Early thermalization of quark-gluon matter with the elastic scattering of ggq and $gg\bar{q}$

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Elastic gluon-gluon-quark (gluon-gluon-antiquark) scattering is studied in perturbative QCD with 123 Feynman diagrams at the tree level. Individually squared amplitudes and interference terms of the Feynman diagrams are derived. With the elastic gluon-gluon-quark scattering and the elastic gluon-gluon-antiquark scattering, transport equations are established. In the thermalization process of initially created quark-gluon matter, this matter is governed by elastic two-to-two scattering and elastic three-to-three scattering. Solutions of the transport equations show that initially created quark-gluon matter takes early thermalization, i.e., thermal states are established rapidly. Different thermalization times of gluon matter and quark matter are obtained.

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I. INTRODUCTION

Quark-gluon plasma is a thermal state and has a temperature, but quark-gluon matter initially produced in Au-Au collisions at high energies of the Relativistic Heavy Ion Collider (RHIC) is not in a thermal state and does not have a temperature. Therefore, how and when initially produced quark-gluon matter establishes a thermal state are key questions in evolution from initially produced quark-gluon matter to quark-gluon plasma, and the evolution is a thermalization process. Hydrodynamic calculations [1,2] in explaining the experimental elliptic flow coefficient of hadrons [3] have revealed that the thermalization time is less than 1 fm/c, i.e., the thermalization is rapid [4]. Initially produced quark-gluon matter has anisotropic distributions of quarks and gluons. The distribution function in the direction of the incoming gold beam is much larger than that in the direction perpendicular to the beam direction [5]. Massless gluons and quarks produced in initial Au-Au collisions move at the velocity of light. Intuitively, such fast gluons and quarks cannot become randomly moving, i.e., a thermal state cannot be established easily. But the early thermalization is against the intuition. Therefore, the early thermalization is a new phenomenon and must be understood. The way to understand the early thermalization is to answer the two questions regarding how initially produced quark-gluon matter establishes a thermal state and when initially produced quark-gluon matter establishes a thermal state.

Let us note that the gluon number density of quark-gluon matter initially produced in central Au-Au collisions at RHIC energies can be very high [6]. A high number density must have its consequences. Indeed, the early thermalization is closely related to the high number density [5,7]. Initially produced gluon matter can be simulated by HIJING [8] and in the collision region corresponds to a gluon number density of 19.4 fm^{-3}. At an interaction range of 0.62 fm, which acts as the radius of a unit volume of sphere, gluon-gluon scattering and gluon-gluon-gluon scattering have occurrence probabilities of 0.3 and 0.2, respectively [9]. This implies substantial occurrence of the gluon-gluon-gluon scattering, and the three-gluon scattering and the two-gluon scattering lead to the early thermalization of initially produced gluon matter [7]. The two-gluon scattering alone cannot cause the early thermalization, and the early thermalization is understood as an effect of many-body scattering.

In Ref. [10] we have studied the thermalization of initially produced quark-gluon matter with elastic two-to-two scattering, which happens no matter what the gluon and quark number densities are [11–15], and elastic three-to-three scattering, which includes the elastic scattering of gluon-gluon-gluon, quark-quark-quark, antiquark-antiquark-antiquark, quark-quark-antiquark, quark-antiquark-antiquark, gluon-quark-quark, gluon-antiquark-antiquark, and gluon-quark-antiquark. Since it is hard to derive squared amplitudes for elastic gluon-gluon-quark scattering and elastic gluon-gluon-antiquark scattering, we can include the two types of elastic scattering only in the present work. Feynman diagrams at the tree level for the elastic gluon-gluon-quark scattering are introduced in Sec. II. A Feynman diagram has a squared amplitude. Two diagrams have two individually squared amplitudes and one interference term. The individually squared amplitudes and the interference term are derived in perturbative QCD in Sec. III. The sum of the individually squared amplitudes and the interference terms of all Feynman diagrams is the squared amplitude for the elastic gluon-gluon-quark scattering. The squared amplitude for the elastic gluon-gluon-antiquark scattering is identical with the one for the elastic gluon-gluon-quark scattering. Including the squared amplitudes for the two types of elastic scattering, transport equations for gluons, quarks, and antiquarks are given in Sec. IV. Numerical solutions of the transport equations and discussions are presented in Sec. V. A summary is presented in the last section.

II. ELASTIC GLUON-GLUON-QUARK SCATTERING

The elastic gluon-gluon-quark scattering brings about 123 Feynman diagrams at the tree level. The 123 diagrams are divided into four classes. The first class consists of

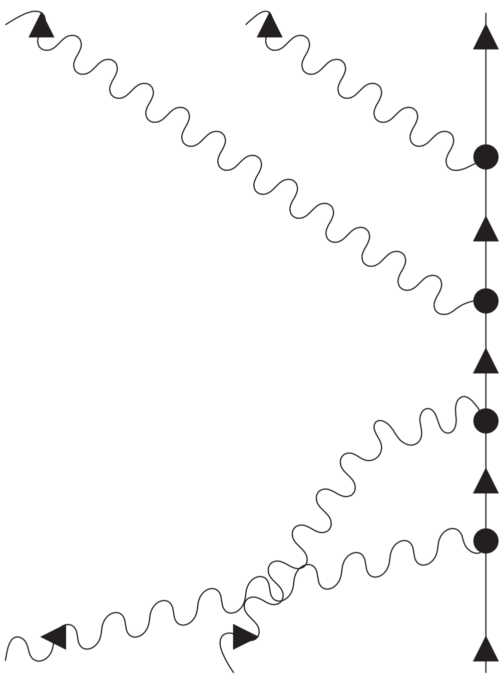

FIG. 1. Elastic gluon-gluon-quark scattering.

24 diagrams that do not contain triple-gluon vertices and four-gluon vertices; the second class 36 diagrams, of which each contains one triple-gluon vertex and no one four-gluon vertex; the third class 38 diagrams, of which each contains either two triple-gluon vertices or one four-gluon vertex; and the fourth class 25 diagrams, of which each contains either three triple-gluon vertices or one four-gluon vertex plus one triple-gluon vertex. Let wiggly lines and solid lines stand for gluons and quarks, respectively. Twenty-one Feynman diagrams are displayed in Figs. 1–6. The other 102 diagrams are derived from the diagrams in Figs. 1–6 as follows.

A. Feynman diagrams in the first class

The Feynman diagrams in the first class involve the absorption processes of initial gluons and the emission processes of final gluons. Every diagram in the first class has four gluon-quark vertices. The Feynman diagram in Fig. 1 indicates that the left initial gluon is absorbed first by the quark, the right initial gluon is then absorbed, the left final gluon is emitted first from the quark, and the right final gluon is then emitted. This is a permutation of the absorption processes of the two initial gluons and the emission processes of the two final gluons. If the right final gluon is emitted first and is followed by the absorption of the right initial gluon, and the left final gluon is the next emitted and is followed by the absorption of the left initial gluon, another permutation of the absorption and emission processes is obtained. The corresponding Feynman diagram is derived from the diagram in Fig. 1 by the exchange of the vertex of the left initial gluon and the quark and the vertex of the right final gluon and the quark. From this example we infer that the diagram C_{1245} leads to 23 new diagrams by exchanging the four gluon-quark vertices. Therefore, the 24 diagrams in the first class correspond to 24 permutations of the absorption amd emission processes.

B. Feynman diagrams in the second class

Any of the Feynman diagrams in the second class involves one triple-gluon vertex and no one four-gluon vertex. The triple-gluon vertex in a diagram in Fig. 2 relates to the two initial gluons, one initial gluon and one final gluon, or the two final gluons. We may derive 33 diagrams from the three diagrams.

The diagram $C_{(12)45}$ in Fig. 2 indicates that the two initial gluons annihilate into one gluon that is absorbed by the quark, and the quark then radiates two gluons. The diagram has three gluon-quark vertices and leads to five new diagrams with permutations of the three vertices. Hence, we have six Feynman diagrams, each of which has a triple-gluon vertex that possesses the two initial gluons.

The diagram $C_{(14)25}$ means that one initial gluon breaks into one final gluon and a virtual gluon that is absorbed by the quark, and, furthermore, the quark absorbs another initial gluon and then radiates another final gluon. Now we take into account the following five cases: (1) the gluon that breaks into two gluons may be the left or right initial gluon; (2) the final gluon that comes from the broken gluon is either the left or right final gluon; (3) the initial gluon absorbed by the quark may be the left or right initial gluon; (4) the final gluon radiated is either the left or right final gluon; (5) the three gluon-quark vertices may take six permutations. Operating along with the five cases, we get 24 diagrams which include the diagram $C_{(14)25}$. Each of the 24 diagrams has a triple-gluon vertex that possesses one initial gluon and one final gluon.

The diagram $C_{21(45)}$ indicates that the quark absorbs the two initial gluons first and then radiates a gluon which breaks into the two final gluons. The quark line has three gluon-quark vertices. Different permutations of the three vertices lead to five new diagrams from the diagram $C_{21(45)}$. Hence, we have six Feynman diagrams, each of which has a triple-gluon vertex that possesses the two final gluons.

C. Feynman diagrams in the third class

Thirty diagrams in the third class involve two triple-gluon vertices individually, and the other eight diagrams one four-gluon vertex. Six of the 30 diagrams are shown in Fig. 3, and 2 of the 8 diagrams in Fig. 4. The diagram $C_{(25)(14)}$ or $C_{(12)(45)}$ uses up the external gluons in the two triple-gluon vertices. In each of the other diagrams there is one external gluon coupled to the quark.

The six diagrams in Fig. 3 are characteristic of having two triple-gluon vertices and two gluon-quark vertices. The permutation of the two gluon-quark vertices generates 6 new diagrams. More diagrams can be generated from the exchange of the two initial gluons in the diagrams $C_{(142)5}$, $C_{2(145)}$, and $C_{2(451)}$ and/or the exchange of the two final gluons in the diagrams $C_{(124)5}$, $C_{(142)5}$, $C_{2(145)}$, and $C_{(25)(14)}$. We thus can derive 24 diagrams from the 6 diagrams in Fig. 3.

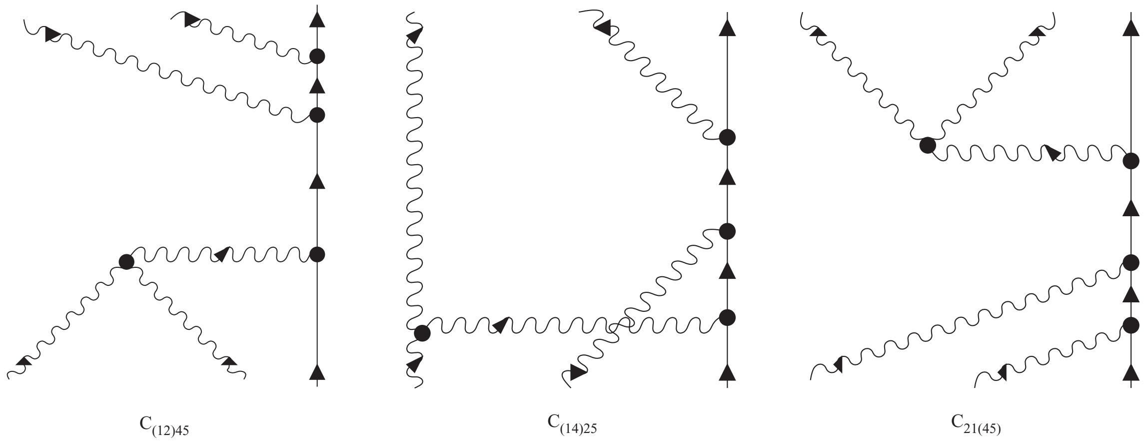

FIG. 2. Elastic gluon-gluon-quark scattering.

The two diagrams in Fig. 4 are characteristic of one four-gluon vertex and two gluon-quark vertices. The two gluon-quark vertices may be exchanged. The two final gluons in the diagram $C_{(124\sim)5}$ can be exchanged, as can the two initial gluons in the diagram $C_{2(145\sim)}$. The four exchanges lead to 6 new diagrams from the two diagrams in Fig. 4.

D. Feynman diagrams in the fourth class

Fifteen diagrams in the fourth class relate to three triple-gluon vertices individually and the other 10 diagrams one triple-gluon vertex plus one four-gluon vertex. Four of the 15 diagrams are shown in Fig. 5, and 5 of the 10 diagrams in Fig. 6. In every diagram the quark line is only coupled to a virtual gluon.

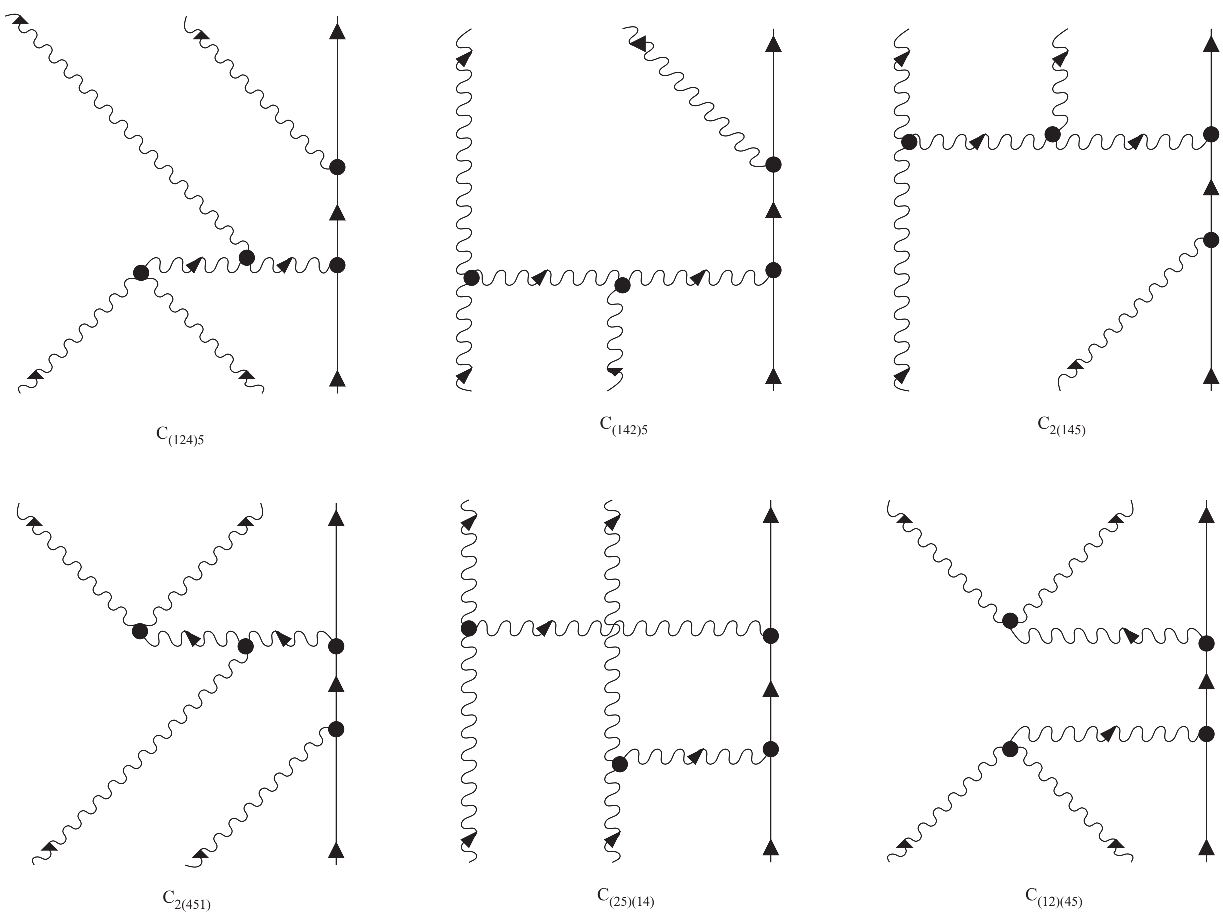

FIG. 3. Elastic gluon-gluon-quark scattering.

In addition to the three triple-gluon vertices every diagram in Fig. 5 has three virtual gluons, but only one virtual gluon is connected to the quark. In the diagram $C_{(12)(45)2}$ the virtual gluon is also connected to the right initial gluon. Of course, the virtual gluon can be connected to the other three external gluons, and three new diagrams are, thus, generated. We cannot derive a diagram from the diagram $C_{(12)(45)M}$ but can derive one more diagram from the diagram $C_{(14)(25)M}$ by the exchange of the two final gluons. One virtual gluon in the diagram C_{14-} connects the quark and the left final gluon. It may also connect the quark and the other three external gluons. The two final gluons may be exchanged. We then can derive 7 diagrams from the diagram C_{14-}. In total, we have derived 11 diagrams from the 4 diagrams in Fig. 5.

Every diagram in Fig. 6 has one triple-gluon vertex and one four-gluon vertex. If the two initial gluons are coupled in a vertex and the two final gluons in another vertex as shown in the diagrams $C_{12\sim}$ and $C_{45\sim}$, no more diagrams can be derived. If three external gluons are coupled in a vertex as shown in the diagrams $C_{(12)45\sim}$ and $C_{12(45)\sim}$, the exchange of the two external gluons mounted in different vertices leads to two new diagrams. If every vertex is mounted with an initial gluon and a final gluon as shown in the diagram $C_{25\sim}$, the exchange of the two initial gluons and/or the exchange of the two final gluons generate three new diagrams. In total, we can derive five diagrams from the five diagrams in Fig. 6.

III. SQUARED AMPLITUDE FOR ELASTIC GLUON-GLUON-QUARK SCATTERING

Amplitudes of the Feynman diagrams of the elastic gluon-gluon-quark scattering $g(p_1)+g(p_2)+q(p_3)\rightarrow g(p_4)+g(p_5)+q(p_6)$ with four-momenta $p_i=(E_i,\vec{p}_i)(i=1,\ldots,6)$ are written in accordance with the Feynman rules in perturbative QCD [16,17]. The amplitudes contain the polarization four-vectors $\epsilon_\mu(\lambda_i)(i=1,2,4,5)$ of external gluons and linear combinations of gluon momenta due to the triple-gluon vertices. A diagram's squared amplitude or an interference term of two diagrams has a structure with one of the following 41 types:

$$\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}\not{u}_{11}\not{u}_{12}\not{u}_{13}\not{u}_{14}\not{u}_{15}\not{u}_{16}),$$

$$\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}\not{u}_{11}\not{u}_{12}\not{u}_{13}\not{u}_{14}),$$

$$u_{15}\cdot\epsilon^*(\lambda_{k_1})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}\not{u}_{11}\not{u}_{12}\not{u}_{13}\not{u}_{14}),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}\not{u}_{11}\not{u}_{12}),$$

$$u_{13}\cdot u_{14}\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}\not{u}_{11}\not{u}_{12}),$$

$$u_{13}\cdot\epsilon(\lambda_{k_1})u_{14}\cdot\epsilon^*(\lambda_{k_2})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}\not{u}_{11}\not{u}_{12}),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})u_{13}\cdot\epsilon^*(\lambda_{k_3})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}\not{u}_{11}\not{u}_{12}),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}),$$

$$u_{11}\cdot u_{12}\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}),$$

$$u_{11}\cdot\epsilon^*(\lambda_{k_1})u_{12}\cdot\epsilon^*(\lambda_{k_2})u_{13}\cdot\epsilon^*(\lambda_{k_3})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}),$$

$$\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})u_{11}\cdot\epsilon^*(\lambda_{k_3})u_{12}\cdot\epsilon^*(\lambda_{k_4})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}),$$

$$u_{11}\cdot u_{12}\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})u_{13}\cdot\epsilon^*(\lambda_{k_3})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_{11}\cdot\epsilon^*(\lambda_{k_5})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8\not{u}_9\not{u}_{10}),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon(\lambda_{k_3})\cdot\epsilon(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})\epsilon^*(\lambda_{k_7})\cdot\epsilon^*(\lambda_{k_8})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$u_9\cdot u_{10}u_{11}\cdot u_{12}\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$u_9\cdot u_{10}\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$u_9\cdot\epsilon(\lambda_{k_1})u_{10}\cdot\epsilon(\lambda_{k_2})u_{11}\cdot\epsilon^*(\lambda_{k_3})u_{12}\cdot\epsilon^*(\lambda_{k_4})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})u_9\cdot\epsilon(\lambda_{k_3})u_{10}\cdot\epsilon^*(\lambda_{k_4})u_{11}\cdot\epsilon^*(\lambda_{k_5})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$u_9\cdot u_{10}\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})u_{11}\cdot\epsilon(\lambda_{k_3})u_{12}\cdot\epsilon(\lambda_{k_4})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_9\cdot\epsilon(\lambda_{k_5})u_{10}\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$u_9\cdot u_{10}\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_{11}\cdot\epsilon(\lambda_{k_5})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})u_9\cdot\epsilon(\lambda_{k_7})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6\not{u}_7\not{u}_8),$$

$$u_7\cdot u_8u_9\cdot u_{10}\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),$$

$$u_7\cdot u_8\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon(\lambda_{k_3})\cdot\epsilon(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})\epsilon^*(\lambda_{k_7})\cdot\epsilon^*(\lambda_{k_8})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),$$

$$u_7\cdot\epsilon(\lambda_{k_1})u_8\cdot\epsilon(\lambda_{k_2})u_9\cdot\epsilon^*(\lambda_{k_3})u_{10}\cdot\epsilon^*(\lambda_{k_4})u_{11}\cdot\epsilon^*(\lambda_{k_5})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),$$

$$\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})u_7\cdot\epsilon(\lambda_{k_3})u_8\cdot\epsilon^*(\lambda_{k_4})u_9\cdot\epsilon^*(\lambda_{k_5})u_{10}\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),$$

$$
\begin{aligned}
&u_7\cdot u_8\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})u_9\cdot\epsilon(\lambda_{k_3})u_{10}\cdot\epsilon(\lambda_{k_4})u_{11}\cdot\epsilon^*(\lambda_{k_5})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),\\
&\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_7\cdot\epsilon(\lambda_{k_5})u_8\cdot\epsilon^*(\lambda_{k_6})u_9\cdot\epsilon^*(\lambda_{k_7})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),\\
&u_7\cdot u_8\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_9\cdot\epsilon(\lambda_{k_5})u_{10}\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),\\
&u_7\cdot u_8u_9\cdot u_{10}\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_{11}\cdot\epsilon^*(\lambda_{k_5})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),\\
&\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon(\lambda_{k_3})\cdot\epsilon(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})u_7\cdot\epsilon^*(\lambda_{k_7})u_8\cdot\epsilon^*(\lambda_{k_8})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),\\
&u_7\cdot u_8\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})u_9\cdot\epsilon(\lambda_{k_7})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\not{u}_5\not{u}_6),\\
&u_5\cdot u_6u_7\cdot u_8\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon(\lambda_{k_3})\cdot\epsilon(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})\epsilon^*(\lambda_{k_7})\cdot\epsilon^*(\lambda_{k_8})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&u_5\cdot\epsilon(\lambda_{k_1})u_6\cdot\epsilon(\lambda_{k_2})u_7\cdot\epsilon(\lambda_{k_3})u_8\cdot\epsilon^*(\lambda_{k_4})u_9\cdot\epsilon^*(\lambda_{k_5})u_{10}\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&\epsilon^*(\lambda_{k_1})\cdot\epsilon^*(\lambda_{k_2})u_5\cdot\epsilon(\lambda_{k_3})u_6\cdot\epsilon(\lambda_{k_4})u_7\cdot\epsilon(\lambda_{k_5})u_8\cdot\epsilon^*(\lambda_{k_6})u_9\cdot\epsilon^*(\lambda_{k_7})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&u_5\cdot u_6\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})u_7\cdot\epsilon(\lambda_{k_3})u_8\cdot\epsilon^*(\lambda_{k_4})u_9\cdot\epsilon^*(\lambda_{k_5})u_{10}\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_5\cdot\epsilon(\lambda_{k_5})u_6\cdot\epsilon(\lambda_{k_6})u_7\cdot\epsilon^*(\lambda_{k_7})u_8\cdot\epsilon^*(\lambda_{k_8})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&u_5\cdot u_6\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_7\cdot\epsilon(\lambda_{k_5})u_8\cdot\epsilon^*(\lambda_{k_6})u_9\cdot\epsilon^*(\lambda_{k_7})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&u_5\cdot u_6u_7\cdot u_8\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})u_9\cdot\epsilon(\lambda_{k_5})u_{10}\cdot\epsilon^*(\lambda_{k_6})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&u_5\cdot u_6u_7\cdot u_8\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})u_9\cdot\epsilon(\lambda_{k_7})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),\\
&u_5\cdot u_6\epsilon(\lambda_{k_1})\cdot\epsilon(\lambda_{k_2})\epsilon^*(\lambda_{k_3})\cdot\epsilon^*(\lambda_{k_4})\epsilon^*(\lambda_{k_5})\cdot\epsilon^*(\lambda_{k_6})u_7\cdot\epsilon(\lambda_{k_7})u_8\cdot\epsilon(\lambda_{k_8})\mathrm{tr}(\not{u}_1\not{u}_2\not{u}_3\not{u}_4),
\end{aligned}
$$

where $u_h(h=1,\ldots,16)$ are linear combinations of four-momenta of external gluons, virtual gluons, external quarks, and virtual quarks and which do not include the 3×3 SU(3) color matrices of the gluon-quark vertex factor and the antisymmetric SU(3) structure constants of the triple-gluon vertex factor and the four-gluon vertex factor. The subscripts $k_1,\ldots,k_8$ label the external gluons and take the values 1, 2, 4, and 5. The traces of products of Dirac matrices are calculated via the relation [18]

$$\mathrm{tr}(\not{c}_1\cdots\not{c}_N)=c_1\cdot c_2\mathrm{tr}(\not{c}_3\cdots\not{c}_N)-c_1\cdot c_3\mathrm{tr}(\not{c}_2\not{c}_4\cdots\not{c}_N)+\cdots+c_1\cdot c_N\mathrm{tr}(\not{c}_2\cdots\not{c}_{N-1}),\tag{1}$$

where N is an even positive integer and $c_h(h=1,\ldots,N)$ are arbitrary four-vectors. The above types of structures turn into the following 12 forms:

$$
\begin{aligned}
&a_1\cdot b_1a_2\cdot b_2a_3\cdot b_3a_4\cdot b_4\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}\text{scalar products of polarization four-vectors},\\
&a_1\cdot b_1a_2\cdot b_2a_3\cdot b_3\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_4\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b_4\times\text{scalar products},\\
&a_1\cdot b_1a_2\cdot b_2\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_3\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b_3a_4\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b_4\times\text{scalar products},\\
&a_1\cdot b_1a_2\cdot b_2a_3\cdot b_3\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_4\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b_4\times\text{scalar products},\\
&a_1\cdot b_1\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_2\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b_2a_3\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b_3a_4\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot b_4\times\text{scalar products},\\
&a_1\cdot b_1a_2\cdot b_2\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_3\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b_3a_4\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot b_4\times\text{scalar products},\\
&a_1\cdot b_1a_2\cdot b_2a_3\cdot b_3\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_4\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot b_4\times\text{scalar products},\\
&\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_1\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b_1a_2\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b_2a_3\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot b_3a_4\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b_4,\\
&a_1\cdot b_1\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_2\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b_2a_3\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b_3a_4\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b_4,\\
&a_1\cdot b_1a_2\cdot b_2\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_3\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b_3a_4\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b_4,\\
&a_1\cdot b_1a_2\cdot b_2\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_3\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b_3a_4\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b_4,\\
&a_1\cdot b_1a_2\cdot b_2a_3\cdot b_3\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}a_4\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b_4,
\end{aligned}
$$

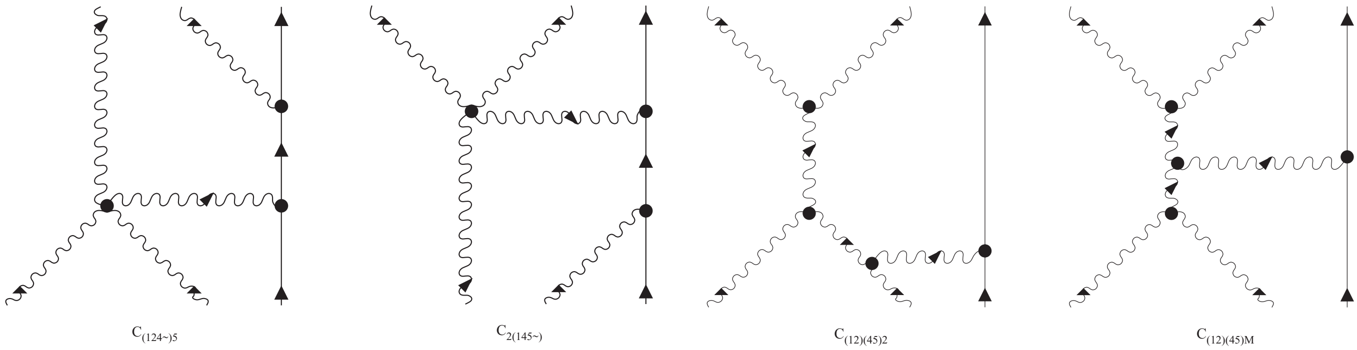

FIG. 4. Elastic gluon-gluon-quark scattering.

where a_h and b_h $(h = 1, 2, 3, 4)$ are linear combinations of four-momenta; the subscripts i, j, m, and n label the external gluons, take the values 1, 2, 4, and 5, and do not equal each other. For an external gluon in the physical transverse state with four-momentum p_i and helicity λ_i, we use the projection [16]

$$\sum_{\lambda_i} \epsilon_\mu(\lambda_i)\epsilon_\nu^*(\lambda_i) = -g_{\mu\nu} + \frac{w_\mu p_{i\nu} + w_\nu p_{i\mu}}{w \cdot p_i} - \frac{w^2 p_{i\mu} p_{i\nu}}{(w \cdot p_i)^2}, \tag{2}$$

where w is an arbitrary four-vector. A convenient choice for w is $w = p_3$ with $p_3^2 = 0$. Equation (2) leads to

$$p_i^\mu \sum_{\lambda_i} \epsilon_\mu(\lambda_i)\epsilon_\nu^*(\lambda_i) = 0, \tag{3}$$

$$p_i^\nu \sum_{\lambda_i} \epsilon_\mu(\lambda_i)\epsilon_\nu^*(\lambda_i) = 0. \tag{4}$$

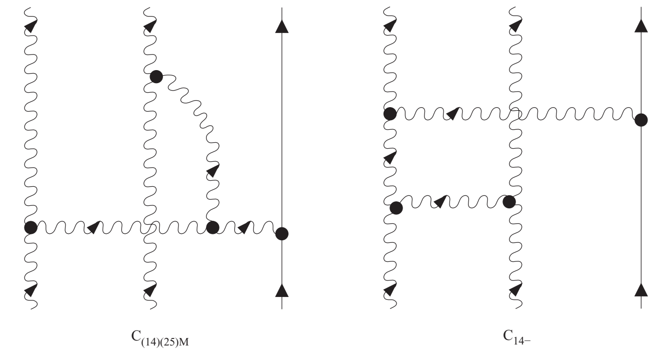

FIG. 5. Elastic gluon-gluon-quark scattering.

The scalar products of polarization four-vectors in the first 7 of the above 12 forms take the following expressions with the values −2 and 2:

$$\sum_{\lambda_i} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_i) = -2, \tag{5}$$

$$\sum_{\lambda_i\lambda_j} \epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j) = \sum_{\lambda_i\lambda_j} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j) = 2, \tag{6}$$

$$\sum_{\lambda_i\lambda_j\lambda_k} \epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot\epsilon(\lambda_i) = -2, \tag{7}$$

$$\sum_{\lambda_i\lambda_j\lambda_k} \epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_k)\epsilon^*(\lambda_k)\cdot\epsilon(\lambda_i) = -2, \tag{8}$$

$$\sum_{\lambda_i\lambda_j\lambda_k} \epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot\epsilon(\lambda_i) = -2, \tag{9}$$

$$\sum_{\lambda_i\lambda_j\lambda_k} \epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_k)\epsilon^*(\lambda_k)\cdot\epsilon(\lambda_i) = -2, \tag{10}$$

$$\sum_{\lambda_i\lambda_j\lambda_k} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot\epsilon^*(\lambda_i) = -2, \tag{11}$$

$$\sum_{\lambda_i\lambda_j\lambda_k} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_k)\epsilon^*(\lambda_k)\cdot\epsilon^*(\lambda_i) = -2, \tag{12}$$

$$\sum_{\lambda_i\lambda_j\lambda_k} \epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot\epsilon^*(\lambda_i) = -2, \tag{13}$$

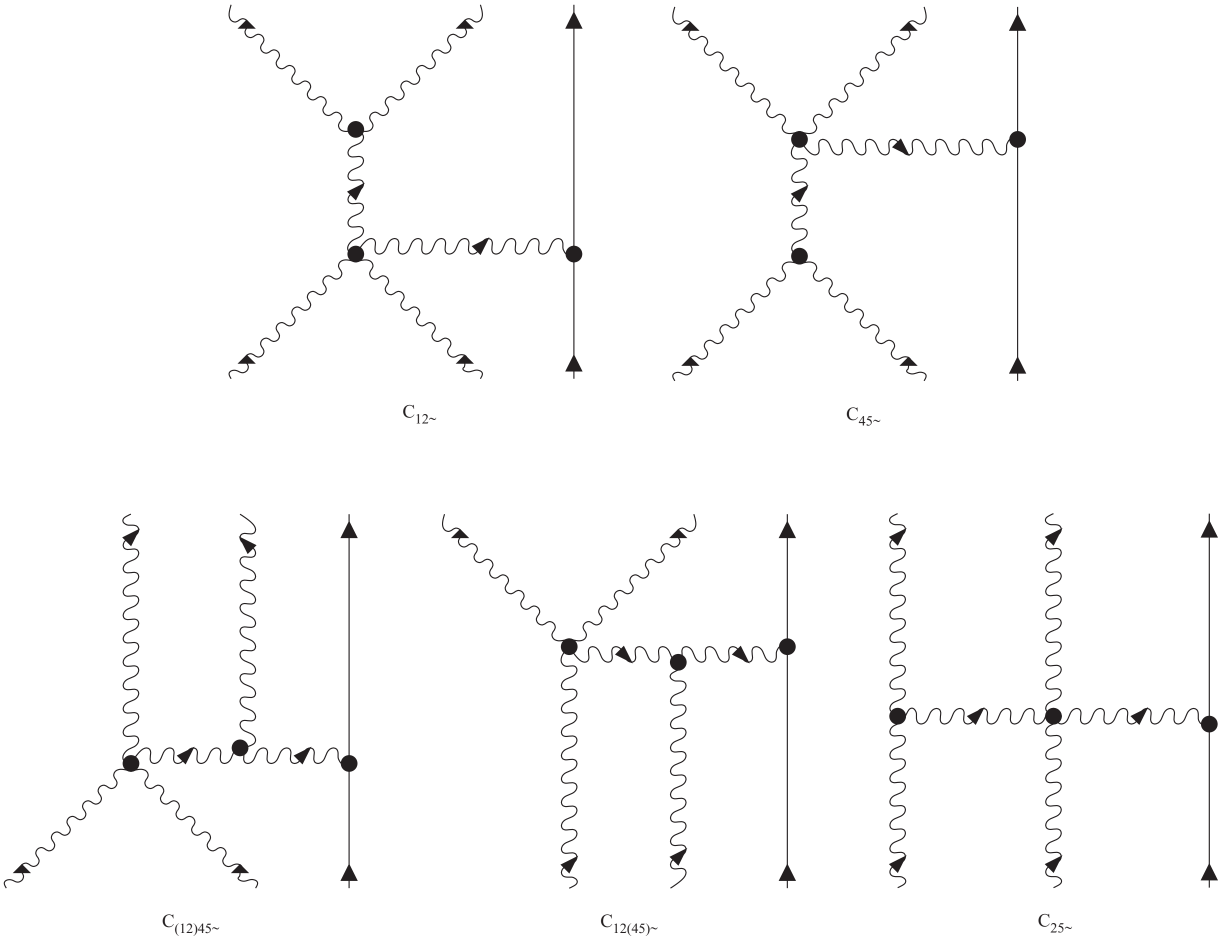

FIG. 6. Elastic gluon-gluon-quark scattering.

$$\sum_{\lambda_i \lambda_j \lambda_k} \epsilon(\lambda_i) \cdot \epsilon(\lambda_j) \epsilon^*(\lambda_j) \cdot \epsilon(\lambda_k) \epsilon^*(\lambda_k) \cdot \epsilon^*(\lambda_i) = -2, \tag{14}$$

$$\sum_{\lambda_i \lambda_j \lambda_m \lambda_n} \epsilon^*(\lambda_i) \cdot \epsilon^*(\lambda_j) \epsilon(\lambda_j) \cdot \epsilon^*(\lambda_m) \epsilon(\lambda_m) \cdot \epsilon^*(\lambda_n) \epsilon(\lambda_n) \cdot \epsilon(\lambda_i) = 2, \tag{15}$$

$$\sum_{\lambda_i \lambda_j \lambda_m \lambda_n} \epsilon^*(\lambda_i) \cdot \epsilon^*(\lambda_j) \epsilon(\lambda_j) \cdot \epsilon^*(\lambda_m) \epsilon(\lambda_m) \cdot \epsilon(\lambda_n) \epsilon^*(\lambda_n) \cdot \epsilon(\lambda_i) = 2, \tag{16}$$

$$\sum_{\lambda_i \lambda_j \lambda_m \lambda_n} \epsilon^*(\lambda_i) \cdot \epsilon^*(\lambda_j) \epsilon(\lambda_j) \cdot \epsilon(\lambda_m) \epsilon^*(\lambda_m) \cdot \epsilon^*(\lambda_n) \epsilon(\lambda_n) \cdot \epsilon(\lambda_i) = 2, \tag{17}$$

$$\sum_{\lambda_i \lambda_j \lambda_m \lambda_n} \epsilon^*(\lambda_i) \cdot \epsilon^*(\lambda_j) \epsilon(\lambda_j) \cdot \epsilon(\lambda_m) \epsilon^*(\lambda_m) \cdot \epsilon(\lambda_n) \epsilon^*(\lambda_n) \cdot \epsilon(\lambda_i) = 2, \tag{18}$$

$$\sum_{\lambda_i \lambda_j \lambda_m \lambda_n} \epsilon^*(\lambda_i) \cdot \epsilon(\lambda_j) \epsilon^*(\lambda_j) \cdot \epsilon^*(\lambda_m) \epsilon(\lambda_m) \cdot \epsilon^*(\lambda_n) \epsilon(\lambda_n) \cdot \epsilon(\lambda_i) = 2, \tag{19}$$

$$\sum_{\lambda_i \lambda_j \lambda_m \lambda_n} \epsilon^*(\lambda_i) \cdot \epsilon(\lambda_j) \epsilon^*(\lambda_j) \cdot \epsilon^*(\lambda_m) \epsilon(\lambda_m) \cdot \epsilon(\lambda_n) \epsilon^*(\lambda_n) \cdot \epsilon(\lambda_i) = 2, \tag{20}$$

$$\sum_{\lambda_i \lambda_j \lambda_m \lambda_n} \epsilon^*(\lambda_i) \cdot \epsilon(\lambda_j) \epsilon^*(\lambda_j) \cdot \epsilon(\lambda_m) \epsilon^*(\lambda_m) \cdot \epsilon^*(\lambda_n) \epsilon(\lambda_n) \cdot \epsilon(\lambda_i) = 2, \tag{21}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot\epsilon(\lambda_i)=2, \tag{22}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{23}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{24}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{25}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{26}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{27}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{28}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{29}$$

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} \epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot\epsilon^*(\lambda_i)=2, \tag{30}$$

where the subscripts i, j, k, m, and n label the external gluons, take the values 1, 2, 4, and 5, and do not equal each other. Aside from the scalar products of polarization four-vectors, the other expressions encountered in the 12 forms before Eq. (2) are expressed in terms of scalar products of four-momenta,

$$\sum_{\lambda_i} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot b = -a\cdot b + \frac{a\cdot p_3 b\cdot p_i + b\cdot p_3 a\cdot p_i}{p_3\cdot p_i}, \tag{31}$$

$$\begin{aligned}
&\sum_{\lambda_i\lambda_j} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b = \sum_{\lambda_i\lambda_j} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot b\\
&= \sum_{\lambda_i\lambda_j} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot b = \sum_{\lambda_i\lambda_j} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot b\\
&= a\cdot b - \frac{b\cdot p_3 a\cdot p_j}{p_3\cdot p_j} - \frac{a\cdot p_3 b\cdot p_i}{p_3\cdot p_i} + \frac{a\cdot p_3 p_i\cdot p_j b\cdot p_3}{p_3\cdot p_i p_3\cdot p_j},
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot b\\
&\quad= \sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_k)\epsilon^*(\lambda_k)\cdot b\\
&\quad= \sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot b\\
&\quad= \sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_k)\epsilon^*(\lambda_k)\cdot b\\
&\quad= \sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot b\\
&\quad= \sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_k)\epsilon^*(\lambda_k)\cdot b\\
&\quad= \sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_k)\epsilon(\lambda_k)\cdot b
\end{aligned}$$

$$= \sum_{\lambda_i\lambda_j\lambda_k} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_k)\epsilon^*(\lambda_k)\cdot b$$
$$= -a\cdot b + \frac{b\cdot p_3 a\cdot p_k}{p_3\cdot p_k} + \frac{a\cdot p_3 b\cdot p_i}{p_3\cdot p_i} - \frac{a\cdot p_3 p_i\cdot p_k b\cdot p_3}{p_3\cdot p_i p_3\cdot p_k}, \tag{33}$$

which is independent of p_j, and

$$\sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon^*(\lambda_i)\epsilon(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon^*(\lambda_j)\epsilon(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon^*(\lambda_m)\epsilon(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon^*(\lambda_n)\epsilon(\lambda_n)\cdot b$$
$$= \sum_{\lambda_i\lambda_j\lambda_m\lambda_n} a\cdot\epsilon(\lambda_i)\epsilon^*(\lambda_i)\cdot\epsilon(\lambda_j)\epsilon^*(\lambda_j)\cdot\epsilon(\lambda_m)\epsilon^*(\lambda_m)\cdot\epsilon(\lambda_n)\epsilon^*(\lambda_n)\cdot b$$
$$= a\cdot b - \frac{b\cdot p_3 a\cdot p_n}{p_3\cdot p_n} - \frac{a\cdot p_3 b\cdot p_i}{p_3\cdot p_i} + \frac{a\cdot p_3 p_i\cdot p_n b\cdot p_3}{p_3\cdot p_i p_3\cdot p_n}, \tag{34}$$

which is independent of p_j and p_m. In Eqs. (31)–(34) a and b are linear combinations of four-momenta.

Every gluon-quark vertex factor has a SU(3) color matrix, every triple-gluon vertex factor has a SU(3) structure constant, and every four-gluon vertex factor has two SU(3) structure constants. From a diagram in the first class to a diagram in the fourth class, the number of gluon-quark vertices decreases and the number of triple-gluon vertices increases. While the number of gluon-quark vertices decreases by one, the number of triple-gluon vertices increases by one. Therefore, in a diagram's amplitude the total number of the SU(3) color matrices and the SU(3) structure constants is four. An individually squared amplitude or an interference term involves the product of n $(0 \leqslant n \leqslant 6)$ SU(3) structure constants and the trace of the product of $8-n$ SU(3) color matrices. FORTRAN code was used to design the products.

Including the average over the spin and color states of the two initial gluons and the initial quark, finally, the spin- and color-summed squared amplitude for a diagram and the spin- and color-summed interference term of two diagrams are given as functions of the scalar products of the four-momenta of external gluons and external quarks. By means of $s_{12}=(p_1+p_2)^2$, $s_{23}=(p_2+p_3)^2$, $s_{31}=(p_3+p_1)^2$, $u_{15}=(p_1-p_5)^2$, $u_{16}=(p_1-p_6)^2$, $u_{24}=(p_2-p_4)^2$, $u_{26}=(p_2-p_6)^2$, $u_{34}=(p_3-p_4)^2$, and $u_{35}=(p_3-p_5)^2$, the 123 individually squared amplitudes and the interference terms of the 123 Feynman diagrams are expressed in terms of the nine independent variables s_{12}, s_{23}, s_{31}, u_{15}, u_{16}, u_{24}, u_{26}, u_{34}, and u_{35}. We denote by $|\mathcal{M}_{ggq\to ggq}|^2$ the squared amplitude for the elastic ggq scattering, which is the sum of the individually squared amplitudes and the interference terms. The amplitude $\mathcal{M}_{gg\bar{q}\to gg\bar{q}}$ for the elastic $gg\bar{q}$ scattering satisfies the relation $|\mathcal{M}_{gg\bar{q}\to gg\bar{q}}|^2=|\mathcal{M}_{ggq\to ggq}|^2$.

IV. TRANSPORT EQUATIONS

In the last section we have obtained the squared amplitudes for the elastic ggq scattering and the elastic $gg\bar{q}$ scattering. The other types of elastic three-to-three scattering have already been studied in Refs. [7,9,10,19] for the evolution of quark-gluon matter. We assume that quark-gluon matter consists of gluons and equal amounts of up quarks, down quarks, up antiquarks, and down antiquarks. Elastic two-to-two scattering and elastic three-to-three scattering involve quarks and antiquarks on an equal footing, and antiquark matter takes the same evolution as quark matter. Let $\mathcal{M}_{gqq'\to gqq'}$ be the amplitude for elastic gluon-quark-quark scattering, and let $\mathcal{M}_{\bar{q}\bar{q}'\to\bar{q}\bar{q}'}$ be the amplitude for elastic antiquark-antiquark scattering, where q (q') is the up or down quark, and $\bar{q}$ ($\bar{q}'$) is the up or down antiquark. In the same way, we understand the notations $\mathcal{M}_{ggg\to ggg}$, $\mathcal{M}_{g\bar{q}\bar{q}'\to g\bar{q}\bar{q}'}$, $\mathcal{M}_{gq\bar{q}'\to gq\bar{q}'}$, $\mathcal{M}_{qqq'\to qqq'}$, $\mathcal{M}_{\bar{q}\bar{q}\bar{q}'\to\bar{q}\bar{q}\bar{q}'}$, $\mathcal{M}_{qq\bar{q}'\to qq\bar{q}'}$, $\mathcal{M}_{qq'\bar{q}\to qq'\bar{q}}$, $\mathcal{M}_{q'\bar{q}\bar{q}\to q'\bar{q}\bar{q}}$, $\mathcal{M}_{q\bar{q}\bar{q}'\to q\bar{q}\bar{q}'}$, $\mathcal{M}_{gg\to gg}$, $\mathcal{M}_{gq\to gq}$, $\mathcal{M}_{g\bar{q}\to g\bar{q}}$, $\mathcal{M}_{qq'\to qq'}$, and $\mathcal{M}_{q\bar{q}'\to q\bar{q}'}$. While quark-gluon matter is governed by all types of elastic two-to-two scattering and elastic three-to-three scattering, the transport equation for gluon matter is

$$
\begin{aligned}
\frac{\partial f_{g1}}{\partial t}+\vec{\mathrm{v}}_1\cdot\vec{\nabla}_{\vec{r}}f_{g1}=&-\frac{1}{2E_1}\int\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\\
&\times\bigg\{\frac{g_G}{2}|\mathcal{M}_{gg\to gg}|^2[f_{g1}f_{g2}(1+f_{g3})(1+f_{g4})-f_{g3}f_{g4}(1+f_{g1})(1+f_{g2})]\\
&+g_Q(|\mathcal{M}_{gu\to gu}|^2+|\mathcal{M}_{gd\to gd}|^2+|\mathcal{M}_{g\bar{u}\to g\bar{u}}|^2+|\mathcal{M}_{g\bar{d}\to g\bar{d}}|^2)\\
&\times[f_{g1}f_{q2}(1+f_{g3})(1-f_{q4})-f_{g3}f_{q4}(1+f_{g1})(1-f_{q2})]\bigg\}\\
&-\frac{1}{2E_1}\int\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}\frac{d^3p_5}{(2\pi)^3 2E_5}\frac{d^3p_6}{(2\pi)^3 2E_6}\\
&\times(2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5-p_6)\bigg\{\frac{g_G^2}{12}|\mathcal{M}_{ggg\to ggg}|^2\\
&\times[f_{g1}f_{g2}f_{g3}(1+f_{g4})(1+f_{g5})(1+f_{g6})-f_{g4}f_{g5}f_{g6}(1+f_{g1})(1+f_{g2})(1+f_{g3})]\\
&+\frac{g_Gg_Q}{2}(|\mathcal{M}_{ggu\to ggu}|^2+|\mathcal{M}_{ggd\to ggd}|^2+|\mathcal{M}_{gg\bar{u}\to gg\bar{u}}|^2+|\mathcal{M}_{gg\bar{d}\to gg\bar{d}}|^2)\\
&\times[f_{g1}f_{g2}f_{q3}(1+f_{g4})(1+f_{g5})(1-f_{q6})-f_{g4}f_{g5}f_{q6}(1+f_{g1})(1+f_{g2})(1-f_{q3})]\\
&+g_Q^2\bigg[\frac{1}{4}|\mathcal{M}_{guu\to guu}|^2+\frac{1}{2}(|\mathcal{M}_{gud\to gud}|^2+|\mathcal{M}_{gdu\to gdu}|^2)+\frac{1}{4}|\mathcal{M}_{gdd\to gdd}|^2\\
&+|\mathcal{M}_{gu\bar{u}\to gu\bar{u}}|^2+|\mathcal{M}_{gu\bar{d}\to gu\bar{d}}|^2+|\mathcal{M}_{gd\bar{u}\to gd\bar{u}}|^2+|\mathcal{M}_{gd\bar{d}\to gd\bar{d}}|^2\\
&+\frac{1}{4}|\mathcal{M}_{g\bar{u}\bar{u}\to g\bar{u}\bar{u}}|^2+\frac{1}{2}(|\mathcal{M}_{g\bar{u}\bar{d}\to g\bar{u}\bar{d}}|^2+|\mathcal{M}_{g\bar{d}\bar{u}\to g\bar{d}\bar{u}}|^2)+\frac{1}{4}|\mathcal{M}_{g\bar{d}\bar{d}\to g\bar{d}\bar{d}}|^2\bigg]
\end{aligned}
$$

$$\times [f_{g1} f_{q2} f_{q3}(1+f_{g4})(1-f_{q5})(1-f_{q6}) - f_{g4} f_{q5} f_{q6}(1+f_{g1})(1-f_{q2})(1-f_{q3})]\bigg\}, \quad (35)$$

and the transport equation for up-quark matter is

$$\begin{aligned}
\frac{\partial f_{q1}}{\partial t} + \vec{\mathrm{v}}_1 \cdot \vec{\nabla}_{\vec{r}} f_{q1} = {} & -\frac{1}{2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \\
& \times \bigg\{ g_G |\mathcal{M}_{ug\to ug}|^2 [f_{q1} f_{g2}(1-f_{q3})(1+f_{g4}) - f_{q3} f_{g4}(1-f_{q1})(1+f_{g2})] \\
& + g_Q\left(\frac{1}{2}|\mathcal{M}_{uu\to uu}|^2 + |\mathcal{M}_{ud\to ud}|^2 + |\mathcal{M}_{u\bar{u}\to u\bar{u}}|^2 + |\mathcal{M}_{u\bar{d}\to u\bar{d}}|^2\right) \\
& \times [f_{q1} f_{q2}(1-f_{q3})(1-f_{q4}) - f_{q3} f_{q4}(1-f_{q1})(1-f_{q2})]\bigg\} \\
& -\frac{1}{2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}\frac{d^3p_5}{(2\pi)^3 2E_5}\frac{d^3p_6}{(2\pi)^3 2E_6} \\
& \times (2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5-p_6)\bigg\{\frac{g_G^2}{4}|\mathcal{M}_{ugg\to ugg}|^2 \\
& \times [f_{q1} f_{g2} f_{g3}(1-f_{q4})(1+f_{g5})(1+f_{g6}) - f_{q4} f_{g5} f_{g6}(1-f_{q1})(1+f_{g2})(1+f_{g3})] \\
& + g_Q g_G\left(\frac{1}{2}|\mathcal{M}_{uug\to uug}|^2 + |\mathcal{M}_{udg\to udg}|^2 + |\mathcal{M}_{u\bar{u}g\to u\bar{u}g}|^2 + |\mathcal{M}_{u\bar{d}g\to u\bar{d}g}|^2\right) \\
& \times [f_{q1} f_{q2} f_{g3}(1-f_{q4})(1-f_{q5})(1+f_{g6}) - f_{q4} f_{q5} f_{g6}(1-f_{q1})(1-f_{q2})(1+f_{g3})] \\
& + g_Q^2\bigg[\frac{1}{12}|\mathcal{M}_{uuu\to uuu}|^2 + \frac{1}{4}(|\mathcal{M}_{uud\to uud}|^2 + |\mathcal{M}_{udu\to udu}|^2) + \frac{1}{4}|\mathcal{M}_{udd\to udd}|^2 \\
& + \frac{1}{2}|\mathcal{M}_{uu\bar{u}\to uu\bar{u}}|^2 + \frac{1}{2}|\mathcal{M}_{uu\bar{d}\to uu\bar{d}}|^2 + |\mathcal{M}_{ud\bar{u}\to ud\bar{u}}|^2 + |\mathcal{M}_{ud\bar{d}\to ud\bar{d}}|^2 \\
& + \frac{1}{4}|\mathcal{M}_{u\bar{u}\bar{u}\to u\bar{u}\bar{u}}|^2 + \frac{1}{2}(|\mathcal{M}_{u\bar{u}\bar{d}\to u\bar{u}\bar{d}}|^2 + |\mathcal{M}_{u\bar{d}\bar{u}\to u\bar{d}\bar{u}}|^2) + \frac{1}{4}|\mathcal{M}_{u\bar{d}\bar{d}\to u\bar{d}\bar{d}}|^2\bigg] \\
& \times [f_{q1} f_{q2} f_{q3}(1-f_{q4})(1-f_{q5})(1-f_{q6}) - f_{q4} f_{q5} f_{q6}(1-f_{q1})(1-f_{q2})(1-f_{q3})]\bigg\}, \quad (36)
\end{aligned}$$

where the massless gluon or the massless up quark has the velocity vector $\vec{\mathrm{v}}_1$ and the position vector $\vec{r}$, and the spin-color degeneracy factors are $g_{\mathrm{G}} = 16$ for the gluon and $g_{\mathrm{Q}} = 6$ for the quark, respectively. For the elastic two-to-two scattering p_1 and p_2 (p_3 and p_4) are the four-momenta of the two initial (final) partons. For the elastic three-to-three scattering p_1, p_2, and p_3 (p_4, p_5, and p_6) are the four-momenta of the three initial (final) partons. E_i is the energy component of p_i. f_{gi} (f_{qi}) is the gluon (quark) distribution function with the variable p_i. Denote the distribution functions for the up quark, the down quark, the up antiquark, and the down antiquark by f_{ui}, f_{di}, $f_{\bar{u}i}$, and $f_{\bar{d}i}$, respectively, and, as assumed, they are identical, i.e., $f_{ui} = f_{di} = f_{\bar{u}i} = f_{\bar{d}i} = f_{qi}$. In the transport equation for up-quark matter f_{q1} is f_{u1}. For the term with $\mathcal{M}_{u\bar{d}\to u\bar{d}}$, f_{q2}, f_{q3}, and f_{q4} are $f_{\bar{d}2}$, f_{u3}, and $f_{\bar{d}4}$, respectively. For the term with $\mathcal{M}_{ud\bar{u}\to ud\bar{u}}$, f_{q2}, f_{q3}, f_{q4}, f_{q5}, and f_{q6} are f_{d2}, $f_{\bar{u}3}$, f_{u4}, f_{d5}, and $f_{\bar{u}6}$, respectively. For the term with $\mathcal{M}_{gu\bar{d}\to gu\bar{d}}$ in the transport equation for gluon matter, f_{q2}, f_{q3}, f_{q5}, and f_{q6} are f_{u2}, $f_{\bar{d}3}$, f_{u5}, and $f_{\bar{d}6}$, respectively. Other terms in the transport equations can be understood in the same way. The transport equation for down-quark matter is obtained from the transport equation for up-quark matter by the replacement $u \leftrightarrow d$ and $\bar{u} \leftrightarrow \bar{d}$; the transport equation for up-antiquark matter by the replacement $u \leftrightarrow \bar{u}$ and $d \leftrightarrow \bar{d}$; the transport equation for down-antiquark matter by the replacement $u \leftrightarrow \bar{d}$ and $d \leftrightarrow \bar{u}$.

V. NUMERICAL RESULTS AND DISCUSSIONS

Quark-gluon matter initially produced in high-energy nucleus-nucleus collisions can be simulated by HIJING [8]. For central Au-Au collisions at $\sqrt{s_{NN}} = 200$ GeV parton distributions inside matter take the form [20]

$$f(k_\perp, y, r, z, t) = \frac{1}{16\pi R_A^2} g(k_\perp, y)\frac{e^{-(z-t\tanh y)^2/2\Delta_k^2}}{\sqrt{2\pi}\,\Delta_k}, \quad (37)$$

in which R_A, $k_\perp$, y, t, z, and r are the gold nucleus radius, transverse momentum, rapidity, time, coordinate in the longitudinal direction, and radius in the transverse direction, respectively. Δ_k and $g(k_\perp, y)$ are written as

$$\Delta_k \approx \frac{2}{k_\perp \cosh y}, \quad g(k_\perp, y) = \frac{(2\pi)^3}{k_\perp \cosh y} \frac{dN}{dy d^2 k_\perp}.$$

The gluon and the quark have different $dN/dyd^2k_\perp$. The distribution given by Eq. (37) is anisotropic because it has different dependence on $k_\perp$ and $\cosh y$. We have assumed in Sec. IV that the quark distribution is symmetric in flavor and identical with the antiquark distribution. Then 1500 gluons, 250 quarks, or 250 antiquarks of the up or down flavor are created from the distribution function in Eq. (37) within $-0.3 < z < 0.3$ fm and $r < R_A$ by use of the rejection method. This is an initial condition of the transport equations at $t = 0.2$ fm/c. The coupling constant in the squared amplitudes is taken to be $\alpha_s = g_s^2/4\pi = 0.5$ in solving the transport equations. The screening mass formulated in Refs. [21–23] is calculated from the gluon and quark distribution functions and is used to regularize propagators in the squared amplitudes for both the elastic two-to-two scattering and the elastic three-to-three scattering.

The 2500 partons are inhomogeneous in coordinate space and anisotropic in momentum space and scatter to have their momenta changed. Scattering of two partons occurs when the two partons have the closest distance less than the square root of the ratio of the cross section for two-to-two scattering to π. The cross section depends on the total energy of the two colliding partons in their center-of-momentum system and can be found in Ref. [10] for the elastic scattering of one gluon and one quark, one gluon and one antiquark, or two gluons and in Ref. [9] for one quark and one antiquark, two quarks, or two antiquarks, where the fraction 8/9 in Eq. (5) should be replaced by 4/9. Scattering of three partons occurs if the three partons are in a sphere of which the center is at the center-of-mass of the three partons and of which the radius $r_{\rm hs}$ is [7]

$$\pi r_{\rm hs}^2 = \frac{1}{n_{\rm f}!} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \frac{d^3p_5}{(2\pi)^3 2E_5} \frac{d^3p_6}{(2\pi)^3 2E_6} \times (2\pi)^4 \delta^4(p_1+p_2+p_3-p_4-p_5-p_6) |\mathcal{M}_{3\to3}|^2, \quad (38)$$

where $\mathcal{M}_{3\to3}$ is the amplitude for the three-parton scattering and $n_{\rm f}$ is the number of identical partons in the final states. For example, $\mathcal{M}_{3\to3}$ is $\mathcal{M}_{ud\bar{u}\to ud\bar{u}}$ for $ud\bar{u} \to ud\bar{u}$ or $\mathcal{M}_{ggu\to ggu}$ for $ggu \to ggu$; $n_{\rm f}$ equals 2 for $ggu \to ggu$, $ggd \to ggd, gg\bar{u} \to gg\bar{u}, gg\bar{d} \to gg\bar{d}, guu \to guu, gdd \to gdd$, $g\bar{u}\bar{u} \to g\bar{u}\bar{u}$, $g\bar{d}\bar{d} \to g\bar{d}\bar{d}$, $u\bar{u}\bar{u} \to u\bar{u}\bar{u}$, or $u\bar{d}\bar{d} \to u\bar{d}\bar{d}$.

Without scattering every parton moves in a straight line and the anisotropy of quark-gluon matter indicated by Eq. (37) remains. Due to scattering, partons' momenta are changed, and the anisotropy is removed. For example, at $t = 0.2$ fm/c the distribution of gluons in the transverse direction differs substantially from that in the longitudinal direction. But such a difference disappears after scattering in a short period of time. This is because the scattering alters the momentum distribution functions in the two directions so both become identical at a moment. Hence, we observe gluon momentum distribution functions in the three directions marked by the three angles, 0°, 45°, and 90°, relative to an incoming beam direction, and show them by use of the dotted, dashed, and dot-dashed curves in Fig. 7 at a time of the order of 0.52 fm/c. The curves overlapped are fitted to the Jüttner distribution shown by the solid curve,

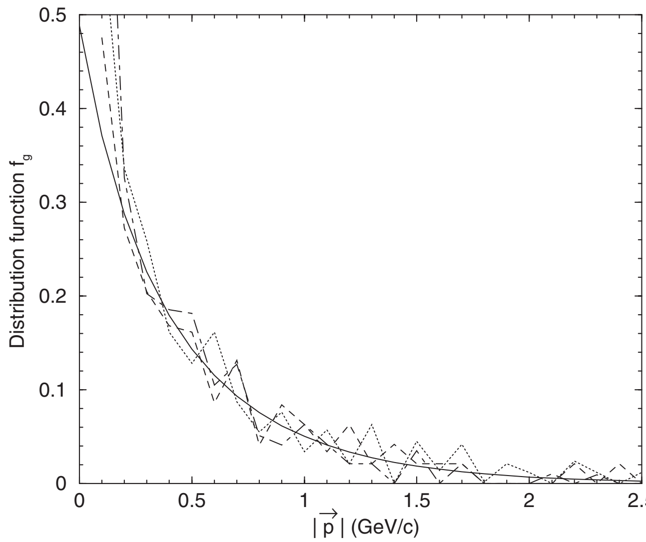

FIG. 7. Gluon distribution functions versus momentum in different directions while gluon matter arrives at a thermal state. The dotted, dashed, and dot-dashed curves correspond to the angles, 0°, 45°, and 90°, relative to one incoming beam direction, respectively. The solid curve represents the thermal distribution function. The thermalization time of gluon matter is 0.32 fm/c.

$$f_g(\vec{p}) = \frac{\lambda_g}{e^{|\vec{p}|/T} - \lambda_g}, \quad (39)$$

with the temperature $T = 0.52$ GeV and the fugacity $\lambda_g = 0.328$. Since such a thermal state of gluon matter with the temperature is established at $t = 0.52$ fm/c, the thermalization time of gluon matter is 0.32 fm/c. Nevertheless, we have to mention that, while the thermal state of gluon matter is established, quark matter has not been in a thermal state. This means that quark matter does not thermalize as quick as gluon matter. We find that the quark momentum distribution functions in the three directions first overlap at a time of the order of 0.86 fm/c, and they are shown in Fig. 8. The thermalization time of quark matter then is 0.66 fm/c. The overlapped curves are also fitted to the Jüttner distribution,

$$f_q(\vec{p}) = \frac{\lambda_q}{e^{|\vec{p}|/T} + \lambda_q}, \quad (40)$$

with $T = 0.46$ GeV and the fugacity $\lambda_q = 0.143$.

While the elastic gluon-gluon-quark scattering and the elastic gluon-gluon-antiquark scattering are absent, a thermalization time of the order of 0.48 fm/c was obtained for gluon matter in Ref. [10]. In the present work the elastic scattering of ggq and $gg\bar{q}$ shortens thermalization time of gluon matter to 0.32 fm/c. In finding the solutions of the transport equations, we know that most partons need to scatter one or two times. At least partons should scatter one time so the thermalization time has a lower limit and cannot be zero.

In Ref. [9] thermalization of quark matter and antiquark matter was studied with the elastic scattering of

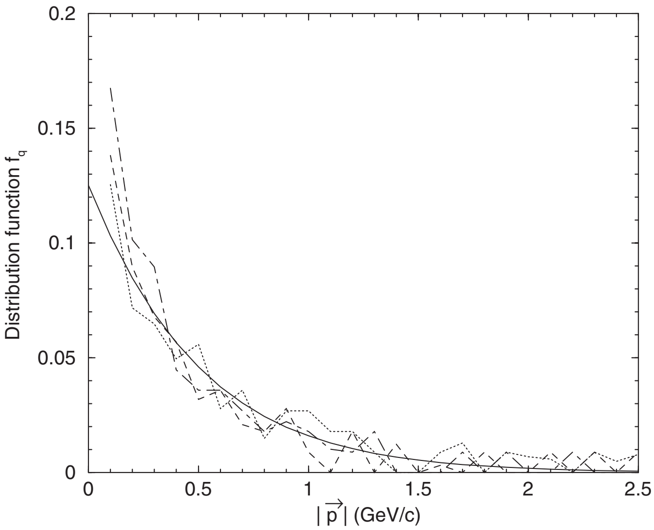

FIG. 8. Same as Fig. 7, except for quark distribution functions while quark matter arrives at a thermal state. The thermalization time of quark matter is 0.66 fm/c.

quark-quark, antiquark-antiquark, quark-antiquark, quark-quark-quark, antiquark-antiquark-antiquark, quark-quark-antiquark, and quark-antiquark-antiquark. The thermalization times of quark matter and antiquark matter were of the order of 1.55 fm/c. Furthermore, in Ref. [10] a thermalization time of the order of 1.36 fm/c was obtained for quark matter while the elastic scattering of gluon-quark-quark, gluon-antiquark-antiquark, and gluon-quark-antiquark is included, and the thermalization time was shortened by 0.19 fm/c. Relative to this thermalization time 1.36 fm/c, the thermalization time 0.66 fm/c in the present work means that the elastic scattering of ggq and $gg\bar{q}$ shortens the thermalization time by the amount 0.7 fm/c. Compared to 0.19 fm/c, the amount is large. This is due to two facts. One is that the gluon number density is 6 times the quark number density in a flavor; another is that the squared amplitude for the elastic gluon-gluon-quark scattering is an order of magnitude larger than that for the elastic gluon-quark-quark scattering.

The thermalization times of gluon matter and quark matter are 0.32 fm/c and 0.66 fm/c, respectively, which indicate early thermalization of both gluon matter and quark matter. Even though the thermalization times of gluon matter and quark matter are both less than 1 fm/c, they differ. The elastic two-gluon scattering and the elastic three-gluon scattering directly cause a change of gluon distribution and indirectly cause changes of quark distribution and antiquark distribution by the elastic scattering of gluon-quark, gluon-antiquark, gluon-quark-quark, gluon-quark-antiquark, gluon-antiquark-antiquark, gluon-gluon-quark, and gluon-gluon-antiquark. The elastic two-body scattering and the elastic three-body scattering inside quark and antiquark matter directly cause changes of quark distribution and antiquark distribution and indirectly cause a change of gluon distribution by the elastic scattering of gluon-quark, gluon-antiquark, gluon-quark-quark, gluon-quark-antiquark, gluon-antiquark-antiquark, gluon-gluon-quark, and gluon-gluon-antiquark. In Ref. [10] we have shown that the elastic scattering of gluon-quark, gluon-antiquark, gluon-quark-quark, gluon-quark-antiquark, and gluon-antiquark-antiquark gives similar contributions to the variation of the gluon distribution function and the variation of the quark (antiquark) distribution function. In the present work $\frac{g_G g_Q}{2}(|\mathcal{M}_{ggu\to ggu}|^2 + |\mathcal{M}_{ggd\to ggd}|^2 + |\mathcal{M}_{gg\bar{u}\to gg\bar{u}}|^2 + |\mathcal{M}_{gg\bar{d}\to gg\bar{d}}|^2)$ in Eq. (35) is three times $\frac{g_G^2}{4}|\mathcal{M}_{ugg\to ugg}|^2$ in Eq. (36). The gluon distribution function f_{gi} is roughly twice the quark distribution function f_{qi}. The variation of the gluon distribution function then is roughly three times the variation of the quark (antiquark) distribution function. Now we can understand the difference of the thermalization times of gluon matter and quark matter from the facts: gluon matter is denser than quark matter, the elastic gluon-gluon (gluon-gluon-gluon) scattering has a larger squared amplitude than the elastic quark-quark or quark-antiquark (quark-quark-quark or quark-quark-antiquark) scattering, and the contribution of the elastic gluon-gluon-quark scattering to the variation of the gluon distribution function is larger than the contribution to the variation of the quark distribution function.

Let us consider one unrealistic case, but it can help us better understand the difference between the thermalization time of gluon matter and that of quark matter. The case is that the quark number equals the gluon number 1500. The thermalization time of quark matter then is reduced by the increase of quark number from 500 to 1500, and the quark distribution function f_{qi} is roughly the gluon distribution function f_{gi}. The difference between the variation of the gluon distribution and the variation of the quark distribution is mainly determined by the spin-color degeneracy factors and the squared amplitudes as shown in Eqs. (35) and (36). We calculate the squared amplitudes for various types of elastic scattering and obtain the following results: $g_Q(|\mathcal{M}_{gu\to gu}|^2 + |\mathcal{M}_{gd\to gd}|^2 + |\mathcal{M}_{g\bar{u}\to g\bar{u}}|^2 + |\mathcal{M}_{g\bar{d}\to g\bar{d}}|^2)$ in Eq. (35) is near $g_G|\mathcal{M}_{ug\to ug}|^2$ in Eq. (36); $g_Q^2[\frac{1}{4}|\mathcal{M}_{guu\to guu}|^2 + \frac{1}{2}(|\mathcal{M}_{gud\to gud}|^2 + |\mathcal{M}_{gdu\to gdu}|^2) + \frac{1}{4}|\mathcal{M}_{gdd\to gdd}|^2 + |\mathcal{M}_{gu\bar{u}\to gu\bar{u}}|^2 + |\mathcal{M}_{gu\bar{d}\to gu\bar{d}}|^2 + |\mathcal{M}_{gd\bar{u}\to gd\bar{u}}|^2 + |\mathcal{M}_{gd\bar{d}\to gd\bar{d}}|^2 + \frac{1}{4}|\mathcal{M}_{g\bar{u}\bar{u}\to g\bar{u}\bar{u}}|^2 + \frac{1}{2}(|\mathcal{M}_{g\bar{u}\bar{d}\to g\bar{u}\bar{d}}|^2 + |\mathcal{M}_{g\bar{d}\bar{u}\to g\bar{d}\bar{u}}|^2) + \frac{1}{4}|\mathcal{M}_{g\bar{d}\bar{d}\to g\bar{d}\bar{d}}|^2]$ is near $g_Q g_G(\frac{1}{2}|\mathcal{M}_{uug\to uug}|^2 + |\mathcal{M}_{udg\to udg}|^2 + |\mathcal{M}_{u\bar{u}g\to u\bar{u}g}|^2 + |\mathcal{M}_{u\bar{d}g\to u\bar{d}g}|^2)$; $\frac{g_G g_Q}{2}(|\mathcal{M}_{ggu\to ggu}|^2 + |\mathcal{M}_{ggd\to ggd}|^2 + |\mathcal{M}_{gg\bar{u}\to gg\bar{u}}|^2 + |\mathcal{M}_{gg\bar{d}\to gg\bar{d}}|^2)$ is three times $\frac{g_G^2}{4}|\mathcal{M}_{ugg\to ugg}|^2$; $\frac{g_G}{2}|\mathcal{M}_{gg\to gg}|^2$ is over three times $g_Q(\frac{1}{2}|\mathcal{M}_{uu\to uu}|^2 + |\mathcal{M}_{ud\to ud}|^2 + |\mathcal{M}_{u\bar{u}\to u\bar{u}}|^2 + |\mathcal{M}_{u\bar{d}\to u\bar{d}}|^2)$; $\frac{g_G^2}{12}|\mathcal{M}_{ggg\to ggg}|^2$ is an order of magnitude larger than $g_Q^2[\frac{1}{12}|\mathcal{M}_{uuu\to uuu}|^2 + \frac{1}{4}(|\mathcal{M}_{uud\to uud}|^2 + |\mathcal{M}_{udu\to udu}|^2) + \frac{1}{4}|\mathcal{M}_{udd\to udd}|^2 + \frac{1}{2}|\mathcal{M}_{uu\bar{u}\to uu\bar{u}}|^2 + \frac{1}{2}|\mathcal{M}_{uu\bar{d}\to uu\bar{d}}|^2 + |\mathcal{M}_{ud\bar{u}\to ud\bar{u}}|^2 + |\mathcal{M}_{ud\bar{d}\to ud\bar{d}}|^2 + \frac{1}{4}|\mathcal{M}_{u\bar{u}\bar{u}\to u\bar{u}\bar{u}}|^2 + \frac{1}{2}(|\mathcal{M}_{u\bar{u}\bar{d}\to u\bar{u}\bar{d}}|^2 + |\mathcal{M}_{u\bar{d}\bar{u}\to u\bar{d}\bar{u}}|^2) + \frac{1}{4}|\mathcal{M}_{u\bar{d}\bar{d}\to u\bar{d}\bar{d}}|^2]$. Hence, the difference between the variation of the gluon distribution and the variation of the quark distribution is mainly caused by the elastic scattering of gluon-gluon, gluon-gluon-gluon, quark-quark, quark-antiquark, quark-quark-quark, quark-quark-antiquark, quark-antiquark-antiquark, gluon-gluon-quark, and gluon-gluon-antiquark, and the difference between the thermalization time of gluon matter

TABLE I. n, $\bar{\lambda}$, and $k_{g\perp\min}$ at the three times which correspond to the formation of quark-gluon matter, the thermal state of gluon matter, and the thermal state of quark matter, respectively.

t (fm/c)	0.2	0.52	0.86
n (fm^{-3})	32.4	17	10.8
$\bar{\lambda}$ (fm)	0.073	0.139	0.218
$k_{g\perp\min}$ (GeV/c)	2.7	1.42	0.91

and that of quark matter certainly exists. In the case that the number of quarks equals the number of gluons, the thermalization time of quark matter approaches that of gluon matter, but quark matter still cannot thermalize faster than gluon matter.

The gluon multiplication process $gg \to ggg$ was used by Biró *et al.* to study chemical equilibration of quark-gluon plasmas [24], and such a process also contributes to the thermalization of quark-gluon matter [13,14]. In Ref. [20] the Landau-Pomeranchuk-Migdal (LPM) effect is taken into account to suppress the gluon radiation in $gg \to ggg$. The LPM effect appears while the radiation formation time is long compared to the mean free path. The effective formation time in QCD is [25]

$$\tau_{\rm QCD} = \frac{C_A}{2C_2}\frac{2\cosh y_g}{k_{g\perp}}, \tag{41}$$

where $C_2 = C_A = 3$ for a gluon and $k_{g\perp}$ and y_g are the transverse momentum and the rapidity of the radiated gluon, respectively. The larger the absolute value of the rapidity, or the smaller the transverse momentum, the longer the effective formation time. At $y_g = 0$, $\tau_{\rm QCD}$ equal to the mean free path $\bar{\lambda}$ gives the minimum of the transverse momentum,

$$k_{g\perp\min} = \frac{1}{\bar{\lambda}}, \tag{42}$$

over which the LPM effect turns up. The mean free path is given by

$$\bar{\lambda} = \frac{1}{\sqrt{2}n\sigma_2}, \tag{43}$$

where n is the quark-gluon number density and σ_2 is the cross section for two-parton scattering. While n or σ_2 increases, the mean free path decreases. We assume that σ_2 is 3 mb, which is smaller than the value 5 mb in Ref. [14]. Quark-gluon matter is produced at the time 0.2 fm/c, and gluon matter and quark matter arrive at their thermal states at 0.52 fm/c and 0.86 fm/c, respectively. In Table I we list the number density, the mean free path, and the minimum of the transverse momentum at the three times. $k_{g\perp\min}$ is 2.7 GeV/c at $t = 0.2$ fm/c, and the gluon radiation is highly suppressed. From $t = 0.2$ fm/c to 0.86 fm/c the minimum of the transverse momentum decreases, and $gg \to ggg$ is gradually involved in thermalization. Thermalization mainly relates to the distribution functions in the momentum region $|\vec{p}| < 1.5$ GeV/c. $k_{g\perp\min}$ is 1.42 GeV/c at 0.52 fm/c. The influence of $gg \to ggg$ on the gluon distribution for $|\vec{p}| < 1.5$ GeV/c should be very limited, so should the influence on the early thermalization of gluon matter. $k_{g\perp\min}$ is 0.91 GeV/c at 0.86 fm/c. Since the radiated gluon will scatter with quarks inside quark-gluon matter, the quark distribution and the early thermalization of quark matter are expected to be moderately affected. Therefore, the gluon multiplication process may slightly modify the thermalization time of gluon matter and may moderately modify the thermalization time of quark matter. The absence of the process is reliable in the early thermalization of gluon matter but approximate in the early thermalization of quark matter.

VI. SUMMARY

We have derived the squared amplitude for the elastic gluon-gluon-quark scattering in perturbative QCD. One hundred twenty-three Feynman diagrams at the tree level are introduced, and a method is presented to derive the individually squared amplitudes and the interference terms. With the squared amplitudes for the elastic ggq scattering and the elastic $gg\bar{q}$ scattering, the transport equations for gluons, up quarks, down quarks, up antiquarks and down antiquarks are solved to get thermal states and thermalization times of gluon matter, quark matter and antiquark matter which are initially produced in central Au-Au collisions at the highest RHIC energy and of which the evolution is governed by elastic two-to-two scattering and elastic three-to-three scattering. Early thermalization of initially produced quark-gluon matter is indicated by the short thermalization times of gluon matter and quark matter. However, the thermalization time of gluon matter differs from that of quark matter. So far, we have answered how and when initially produced quark-gluon matter establishes a thermal state.

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