expected to be proportional to θ_0, the amplitude of the zero-point oscillation, which in the rare-earth region is of order 10^{-1}.

We note that excitation amplitudes in the two-rotor model are proportional to powers of θ_0. Now $B(M1) \uparrow_{\text{scissors}} \sim 1/\theta_0^2$, but $B(E2) \uparrow_{\text{scissors}} \sim \theta_0^2$, the strength which was expected for the 2^+ overtone. All the other methods, the schematic random-phase approximation [6], the interacting boson model [7], the sum rule method [8], and a geometrical model [9], give similar results for the scissors modes.

Recently, nevertheless, we decided [5] to study the structure and the electromagnetic strengths of the 2^+ overtone in the two-rotor model. We disregarded the 0^+ one, because we did not know how to excite it in the framework of the two-rotor model. We found that the dominant term of the $B(E2)$ is of

zero order in the expansion with respect to θ_0, and precisely

$$B(E2, 0^+ \to 2_1^+) = \tfrac{3}{64} e^2 Q_{20}^2, \tag{1}$$

where the 2^+ overtone is denoted 2_1^+ to distinguish it from the 2^+ scissors mode (see Table I). This value must be compared with

$$B(E2) \uparrow_{\text{scissors}} = B(E2, 0^+ \to 2^+) = 3\theta_0^2 e^2 Q_{20}^2. \tag{2}$$

The numerical factor in Eq. (1) largely compensates for the presence of the small factor θ_0^2 in Eq. (2), but the strength of the overtone remains almost a factor of 2 larger than that of the $J = 2^+$ scissors mode in the rare-earth region. We expect that this result should be confirmed at least in the interacting boson model, whose Hamiltonian in the semiclassical approximation reproduces [10] that of the two-rotor model, with one qualification to be mentioned later.

Concerning experimental observation, as far as we know, only recently there have been investigations aimed at getting systematic information [11] on scissors modes in Gd isotopes up to about 6 MeV. These experiments are very interesting not only because they show that the $M1$ strength from scissors modes is required to explain their results, but also because they suggest the existence of scissors modes in excited states. They do not give, however, any evidence of our overtones. In our opinion this cannot be taken as a conclusive proof that the 2^+ state does not exist. The cited works, in fact, study neutron-photon reactions that cannot be analyzed in the framework of the two-rotor model, so that we do not know what is the probability of excitation of the 2^+ overtone after neutron capture. The smallness of this probability might explain the experimental result.

In order to assess this crucial point, from the theoretical point of view we need a microscopic model, and probably the most suitable one is the fermion-boson interacting model, which in the boson sector in the semiclassical limit exactly reproduces [10] the two-rotor model Hamiltonian. From the experimental side we need experiments with electromagnetic interactions.

If the 2_1^+ exists, however, it could provide a way for a direct observation of the 0_1^+ overtone because also the $B(E2, 2_1^+ \to 0_1^+)$ is of zero order in θ_0. This process should

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TABLE I. Quantum numbers of the positive parity states of the Two-Rotor Model.

Quantum numbers	J_σ^π	K	n	Energy
Ground state	0^+	0	0	0
Scissors modes	1^+	1	0	$\hbar\omega + \hbar^2/\mathcal{I}$
	2^+	1	0	$\hbar\omega + 3\hbar^2/\mathcal{I}$
Breathing overtone	0_1^+	0	1	$2\hbar\omega$
2^+ overtone	2_1^+	2	0	$2\hbar\omega + 3\hbar^2/\mathcal{I}$
		0	1	

occur via radiation of photons of energy

$$E = \frac{3\hbar^2}{\mathcal{I}}, \qquad (3)$$

where $\mathcal{I}$ is the moment of inertia of the nucleus. It is then interesting to study this decay, first, in order to establish whether the isovector breathing mode can actually be seen. Needless to say, its observation in spite of the modest collectivity and substantial fragmentation of the scissors modes would be of much consequence for the picture of scissors modes provided by the two-rotor model, the more so in view of the very specific structure of the 2_1^+ overtone. Second, the existence, but also the absence, of such a decay might be relevant to the identification of the 2_1^+ overtone. In any case the evaluation of the $B(E2)$ for such a process is necessary for the knowledge of the total electromagnetic width of this level. A further reason of interest is related to the other electrically charged systems for which scissors modes have been predicted [12].

II. DECAY OF THE 2+ OVERTONE TO THE BREATHING ONE

In the present paper we report the evaluation of the $B(E2)$ for the decay of the 2_1^+ overtone into the breathing one, and the $M1$ strength for the subsequent decay of the breathing mode into the 1^+ scissors mode. This second strength turns out to be equal to the strength for exciting the 1^+ scissors mode from the ground state. To our surprise, instead, the first one is negligible due to a cancellation that appears totally accidental. The breathing overtone remains therefore elusive in the framework of the two-rotor model.

In order to make the paper self-contained, we report the relevant features of the two-rotor Model. Its Hamiltonian is

$$H = \frac{\vec{J}^2}{2\mathcal{I}} + H_{\text{intr}}, \qquad (4)$$

where the first term is the total rotational energy of the nucleus, $\vec{J}$ is the total angular momentum, and the second term is the intrinsic Hamiltonian which, in the slightly modified form of Ref. [13], neglecting terms of order θ_0, is

$$H_{\text{intr}} = \frac{1}{2\mathcal{I}}\left[-\frac{d^2}{d\theta^2} - (2 + \cot^2(2\theta)) \right.$$
$$\left. + \cot^2\theta\, J_\zeta^2 + \tan^2\theta\, J_\eta^2 \right] + V(\theta). \qquad (5)$$

θ is half the angle between the proton neutron axes; J_ξ, $J\eta$, and J_ζ are the angular momentum components along the axes of the body-fixed frame; and V is the proton-neutron interaction potential. The range of θ can be separated into two regions,

$$s_I = s(\theta)s\left(\frac{\pi}{4} - \theta\right), \quad s_{II} = s\left(\frac{\pi}{2} - \theta\right)s\left(\theta - \frac{\pi}{4}\right), \qquad (6)$$

where $s(x)$ is the step function: $s(x) = 1$, $x > 0$ and zero otherwise. They are obtained from each other by the reflection of θ with respect to $\pi/2$. It is convenient to introduce the notation

$$R_\theta f(\theta) = \overset{\circ}{f}(\theta), \qquad (7)$$

where

$$\overset{\circ}{f}(\theta) = f\left(\frac{\pi}{2} - \theta\right), \qquad (8)$$

so that $\overset{\circ}{s}_I = s_{II}$. We assume $\overset{\circ}{V} = V$, as appropriate to the geometry of the system. Since we know that the angle between the neutron-proton axes is very small we can assume for the potential a quadratic approximation

$$V = \tfrac{1}{2}C\,\theta_0^2\,x^2 s_I + \tfrac{1}{2}C\,\theta_0^2\,y^2 s_{II}, \qquad (9)$$

where

$$\theta_0 = (\mathcal{I}C)^{-\frac{1}{4}}, \quad x = \frac{\theta}{\theta_0}, \quad y = \frac{\frac{\pi}{2} - \theta}{\theta_0}. \qquad (10)$$

The intrinsic Hamiltonian is then invariant with respect to the transformation

$$R = R_\xi\left(\frac{\pi}{2}\right)R_\theta, \qquad (11)$$

where R_ξ is the rotation operator around the ξ axis, so that we can study the eigenvalue equation separately in the regions I and II.

The qualification mentioned at the beginning relative to the semiclassical limit of the interacting boson model is that this limit has been performed [10] only for region I. We certainly do not expect any surprise for region II, but the symmetries of the Hamiltonian with the consequent constraints on the states remain to be worked out in this model.

The eigenfunctions and eigenvalues of H_{intr} in region I are [13]

$$\varphi_{Kn}(x) = \sqrt{\frac{n!}{(n+K)!\,\theta_0}}\, x^{K+\frac{1}{2}} L_n^K(x^2) e^{-\frac{1}{2}x^2}, \qquad (12)$$

$$\epsilon_{Kn} = \omega(2n + K + 1), \quad \omega = \sqrt{\frac{C}{\mathcal{I}}}, \qquad (13)$$

where the L_n^K are Laguerre polynomials and the wave functions φ are normalized according to

$$\int_0^\infty dx\, [\varphi_{Kn}(x)]^2 = \frac{1}{2}. \qquad (14)$$

Enforcing the symmetries of the Hamiltonian and of the system (separate rotations of the proton and neutron bodies through π around the ξ axis), one finds the following results [1,13]. Restricting ourselves to states of positive parity we found

$$\Psi_{JM\sigma} = \sum_{K \geqslant 0} \mathcal{F}_{MK}^J(\alpha, \beta, \gamma)\Phi_{JK\sigma}(\theta), \qquad (15)$$

where

$$\mathcal{F}^J_{MK} = \sqrt{\frac{2J+1}{16(1+\delta_{K0})\pi^2}} \left(\mathcal{D}^J_{MK} + (-1)^J \mathcal{D}^J_{M-K} \right). \quad (16)$$

J, M are the nucleus angular momentum and its component on the z axis of the laboratory system, and σ labels the different states with the same J. When it is zero it will be omitted. We make an assignment of this quantum number differently from previous papers, in which we ignored the $J = 0^+$ overtone. We impose the normalization

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\theta \, |\Psi_{IM\sigma}|^2 = 1. \quad (17)$$

Second, the expressions of the intrinsic functions φ in regions I and II are related according to

$$\Phi_{00} = \varphi_{00}\, s_I + \overset{\circ}{\varphi}_{00}\, s_{II},$$

$$\Phi_{11} = \Phi_{210} = \varphi_{10}\, s_I - \overset{\circ}{\varphi}_{10}\, s_{II},$$

$$\Phi_{001} = \varphi_{01}\, s_I + \overset{\circ}{\varphi}_{01}\, s_{II}, \quad (18)$$

$$\Phi_{201} = \frac{1}{\sqrt{2}} \left[\varphi_{01}\, s_I - \frac{1}{2}(\sqrt{3}\, \overset{\circ}{\varphi}_{20} + \overset{\circ}{\varphi}_{01}) s_{II} \right]$$

$$\Phi_{221} = \frac{1}{\sqrt{2}} \left[\varphi_{20}\, s_I + \frac{1}{2}(\overset{\circ}{\varphi}_{20} - \sqrt{3}\, \overset{\circ}{\varphi}_{01}) s_{II} \right].$$

Even if the nucleus in its ground state has axial symmetry, this symmetry is in general lost in the excited states, so that the component of angular momentum along any body-fixed axis is not conserved, resulting in a superposition of intrinsic states with different K and n quantum numbers. All the above states, however, with the exception of the $J = 2^+$ overtone, are pure K and n states. The ground state 0^+ has quantum numbers $J = \sigma = K = n = 0$, the scissors modes $1^+, 2^+$ have quantum numbers $\sigma = 0$, $K = 1$, $n = 0$, and $J = 1, 2$ respectively, and the breathing overtone 0^+_1 has quantum numbers $J = 0$, $\sigma = 1$, $K = 0$, $n = 1$. The other overtone 2^+_1 has $J = 2$, $\sigma = 1$ and coupled intrinsic components with $K = 2, n = 0$ and $K = 0, n = 1$.

The collective motion of the 0^+ overtone is a kind of isovector breathing mode. The 2^+ overtone is a superposition of the breathing mode in region I (the state φ_{01}) and of the state φ_{20}, which is a relative rotation of the neutron-proton axes as in the scissors mode but with angular momentum $K = 2$. Such mixing is determined by the different forms that the intrinsic Hamiltonian takes in regions I and II. Because of it the 2^+ overtone might be called scissors-breathing mode. All these states with their quantum numbers are reported in Table I.

We already mentioned that the spectrum of the intrinsic part of the two-rotor model is identical to that of the planar harmonic oscillator with some constraints due to various symmetries. These constraints reduce the degeneracy of the first excited states of the planar harmonic oscillator from 2 to 1 (the scissors mode) and of the higher levels from 3 to 2 (the two overtones).

Notice that the normalization of the Φ in Eq. (15) is different from that in Ref. [13].

As we said at the beginning, the quadrupole operator to zero order [5] in θ_0 is

$$M(E2,\mu) = e\, Q_{20} \left[\mathcal{D}^2_{\mu 0} \left(s_I - \frac{1}{2}\, s_{II} \right) \right.$$

$$\left. + \frac{1}{2}\sqrt{\frac{3}{2}} (\mathcal{D}^2_{\mu 2} + \mathcal{D}^2_{\mu -2}) s_{II} \right], \quad (19)$$

where $e\, Q_{20}$ is the quadrupole moment in the intrinsic frame. We then see that while to zero order in θ_0 we cannot excite the breathing overtone from the ground state, we could reach it by the decay of the $J = 2^+$ overtone with the amplitude

$$\langle \Psi_{2M1} | M(E2,\mu) | \Psi_{001} \rangle = \frac{1}{2\sqrt{10}}\, e\, Q_{20} C^{2M}_{002\mu}$$

$$\times (\langle \varphi_{01}|\varphi_{01}\rangle + \sqrt{3}\langle \phi_{20}|\phi_{01}\rangle),$$

where $C^{2M}_{002\mu}$ is a Clebsch-Gordan coefficient. Because

$$\langle \varphi_{01}|\varphi_{01}\rangle = -\sqrt{2}\langle \phi_{20}|\phi_{01}\rangle = \tfrac{1}{2} \quad (20)$$

the contributions of the $K = 0$ and $K = 2$ components of the $J = 2^+$ overtone almost cancel out with each other, at variance with the $B(E2, 0^+ \to 2^+_1)$ that is entirely due to its $K = 0$ component, namely its breathing component [5]. Such cancellation appears completely accidental, but it overcompensates the absence of the small factor θ_0^2, so that the transition strength

$$B(E2, 2^+_1 \to 0^+_1) = \frac{1}{32}\left(1 - \sqrt{\frac{3}{2}}\right)^2 e^2\, Q^2_{20} \quad (21)$$

turns out to be negligibly small.

In conclusion the electromagnetic width of the 2^+_1 overtone is entirely due to its $M1$ decay to the scissors mode 1^+ and to its $E2$ decay to the ground state [5], and the breathing overtone cannot be reached through its decay.

III. DECAY OF THE BREATHING OVERTONE TO THE SCISSORS MODE 1^+

For the case in which the breathing mode could be excited by other means or appear in intermediate states, we report its $B(M1)$ strength for decaying into the 1^+ scissors mode. The expression of the magnetic dipole operator is [5]

$$M(M1,\nu) = -\frac{1}{\sqrt{2}}(\mathcal{D}^1_{\nu 1} - \mathcal{D}^1_{\nu -1})\mathcal{M}$$

$$\times (s_I - s_{II})\left(\frac{d}{d\theta} - \frac{1}{2\theta}\right), \quad (22)$$

where

$$\mathcal{M} = i\sqrt{\frac{3}{16\pi}}(g_p - g_n)\frac{e}{2m_p}, \quad (23)$$

g_n, g_p being the orbital gyromagnetic factors of neutrons and protons respectively and m_p is the proton mass.

By a straightforward calculation we get

$$\langle \Psi_{001}|M(M1,\nu)|\Psi_{1M}\rangle = 2\sqrt{3}\,\mathcal{M}\langle \phi_{01}|\nabla_\theta|\phi_{10}\rangle$$
$$\times C^{00}_{1M1\nu}C^{00}_{1-111}. \qquad (24)$$

Since

$$\langle \phi_{01}|\nabla_\theta|\phi_{10}\rangle = \frac{1}{2\theta_0} \qquad (25)$$

we find

$$B(M1, 0_1^+ \to 1^+) = \frac{1}{\theta_0^2}\,|\mathcal{M}|^2, \qquad (26)$$

which is equal to the $B(M1)$ strength for excitation of the 1^+ scissors mode from the ground state.

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