

## Baryon photo-decay amplitudes at the pole

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We derive relations for baryon photo-decay amplitudes both for the Breit-Wigner and the pole positions. With an updated SAID partial wave analysis, technically similar to the earliest Virginia Tech analysis of photoproduction data, we compare photo-decay amplitudes at both resonance positions for a few selected nucleon resonances. Comparisons are made and a qualitative similarity, seen between the pole and Breit-Wigner values extracted by the Bonn-Gatchina group, is confirmed in the present study.

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*Introduction.* Baryon resonance properties, evaluated at the pole position, are beginning to supersede and replace quantities which have generally been determined using the Breit-Wigner (BW) approach plus background parametrizations. This is reflected in the most recent Review of Particle Properties [1], with many pole values coming from the recent Bonn-Gatchina multichannel analyses [2]. While the pole extraction is well-defined and less model dependent than the Breit-Wigner approach, the continuation of fit amplitudes to the pole is itself a possible source of error. This has motivated numerous studies involving speed plots, Laurent series representations, regularization methods, and contour integration [3–8]. Here we will compare Breit-Wigner and pole extractions, using an early SAID fit form, with the focus on  $N^*$  photo-decay amplitudes.

As the amplitude itself becomes infinite at the pole, we are interested in residues. We first clarify the connection between multipole residues and the photo-decay amplitudes. This can be related to a result published with the first SAID photoproduction fits [9]. A comparison of recent results and the first attempts tabulated in Ref. [9] reveals some large discrepancies. We study and resolve this problem below.

The photo-decay amplitudes determined via Breit-Wigner and pole methods, as given by the Bonn-Gatchina group [2], tend to be very similar in modulus. We reproduce this trend within the original SAID photoproduction model.

*Breit-Wigner versus pole quantities.* The total cross section of pion photoproduction can be written in terms of helicity multipoles by

$$\sigma_{\gamma,\pi} = \frac{1}{2} (\sigma_{\gamma,\pi}^{1/2} + \sigma_{\gamma,\pi}^{3/2}), \quad (1)$$

$$\sigma_{\gamma,\pi}^h = 4\pi \frac{q}{k} \sum_{\alpha(\ell,J,I)} (2J+1) |A_\alpha^h|^2 C^2, \quad (2)$$

with  $q$  and  $k$  being the center-of-mass pion and photon momenta. The factor  $C$  is  $\sqrt{2/3}$  for isospin  $3/2$  and  $-\sqrt{3}$  for isospin  $1/2$ . The helicity multipoles are given in terms of electric and magnetic multipoles

$$A_{\ell+}^{1/2} = -\frac{1}{2} [(\ell+2)E_{\ell+} + \ell M_{\ell+}], \quad (3)$$

$$A_{\ell+}^{3/2} = \frac{1}{2} \sqrt{\ell(\ell+2)} [E_{\ell+} - M_{\ell+}], \quad (4)$$

$$A_{(\ell+)-}^{1/2} = -\frac{1}{2} [\ell E_{(\ell+)-} - (\ell+2)M_{(\ell+)-}], \quad (5)$$

$$A_{(\ell+)-}^{3/2} = -\frac{1}{2} \sqrt{\ell(\ell+2)} [E_{(\ell+)-} + M_{(\ell+)-}], \quad (6)$$

with  $J = \ell + 1/2$  for ‘+’ multipoles and  $J = (\ell + 1) - 1/2$  for ‘-’ multipoles, all having the same total spin  $J$ .

Comparing with the definition of the cross section from a unitary amplitude [10]

$$\sigma_{i,f} = \frac{4\pi}{k^2} \frac{2J+1}{(2s_1+1)(2s_2+1)} |T_{i,f}^h|^2, \quad (7)$$

where  $k$  is the c.m. momentum in the initial state and  $s_1$  and  $s_2$  are the spins of the two incoming particles, allows us to compare the  $(\gamma N)$  channel, in a consistent way, to other inelastic channels.

For the polarized photoproduction cross section with helicity  $h$  we have

$$\sigma_{\gamma,\pi}^h = \frac{2\pi}{k^2} (2J+1) |T_{\gamma,\pi}^h|^2, \quad (8)$$

leading to the relation between unitary and helicity amplitudes

$$T_{\gamma,\pi}^h = \sqrt{2kq} A_\alpha^h C. \quad (9)$$

For a better understanding of the difference between Breit-Wigner parameters and pole parameters, the unitary amplitude can be written in terms of a propagator and initial and final partial widths [10],

$$T_{\gamma,\pi}^h(W) = \frac{(\Gamma_h/2)^{1/2} (\Gamma_\pi/2)^{1/2}}{M - W - i\Gamma/2}. \quad (10)$$

At the Breit-Wigner resonance position  $W_r = M$  the amplitude becomes purely imaginary and the BW resonance amplitude is defined as

$$\tilde{T}_{\gamma,\pi}^h = \text{Im } T_{\gamma,\pi}^h(W_r) \quad (11)$$

$$= \frac{\Gamma_h^{1/2} \Gamma_\pi^{1/2}}{\Gamma}. \quad (12)$$

At the pole position,  $W_p = M - i\Gamma/2$ , the amplitude becomes infinite and the pole parameter is defined as the residue

$$R_{\gamma,\pi}^h = \text{Res } T_{\gamma,\pi}^h(W_p) \quad (13)$$

$$= \frac{\Gamma_h^{1/2} \Gamma_\pi^{1/2}}{2}, \quad (14)$$

which is in general complex. However, in our illustrative example, the widths can be considered as constant, in which case the residues are real. Note that, traditionally, the residues of baryon resonances have been defined with a relative minus sign compared to the standard mathematical definition. The residue for  $\gamma N \rightarrow \pi N$  can be factorized into residues for  $\gamma N \rightarrow \gamma N$  and  $\pi N \rightarrow \pi N$

$$R_{\gamma,\pi}^h = \sqrt{\text{Res} T_{\gamma N}^h(W_p) \text{Res} T_{\pi N}(W_p)}. \quad (15)$$

For these residues we will use in the following the short-hand notation  $\text{Res}_{\gamma(h)N}$  and  $\text{Res}_{\pi N}$ .

At the BW position, the total photoproduction cross section with helicity  $h$  is

$$\sigma_{\gamma,\pi}^h = \frac{2\pi}{k^2} (2J+1) \frac{\Gamma_h \Gamma_\pi}{\Gamma^2} \quad (16)$$

and the unpolarized cross section is

$$\sigma_{\gamma,\pi} = \frac{\pi}{k^2} (2J+1) (\Gamma_{1/2} + \Gamma_{3/2}) \frac{\Gamma_\pi}{\Gamma^2}. \quad (17)$$

With the relation [11,12] between the electromagnetic width and the photo-decay amplitudes  $A_{1/2}, A_{3/2}$

$$\Gamma_\gamma(M_r) = \frac{k^2}{\pi} \frac{2}{2J+1} \frac{m_N}{M_r} (|A_{1/2}|^2 + |A_{3/2}|^2), \quad (18)$$

the total cross section takes the form

$$\sigma_{\gamma,\pi}(M_r) = \frac{2m_N \Gamma_{\pi,r}}{M_r \Gamma_r^2} (|A_{1/2}|^2 + |A_{3/2}|^2), \quad (19)$$

where  $\Gamma_r$  and  $\Gamma_{\pi,r}$  are widths evaluated at the BW resonance energy  $M_r$  and  $m_N$  is the nucleon mass.

Equation (18) can be used as a definition for the photo-decay amplitudes

$$A_h(W) = \sqrt{\frac{\pi(2J+1)M}{2k^2 m_N}} \Gamma_h^{1/2}(W), \quad (20)$$

and by comparison with Eqs. (9) and (12) we obtain the amplitudes at the Breit-Wigner position

$$A_h^{\text{BW}} = C \sqrt{\frac{q_r}{k_r} \frac{\pi(2J+1)M_r \Gamma_r^2}{m_N \Gamma_{\pi,r}}} \mathcal{A}_\alpha^h. \quad (21)$$

Similarly, a comparison with Eqs. (9), (14), and (15) leads to the amplitudes at the pole position

$$A_h^{\text{pole}} = C \sqrt{\frac{q_p}{k_p} \frac{2\pi(2J+1)W_p}{m_N \text{Res}_{\pi N}}} \text{Res} \mathcal{A}_\alpha^h, \quad (22)$$

where the subscript  $p$  denotes quantities evaluated at the pole position.

Finally, normalized residues, partial widths, and branching ratios at the pole can also be determined in accordance to the conventions of the PDG. The normalized residues are the residues divided by the half-width at the pole,

$$R_{\text{norm},\gamma,\pi}^h = \frac{R_{\gamma,\pi}^h}{\Gamma_p/2} = \frac{\Gamma_h^{1/2} \Gamma_\pi^{1/2}}{\Gamma_p}, \quad (23)$$

and obtain complex values, whereas the partial widths of the  $\pi N$  and  $\gamma N$  channels,

$$\Gamma_{\pi,p} = 2|\text{Res}_{\pi N}|, \quad (24)$$

$$\Gamma_{h,p} = 2|\text{Res}_{\gamma(h)N}| = \frac{2|R_{\gamma,\pi}^h|^2}{|\text{Res}_{\pi N}|}, \quad (25)$$

and the branching ratios at the pole

$$B_{\text{pole}}(\pi N) = \frac{\Gamma_{\pi,p}}{\Gamma_p} = \frac{|\text{Res}_{\pi N}|}{\Gamma_p/2}, \quad (26)$$

$$B_{\text{pole}}(\gamma^h N) = \frac{\Gamma_{h,p}}{\Gamma_p} = \frac{|R_{\gamma,\pi}^h|^2}{|\text{Res}_{\pi N}| \Gamma_p/2} \quad (27)$$

acquire real and positive numerical values.

*A simple model test.* In the first SAID analysis of pion photoproduction data [9], multipoles were fitted using the form

$$M_\ell = [\text{Born} + B(W)](1 + iT_{\pi N}^\ell) + C(W)T_{\pi N}^\ell, \quad (28)$$

based on a simple  $K$  matrix approach [13]. This form had the advantage that only the elastic  $\pi N$   $T$  matrix was required ( $T_{\pi N}^\ell$ ), as connected to photoproduction via Watson's theorem below the  $\pi\pi N$  threshold, and continuing smoothly from this constraint as the  $\pi N$  partial waves became inelastic. In the above,  $\ell$  is the relative  $\pi N$  angular momentum. Labels for isospin and total spin have been suppressed. The phenomenological pieces,  $B(W)$  and  $C(W)$ , were polynomials in energy with the required threshold behavior and were fitted for each partial wave.

In deriving Eq. (28), the inelasticity was assumed to be dominated by a single channel. This simple approach has now been improved [14]. However, given a known set of elastic residues and pole positions for the underlying  $\pi N$  reaction, the above form provides a simple test case for extracting pole-related quantities in pion photoproduction, while giving a reasonable fit to the data. This fit has been reproduced for the present study.

The  $\pi N$   $T$ -matrix terms in Eq. (28) contain information regarding included resonances and opening thresholds [15,16]. As a result, the energy-dependent pre-factors are quite smoothly varying and can be represented by low-order polynomials in energy. Here, and in Ref. [9], the multipole residues were extracted from the known  $\pi N$  pole positions and residues, and a straightforward evaluation of the energy-dependent pre-factors at the pole position.

Beyond being just a toy model, the form in Eq. (28) was fitted to data from the  $\pi^+n$  threshold to a laboratory photon energy of 2 GeV, sufficient to compare with the results of Ref. [9] and other more recent determinations [2]. Results for both Breit-Wigner plus background and pole determinations are given in Table I. The form of background-resonance separation is very similar to that used in the MAID fits [17] and is detailed in Ref. [18]. Errors for the Breit-Wigner fits were determined by fitting the multipoles from the form of Eq. (28), using a Breit-Wigner resonance, over varying energy ranges. For the pole determinations, the Born +  $B(W)$  and  $C(W)$  were represented by two polynomials,  $\alpha(W)$  and  $\beta(W)$ ,

TABLE I. Breit-Wigner and pole values for selected nucleon resonances. Masses, widths, and residues are given in units of MeV, the helicity 1/2 and 3/2 photo-decay amplitudes in units of  $10^{-3}(\text{GeV})^{-1/2}$ . Errors on the phases are generally 2–5 degrees. For isospin 1/2 resonances the values of the proton target are given.

Resonance	Breit-Wigner values				Pole values			
	(Mass, width)	$\Gamma_{\pi/2}$	$A_{1/2}$	$A_{3/2}$	(Re $W_p$ , $-2 \text{ Im } W_p$ )	$R_{\pi}$	$A_{1/2}$	$A_{3/2}$
$\Delta(1232) 3/2^+$	(1233, 119)	60	$-141 \pm 3$	$-258 \pm 5$	(1211, 99)	52 $[-47^\circ]$	$-136 \pm 5 [-18^\circ]$	$-255 \pm 5 [-6^\circ]$
$N(1440) 1/2^+$	(1485, 284)	112	$-60 \pm 2$		(1359, 162)	38 $[-98^\circ]$	$-66 \pm 5 [-38^\circ]$	
$N(1520) 3/2^-$	(1515, 104)	33	$-19 \pm 2$	$+153 \pm 3$	(1515, 113)	38 $[-5^\circ]$	$-24 \pm 3 [-7^\circ]$	$+157 \pm 6 [+10^\circ]$
$N(1535) 1/2^-$	(1547, 188)	34	$+92 \pm 5$		(1502, 95)	16 $[-16^\circ]$	$+77 \pm 5 [+4^\circ]$	
$N(1650) 1/2^-$	(1635, 115)	58	$+35 \pm 5$		(1648, 80)	14 $[-69^\circ]$	$+35 \pm 3 [-16^\circ]$	

of varying orders, over a range of energies sufficient for extrapolation to the pole. The stability of these results and the errors from the  $\pi N$  elastic pole determinations were combined in a representative error.

*Results and conclusions.* From Table I, we see that the pole and Breit-Wigner determinations, for the states considered in Ref. [9] plus a nearby state, are quite similar in modulus. In the earlier determination, however, the pole widths, constructed from squares of the helicity amplitudes, were found to be qualitatively similar for the  $\Delta(1232)$  and  $N(1520)$ , but radically different for the  $N(1440)$  and  $N(1535)$ , differing in the latter cases by factors of about 2 and 5, respectively. A possible cause of the discrepancy is seen in the  $N(1440)$  and  $N(1535)$  pole positions and elastic residues. Since 1990, the modulus of the elastic residue has shifted from 108 to 38 MeV for  $N(1440)$ , and from 54 to 16 MeV for the  $N(1535)$ . For these states, the pole positions have also shifted significantly.

For the pole amplitudes extracted in Table I, results are generally quite similar to those from the Bonn-Gatchina group [2]. An exception is the  $N(1535)$ , where the simple model of Ref. [9] is known to differ significantly from a more sophisticated approach [2,14]. A substantial benefit from the pole extraction is found for the nearby  $N(1650)$ . BW fits to the underlying  $\pi N$  amplitude have produced unreliable width and elasticity values, which, in turn, have made BW fits to photoproduction multipoles difficult and similarly unreliable. None of these issues affect the pole determination.

Prior to the recent work of the Bonn-Gatchina group [2], significant differences were seen in comparing photoproduction amplitudes determined through BW and pole determinations. These included the early fits of Ref. [9] as well as determinations of the  $E2/M1$  ratio [19,20] for the  $\Delta(1232)$ . In the latter case, stability of this ratio at the pole was found to be better than associated BW fits. Here we have repeated the study of Ref. [9], finding results in qualitative if not quantitative agreement with those of Ref. [2]. Results are often quite similar to BW determinations, apart from a phase. In cases where these values differ, the pole determination is more reliable.

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## APPENDIX: EXAMPLES

The following examples will illustrate the derivation of the photo-decay amplitudes at the pole position.

- (i) For the  $\Delta(1232)$  resonance, we obtain a pole position of  $W_p = (1.211 - i0.099/2)$  GeV with an elastic residue of  $\text{Res}_{\pi N} = 52 e^{-i47^\circ}$  MeV. For photoproduction, there are two isospin 3/2 multipoles, for which we find the residues

$$\text{Res } M_{1+}^{3/2} = 2.96 e^{-i30^\circ} \text{ mfm GeV}, \quad (\text{A1})$$

$$\text{Res } E_{1+}^{3/2} = -0.16 e^{i35^\circ} \text{ mfm GeV}. \quad (\text{A2})$$

With Eqs. (3) and (4) we obtain the residues of the helicity multipoles

$$\text{Res } \mathcal{A}_{1+}^{1/2} = -\frac{1}{2} (\text{Res } M_{1+}^{3/2} + 3 \text{Res } E_{1+}^{3/2}) \quad (\text{A3})$$

$$= -1.40 e^{-i39^\circ} \text{ mfm GeV}, \quad (\text{A4})$$

$$\text{Res } \mathcal{A}_{1+}^{3/2} = -\frac{\sqrt{3}}{2} (\text{Res } M_{1+}^{3/2} - \text{Res } E_{1+}^{3/2}) \quad (\text{A5})$$

$$= -2.63 e^{-i27^\circ} \text{ mfm GeV}. \quad (\text{A6})$$

To obtain the photo-decay amplitudes, these residues must be multiplied by a complex factor depending on spin, isospin, kinematics at the pole, and the elastic  $\pi N$  residue,

$$A_h^{\text{pole}} = N \frac{\text{Res } \mathcal{A}_{1+}^h}{197 \text{ mfm GeV}}, \quad (\text{A7})$$

$$N = C \sqrt{\frac{q_p 2\pi(2J+1)W_p}{k_p m_N \text{Res}_{\pi N}}}. \quad (\text{A8})$$

With the isospin factor  $C = \sqrt{2/3}$ ,  $q_p/k_p = 0.88 e^{-i3^\circ}$ ,  $J = 3/2$ , the nucleon mass  $m_N$ , giving  $N = 19.2 e^{i21^\circ} \text{ GeV}^{-1/2}$ , we obtain the photo-decay

amplitudes at the pole

$$A_{1/2}^{\text{pole}} = -0.136 e^{-i18^\circ} \text{ GeV}^{-1/2}, \quad (\text{A9})$$

$$A_{3/2}^{\text{pole}} = -0.255 e^{-i6^\circ} \text{ GeV}^{-1/2}. \quad (\text{A10})$$

The magnitudes are very close to the Breit-Wigner values; the phases are considerably smaller than the phases of the residues themselves, because a large phase of the elastic residue is already taken out.

- (ii) For the Roper resonance  $N(1440)$ , we obtain a pole position of  $W_p = (1.359 - i0.162/2) \text{ GeV}$  with an elastic residue of  $\text{Res}_{\pi N} = 38 e^{-i98^\circ} \text{ MeV}$ . For photo-production, there is only one isospin 1/2 multipole, for which we find the residue

$$\text{Res } M_{1-}^{1/2} = 0.35 e^{-i85^\circ} \text{ mfm GeV}. \quad (\text{A11})$$

With Eq. (5) we obtain the residue of the helicity 1/2 multipole as

$$\text{Res } \mathcal{A}_{1-}^{1/2} = \text{Res } M_{1-}^{1/2}. \quad (\text{A12})$$

With the isospin factor  $C = -\sqrt{3}$  and  $q_p/k_p = 0.95 e^{-i1^\circ}$ ,  $J = 1/2$ , the pole mass  $M_p = 1.359 \text{ GeV}$ , giving  $N = -37 e^{i47^\circ} \text{ GeV}^{-1/2}$  we obtain the photo-decay amplitude at the pole

$$A_{1/2}^{\text{pole}} = -0.066 e^{-i38^\circ} \text{ GeV}^{-1/2}, \quad (\text{A13})$$

again a value with a magnitude close to the BW value and a much smaller phase compared to the multipole residue.

- (iii) For the  $D_{13}$  resonance  $N(1520)$ , we obtain a pole position of  $W_p = (1.515 - i0.113/2) \text{ GeV}$  with an elastic residue of  $\text{Res}_{\pi N} = 38 e^{-i5^\circ} \text{ MeV}$ . For photo-production, there are two isospin 1/2 multipoles, for which we find the residues

$$\text{Res } E_{2-}^{1/2} = 0.442 e^{i10.5^\circ} \text{ mfm GeV}, \quad (\text{A14})$$

$$\text{Res } M_{2-}^{1/2} = 0.196 e^{i4.5^\circ} \text{ mfm GeV}. \quad (\text{A15})$$

With Eqs. (5) and (6) we obtain the residues of the helicity 1/2 and 3/2 multipoles as

$$\text{Res } \mathcal{A}_{2-}^{1/2} = -\frac{1}{2} (\text{Res } E_{2-}^{1/2} - 3 \text{Res } M_{2-}^{1/2}) \quad (\text{A16})$$

$$= 0.078 e^{-i13^\circ} \text{ mfm GeV}, \quad (\text{A17})$$

$$\text{Res } \mathcal{A}_{2-}^{3/2} = -\frac{\sqrt{3}}{2} (\text{Res } E_{2-}^{1/2} + \text{Res } M_{2-}^{1/2}) \quad (\text{A18})$$

$$= -0.55 e^{i9^\circ} \text{ mfm GeV}. \quad (\text{A19})$$

With the isospin factor  $C = -\sqrt{3}$ ,  $q_p/k_p = 0.97 e^{-i0^\circ}$ , giving  $N = -56 e^{i1^\circ} \text{ GeV}^{-1/2}$  we obtain the photo-decay amplitudes at the pole

$$A_{1/2}^{\text{pole}} = -0.024 e^{-i7^\circ} \text{ GeV}^{-1/2}, \quad (\text{A20})$$

$$A_{3/2}^{\text{pole}} = +0.157 e^{i10^\circ} \text{ GeV}^{-1/2}, \quad (\text{A21})$$

also values with magnitudes very similar to the BW values.

- (iv) For the  $S_{11}$  resonance  $N(1535)$ , we obtain a pole position of  $W_p = (1.502 - i0.095/2) \text{ GeV}$  with an elastic residue of  $\text{Res}_{\pi N} = 16 e^{-i16^\circ} \text{ MeV}$ . For photo-production, there is only one isospin 1/2 multipole, for which we find the residue

$$\text{Res } E_{0+}^{1/2} = 0.25 e^{-i3^\circ} \text{ mfm GeV}, \quad (\text{A22})$$

With Eq. (3) we obtain the residue of the helicity 1/2 multipole as

$$\text{Res } \mathcal{A}_{0+}^{1/2} = -\text{Res } E_{0+}^{1/2}. \quad (\text{A23})$$

With the isospin factor  $C = -\sqrt{3}$ ,  $q_p/k_p = 0.97 e^{-i0^\circ}$ , giving  $N = -60 e^{i7^\circ} \text{ GeV}^{-1/2}$  we obtain the photo-decay amplitude at the pole

$$A_{1/2}^{\text{pole}} = 0.077 e^{i4^\circ} \text{ GeV}^{-1/2}, \quad (\text{A24})$$

a value with a magnitude which differs about 20% from the BW value.

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