Nonlocality in the adiabatic model of $A(d, p)B$ reactions

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In a previous publication [Phys. Rev. Lett. **110**, 112501 (2013)] we have proposed a generalization of the adiabatic model of (d, p) reactions that allows the nonlocality of the nucleon optical potential to be included in a consistent way together with the deuteron breakup. In this model an effective local d-A potential is constructed from local nucleon optical potentials taken at an energy shifted by ∼40 MeV with respect to the widely used $E_d/2$ value, where E_d is the deuteron incident energy. The effective d-A potential is shallower than that traditionally used in the analysis of (d, p) reactions within the adiabatic distorted wave approximation and this affects the calculated cross sections and the nuclear structure quantities obtained from their comparison with experimental data. In the present paper we give full derivation of the deuteron effective potential, consider its leading-order term within the local-energy approximation and discuss corrections to the leading-order term. The new method is applied to (d, p) reactions on ^{16}O, ^{36}Ar, and ^{40}Ca targets and the influence of the deviation from the $E_d/2$ rule on the calculated cross sections is quantified.

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I. INTRODUCTION

Exploring shell evolution and changing magic numbers in the neutron- and proton-rich region of nuclear chart has become a major focus of nuclear structure research in the last decade. Many structure models are being developed that aim to relate shell structure to realistic nucleon-nucleon forces. On the experimental side, shell structure is probed by one-nucleon removal reactions aimed at determining the spectroscopic strength of nucleon orbits. The extraction of this information relies on the availability of theoretical models of one-nucleon removal cross sections that provide a transparent link between spectroscopic information and the reaction dynamics.

The $d(A, B)p$ reactions measured at radioactive beam facilities in inverse kinematics are well suited for shell structure studies beyond the edge of stability. The mechanism of the (d, p) reaction is believed to be reasonably well understood. The (d, p) reaction has been successfully described in terms of a model in which only the degrees of freedom associated with the neutron and proton in the incident deuteron are treated explicitly. The model is imbedded in a many-body theory in which nuclei A and B enter with their full many-body wave functions through the concept of overlap functions. Channels in which the d-A system is excited into the continuum of broken-up states $n + p + A$ play an important role in the theory and are accounted for by use of adiabatic distorted wave approximation (ADWA) [1].

The standard way of treating (d, p) reactions within the ADWA is to use local nucleon optical potentials taken at energy $E_d/2$ [2,3], where E_d is the energy of the incoming deuteron. For elastic deuteron scattering this idea is a consequence of the loose binding of the deuteron [4]. The $E_d/2$ result can also be derived [4] for the folding d-A potential in the $n + p + A$ model by using nonlocal n-A and p-A potentials with a realistic nonlocality range, i.e., a range that is much smaller than the size of the deuteron. However, despite wide use of the $E_d/2$ rule in the ADWA analysis of (d, p) reactions its legitimacy has not yet been investigated.

The need to include explicit energy dependence of the nucleon optical potentials into three-body description of (d, p) reactions has been highlighted in Ref. [5]. This can be done either by treating the energy dependence explicitly [5] or by assuming that it is a consequence of the nonlocality of the N-A interactions [6]. In both cases the deviation of the (d, p) cross sections from the standard $E_d/2$ treatment is significant. These conclusions have been based on calculations using Faddeev three-body theory. Although this theory provides exact solution for the three-body wave function it has a drawback of neglecting the internal structure of A and ignoring many-body aspects of the $n + p + A$ wave function in the proton channel. As the result, the ratio of the measured and calculated cross sections, which is traditionally used to give nuclear structure information in the form of spectroscopic factors and asymptotic normalization coefficients, cannot serve this purpose anymore. Reference [7] suggests a way that the missing many-body information can be included within the Faddeev framework by generalizing it to include couplings to excited target states and modifying the asymptotic conditions at small p-A and n-A distances to account for realistic values of coupling constants in two-body decay vertices. However, the result is a formalism that is too complicated for wide use by nuclear experimentalists and, in fact, no applications have been made so far.

The new Faddeev formalism requires the use of nonlocal nucleon optical potentials. The nonlocality arises because the model treats only a sub-set of the many-body degrees of freedom explicitly so that transitions from a point in the explicit space to intermediate points in the implicit space and back is manifest in a set of parameters of nonlocal potentials.

Recently, we have proposed a simple method to account for nonlocality of the N-A optical potentials in (d, p) reactions within the ADWA [8]. The new model reduces all the complexity of the nonlocal and breakup effects in (d, p) reactions to a simple effective local d-A potential. The latter should be calculated using the local p-A and n-A optical potentials taken at the energy shifted with respect to the $E_d/2$

value by ∼40 MeV. This shift arises as a consequence of the dominance of the contribution to the (d, p) reaction amplitude

$$T_{(d,p)} = \langle \psi^{(-)}_{pB} | V_{np} | \Psi^{(+)} \rangle \tag{1}$$

from the small n-p distances corresponding to the large relative n-p kinetic energy because of the uncertainty principle. The state $\psi^{(-)}_{pB}$ describes proton scattering by B but with V_{np} switched off. This exact formula was first derived by Goldberger and Watson [9] for $A \to \infty$ when $\psi^{(-)}_{pB}$ factorizes into a proton wave distorted by V_{pA} and the final nucleus wave function ϕ_B. For finite A the formula was first discussed quantitatively in Ref. [10] and deduced from the Alt-Grassberger-Sandhas equations in Ref. [11]. For the purpose of the present work the recoil excitation effects [10] that contribute to $\psi^{(-)}_{pB}$ for finite A can be neglected. Other corrections involving paths between the initial and final states arising from the excitation of A and B are outside the scope of this work.

In the present paper, we give more details of the derivation of the effective d-A potential and consider the leading-order corrections to this potential. In Sec. II we recall the general properties of the two-body nonlocal model. In Sec. III we derive a general expression for the effective adiabatic d-A potential with nonlocal optical n-A potentials. In Secs. IV A and IV B we discuss the effective d-A potential in the leading-order approximation, deduce the energy shift, and clarify the its physical origin. The corrections to the local-energy approximation and the first-order expansion of the n-A potential are considered in Secs. IV C and V respectively. Applications to some selected (d, p) reactions are described in Sec. VI and the conclusions are drawn in Sec. VII. Some derivations of key formulas are given in the Appendix.

II. NONLOCAL TWO-BODY MODEL

We start with a brief reminder of the two-body N-A model with a central energy-independent nonlocal N-A interaction V_{NA}. For such a potential the scattering wave function for N-A scattering satisfies

$$(T - E)\psi(\boldsymbol{r}) = -\int d\boldsymbol{r}' V_{NA}(\boldsymbol{r}, \boldsymbol{r}')\psi(\boldsymbol{r}'), \tag{2}$$

where $\boldsymbol{r} = \boldsymbol{r}_A - \boldsymbol{r}_N$ is the radius vector between A and N, T is the corresponding kinetic energy operator and E is the energy of the N-A system in the center of mass. We note that Eq. (2) is written in a translation-invariant way. The nonlocal potential V_{NA} is assumed to have the Perey-Buck form [12],

$$V_{NA}(\boldsymbol{r}, \boldsymbol{r}') = H(\boldsymbol{r} - \boldsymbol{r}')U_{NA}[(\boldsymbol{r} + \boldsymbol{r}')/2] \tag{3}$$

with the nonlocality factor

$$H(\boldsymbol{x}) = \pi^{-3/2}\beta^{-3}e^{-(\frac{x}{\beta})^2}, \tag{4}$$

where β is the range of nonlocality. Perey and Buck show that a local potential $U_{\rm loc}$ that to a good approximation gives the same scattering phase shifts as the nonlocal potential can be obtained as the solution the transcendental equation

$$U_{\rm loc}(r) = U_{NA}(r)\exp\left\{-\frac{\mu_N\beta^2}{2\hbar^2}[E - U_{\rm loc}(r)]\right\}, \tag{5}$$

where μ_N is the reduced mass of the $N + A$ system. This equation must be corrected for proton scattering, by reducing the center-of-mass energy E in the right-hand side of Eq. (5) by the local Coulomb interaction $V_{\rm coul}(r)$. For practical purposes $V_{\rm coul}(r)$ can be replaced by a constant, for example, by the one given in Ref. [13] by

$$\bar{V}_{\rm coul} = -1.08 + 1.35[(Z - 1)/A^{1/3}] \ \text{MeV}. \tag{6}$$

The wave function ψ obtained from the nonlocal equation (2) is smaller than the solution φ of the local Schrödinger equation

$$(T + U_{\rm loc} - E)\varphi = 0 \tag{7}$$

by the Perey factor [14]

$$f(r) = \exp\left(\frac{\mu_N\beta^2}{4\hbar^2}U_{\rm loc}\right). \tag{8}$$

The corrections to this equivalent local model have been considered in Ref. [15]. These corrections are expressed via $U_{\rm loc}$ and its derivatives.

III. THREE-BODY SCHRÖDINGER EQUATION WITH NONLOCAL *n*-*A* AND *p*-*A* POTENTIALS IN THE ADIABATIC APPROXIMATION

We start with the three-body $p + n + A$ model with the n-p interaction V_{np} and central nonlocal N-A interactions V_{NA}. The Schrödinger equation for the three-body wave function $\Psi(\boldsymbol{r}, \boldsymbol{R})$ is

$$\begin{aligned}
&(T_r + T_R + V_{np}(\boldsymbol{r}) - E)\Psi(\boldsymbol{r}, \boldsymbol{R}) \\
&= -8\left(\frac{A+1}{A+2}\right)^3\left[\int d\boldsymbol{R}'\, V_{nA}\left(\frac{\boldsymbol{r}}{2} - \boldsymbol{R}' - \frac{A}{A+2}(\boldsymbol{R}' - \boldsymbol{R}), \frac{\boldsymbol{r}}{2} - \boldsymbol{R}\right)\Psi\left(\boldsymbol{r} - \frac{2A}{A+2}(\boldsymbol{R}' - \boldsymbol{R}), \boldsymbol{R}'\right)\right. \\
&\quad \left. + V_{pA}\left(-\frac{\boldsymbol{r}}{2} - \boldsymbol{R}' - \frac{A}{A+2}(\boldsymbol{R}' - \boldsymbol{R}), -\frac{\boldsymbol{r}}{2} - \boldsymbol{R}\right)\Psi\left(\boldsymbol{r} + \frac{2A}{A+2}(\boldsymbol{R}' - \boldsymbol{R}), \boldsymbol{R}'\right)\right] - V^{\rm coul}_{pA}\left(-\frac{\boldsymbol{r}}{2} - \boldsymbol{R}\right)\Psi(\boldsymbol{r}, \boldsymbol{R}),
\end{aligned} \tag{9}$$

where $\boldsymbol{r} = \boldsymbol{r}_n - \boldsymbol{r}_p$, $\boldsymbol{R} = \boldsymbol{r}_A - (\boldsymbol{r}_n + \boldsymbol{r}_p)/2$, T_r (T_R) is the kinetic energy operator associated with the coordinate $\boldsymbol{r}$ ($\boldsymbol{R}$) and E is the three-body kinetic energy in the center of mass. This form guarantees that the neutron coordinate with respect to the p-A center of mass does not change when the proton is interacting with A and vice versa. The derivation of this formula is given in the Appendix and is similar to the one used in Ref. [4] to study the d-A folding potential with nonlocal nucleon optical potentials.

We emphasise that by solving Eq. (9) with incoming deuteron boundary conditions we generate a model for the projection onto the ground state of A of the full many-body wave function that describes everything that happens when a deuteron in its ground state is incident on A. The $\Psi(\boldsymbol{r}, \boldsymbol{R})$ thus derived is designed to be used in a (d, p) matrix element in which it appears multiplied by the true ground-state wave function Ψ_A of A.

The series of approximations to the full $n + p + A$ many-body problem that lead to this model are discussed in Section 3.1, pp. 136–137, of Ref. [1], where it is also shown how in the resulting (d, p) matrix element the detailed structure of A and B appears in an overlap function that is only a function of the stripped neutron coordinates. This $A(d, p)B$ reaction amplitude involves values of $\Psi(\boldsymbol{r}, \boldsymbol{R})$ for values of small r within the range of the n-p interaction only. Therefore, we attempt to find the solution of Eq. (9) only in this particular region. This is achieved by expanding the three-body wave function $\Psi(\boldsymbol{r}, \boldsymbol{R})$ onto the set of s-wave Weinberg states $\phi_i(\boldsymbol{r})$ [2]

$$\Psi(\boldsymbol{r}, \boldsymbol{R}) = \sum_{i=0}^{\infty} \phi_i(\boldsymbol{r})\chi_i(\boldsymbol{R}) \tag{10}$$

and keeping only the first term of this expansion. Then multiplying Eq. (9) by $V_{np}\phi_0$ and integrating over $\boldsymbol{r}$ we reduce the three-body nonlocal equation (9) to the nonlocal two-body equation for the channel function $\chi_0(\boldsymbol{R}) \equiv \chi(\boldsymbol{R})$,

$$(T_R + U_C(R) - E_d)\chi(\boldsymbol{R}) = -\int d\boldsymbol{R}'\, U_{dA}(\boldsymbol{R}', \boldsymbol{R})\chi(\boldsymbol{R}'), \tag{11}$$

where $U_C(R)$ is the local Coulomb d-A potential, $E_d = E - \epsilon_d$ is the deuteron center-of-mass energy, ϵ_d is the deuteron binding energy and

$$U_{dA}(\boldsymbol{R}', \boldsymbol{R}) = 8\left(\frac{A+1}{A+2}\right)^3 \int d\boldsymbol{r}\, \phi_1(\boldsymbol{r})\Bigg[V_{nA}\left(\frac{\boldsymbol{r}}{2} - \boldsymbol{R}' - \frac{A}{A+2}(\boldsymbol{R}' - \boldsymbol{R}), \frac{\boldsymbol{r}}{2} - \boldsymbol{R}\right)\phi_0\left(\boldsymbol{r} - \frac{2A}{A+2}(\boldsymbol{R}' - \boldsymbol{R})\right) + V_{pA}\left(-\frac{\boldsymbol{r}}{2} - \boldsymbol{R}' - \frac{A}{A+2}(\boldsymbol{R}' - \boldsymbol{R}), -\frac{\boldsymbol{r}}{2} - \boldsymbol{R}\right)\phi_0\left(\boldsymbol{r} + \frac{2A}{A+2}(\boldsymbol{R}' - \boldsymbol{R})\right)\Bigg]. \tag{12}$$

Here ϕ_0 is the deuteron ground-state wave function and

$$\phi_1(\boldsymbol{r}) = \frac{V_{np}(\boldsymbol{r})\phi_0(\boldsymbol{r})}{\langle\phi_0|V_{np}|\phi_0\rangle}. \tag{13}$$

To simplify the right-hand side of Eq. (12) we use the Perey-Buck presentation of the nonlocal potentials V_{nA} and V_{pA} given by Eq. (3). We introduce new variables

$$\boldsymbol{s} = 2\frac{A+1}{A+2}(\boldsymbol{R}' - \boldsymbol{R}), \quad \boldsymbol{x} = \boldsymbol{r} - \frac{2A}{A+2}(\boldsymbol{R}' - \boldsymbol{R}) \tag{14}$$

and neglect the small terms $(\boldsymbol{R}' - \boldsymbol{R})/(A+2)$ arising in the arguments of the p-A and N-A potentials. Eq. (11) can then be rewritten as

$$(T_R + U_C(R) - E_d)\chi(\boldsymbol{R}) = -\int d\boldsymbol{s}\, d\boldsymbol{x}\, H(\boldsymbol{s})\Bigg[\phi_1(\boldsymbol{x} + \alpha_1\boldsymbol{s})U_{nA}\left(\frac{\boldsymbol{x}}{2} - \boldsymbol{R}\right) + \phi_1(\boldsymbol{x} - \alpha_1\boldsymbol{s})U_{pA}\left(-\frac{\boldsymbol{x}}{2} - \boldsymbol{R}\right)\Bigg]\phi_0(\boldsymbol{x})\chi\left(\frac{\alpha_2\boldsymbol{s}}{2} + \boldsymbol{R}\right), \tag{15}$$

where

$$\alpha_1 = \frac{A}{A+1}, \quad \alpha_2 = \frac{A+2}{A+1}. \tag{16}$$

In the $\beta \to 0$ limit Eq. (15) transforms into a formula for the Johnson-Tandy effective deuteron potential [2] with energy-independent form factors U_{nA} and U_{pA}.

Our aim is to find an equivalent local potential model that gives the same phase shifts as the nonlocal model (15). For this purpose we express $\chi(\frac{\alpha_2\boldsymbol{s}}{2} + \boldsymbol{R})$ in the form

$$\chi\left(\frac{\alpha_2\boldsymbol{s}}{2} + \boldsymbol{R}\right) = e^{\frac{i}{2}\alpha_2\boldsymbol{s}\cdot\boldsymbol{K}_R}\chi(\boldsymbol{R}) = 4\pi\sum_{\lambda\mu} i^\lambda j_\lambda\left(\frac{1}{2}\alpha_2\, s\, K_R\right)Y^*_{\lambda\mu}(\hat{\boldsymbol{s}}) \times Y_{\lambda\mu}(\hat{\boldsymbol{K}}_R)\chi(\boldsymbol{R}), \tag{17}$$

where $\boldsymbol{K}_R$ is the momentum operator associated with the $\boldsymbol{R}$ degree of freedom ($\boldsymbol{K}_R = \frac{1}{i}\nabla_R$ in configuration space) and j_λ is the spherical Bessel function. The Bessel function has the expansion (see Ref. [16], Eqs. 10.1.2 and 10.1.47),

$$j_\lambda(z) = z^\lambda \sum_{n=0}^{\infty} \frac{(-)^n}{n!(2\lambda + 2n + 1)!!}\left(\frac{z^2}{2}\right)^n. \tag{18}$$

Using this expansion the operator containing $\boldsymbol{K}_R$ in Eq. (17) can be written

$$i^\lambda j_\lambda\left(\frac{1}{2}\alpha_2 s K_R\right) Y_{\lambda\mu}(\hat{\boldsymbol{K}}_R) = \left(\frac{\alpha_2 s}{2}\right)^\lambda \mathcal{Y}_{\lambda\mu}(\boldsymbol{K}_R) \sum_{n=0}^{\infty} \frac{(-)^n}{n!(2\lambda+2n+1)!!} \times \left(\frac{\mu_d \alpha_2^2 s^2}{4\hbar^2}\right)^n T_R^n, \quad (19)$$

where μ_d is reduced mass of the $d+A$ system and

$$\mathcal{Y}_{\lambda\mu}(\boldsymbol{K}_R) = K_R^\lambda Y_{\lambda\mu}(\hat{\boldsymbol{K}}_R). \quad (20)$$

Equations (15), (17), and (19) provide a basis for deriving an equivalent local $d+A$ adiabatic model. They show that the action of the nonlocal potential reduces to a infinite sum of momentum-dependent potentials. As will be shown below, only few terms from this sum are actually important for (d, p) reactions at energies of current interest.

IV. PRACTICAL EVALUATION OF THE EQUIVALENT LOCAL DISTORTING POTENTIAL IN THE DEUTERON CHANNEL

In the first subsection below we evaluate the equivalent local potential obtained when the approximation $U_{NA}(\pm\frac{\boldsymbol{x}}{2} - \boldsymbol{R}) \approx U_{NA}(\boldsymbol{R})$ is made to evaluate the x integration in Eq. (15). We will refer to this as the zero-range approximation. However, we do not mean that the n-p interaction described by the coordinate x has zero range. We note that the variable x is restricted to be within the range of V_{np} by the factor ϕ_1. The approximation is therefore that the effects of variations in the nucleon optical potentials U_{NA} over the relevant range of x are small. For local nucleon potentials the same approximation leads to the Johnson-Soper potential [3], which is the sum of the proton and neutron optical potentials. The finite-range corrections to the Johnson-Soper potential are usually small although they may become significant at high deuteron incident energies [17].

For nonlocal potentials, using the zero-range approximation we find a very large energy shift that produces significant corrections to the $E_d/2$ prescription, but the modification of the associated distorted wave by the Perey effect generated by corrections to the local energy approximation is very small. A larger Perey effect arising from the first-order corrections to the zero-range approximation is obtained in Sec. V.

A. Local equivalent potential in the zero-range approximation

In the zero-range approximation only the $\lambda = 0$ term remains in expansion (17) so that the Schrödinger equation (15) for χ becomes

$$(T_R + U_C(R) - E_d)\chi(\boldsymbol{R}) = -U_{dA}(R)\tilde{H}_0(T_R)\chi(\boldsymbol{R}), \quad (21)$$

where $U_{dA}(R) = U_{nA}(R) + U_{pA}(R)$,

$$\tilde{H}_0(T_R) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!}\left(\frac{\mu_d\alpha_2^2}{4\hbar^2}\right)^n \frac{M_{2n}^{(0)}}{(2n+1)!!} T_R^n \quad (22)$$

and the moments $M_{2n}^{(0)}$ are defined by

$$M_{2n}^{(0)} = \int ds d\boldsymbol{x}\, s^{2n} H(s)\phi_1(\boldsymbol{x} - \alpha_1 \boldsymbol{s})\phi_0(\boldsymbol{x}). \quad (23)$$

Equation (21) becomes local in the local-energy approximation used in reaction and nuclear matter theories [18],

$$T_R = T_0, \quad (24)$$

$$T_0(R) = E_d - U_{\rm loc}^0(R) - U_C(R), \quad (25)$$

where the local potential $U_{\rm loc}^0(R)$ is defined as

$$U_{\rm loc}^0(R) = U_{dA}(R)\tilde{H}_0(T_0(R)). \quad (26)$$

To calculate $M_{2n}^{(0)}$ we employ a simple model of the n-p interaction, given by the Hultén potential [19], often used in adiabatic (d, p) calculations, for which ϕ_0 and ϕ_1 have simple analytical forms (see Appendix). The resulting moments $M_{2n}^{(0)}$ reveal a specific behavior with increasing n that can be approximated by $(\text{constant})^{2n}(2n+1)!!$. This suggests that we can introduce new coefficients $\beta_n^{(0)}$, related to $M_{2n}^{(0)}$ by

$$\beta_n^{(0)} = \frac{1}{\sqrt{2}}\left[\frac{M_{2n}^{(0)}}{(2n+1)!!M_0^{(0)}}\right]^{\frac{1}{2n}}, \quad (27)$$

which depend on n only very weakly. The normalization of $\beta_n^{(0)}$ is chosen in such a way that $\beta_n^{(0)} \to \beta/2$ when the nonlocality range β is smaller than the range of V_{np} and, therefore, when $\phi_1(\boldsymbol{x} - \alpha_1\boldsymbol{s}) \approx \phi_1(\boldsymbol{x})$. The value $\beta/2$ was obtained for β_d in the calculation of the deuteron folding potential appropriate for elastic scattering in Ref. [4] where the short-range function ϕ_1 in Eq. (23) is replaced by the deuteron wave function ϕ_0 that has a range much larger than β.

We show $\beta_n^{(0)}$ in Table I for several target masses A calculated with two values of β: $\beta = 0.85$ fm from Ref. [12] and $\beta = (0.022\hbar^2/\mu_N)^{1/2}$ from Ref. [13]. Table I confirms that for a given A and β coefficients $\beta_n^{(0)}$ change very little with n. Thus we can assume in a good approximation that $\beta_n^{(0)} \approx \beta_d \equiv \beta_1^{(0)}$. This allows the summation in Eq. (22) to be done analytically resulting in

$$\tilde{H}_0(T_R) = M_0^{(0)} \exp\left(-\frac{\mu_d\alpha_2^2\beta_d^2}{2\hbar^2} T_R\right). \quad (28)$$

In other words, for a Gaussian nucleon nonlocality factor the nonlocality factor for the effective deuteron potential appropriate to (d, p) reactions is also Gaussian but with a range parameter differing from that appropriate to elastic deuteron scattering. In the $A \to \infty$ limit this range deviates from the $\beta/2$ value by $\sim$6% but for finite A this deviation may be smaller because of the center-of-mass correction α_2.

Equations (25), (26), and (28), combined together, result in the transcendental equation

$$U_{\rm loc}^0 = M_0^{(0)} U_{dA} \exp\left[-\frac{\mu_d\alpha_2^2\beta_d^2}{2\hbar^2}\left(E_d - U_{\rm loc}^0 - U_C\right)\right] \quad (29)$$

TABLE I. Coefficients $\beta_n^{(\lambda)}$ (in fm), moments $M_0^{(\lambda)}$ (in fm$^{2\lambda}$) for $\lambda = 0, 1$, additional energies E_0 (in MeV), and effective nonlocality range $\tilde{\beta}_d$ for the Perey factor (in fm) calculated for a few target masses A with two nonlocality ranges β.

n	$\beta = 0.85$ fm [12]				$\beta = (0.022\hbar^2/\mu_N)^{1/2}$ [13]			
	$A = 16$	$A = 40$	$A = 208$	$A = \infty$	$A = 16$	$A = 40$	$A = 208$	$A = \infty$
$\beta_1^{(0)}$	0.4005	0.3993	0.3987	0.3985	0.4581	0.4492	0.4443	0.4432
$\beta_2^{(0)}$	0.4020	0.4009	0.4003	0.4002	0.4603	0.4514	0.4466	0.4454
$\beta_3^{(0)}$	0.4032	0.4022	0.4016	0.4015	0.4621	0.4532	0.4484	0.4472
$\beta_4^{(0)}$	0.4042	0.4032	0.4027	0.4026	0.4636	0.4547	0.4499	0.4487
$\beta_5^{(0)}$	0.4050	0.4042	0.4037	0.4036	0.4648	0.4560	0.4512	0.4500
$\beta_6^{(0)}$	0.4058	0.4050	0.4045	0.4044	0.4660	0.4571	0.4523	0.4512
$M_0^{(0)}$	0.797	0.787	0.781	0.780	0.755	0.749	0.746	0.745
E_0	35.3	40.1	42.7	43.3	33.5	37.5	39.8	40.3
$M_0^{(1)}$	0.155	0.153	0.152	0.152	0.194	0.186	0.181	0.180
$\beta_1^{(1)}$	0.4053	0.4044	0.4039	0.4038	0.4650	0.4562	0.4513	0.4502
$\beta_2^{(1)}$	0.4063	0.4054	0.4050	0.4049	0.4665	0.4576	0.4528	0.4516
$\beta_3^{(1)}$	0.4071	0.4063	0.4059	0.4057	0.4676	0.4588	0.4540	0.4528
$\beta_4^{(1)}$	0.4078	0.4070	0.4066	0.4065	0.4687	0.4597	0.4550	0.4539
$\beta_5^{(1)}$	0.4084	0.4077	0.4073	0.4072	0.4697	0.4608	0.4560	0.4548
$\beta_6^{(1)}$	0.4089	0.4083	0.4079	0.4078	0.4704	0.4616	0.4568	0.4556
$\tilde{\beta}_d$	0.509	0.509	0.509	0.510	0.585	0.575	0.569	0.568

that determines the local potential $U_{\rm loc}^0$. Here and below, the arguments of $U_{\rm loc}^0$, U_{dA}, and U_C are omitted for simplicity of presentation.

Equation (29) differs from the Perey-Buck equation (5) by the renormalization factor $M_0^{(0)}$ and has the same form as the equation for the d-A folding potential derived with nonlocal nucleon optical potentials in Ref. [4]. However, because the renormalization factor for the folding potential appropriate to elastic d scattering is close to one and the deuteron nonlocality range is equal to $\beta/2$ in a good approximation the folding potential is just the sum of the neutron and proton optical potentials taken at half the deuteron energy [4]. In contrast, the renormalization factor $M_0^{(0)}$ that enters Eq. (29) is about 0.75–0.8 (see Table I) since the range of V_{np} is comparable to the range of nonlocality β of the nucleon optical potential. This causes a significant deviation of $U_{\rm loc}^0$ from the sum of neutron and proton potentials taken at half the deuteron energy.

The transcendental equation (29) can be solved directly using numerical methods. Alternatively, only a few terms $n \leqslant n_{\rm max}$ can be kept in the expansion (22) of $\tilde{H}_0(T_0)$. In particular, $n_{\rm max} = 0$ gives $U_{\rm loc}^0 = M_0^{(0)} U_{dA}$, while for $n_{\rm max} = 1, 2, \ldots U_{\rm loc}^0$ is a solution of a linear, quadratic, etc. equation, respectively. Figure 1 shows that for the case of $d + {}^{40}$Ca solving the quadratic equation (keeping $n_{\rm max} = 2$) results in a $U_{\rm loc}^0$, which is very close to the exact numerical solution.

An illuminating solution of Eq. (29) exists when $U_C(R)$ in $\tilde{H}_0(T_0)$ is replaced by a constant $\bar{U}_C$ independent of R (we have checked that this replacement is reasonable for the particular cases considered below) and if the proton $U_{pA}^{\rm loc}$ and neutron $U_{nA}^{\rm loc}$ local energy-dependent potentials are the same: $U_{pA}^{\rm loc}(E_n + \bar{V}_{\rm coul}) = U_{nA}^{\rm loc}(E_n)$, where $\bar{V}_{\rm coul}$ is the p-A Coulomb potential and is also a constant. This solution is obtained if a new parameter E_0 is introduced, defined by the equation

$$\alpha_2 \left(\frac{2\beta_d}{\beta}\right)^2 M_0^{(0)} = \exp\left(-\frac{\mu_N \beta^2 E_0}{2\hbar^2}\right), \tag{30}$$

which has the meaning of some additional energy. Then $U_{\rm loc}^0(E_d)$ satisfies

$$U_{\rm loc}^0(E_d) = \frac{1}{\alpha_2}\left(\frac{\beta}{2\beta_d}\right)^2 \left(U_{nA}^{\rm loc}(E_n) + U_{pA}^{\rm loc}(E_p)\right), \tag{31}$$

where the center-or-mass energies E_p and E_n are given by expressions

$$E_n = \frac{\alpha_2}{2}\left(\frac{2\beta_d}{\beta}\right)^2 (E_d - \bar{U}_C) + E_0, \tag{32}$$

$$E_p = \frac{\alpha_2}{2}\left(\frac{2\beta_d}{\beta}\right)^2 (E_d - \bar{U}_C) + E_0 + \bar{V}_{\rm coul}, \tag{33}$$

in which

$$E_0 = -\frac{2\hbar^2}{\mu_N \beta^2} \ln\left[\alpha_2 \left(\frac{2\beta_d}{\beta}\right)^2 M_0^{(0)}\right]. \tag{34}$$

The calculated energies E_0 are shown in Table I for two values of the nonlocality range β. They show some dependence on target mass A and have an average value about 40 MeV.

Equation (31) tells us that the local adiabatic deuteron potential needed to solve the two-body Schrödinger equation at energy E_d is a normalized sum of the neutron and proton potentials taken at energies that are ~40 MeV larger than the $E_d/2$ value, widely used in the analysis of (d, p) reactions. Since the depth of the real part of the nucleon optical potential decreases with energy, the real part of the local adiabatic potential $U_{\rm loc}^0$ calculated at an energy much larger than $E_d/2$, will be shallower than the adiabatic deuteron Johnson-Soper potential $U_{\rm JS}$, calculated at $E_d/2$. In the particular case of

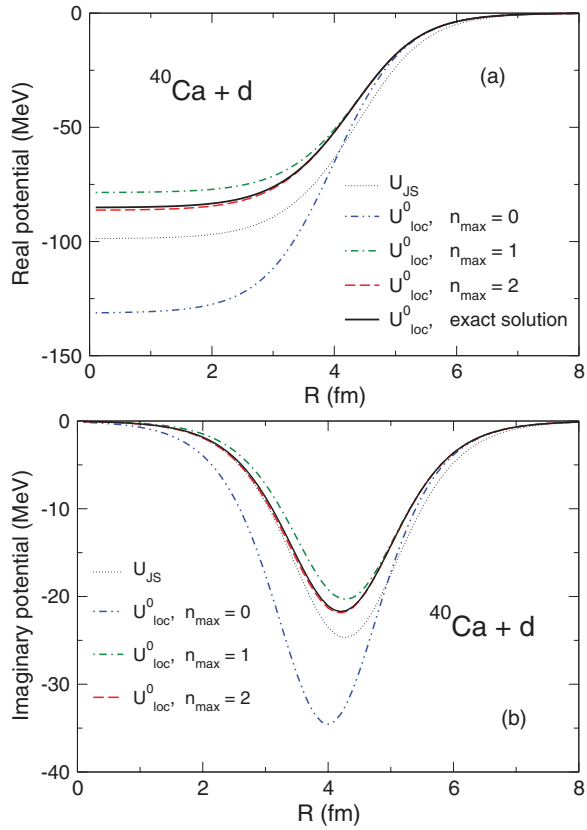

FIG. 1. (Color online) Equivalent local potential $U^0_{\rm loc}$ calculated for $d + {}^{40}$Ca at incident deuteron laboratory energy of 11.8 MeV using global nonlocal potential from Ref. [13] and assuming $A \to \infty$. The Coulomb interaction $U_C(R)$ in Eq. (29) was replaced by a constant $\bar{V}_{\rm coul}$ given by Eq. (6) according to prescription of Ref. [13]. (a) and (b) refer to the real and imaginary parts of $U^0_{\rm loc}$ obtained by solution of the transcendental equation (29) either by expanding its right-hand side up to $n_{\rm max}$ terms or by using exact numerical methods. The Johnson-Soper potential U_{JS} traditionally used in adiabatic (d, p) calculations is also shown for comparison.

$d+{}^{40}$Ca the reduction of the depth of real potential due to E_0 is about 26% but the renormalization by the factor $[\beta/(2\beta_d)]^2/\alpha_2$ in Eq. (31) increases it by 10% so that the resulting real depth of the potential $U^0_{\rm loc}$ is reduced by 14% with respect to $U_{\rm JS}$ (see Fig. 1).

B. Physical origin of E_0 and the energy shift ΔE

The energy E_0 that enters E_n and E_p is very large (see Table I) and at first sight it is not obvious where an energy of this magnitude might come from. We can get some idea of the key factors determining E_0, and therefore $U^0_{\rm loc}$, by considering the limit when the range of nonlocality, β, is small compared with the range of the n-p interaction. For this purpose, we assume that $\beta_d = \beta^0_1$ and $A \to \infty$. Then from Eq. (27) we deduce

$$\left(\frac{2\beta_d}{\beta}\right)^2 M^{(0)}_0 = \frac{2}{3}\frac{M^{(0)}_2}{\beta^2}. \tag{35}$$

To get the $M^{(0)}_2$ behavior at small β we expand $\phi_1(\boldsymbol{x}-\boldsymbol{s})$ around $s = 0$ keeping only the terms up to the second order. This gives

$$M^{(0)}_2 \approx \tfrac{3}{2}\beta^2 + \tfrac{5}{8}\beta^4\langle\nabla^2_{np}\rangle, \tag{36}$$

where

$$\langle\nabla^2_{np}\rangle = \int d\boldsymbol{x}\phi_1(x)\nabla^2_x\phi_0(x), \tag{37}$$

is a measure of the squared momentum associated with the n-p separations within the range of $\phi_1(x)$, i.e., the range of V_{np}. From Eq. (34) we obtain

$$E_0 = -\frac{5}{12}\frac{\hbar^2}{M_N}\langle\nabla^2_{np}\rangle = \frac{5}{12}\langle T_{np}\rangle, \tag{38}$$

which is independent of the value of the nucleon nonlocality range β and is equal to 5/12 of the relative n-p kinetic energy T_{np} averaged over the (short) range of the n-p potential. This average is dominated by high momentum n-p components because of the Heisenberg uncertainty principle. In the particular case of the Hultén potential we have $\langle T_{np}\rangle = 114$ MeV and $E_0 = 47.5$ MeV.

To get $U^0_{\rm loc}$ in Eq. (31) we use the approximation

$$\left(\frac{\beta}{2\beta_2}\right)^2 \approx \left(1 - \frac{1}{6}\beta^2\langle\nabla^2_{np}\rangle\right) \tag{39}$$

and replace $U^{\rm loc}_{nA}$ and $U^{\rm loc}_{pA}$ by the approximations obtained from Eq. (5) by retaining only terms linear on β^2

$$U^{\rm loc}_{NA}(E_N) = U_{NA}\frac{1 - \frac{\mu_N\beta^2}{2\hbar^2}E_N}{1 - \frac{\mu_N\beta^2}{2\hbar^2}U_{NA}}. \tag{40}$$

Neglecting β^4 terms and assuming that $\bar{U}_C \approx \bar{V}_{\rm coul}$ we obtain the approximate $U^0_{\rm loc}$,

$$U^0_{\rm loc} \approx 2U_{NA}\frac{1 - \frac{\mu_N\beta^2}{2\hbar^2}\left(\frac{E_d}{2} + E_0 - \frac{\hbar^2}{12\mu_N}\langle\nabla^2_{np}\rangle\right)}{1 - \frac{\mu_N\beta^2}{2\hbar^2}U_{NA}}. \tag{41}$$

Comparing (40) and (41) we notice that $U^0_{\rm loc}$ is a sum of neutron and proton potentials taken at energy

$$E_N = \frac{E_d}{2} + \Delta E, \tag{42}$$

where

$$\Delta E = E_0 - \frac{\hbar^2}{12\mu_N}\langle\nabla^2_{np}\rangle = \frac{1}{2}\langle T_{np}\rangle. \tag{43}$$

This is exactly the results obtained in our Letter [8].

The large energy shift we obtain thus reflects the fact that a main feature of the (d, p) reaction amplitude is that it is sensitive only to the short-range (and high relative kinetic energy energy) n-p components of the three-body wave function and is not very sensitive to the precise value of the range of nonlocality of the nucleon optical potentials. The role of nucleon nonlocality is just to give the nucleon optical potentials their correct energy dependence and ensures that the optical potentials at the shifted energy are not the same as at $E_d/2$.

In the Appendix we give a discussion of the case of a deuteron propagating in nuclear matter for which some of the subtleties of the finite target effects disappear and reveal how the energy shift ΔE arises in a transparent fashion.

C. Correction to the local-energy approximation in the zero-range case

In this section we derive modifications to the model described in the previous subsection arising from corrections to the local-energy approximation. We include the leading correction term linear in the kinetic energy operator T_R. Using results from the Appendix [Eq. (A38)] we get from Eq. (21)

$$(T_R + U_C - E_d)\chi(\boldsymbol{R}) = -\left(U_{\rm loc}^{\rm mod} - \gamma U_{\rm loc}^0 T_R\right)\chi(\boldsymbol{R}) + \frac{\hbar^2\gamma^2}{2\mu_d} U_{\rm loc}^0 \nabla T_0 \cdot \nabla\chi(\boldsymbol{R}), \quad (44)$$

where γ is given by Eq. (A30), $U_{\rm loc}^0$ is given by Eq. (26), and

$$U_{\rm loc}^{\rm mod} = U_{\rm loc}^0\left[1 + \gamma(T_0 - \Delta)\right]. \quad (45)$$

The energy Δ, given by Eq. (A39), arises because T_R and T_0 do not commute. The solution of Eq. (44) is the product

$$\chi(\boldsymbol{R}) = f_0(\boldsymbol{R})\varphi(\boldsymbol{R}), \quad (46)$$

where the function f_0 satisfies the first-order differential equation

$$\frac{\nabla f_0}{f_0} = g(R)\nabla T_0, \quad (47)$$

with

$$g(R) = -\frac{\gamma^2}{2}\frac{U_{\rm loc}^0(R)}{1 - \gamma U_{\rm loc}^0(R)} \quad (48)$$

and the boundary condition $f(R) \to 1$ at $R \to \infty$. The solution of this equation is

$$f_0(R) = \exp\left(-\int_R^\infty dR'\, g(R')T_0'(R')\right), \quad (49)$$

which in the case when $U_C(R)$ in $\tilde{H}_0(T_0)$ is replaced by constant is given by a simple formula

$$f_0(R) = \frac{\exp\left(-\gamma U_{\rm loc}^0/2\right)}{\sqrt{1 - \gamma U_{\rm loc}^0}}. \quad (50)$$

The function f_0 plays the same role as the well-known Perey factor. It modifies the scattering wave φ, that satisfies the local equation

$$(T_R + U_C - E_d)\varphi = -\left(U_{\rm loc}^0 + \Delta U_0\right)\varphi, \quad (51)$$

in the nuclear interior. In Eq. (51), the ΔU_0 is the correction to the local potential $U_{\rm loc}^0$ derived in Sec. IV and is given by

$$\Delta U_0 = \frac{\hbar^2}{\mu_d}\left[\left(\frac{\nabla f_0}{f_0}\right)^2 - \frac{1}{2}\frac{\nabla^2 f_0}{f_0}\right] - \frac{\gamma U_{\rm loc}^0}{1 - \gamma U_{\rm loc}^0}\Delta = \frac{\hbar^2\gamma^2 T_0'}{24\mu_d}\frac{6\left(U_{\rm loc}^0\right)' + \gamma U_{\rm loc}^0\left(4 - \gamma U_{\rm loc}^0\right)T_0'}{\left(1 - \gamma U_{\rm loc}^0\right)^2}. \quad (52)$$

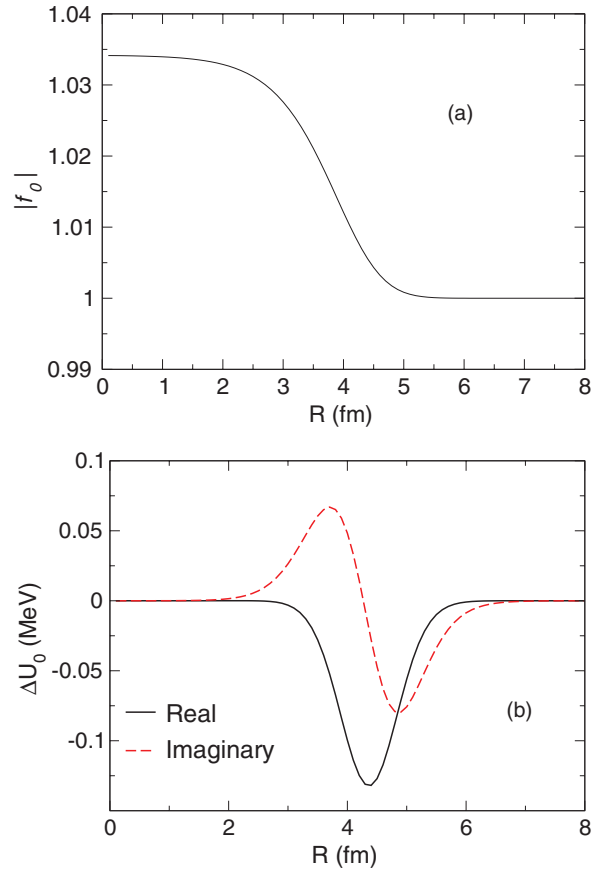

FIG. 2. (Color online) (a) Perey factor f_0 and (b) correction ΔU_0 to the equivalent local potential $U_{\rm loc}^0$ for $d + {}^{40}$Ca (see text). These results obtained under the same input conditions explained in the caption of Fig. 1.

Equations (50) and (52) tell us that corrections beyond the local-energy approximation are usually small. For f_0 this follows from the fact that for small $\gamma U_{\rm loc}^0$ both the numerator and denominator have exactly the same linear terms in the series expansion so that the deviation of f_0 from one arises from the terms quadratic in $\gamma U_{\rm loc}^0$. The ΔU_0 is also determined by $(\gamma U_{\rm loc}^0)^2$. Thus, both f_0 and ΔU_0 are the fourth-order effect of the nucleon nonlocality parameter β.

The Perey factor f_0 and the correction ΔU_0 to the equivalent local potential $U_{\rm loc}^0$ are shown in Fig. 2 for $d + {}^{40}$Ca at deuteron incident energy of 11.8 MeV. The Perey factor increases the scattering wave in the nuclear interior by less than 4% while ΔU_0 does not exceed 140 keV. Such corrections are small indeed and can be safely neglected. However, we should keep in mind that other corrections of the β^4 order, which arise due to the next term of expansion in Eqs. (A31)–(A36) (see Appendix), may give a similar contribution. We do not study any of these corrections here.

V. BEYOND THE ZERO-RANGE APPROXIMATION: FIRST-ORDER CORRECTIONS

In the previous sections we considered the leading-order approximation arising when the nonlocal form factor $U(\frac{\boldsymbol{x}}{2} - \boldsymbol{R})$ is replaced by $U(\boldsymbol{R})$. Here we derive the corrections

to this approximation arising when the next term in the Taylor series expansion of the central potential $U_{NA}(\pm\frac{\boldsymbol{x}}{2}-\boldsymbol{R})$ is retained

$$U\left(\pm\frac{\boldsymbol{x}}{2}-\boldsymbol{R}\right)\approx U(\boldsymbol{R})\mp\frac{1}{2}\boldsymbol{x}\cdot\boldsymbol{\nabla}U(\boldsymbol{R}). \tag{53}$$

In this case, two terms remain in expansion (17), $\lambda=0$ and $\lambda=1$, so that we obtain

$$\begin{aligned}&(T_R+U_C(R)-E_d)\chi(\boldsymbol{R})\\&\quad=-U_{dA}(R)\tilde{H}_0(T_R)\chi(\boldsymbol{R})-\boldsymbol{\nabla}[U_{dA}(R)]\tilde{H}_1(T_R)\boldsymbol{\nabla}\chi(\boldsymbol{R}),\end{aligned} \tag{54}$$

where

$$\tilde{H}_1(T_R)=\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\left(\frac{\mu_d\alpha_2^2}{4\hbar^2}\right)^n\frac{M_{2n}^{(1)}}{(2n+3)!!}T_R^n \tag{55}$$

and the moment $M_{2n}^{(1)}$ is defined as

$$M_{2n}^{(1)}=\int d\boldsymbol{s}d\boldsymbol{x}\,s^{2n}H(s)\phi_1(\boldsymbol{x}-\alpha_1\boldsymbol{s})\phi_0(\boldsymbol{x})\frac{\alpha_2\boldsymbol{s}\cdot\boldsymbol{x}}{4}. \tag{56}$$

As in Sec. III, the new coefficients

$$\beta_n^{(1)}=\frac{1}{\sqrt{2}}\left[\frac{3M_{2n}^{(1)}}{(2n+3)!!M_0^{(1)}}\right]^{\frac{1}{2n}} \tag{57}$$

are practically independent of n (see Table I). If we ignore the n dependence completely for all n the summation in Eq. (55) can be evaluated to give a Gaussian nonlocality factor of the Perey-Buck form with the same nonlocality range β_d as in $H_0(T_0)$ but with a different normalization

$$\tilde{H}_1(T_R)=\frac{M_0^{(1)}}{3}\exp\left(-\frac{\mu_2\alpha_2^2\beta_d^2}{2\hbar^2}T_R\right)=\frac{M_0^{(1)}}{3M_0^{(0)}}\tilde{H}_0(T_R). \tag{58}$$

We further assume $\tilde{H}_1(T_R)=\tilde{H}_1(T_0)$ because we have shown above that corrections to the local energy approximations are small.

The solution of Eq. (54) is the product $\chi(\boldsymbol{R})=f(\boldsymbol{R})\varphi(\boldsymbol{R})$, where φ satisfies the local equation

$$(T_R+U_C-E_d)\varphi=-\left(U_{\rm loc}^0+\Delta U_1\right)\varphi, \tag{59}$$

where

$$\Delta U_1=\frac{\hbar^2}{\mu_d}\left[\left(\frac{\boldsymbol{\nabla}f}{f}\right)^2-\frac{1}{2}\frac{\boldsymbol{\nabla}^2f}{f}\right] \tag{60}$$

and the function f (the Perey factor) is the solution of the first order differential equation

$$\frac{\boldsymbol{\nabla}f}{f}=\frac{\mu_d}{\hbar^2}\tilde{H}_1(T_0)\boldsymbol{\nabla}U_{dA} \tag{61}$$

with the boundary condition $f(R)\to 1$ at $R\to\infty$. If the Coulomb interaction $U_C(R)$ entering $\tilde{H}_1(T_0)$ is a constant then the solution of Eq. (61) is

$$f=f_{dA}\Delta f. \tag{62}$$

The first factor,

$$f_{dA}=\exp\left[\frac{\mu_d\tilde{\beta}_d^2}{4\hbar^2}U_{\rm loc}^0\right], \tag{63}$$

where $\tilde{\beta}_d^2=4M_0^{(1)}/(3M_0^{(0)})$, is the same as the Perey factor for the d-A scattering used in many transfer reaction codes developed for distorted-wave Born approximation calculations. These codes use a range of nonlocality in the deuteron channel roughly equal to 0.55 fm. This value is consistent with the observed energy dependence of phenomenological local deuteron optical potentials but is not appropriate for calculating the deuteron adiabatic potential for (d,p) reactions. The calculated new range $\tilde{\beta}_d$ appropriate for Perey factor in the adiabatic model of (d,p) reactions is between 0.48 to 0.57 fm depending on the target mass and the nucleon nonlocality range β (see Table I). These numbers are either comparable or slightly smaller than those used in the distorted-wave Born approximation codes.

The second factor in (62),

$$\Delta f=\exp\left[-\left(\frac{\mu_d\alpha_2\beta_d\tilde{\beta}_dU_{\rm loc}^0}{4\hbar^2}\right)^2\right], \tag{64}$$

is of the order of $\exp[-(0.003*U_{\rm loc}^0)^2]$, where $U_{\rm loc}^0$ is measured in MeV. In the nuclear interior, where $U_{\rm loc}^0\sim 90$ MeV, Δf corrects f_{dA} by about 7%. In the nuclear surface, where $U_{\rm loc}^0=\frac{1}{2}U_{\rm loc}^0(0)$, this correction is about 2% decreasing rapidly with farther distance from A. Δf is comparable to the contribution of f_0 from corrections to the local-energy approximation considered in the previous section [see Fig. 3(a)] because it is also determined by β^4 terms. These two different contributions beyond the standard Perey factor f_{dA} tend to compensate each other. As f is a slowly varying function the correction ΔU_1 to the local potential $U_{\rm loc}^0$ is small. In the particular case of $d+{}^{40}$Ca this correction does not exceed 1 MeV [see Fig. 3(b)]. This correction is about 1.5% and 3%, respectively, of the real and imaginary parts of $U_{\rm loc}^0$ at $r\sim 3-4$ fm.

VI. APPLICATION TO SOME SELECTED (d,p) REACTIONS

In this section we investigate how the change in the depth of the effective deuteron potential arising in our model affects the (d,p) cross sections. We study this for ${}^{16}\mathrm{O}(d,p)^{17}\mathrm{O}$, ${}^{36}\mathrm{Ar}(d,p)^{37}\mathrm{Ar}$, and ${}^{40}\mathrm{Ca}(d,p)^{41}\mathrm{Ca}$ reaction using the global potential fitted by Giannini and Ricco [13] to proton scattering data and single-particle form factors in a few $Z=N$ nuclei. The same potential is suitable for neutron scattering. We study the reactions chosen at the deuteron incident energies between 9 and 15 MeV. These energies (in inverse kinematics) are typical of modern radioactive nuclear beam facilities (such as TRIUMF, ISOLDE) used for nuclear structure and nuclear astrophysics studies. At these energies, spin-orbit and deuteron d-state effects, omitted in the present paper, are negligible and the application of the zero-range approximation to the evaluation of both the (d,p) amplitude and the Johnson-Tandy potential is justified [17]. The specific examples considered below are calculated at deuteron laboratory energies of

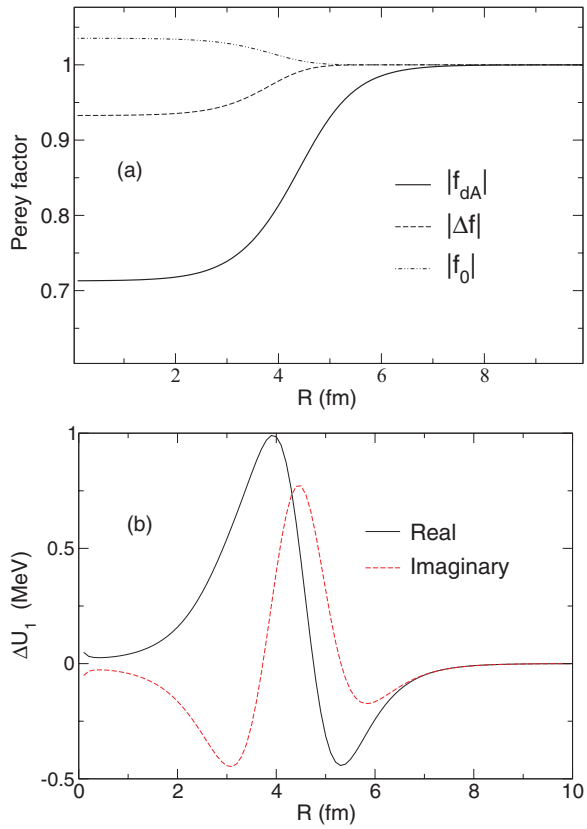

FIG. 3. (Color online) (a) Perey factor f_{dA} and corrections Δf and f_0. (b) First-order correction ΔU_1 to the equivalent local potential $U^0_{\rm loc}$. These results are obtained for $d + {}^{40}$Ca under the same input conditions explained in the caption of Fig. 1.

E =15, 9.162 and 11.8 MeV for ^{16}O, ^{36}Ar, and ^{40}Ca targets respectively for which experimental data are available.

The effective local potentials for $d + {}^{16}$O, $d + {}^{36}$Ar, and $d + {}^{40}$Ca were obtained using the Giannini-Ricco global nonlocal energy-independent nucleon optical potential. They all have the typical behavior shown in Fig. 4(a) for $d + {}^{16}$O where they are compared to the Johnson-Soper adiabatic potential. This figure demonstrates that the leading-order approximation contains most of the effect of the shift in the nucleon energies. All other corrections discussed above are small.

The effective local deuteron potentials obtained were read into the TWOFNR code [20] to predict the (d, p) cross sections in the zero-range approximation. Since the main aim of the present paper is to quantify the influence of the effective potential in the d-A channel, we approximate the $\psi^{(-)}_{pB}$ of Eq. (1) by $\psi_B \chi^{(-)}_{pB}$ where χ_{pB} is the usual proton distorted wave function in the p-B channel generated by the p-B optical model. This allows us to include nonlocality in the proton channel using known procedure in which χ_{pB} is modified by the Perey factor. Here, we used the p-B optical potentials and the nonlocality range for the Perey factor from the Giannini-Ricco systematics [13]. Also, the overlap functions $\langle {}^{16}\mathrm{O}|{}^{17}\mathrm{O}\rangle$, $\langle {}^{36}\mathrm{Ar}|{}^{37}\mathrm{Ar}\rangle$, and $\langle {}^{40}\mathrm{Ca}|{}^{41}\mathrm{Ca}\rangle$ were generated by the standard prescription in which the well depth of the neutron potential is fitted to reproduce the neutron separation energies. The radius $r_0 = 1.25$ fm and the diffuseness $a = 0.65$ fm of the neutron potential well were fixed and the depth of the neutron spin-orbit potential was 5 MeV. The exact shape of this potential is not important because these reactions are peripheral: any change in the shape of overlap function produces a change in the magnitude of the calculated differential cross sections that reflects the change in the single-particle neutron asymptotic normalization coefficients (ANCs). Peripherality is also confirmed by comparison between the calculations with and without the Perey factor in the proton and deuteron distorted scattering waves: the Perey damping in the nuclear interior has little effect because this region does not contribute much to the reaction amplitude.

We quantify the influence of nonlocal (NL) effects in Table II where the differential cross sections at the main peak, $\sigma_{\rm NL}$(peak), are given as ratios to the standard Johnson-Soper cross sections $\sigma_{\rm JS}$(peak). We see that reduction in the deuteron potential depth in $U^0_{\rm loc}$ increases the (d, p) cross sections by the fraction ranging from 5 to 27%. This would decrease the spectroscopic factors and ANCs, obtained from comparison of the theoretical and experimental cross sections, by the same proportions. Table II also shows that the center-of-mass corrections and the first-order corrections, given by ΔU_1, change these ratios by no more than 2% and 4% respectively. The Perey effect further increases this ratio by no more than 3%, thus confirming the peripheral nature of these reactions.

The theoretical angular distributions obtained both with the adiabatic local-equivalent and the Johnson-Soper potentials are shown in Figs. 4(b)–4(f) in comparison with experimental data taken from Refs. [21–23]. For ^{16}O$(d, p)^{17}$O and ^{40}Ca$(d, p)^{41}$Ca spectroscopic factors equal to one were used to normalize the calculations while for ^{36}Ar$(d, p)^{37}$Ar the spectroscopic factor was set to 0.5. The standard zero-range normalization factor $D_0^2 = 2.44 \times 10^8$ MeV^{2}fm^3 was also used. Figures 4(b) and 4(c) demonstrate again that the leading-order approximation to nonlocality produces the most important changes in the cross sections as compared to the Johnson-Soper calculations. Figures 4(d)–4(f) demonstrate that reduction of the potential depth arising from nonlocality leads to a renormalization of the angular distributions in the main peak where all the determinations of spectroscopic factors and ANCs are usually done. The changes in shape are more noticeable at larger angles where the cross sections are small but even there these changes are not very large and we don't show them here.

VII. SUMMARY AND CONCLUSION

A basic property of nucleon optical potentials, their nonlocality, together with deuteron breakup effects can be accounted for in the ADWA theory of $A(d, p)B$ reactions in terms of an effective local deuteron potential that can be constructed in a very simple way. We have shown that if the neutron and proton optical potentials are the same then to a good approximation this potential is just the renormalized sum of the local neutron and proton optical potentials (31) taken at an energy shifted from the widely used $E_d/2$ value by an additional energy E_0 given by Eqs. (33), (34). This

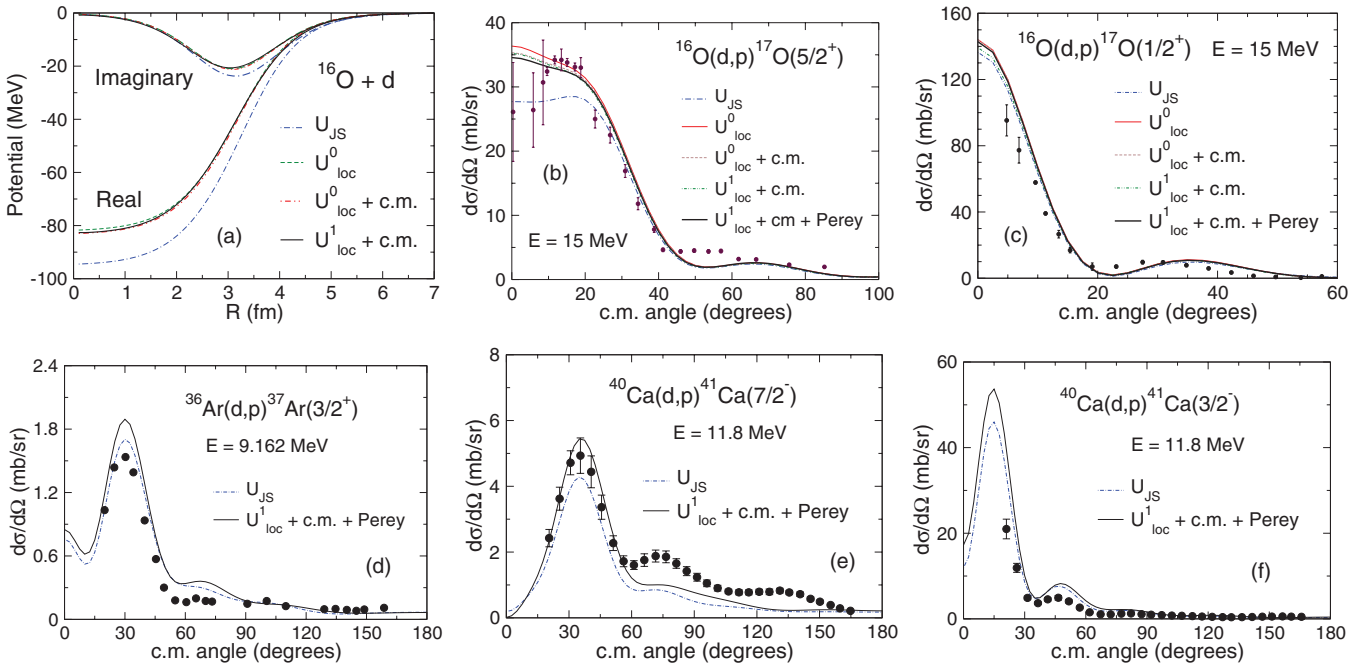

FIG. 4. (Color online) (a) Equivalent local potentials for $d + {}^{16}$O calculated in the leading-order approximation without and with corrections for the center-of-mass and the first-order effects in comparison to the Johnson-Soper potential $U_{\rm JS}$; (b), (c) Differential cross sections for ^{16}O$(d, p)^{17}$O, reactions calculated with these potentials without and with the Perey factor f_{dA}; (d)–(f) Differential cross sections for ^{36}Ar$(d, p)^{37}$Ar (divided by two) and ^{40}Ca$(d, p)^{41}$Ca calculated with the center-of-mass and ΔU_1 correction and the Perey effect in comparison with Johnson-Soper adiabatic model.

energy is related to the n-p kinetic energy averaged over the range of the n-p interaction. E_0 ranges from 30 to 40 MeV depending on the nonlocality range and the mass of the target. Since the depth of the real part of the nucleon optical potential decreases with energy, the depth of the effective local deuteron adiabatic potentials is reduced with respect to the Johnson-Soper potential calculated at $E_d/2$. Corrections due to asymmetry in neutron and proton potentials can be derived, but if necessary the local deuteron potential can be obtained directly from the solution of transcendental equation (29) without appealing to the concept of energy shift. For all cases considered above, in which any asymmetry of the nucleon potentials was absent, we have checked that $U^0_{\rm loc}$ obtained from Eq. (29) and Eq. (31) are exactly the same both in the $A \to \infty$ limit and at finite A.

TABLE II. The ratio of differential cross sections at the main peak, $\sigma_{\rm NL}$(peak), calculated for several (d, p) reactions in the adiabatic nonlocal model, to the cross sections $\sigma_{\rm JS}$(peak) obtained in the Johnson-Soper adiabatic model. The contributions from the leading-order approximation ($U^0_{\rm loc}$) without and with full center-of-mass treatment are presented in columns 2 and 3, respectively. The contributions from $U^1_{\rm loc} = U^0_{\rm loc} + \Delta U_1$ are shown in columns 4 and 5 without and with Perey factor f_{dA}, respectively.

Reaction	$\sigma_{\rm NL}$(peak)/$\sigma_{\rm JS}$(peak)			
	no c.m.	c.m. included		
	$U^0_{\rm loc}$	$U^0_{\rm loc}$	$U^1_{\rm loc}$	$U^1_{\rm loc}$ & f_{dA}
^{16}O$(d, p)^{17}$O$(\frac{5}{2}^+)$	1.16	1.14	1.13	1.13
^{16}O$(d, p)^{17}$O$(\frac{1}{2}^+)$	1.06	1.06	1.02	1.05
^{36}Ar$(d, p)^{37}$Ar$(\frac{3}{2}^+)$	1.12	1.11	1.09	1.11
^{40}Ca$(d, p)^{41}$Ca$(\frac{7}{2}^-)$	1.30	1.29	1.27	1.27
^{40}Ca$(d, p)^{41}$Ca$(\frac{3}{2}^-)$	1.18	1.17	1.15	1.16

We have shown how the leading-order effective potential $U^0_{\rm loc}$ can be corrected for effects arising from deviations from the local-energy approximation and our neglect of variation of the nucleon optical potentials over the range of nonlocality. However, we found that these corrections are small both with respect to $U^0_{\rm loc}$ and to the difference $U_{\rm JS} - U^0_{\rm loc}$. The change in the cross sections due to these effects does not exceed 4% in the main peak cross sections from which the nuclear structure information is usually determined.

An attractive feature of our treatment of nonlocality is that it allows all the complexity of the three-body physics with nonlocal potentials to be accommodated in simple effective deuteron potentials that can be calculated externally and then read in to widely used transfer reaction codes. This opens the possibility for all experimental groups involved in the (d, p) measurements to take the new effect routinely into account. As a bonus, the Perey effect arising in the three-body model is described by the same factor that is incorporated in many transfer reaction codes so that it can also be accounted for if necessary.

In the few cases considered here, the energy-shift effect reduced the spectroscopic factors and the ANCs by 5–27%. These changes can be larger than the uncertainties of the experimental data and thus can influence any conclusions

about these nuclear structure quantities made on the basis of comparison between theory and experiment. In the few cases we have studied we have found that nonlocality reduces spectroscopic factors deduced by comparison with experimental cross sections. It should be noted that we do not claim here that a reduction as opposed to an increase is a general result. The systematics of the nonlocality effect remain to be investigated.

Finally, our quantitative conclusions were based on a simple model on the n-p interaction given by the Hultén potential, which is designed to fit low-energy s-wave n-p phase shifts only. However, as we showed above, the effective local potential $U^0_{\rm loc}$ is determined by an energy shift that is mainly determined by the n-p kinetic energy in the deuteron averaged over the range of V_{np}. Provided the simple n-p model predicts this average accurately our conclusions about $U^0_{\rm loc}$ should be valid. To illustrate this point, we have calculated $M^{(0)}_0$ and β_d in the $A \to \infty$ limit using the s-wave deuteron wave function given by the realistic phenomenological potential AV18 [24]. We obtain $M^{(0)}_0 = 0.74$ and $\beta_d = 0.404$ fm when the Perey value of nucleon nonlocality $\beta = 0.85$ fm is used, which is very close to the corresponding values of 0.78 and 0.399 fm obtained in the Hultén model. With AV18 values of $M^{(0)}_0$ and β_d an energy shift of $E_0 = 46$ MeV is obtained, close to the Hultén value of 40 MeV. The effective deuteron local potential and the corresponding (d, p) cross sections will be similar to those calculated above and the conclusions made on the basis of the Hultén model will be unchanged.

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APPENDIX

1. Three-body Schrödinger equation in laboratory and center-of-mass frames

In the laboratory frame, the Schrödinger equation for the wave function $\Psi_{\rm lab}(\boldsymbol{r}_n, \boldsymbol{r}_p, \boldsymbol{r}_A)$ of the $n + p + A$ system with the local n-p interaction $V_{np}(\boldsymbol{r}_n - \boldsymbol{r}_p)$ and nonlocal p-A and n-A interactions is written as

$$
\begin{aligned}
&(T_n + T_p + T_A + V_{np}(\boldsymbol{r}_n - \boldsymbol{r}_p) - E_3)\Psi_{\rm lab}(\boldsymbol{r}_n, \boldsymbol{r}_p, \boldsymbol{r}_A) \\
&\quad = -\int d\boldsymbol{r}'_n d\boldsymbol{r}'_p d\boldsymbol{r}'_A \left[\delta\left(\boldsymbol{r}_p - \frac{\boldsymbol{r}_n + A\boldsymbol{r}_A}{A+1} - \boldsymbol{r}'_p + \frac{\boldsymbol{r}'_n + A\boldsymbol{r}'_A}{A+1}\right) V_{nA}(\boldsymbol{r}'_n - \boldsymbol{r}'_A, \boldsymbol{r}_n - \boldsymbol{r}_A) + V_{pA}(\boldsymbol{r}'_p - \boldsymbol{r}'_A, \boldsymbol{r}_p - \boldsymbol{r}_A)\right. \\
&\qquad \left. \times\, \delta\left(\boldsymbol{r}_n - \frac{\boldsymbol{r}_p + A\boldsymbol{r}_A}{A+1} - \boldsymbol{r}'_n + \frac{\boldsymbol{r}'_p + A\boldsymbol{r}'_A}{A+1}\right)\right] \delta\left(\frac{\boldsymbol{r}_n + \boldsymbol{r}_p + A\boldsymbol{r}_A}{A+2} - \frac{\boldsymbol{r}'_n + \boldsymbol{r}'_p + A\boldsymbol{r}'_A}{A+2}\right) \Psi_{\rm lab}(\boldsymbol{r}'_n, \boldsymbol{r}'_p, \boldsymbol{r}'_A),
\end{aligned}
\tag{A1}
$$

where E_3 is the three-body energy in the laboratory system. Separating the center-of-mass motion,

$$
\begin{aligned}
&\Psi_{\rm lab}(\boldsymbol{r}_n, \boldsymbol{r}_p, \boldsymbol{r}_A) \\
&\quad = \phi_{\rm c.m.}\left(\frac{\boldsymbol{r}_n + \boldsymbol{r}_p + A\boldsymbol{r}_A}{A+2}\right) \Psi\left(\boldsymbol{r}_n - \boldsymbol{r}_p, \boldsymbol{r}_A - \frac{\boldsymbol{r}_p + \boldsymbol{r}_n}{2}\right),
\end{aligned}
\tag{A2}
$$

then introducing variables

$$
\begin{aligned}
\boldsymbol{r} &= \boldsymbol{r}_n - \boldsymbol{r}_p, \\
\boldsymbol{R} &= \boldsymbol{r}_A - \frac{\boldsymbol{r}_p + \boldsymbol{r}_n}{2}, \\
\boldsymbol{R}_{\rm c.m.} &= \frac{\boldsymbol{r}_n + \boldsymbol{r}_p + A\boldsymbol{r}_A}{A+2}
\end{aligned}
\tag{A3}
$$

and integrating over $\boldsymbol{r}'$ and $\boldsymbol{R}'_{\rm c.m.}$ in the right-hand side of Eq. (A1) we get Eq. (9) in which $E = E_3 - E_{\rm c.m.}$.

2. Moments $M^{(\lambda)}_{2n}$

The moments $M^{(0)}_{2n}$ and $M^{(1)}_{2n}$ given by Eqs. (23) and (56) can be calculated using $\phi_0(q)$ and $\phi_1(q)$ in momentum space. In this case, explicit dependence on the shape of the n-p potential disappears and the measured wave function can be used. The expression for $M^{(0)}_{2n}$ then becomes

$$
M^{(0)}_{2n} = \frac{\int d\boldsymbol{q}(q^2 + \kappa^2)\phi^2_0(q) H_{2n}(\alpha_1 q)}{\int d\boldsymbol{q}(q^2 + \kappa^2)\phi^2_0(q)},
\tag{A4}
$$

where

$$
\begin{aligned}
H_{2n}(\alpha_1 q) &= \int d\boldsymbol{s}\, e^{i\alpha_1 \boldsymbol{q}\boldsymbol{s}} s^{2n} H(s) \\
&= n!\beta^{2n} e^{-\frac{\alpha_1^2\beta^2 q^2}{4}} L^{1/2}_n\left(\frac{\alpha_1^2\beta^2 q^2}{4}\right)
\end{aligned}
\tag{A5}
$$

and $L^{1/2}_n$ is the Laguerre polynomial of order n. The expression for $M^{(1)}_{2n}$ can be rewritten via derivatives of ϕ_0 and H_{2n} in momentum space

$$
M^{(1)}_{2n} = \frac{\alpha_2}{4} \frac{\int d\boldsymbol{q}(q^2 + \kappa^2)\phi_0(q)\phi'_0(q) H'_{2n}(\alpha_1 q)}{\int d\boldsymbol{q}(q^2 + \kappa^2)\phi^2_0(q)},
\tag{A6}
$$

where

$$
\begin{aligned}
H'_{2n}(\alpha_1 q) &= \left(\frac{2n}{\alpha_1 q} - \frac{\alpha_1 q \beta^2}{2}\right) H_{2n}(\alpha_1 q) \\
&\quad - 2n\left(n + \frac{1}{2}\right)\frac{\beta^2}{\alpha_1 q} H_{2n-2}(\alpha_1 q).
\end{aligned}
\tag{A7}
$$

We note that the prime in Eq. (A7) indicates differentiation with respect to $\alpha_1 q$. We calculate $M_{2n}^{(0)}$ in a simple model of the n-p interaction, given by the Hultén potential [19], often used in adiabatic (d, p) calculations

$$V_{np}(x) = V_0/(e^{(\gamma-\kappa)x} - 1), \tag{A8}$$

where

$$V_0 = \frac{\hbar^2}{2\mu_{np}}(\gamma^2 - \kappa^2), \tag{A9}$$

$\kappa = 0.232$ fm^{-1} and $\gamma = 6.255\kappa$. The corresponding wave function ϕ_0 reads

$$\phi_0(\boldsymbol{r}) = \mathcal{N}(e^{-\kappa r} - e^{-\gamma r})Y_{00}(\hat{\boldsymbol{r}})/r, \tag{A10}$$

where

$$\mathcal{N} = \sqrt{2\kappa\gamma(\kappa+\gamma)}/(\gamma-\kappa). \tag{A11}$$

Then

$$\begin{aligned}\phi_0(\boldsymbol{q}) &= \phi_0(q)Y_{00}(\hat{\boldsymbol{q}}) \\ &= 4\pi\mathcal{N}\left(\frac{1}{q^2+\kappa^2} - \frac{1}{q^2+\gamma^2}\right)Y_{00}(\hat{\boldsymbol{q}})\end{aligned} \tag{A12}$$

and

$$\phi_0'(q) = 8\pi\mathcal{N}q\left[\frac{1}{(q^2+\gamma^2)^2} - \frac{1}{(q^2+\kappa^2)^2}\right]. \tag{A13}$$

3. Energy shift in the effective deuteron distorting potential in uniform nuclear matter

We give a simplified derivation of the energy dependence of the distorting potential for use in the incoming channel in the ADWA theory of (d, p) reactions.

We first clarify the general connection between energy dependence and nonlocality by considering a particle of mass m_1 and incident kinetic energy E_1 moving in uniform nuclear matter in a momentum-dependent potential

$$V_1 = -V_1^0 + \alpha_1^0\frac{\boldsymbol{p}_1^2}{2m_1}, \tag{A14}$$

where $\boldsymbol{p_1}$ is the particle's momentum operator. This is a simple example of a nonlocal potential. In this case the nonlocality arises because the value of the potential energy depends on the second derivatives of the particle's wave function at some point and not just the value of the wave function at that point. The potential V_1^0 and the constant α_1^0 are independent of position because we are considering uniform nuclear matter. This functional form is widely used as a simplified model of the nuclear mean field and is equivalent to the effective mass approximation (see Ref. [25] Sec. 4.2, p. 29).

By manipulating the Schrödinger equation

$$\left(\frac{\boldsymbol{p}_1^2}{2m_1} + V_1\right)\Psi_1 = E_1\Psi_1, \tag{A15}$$

it is easy to show that an exactly equivalent form is

$$\left(\frac{\boldsymbol{p}_1^2}{2m_1} + U_1(E_1)\right)\Psi_1 = E_1\Psi_1, \tag{A16}$$

where

$$U_1(E_1) = -U_1^0 + \alpha_1 E_1, \tag{A17}$$

with

$$U_1^0 = \frac{V_1^0}{1+\alpha_1^0}, \quad \alpha_1 = \frac{\alpha_1^0}{1+\alpha_1^0}. \tag{A18}$$

This example shows how nonlocality leads to a definite energy dependence of a local equivalent potential $U_1(E_1)$. A connection can be made between the coefficient α_1^0 and the nonlocality range β used in the text but this detail is not needed for the current discussion (α_1^0 is proportional to β^2).

As discussed in the main text, in the ADWA method for (d, p) reactions the distorting potential in the incident channel is calculated from the neutron and proton potentials V_n and V_p by the formula

$$U_d = \langle\phi_1|V_n + V_p|\phi_0\rangle. \tag{A19}$$

The notation here implies an integration over the internal degrees of freedom (coordinate $\boldsymbol{r}$) of the n-p system to leave an operator acting on the n-p center of mass degree of freedom $\boldsymbol{R}$. If V_n and V_p are nonlocal then so will be U_d. We calculate this operator when V_n and V_p have the form of Eq. (A14), i.e., for $N = n, p$

$$V_N = -V_N^0 + \alpha_N^0\frac{\boldsymbol{p}_N^2}{2m}, \tag{A20}$$

where m is the nucleon mass and we will assume here that $V_n^0 = V_p^0$ and $\alpha_n^0 = \alpha_p^0$.

We have

$$\begin{aligned}\boldsymbol{p}_n &= \tfrac{1}{2}\boldsymbol{p}_R + \boldsymbol{p}_r \\ \boldsymbol{p}_p &= \tfrac{1}{2}\boldsymbol{p}_R - \boldsymbol{p}_r,\end{aligned} \tag{A21}$$

where $\boldsymbol{p}_R$ is the momentum of the n-p center of mass and $\boldsymbol{p}_r$ is the momentum of the neutron in the p-n center-of mass system. Hence

$$\boldsymbol{p}_n^2 + \boldsymbol{p}_p^2 = \tfrac{1}{2}\boldsymbol{p}_R^2 + 2\boldsymbol{p}_r^2. \tag{A22}$$

Using these results in expression (A19) we obtain

$$\begin{aligned}U_d &= -2V_n^0 + \alpha_n^0\langle\phi_1|\frac{\boldsymbol{p}_R^2}{4m} + \frac{\boldsymbol{p}_r^2}{m}|\phi_0\rangle \\ &= -2V_n^0 + \alpha_n^0\langle\phi_1|T_r|\phi_0\rangle + \alpha_n^0\frac{\boldsymbol{p}_R^2}{4m}.\end{aligned} \tag{A23}$$

We see that this has exactly the general form of nonlocal potential given in Eq. (A14) for a particle of mass $m_d = 2m$. We can therefore immediately write down the equivalent local potential for a deuteron of energy E_d from Eq. (A24). We obtain

$$U_d(E_d) = U_d^0 + \alpha_d E_d, \tag{A24}$$

with

$$\begin{aligned}U_d^0 &= \frac{-2V_n^0 + \alpha_n^0\langle\phi_1|T_r|\phi_0\rangle}{1+\alpha_n^0} \\ \alpha_d &= \frac{\alpha_n^0}{1+\alpha_n^0}.\end{aligned} \tag{A25}$$

We compare this result with the corresponding formula for the equivalent local potential for a nucleon of energy $E_d/2$, which is

$$U_n\left(\frac{E_d}{2}\right) = \frac{-V_n^0}{1+\alpha_n^0} + \frac{\alpha_n^0}{1+\alpha_n^0}\frac{E_d}{2}. \tag{A26}$$

Equation (A25) gives the exact result

$$U_d(E_d) = 2U_n\left(\frac{E_d}{2} + \frac{\langle\phi_1|T_r|\phi_0\rangle}{2}\right). \tag{A27}$$

The result (A27) means that the distorting potential in the deuteron channel should be calculated using local nucleon optical potentials evaluated at a nucleon energy shifted from the usual $E_d/2$ prescription by $\Delta E = \frac{1}{2}\langle T_r\rangle$ where

$$\langle T_r\rangle = \langle\phi_1|\frac{\boldsymbol{p}_r^2}{m}|\phi_0\rangle. \tag{A28}$$

One merit of this derivation is that does not involve any need to discuss further approximations, given a nucleon potential in nuclear matter is quadratic in momentum.

4. Expansion of $\tilde{H}_0(T_R)$ in power series in $A = -\gamma(T_R - T_0)$

We would like to expand the Perey-Buck nonlocality factor

$$\tilde{H}_0(T_R) = M_0^{(0)}\exp(-\gamma T_R), \tag{A29}$$

where $T_R = -\frac{\hbar^2}{2\mu_d}\nabla_R^2$ and

$$\gamma = \frac{\mu_d\alpha_2^2\beta_d^2}{2\hbar^2}, \tag{A30}$$

about a local value of T_R equal to $T_0(R)$. We do this by using the following result.

For a pair of operators A and B that may not commute

$$\exp(A+B) = \exp(B)\left[1 + \int_0^1 dx A(x) + \int_0^1 dx\, A(x)\int_0^x dx' A(x') + \cdots\right], \tag{A31}$$

where the operator $A(x)$ is defined by

$$A(x) = \exp(-xB)A\exp(xB). \tag{A32}$$

Equation (A31) is an expansion in powers of A. It can be derived by first defining the function

$$F(x) = \exp(-xB)\exp[x(A+B)]. \tag{A33}$$

This function has the first derivative with respect to x

$$\frac{dF(x)}{dx} = A(x)F(x), \tag{A34}$$

and therefore satisfies the integral equation

$$F(x) = F(0) + \int_0^x dx'\frac{dF(x')}{dx'} = 1 + \int_0^x dx' A(x')F(x'). \tag{A35}$$

Iterating this equation and putting $x = 1$ gives the result given in Eq. (A31).

Putting $B = -\gamma T_0$ and $A = -\gamma(T_R - T_0)$ we obtain an expansion in powers of $T_R - T_0$. To first order

$$\tilde{H}(T_R) = \tilde{H}(T_0)\left[1 - \gamma\int_0^1 dx\exp(\gamma x T_0)\times(T_R - T_0)\exp(-\gamma x T_0) + \cdots\right], \tag{A36}$$

We have

$$T_R\exp(-\gamma x T_0) = \frac{\hbar^2}{2\mu_d}\exp(-\gamma x T_0)\left[-\nabla_R^2 + 2\gamma x\boldsymbol{\nabla}T_0\cdot\boldsymbol{\nabla}_R + \gamma x\left(T_0'' + \frac{2}{R}T_0'\right) - \gamma^2x^2T_0'^2\right], \tag{A37}$$

where we have assumed T_0 is a function of R only and T_0' means $\frac{dT_0(R)}{dR}$, etc. Using (A37) in (A36) and performing the integration over x we obtain

$$\tilde{H}(T_R) = \tilde{H}(T_0)\left[1 - \gamma(T_R - T_0 + \Delta) - \frac{\hbar^2\gamma^2}{2\mu_d}\boldsymbol{\nabla}T_0\cdot\boldsymbol{\nabla}_R + \cdots\right], \tag{A38}$$

where

$$\Delta = \frac{\hbar^2\gamma}{2\mu_d}\left(\frac{T_0''}{2} + \frac{T_0'}{R} - \frac{\gamma}{3}T_0'^2\right). \tag{A39}$$

Note that the term $T_R - T_0$ is what we obtain for the first-order term if we ignore the fact that T_R and T_0 do not commute. The correction Δ and the last term in Eq. (A38) arise from the breakdown of that assumption.

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