Rotational constants of multi-phonon bands in an effective theory for deformed nuclei

Jialin Zhang[1] and T. Papenbrock[1,2]
[1]*Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee 37996, USA*
[2]*Physics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831, USA*
(Received 18 February 2013; revised manuscript received 13 March 2013; published 25 March 2013)

We consider deformed nuclei within an effective theory that exploits the small ratio between rotational and vibrational excitations. For even-even nuclei, the effective theory predicts small changes in the rotational constants of bands built on multi-phonon excitations that are linear in the number of excited phonons. In ^{166,168}Er, this explains the main variations of the rotational constants of the two-phonon γ vibrational bands. In ^{232}Th, the effective theory correctly explains the trend that the rotational constants decrease with increasing spin of the bandhead. We also study the effective theory for deformed odd nuclei. Here, time-odd terms enter the Lagrangian and generate effective magnetic forces that yield the high level densities observed in such nuclei.

DOI: 10.1103/PhysRevC.87.034323 PACS number(s): 21.60.Ev, 21.10.Re, 03.65.Vf

I. INTRODUCTION

Deformed nuclei exhibit rotational bands as their lowest excitations, with actinides and rare earth nuclei being the most prominent and best studied examples [1,2]. The theoretical description and understanding of these nuclei largely rests on the Bohr Hamiltonian [3], the collective model by Bohr and Mottelson [4,5], its extension within the general geometric models [6–9], and algebraic models [10,11]. For even-even nuclei the geometrical models employ rotations and shape parameters as the relevant degrees of freedom, while algebraic models utilize bosonic degrees of freedom. The theoretical approach to odd-mass nuclei is more cumbersome and is based on coupling the odd nucleon to an even-even nucleus [12,13]. More microscopic approaches to deformed nuclei can be based on mean-field calculations [14,15] and shell-model studies [16]. Being solidly based on fermionic degrees of freedom, the microscopic models can properly illuminate interesting phenomena such as, e.g., the effect of pairing on nuclear moments of inertia [17–21].

The collective models are particularly successful in certain symmetry limits of the Hamiltonian (or for certain choices of the potential energy) where analytical solutions are available. Away from these limits, generalizations of collective models employ expansions of kinetic and potential terms, or expansions in the number of boson operators. Such approaches can be systematic but lack a power counting; i.e., higher-order terms in the Hamiltonian are not guaranteed to yield smaller contributions than low-order terms. This difficulty compounds the adjustment of model parameters [9]. Recently, a computationally tractable approach to the collective model was proposed by Rowe [22] and applied to the Bohr model [23]. Some of the challenges in the theory of deformed nuclei are described in Ref. [24].

An alternative approach to deformed nuclei can be formulated as an effective theory [25]. This approach employs degrees of freedom similar to those of the Bohr Hamiltonian, and its highlights are the nonlinear realization of rotational symmetry (as a consequence of the spontaneous symmetry breaking associated with nuclear deformation) and a power counting. It is thus similar in spirit to other effective field theories [26,27] that have been employed to describe nuclear interactions [28–31], halo nuclei [32,33], and dilute Fermi systems [34–37].

At next-to-leading order, the effective theory for deformed even-even nuclei yields spectra that agree (to this order) with those from the Bohr Hamiltonian; i.e., vibrational states serve as bandheads of rotational bands, with all bands exhibiting the same moment of inertia [25]. However, the phenomenology is richer and more complicated. Deformed nuclei typically exhibit small variations in the rotational constants of individual bands, and accounting for the observation [1] that rotational constants decrease with increasing energy of the bandhead is a longstanding problem for the traditional collective models for well-deformed [23,38–44] and transitional nuclei [45–47]. To address this problem, we extend the effective theory of deformed nuclei to next-to-next-to-leading order.

Another interesting problem concerns deformed odd-mass nuclei. Though they account for half of all deformed nuclei, our understanding of them is much more limited, and the theoretical approach is more complicated than is the case for even-even nuclei. Within the collective models such nuclei are described by coupling a nucleon to an even-even nucleus [7,12] or within boson-fermion models [13]. The presence of the odd fermion compounds the description of odd-mass nuclei considerably. The question thus arises whether the odd nucleon really is a degree of freedom that is relevant at low energies or to what extent collective vibrations and rotations alone are sufficient to describe low-energy phenomena of odd nuclei. In this paper, we will address this question by constructing the effective theory for deformed odd-mass nuclei at next-to-leading order.

This paper is organized as follows. Section II introduces the effective theory for deformed nuclei. In Sec. III we derive the couplings between rotations and vibrations at next-to-next-to leading order, and we compute the resulting spectrum. We confront theory with data in Sec. IV. Section V extends the effective theory for odd-mass nuclei to next-to-leading order. Finally, a summary of our results is presented in Sec. VI.

II. EFFECTIVE THEORY FOR DEFORMED NUCLEI

An effective theory for deformed nuclei with axial symmetry was derived in Ref. [25]. Here we summarize the essential ingredients of the theory and contrast it to the collective model.

The effective theory is based on quadrupole degrees of freedom $\phi_\mu(t)$, $\mu = -2, -1, \ldots, 2$, because these are sufficient to reproduce the spins and parities of low-lying states in even-even nuclei. The reality condition $\phi_{-\mu} = (-1)^\mu \phi_\mu^*$ expresses invariance under time reversal and implies that we deal with five real degrees of freedom. We assume the spontaneous breaking of rotational symmetry and a nonzero expectation value $\langle \phi_0 \rangle = v > 0$. This implies the existence of two Nambu-Goldstone modes, which may be chosen as the Euler angles $\alpha(t)$ and $\beta(t)$ that change the orientation of the axially symmetric nucleus. The three remaining degrees of freedom are chosen as the complex "field" $\phi_2(t)$ and the real "field" $\phi_0(t)$:

$$\phi = \begin{pmatrix} \phi_2 \\ 0 \\ \phi_0 \\ 0 \\ \phi_{-2} \end{pmatrix}. \tag{1}$$

Thus, the complex "field" $\phi_1(t)$ is replaced by the two Nambu-Goldstone bosons. This is consistent with the choice of ϕ_0 having a nonzero expectation value v [48]: An infinitesimal rotation of the configuration with components $\phi_\mu = v\delta_{0\mu}$ will generate nonzero components $\phi_{\pm 1}$. It is convenient to rewrite ϕ_0 in terms of its vacuum expectation value v and a small fluctuating part φ_0 as

$$\phi_0(t) = v + \varphi_0(t). \tag{2}$$

We must assume that $|\varphi_0| \ll v$ because of the spontaneous breaking of rotational symmetry.

Due to the spontaneous symmetry breaking, the rotational symmetry is realized nonlinearly, and quantities with proper transformation properties are

$$E_x = \dot{\alpha} \sin\beta, \quad E_y = -\dot{\beta}. \tag{3}$$

Under a general rotation by the Euler angles $(\varphi_1, \varphi_2, \varphi_3)$, the quantities E_x and E_y transform as the x and y components, respectively, of a vector under a rotation around the z axis by a complicated angle $\eta(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta)$. The exact transformation is of no interest here but can be found in Ref. [25]. Thus, the linear combinations

$$E_\pm = E_x \mp i E_y \tag{4}$$

transform under a rotation as $E_\pm \to e^{\mp i\eta} E_\pm$.

Likewise, the quadrupole fields transform as $\phi_\mu \to e^{-i\mu\eta}\phi_\mu$ under a rotation. The covariant derivative

$$D_t \equiv \partial_t - i E_z J_z, \tag{5}$$

with

$$E_z = -\dot{\alpha}\cos\beta, \tag{6}$$

is invariant under rotations because E_z transforms as a gauge field. Here, J_z is the z component of the angular momentum, i.e., $J_z E_\pm = \pm E_\pm$ and $J_z \phi_\mu = \mu\phi_\mu$.

Due to the nonlinear realization of rotational symmetry, any Lagrangian that consists of $E_\pm$, $\phi_{\pm 2}$, ϕ_0, D_t and is formally invariant under axial [i.e., SO(2)] symmetry is indeed invariant under full rotational [i.e., SO(3)] symmetry.

For the systematic construction of Lagrangians one needs to establish a power counting. We denote the energy scale of rotational excitations as ξ and that of vibrational excitations as Ω. One has $\xi \ll \Omega$ with typical values of $\xi \approx 100$ keV and $\Omega \approx 1$ MeV in rare earth nuclei. For actinides, the typical values for ξ are smaller by about a factor of 2. We also have to identify a breakdown scale Λ of our effective theory. The complete spectroscopy of low-lying levels in deformed nuclei has been reported for ^{168}Er [1] and ^{162}Dy [2]. The existence of negative-parity bands in these nuclei (which would require the introduction of octupole degrees of freedom) and the absence of clear signatures for multi-phonon vibrations indicate that $\Lambda = \kappa\Omega$ with $\kappa \approx 2$ or 3. For the quantities introduced so far the power counting is

$$\begin{aligned} E_\pm \sim E_z \sim \xi, \quad D_t\phi_0 \sim D_t\phi_2 \sim \Omega^{1/2}, \\ \varphi_0 \sim \phi_2 \sim \Omega^{-1/2}, \quad \phi_0 \sim v \sim \xi^{-1/2}. \end{aligned} \tag{7}$$

This power counting is based on the following rationale: The angles α and β are dimensionless, and a time derivative of these fields (as in $E_\pm$ and E_z) must scale as the low-energy scale ξ. Likewise, a time derivative on the field ϕ must scale as Ω, and the scaling of the fields ϕ_2, φ_0 itself ensures that the kinetic term $(D_t\phi)^2$ scales as Ω. Finally, the expectation value v is associated with the spontaneous symmetry breaking and must thus scale as $\xi^{-1/2}$. In an infinite system, we would have $\xi \to 0$, correctly implying both the divergence of the vacuum expectation value v and zero-energy Nambu-Goldstone modes.

Let us briefly recapitulate the effective theory for deformed nuclei at next-to-leading order for even-even nuclei [25]. At leading order (LO), i.e., at order Ω, we have only vibrations, and we note that

$$(D_t\phi_2)(D_t\phi_{-2}) = \dot{\phi}_2\dot{\phi}_2^* - 4\mathrm{Im}(\dot{\phi}_2\phi_2^*)E_z + 4\phi_2\phi_2^* E_z^2 \tag{8}$$

consists of three terms that are suppressed by subsequent factors of ξ/Ω when going from left to right.

The Lagrangian at LO is

$$L_{\mathrm{LO}} = \frac{1}{2}\dot{\phi}_0^2 + \dot{\phi}_2\dot{\phi}_{-2} - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}\phi_2\phi_{-2}. \tag{9}$$

Here, we assume that $\omega_0 \sim \omega_2 \sim \Omega$. We use $\phi_2 = \varphi_2 e^{i\gamma}$ with real φ_2 and γ, and perform the Legendre transformation

$$p_0 = \frac{\partial L_{\mathrm{LO}}}{\partial\dot{\varphi}_0}, \quad p_2 = \frac{\partial L_{\mathrm{LO}}}{\partial\dot{\varphi}_2}, \quad p_\gamma = \frac{\partial L_{\mathrm{LO}}}{\partial\dot{\gamma}}. \tag{10}$$

The Hamiltonian is

$$H_{\mathrm{LO}} = \frac{p_0^2}{2} + \frac{\omega_0^2}{2}\varphi_0^2 + \frac{1}{4}\left(p_2^2 + \frac{p_\gamma^2}{\varphi_2^2}\right) + \frac{\omega_2^2}{4}\varphi_2^2, \tag{11}$$

and the spectrum is thus equal to that of an axially symmetric harmonic oscillator in three spatial dimensions with energies

$$E_{\mathrm{LO}}(n_0, n_2, l_2) = \omega_0(n_0 + 1/2) + \frac{\omega_2}{2}(2n_2 + |l_2| + 1). \tag{12}$$

With view on the breakdown scale Λ of the effective theory, we limit ourselves to the ground state with quantum numbers

$(n_0, n_2, l_2) = (0, 0, 0)$ and the two lowest vibrational states with quantum numbers (1, 0, 0) and (0, 0, 1), respectively. The eigenfunctions are products

$$\Psi_{\rm LO}(\gamma, \varphi_0, \varphi_2) = e^{-il_2\gamma}\psi_{n_0}(\varphi_0)\chi_{n_2 l_2}(\varphi_2). \tag{13}$$

Here, $\psi_{n_0}(\varphi_0)$ is the eigenfunction of the one-dimensional harmonic oscillator with frequency ω_0, while $\chi_{n_2 l_2}(\varphi_2)$ is the radial eigenfunction of the two-dimensional isotropic oscillator with frequency ω_2.

At next-to-leading order (NLO), the Nambu-Goldstone modes enter in addition to higher-order corrections in the kinetic energy (8), and the Lagrangian becomes

$$\begin{aligned} L_{\rm NLO} &= L_{\rm LO} + \Delta L_{\rm NLO}, \\ \Delta L_{\rm NLO} &= \frac{C_0}{2}E_+E_- - 4{\rm Im}(\dot\phi_2\phi_2^*)E_z \\ &= \frac{C_0}{2}(\dot\beta^2 + \dot\alpha^2\sin^2\beta) + 4\varphi_2^2\dot\gamma\dot\alpha\cos\beta. \end{aligned} \tag{14}$$

Here, we assume that $C_0 \sim \xi^{-1}$, and the NLO correction is thus of order ξ. Note that we neglected next-to-leading order corrections ("anharmonicities") to the vibrational potential. Such anharmonicities would affect higher-lying vibrational states (which are at or beyond the breakdown scale Λ of the effective theory) and transition matrix elements (which are not the interest of this work). The Hamiltonian at NLO thus becomes

$$\begin{aligned} H_{\rm NLO} = &\frac{1}{2}p_0^2 + \frac{1}{4}p_2^2 + \frac{p_\gamma^2}{4\varphi_2^2} + \frac{\omega_0^2}{2}\varphi_0^2 + \frac{\omega_2^2}{4}\varphi_2^2 \\ &+ \frac{1}{2C_0}\left(p_\beta^2 + \frac{1}{\sin^2\beta}(p_\alpha - 2p_\gamma\cos\beta)^2\right). \end{aligned} \tag{15}$$

The corresponding energy spectrum is

$$E_{\rm NLO}(n_0, n_2, l_2, I) = E_{\rm LO}(n_0, n_2, l_2) + \frac{I(I+1) - (2l_2)^2}{2C_0}, \tag{16}$$

and the eigenfunctions are

$$\Psi_{\rm NLO}(\alpha, \beta, \gamma, \varphi_0, \varphi_2) = e^{-im\alpha}d^I_{m,2l_2}(\beta)\Psi_{\rm LO}(\gamma, \varphi_0, \varphi_2). \tag{17}$$

Here, $I \geqslant |2l_2|$ denotes the angular momentum, and m is the angular-momentum projection with $-I \leqslant m \leqslant I$. The eigenfunction $d^I_{\mu,\nu}(\beta)$ is part of the Wigner D function $D^I_{\mu,\nu}(\alpha, \beta, \gamma) = e^{-i\mu\alpha}d^I_{\mu,\nu}(\beta)e^{-i\nu\gamma}$. Thus, we can rewrite

$$\Psi_{\rm NLO}(\alpha, \beta, \gamma, \varphi_0, \varphi_2) = D^I_{m,2l_2}(\alpha, \beta, \gamma)\psi_{n_0}(\varphi_0)\chi_{n_2 l_2}(\varphi_2). \tag{18}$$

The spectrum (16) consists of rotational bands (labeled by the angular momentum I) on top of the vibrational bandheads (labeled by the quantum numbers n_0, n_2, l_2). Note that the moment of inertia C_0 is identical for every rotational band.

Let us also compare the effective theory with the Bohr model. Recall that the Bohr model starts from five quadrupole degrees of freedom, and a transformation to the body-fixed coordinate system yields three Euler angles and two shape parameters (usually denoted as β and γ). The β degree of freedom corresponds to axially symmetric oscillations around the static deformation while γ accounts for triaxial deformations. In the Bohr Hamiltonian, the vibrational and rotational degrees of freedom are coupled via the moment of inertia, while the effective theory is less constrained. Bohr's β degree of freedom corresponds to φ_0 in the effective theory. One can combine Bohr's γ degree of freedom and Bohr's rotational angle ψ into a two-dimensional harmonic oscillator [49]. In this combination, these two degrees of freedom correspond to the complex ϕ_2 (or φ_2 and γ) in the effective theory. Let us introduce

$$K \equiv 2l_2 \tag{19}$$

for the third quantum number of the axially symmetric rotor. With this notation, the effective theory at NLO is in agreement with the spectra and wave functions obtained for the collective model (cf. Chapter 6 of Ref. [7]). This agreement is expected.

III. EVEN-EVEN NUCLEI AT NEXT-TO-NEXT-TO-LEADING ORDER

At next-to-next-to-leading order (NNLO) we have to include terms of the size ξ^2/Ω. As before, we focus on the terms that couple rotations and vibrations. This is perhaps one of the main differences between the collective model and the effective theory. In the former, most authors have restricted themselves to study higher-order corrections to the vibrational potential. This is presumably due to the difficulty in writing down (and working with) higher-order corrections to the kinetic terms. In the effective theory, this task is straightforward and yields [25]

$$L_{\rm NNLO} = L_{\rm NLO} + 4\phi_2\phi_2^*E_z^2 + \Delta L_{\rm NNLO}, \tag{20}$$

$$\begin{aligned} \Delta L_{\rm NNLO} = &D_0(E_+E_-)\varphi_0^2 + F_0(E_+E_-)\dot\varphi_0{}^2 \\ &+ D_2(E_+E_-)|\phi_2|^2 + F_2(E_+E_-)|D_t\phi_2|^2 \\ &+ D_1\varphi_0(\phi_2E_-^2 + \phi_{-2}E_+^2) \\ &+ F_1\dot\varphi_0(E_+^2D_t\phi_{-2} + E_-^2D_t\phi_{+2}). \end{aligned} \tag{21}$$

Here, $\Delta L_{\rm NNLO}$ denotes the rotation-vibration interaction at NNLO. Each term in $\Delta L_{\rm NNLO}$ has order of magnitude $O(\xi^2/\Omega)$, making the undetermined coefficients scale as

$$D_0 \sim D_1 \sim D_2 \sim \mathcal{O}(1), \quad F_0 \sim F_1 \sim F_2 \sim \Omega^{-2}. \tag{22}$$

The correctness of these scaling relations should be validated by fitting the derived spectrum to the experimental level schemes.

The Lagrangian $L_{\rm NNLO}$ expanded in terms of the polar coordinates φ_2 and γ and the Euler angles α and β is

$$\begin{aligned} L_{\rm NNLO} = &\frac{1}{2}\dot\varphi_0^2 + \dot\varphi_2^2 + \varphi_2^2\dot\gamma^2 - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}\varphi_2^2 \\ &+ 4\varphi_2^2(\dot\gamma + \dot\alpha\cos\beta)\dot\alpha\cos\beta \\ &+ \frac{C_0}{2}(\dot\beta^2 + \dot\alpha^2\sin^2\beta) + \Delta L_{\rm NNLO}, \end{aligned} \tag{23}$$

with

$$\begin{aligned}\Delta L_{\rm NNLO} = (\dot\beta^2 + \dot\alpha^2 \sin^2\beta)\big[D_0\varphi_0^2 + F_0\dot\varphi_0{}^2 \\ + D_2\varphi_2^2 + F_2\big(\dot\varphi_2{}^2 + \varphi_2^2\dot\gamma^2\big)\big] \\ + 2(\dot\alpha^2\sin^2\beta - \dot\beta^2)[D_1\varphi_0\varphi_2\cos\gamma \\ + F_1\dot\varphi_0(\dot\varphi_2\cos\gamma - \varphi_2\dot\gamma\sin\gamma)] \\ + 4\dot\alpha\dot\beta\sin\beta[D_1\varphi_0\varphi_2\sin\gamma \\ + F_1\dot\varphi_0(\dot\varphi_2\sin\gamma + \varphi_2\dot\gamma\cos\gamma)].\end{aligned} \tag{24}$$

It is difficult to perform the Legendre transformation rigorously on $L_{\rm NNLO}$, because $\Delta L_{\rm NNLO}$ admixes the Nambu-Goldstone modes and quadrupole fields and the velocity-momentum inversions always involve quadratic terms. Fortunately, we do not need to perform the Legendre transformation of the Lagrangian (23) exactly but rather can employ perturbation theory for this task.

For this purpose we follow Fukuda and co-workers [50] who applied perturbative Legendre transformations to several physics problems [51,52]. Fukuda's inversion method expands the generalized velocities perturbatively order by order in the small quantity ξ/Ω. For instance, $\dot\varphi_0$ is expanded as

$$\dot\varphi_0 = \dot\varphi_0^{(0)} + \dot\varphi_0^{(1)} + \dot\varphi_0^{(2)} + \cdots. \tag{25}$$

Here, $\dot\varphi_0{}^{(0)}$ has the same order of magnitude as $\dot\varphi_0$ and is of leading order. Higher-order corrections scale as

$$\dot\varphi_0^{(i+1)} \sim \dot\varphi_0^{(i)}\frac{\xi}{\Omega}. \tag{26}$$

The key step consists of assuming the generalized momenta to be of leading order (and with no further corrections). Thus, the leading-order relation between the momenta and velocities of the Lagrangian (23) is

$$\begin{aligned}&p_0 = \dot\varphi_0^{(0)}, \quad p_2 = 2\dot\varphi_2^{(0)}, \quad p_\gamma = 2\varphi_2^2\dot\gamma^{(0)}, \\ &p_\alpha = C_0\dot\alpha^{(0)}\sin^2\beta + 4\varphi_2^2\dot\gamma^{(0)}\cos\beta, \quad p_\beta = C_0\dot\beta^{(0)}.\end{aligned} \tag{27}$$

It is straightforward to invert these equations. The higher-order corrections of the velocities now fulfill homogeneous equations (as the momenta consist only of leading-order terms) and can be solved perturbatively to the desired order. In what follows, we only present the result of the Legendre transformation of the Lagrangian given by Eq. (23) using Fukuda's inversion method, and we refer the reader to Ref. [52] for more details.

The Legendre transformation yields the Hamiltonian

$$H_{\rm NNLO} = H_{\rm NLO} - \Delta L^{(0)}_{\rm NNLO}. \tag{28}$$

Here $H_{\rm NLO}$ is the NLO Hamiltonian given in Eq. (15), and the term $\Delta L^{(0)}_{\rm NNLO}$ is from Eq. (24) with all leading-order velocities re-expressed in terms of momenta (27) and all higher-order velocities dropped in this term.

The eigenvalues of $H_{\rm NLO}$ are given in Eq. (16) and the small contribution of $\Delta L^{(0)}_{\rm NNLO}$ to the spectrum can be worked out in perturbation theory by computing the expectation value of $\Delta L^{(0)}_{\rm NNLO}$ in the eigenstates (18) of the Hamiltonian (15). For computation of the expectation value $\langle(\dot\alpha^{(0)})^2\sin^2\beta + (\dot\beta^{(0)})^2\rangle$ we note that

$$\begin{aligned}&(\dot\alpha^{(0)})^2\sin^2\beta + (\dot\beta^{(0)})^2 \\ &= \frac{1}{C_0^2}\left(\frac{1}{\sin^2\beta}(p_\alpha - 2p_\gamma\cos\beta)^2 + p_\beta^2\right) \\ &= \frac{1}{C_0^2}[I(I+1) - (2l_2)^2].\end{aligned} \tag{29}$$

For the expectation values involving the quadrupole vibrations we have

$$\begin{aligned}\langle\varphi_0\rangle &= \langle\dot\varphi_0^{(0)}\rangle = 0, \\ \langle\varphi_0^2\rangle &= \frac{1}{\omega_0}\left(n_0 + \frac{1}{2}\right), \\ \langle(\dot\varphi_0^{(0)})^2\rangle &= \omega_0\left(n_0 + \frac{1}{2}\right), \\ \langle\varphi_2^2\rangle &= \frac{1}{\omega_2}(2n_2 + |l_2| + 1), \\ \langle(\dot\varphi_2{}^{(0)})^2 + \varphi_2^2(\dot\gamma^{(0)})^2\rangle &= \frac{\omega_2}{4}(2n_2 + |l_2| + 1).\end{aligned} \tag{30}$$

Hence, we find

$$\begin{aligned}\langle\Delta L^{(0)}_{\rm NNLO}\rangle = \frac{I(I+1) - (2l_2)^2}{2C_0} \\ \times\left[\left(n_0 + \frac{1}{2}\right)R + (2n_2 + |l_2| + 1)S\right].\end{aligned} \tag{31}$$

Here, we used the shorthand notation

$$R \equiv \frac{2}{C_0}\left(\frac{D_0}{\omega_0} + F_0\omega_0\right), \quad S \equiv \frac{2}{C_0}\left(\frac{D_2}{\omega_2} + \frac{1}{4}F_2\omega_2\right). \tag{32}$$

Thus, the next-to-next-to-leading order correction to the energies (16) is the small shift (31) of $O(\xi^2/\Omega)$. This shift yields corrections to the moments of inertia of the different rotational bands and depends on the quantum numbers (n_0, n_2, l_2) of the bandhead. In particular, the moment of inertia of the β band depends on R while that of the γ band depends on S. Thus, the rotational bands of multi-phonon excitations have rotational constants

$$A_{\rm theo} = \frac{1 - \left(n_0 + \frac{1}{2}\right)R - (2n_2 + |l_2| + 1)S}{2C_0}. \tag{33}$$

In practice it is useful to rewrite this expression as

$$A_{\rm theo} = A_{\rm g.s.} - a_\beta n_0 - a_\gamma(2n_2 + |K|/2). \tag{34}$$

Here, $A_{\rm g.s.}$ is the rotational constant of the ground-state band, and a_β and a_γ denote the small corrections for bands built on multi-phonon excitations. We used the relation (19). As usual, $A_{\rm theo}[I(I+1) - K^2]$ describes the energy levels of rotational bands. Note that the change in the rotational constants is linear in the number of excited phonons. This is one of the main results of this paper. The small correction to the moment of inertia depends on the parameters a_β and a_γ (or R and S) and can be determined by a fit to data. Note that the terms in Eq. (24) proportional to D_1 and F_1 do not affect the spectrum at next-to-next-to leading order because of the zero expectation values of the position φ_0 and velocity $\dot\varphi_0$ of the harmonic oscillator.

TABLE I. Experimental excitation energies E (in keV) and spins K of γ vibrational bandheads in ^{168,166}Er and ^{232}Th. The rotational constants A (in keV) are deduced from the first level spacing of the rotational band. In the theoretical description, the γ vibrational states have quantum numbers $n_0 = 0 = n_2$ and $l_2 = K/2$. The theoretical result $A_{\rm theo}$ (in keV) for the rotational constant is determined by a fit to the $K = 0$ and $K = 2$ bands and is a prediction for the $K = 4$ states.

	^{168}Er			^{166}Er			^{232}Th		
E	0	821	2056	0	786	2028	0	785	1414
K	0	2	4	0	2	4	0	2	4
A	13.17	12.33	11.37	13.43	12.25	10.56	8.23	7.38	7.27
$A_{\rm theo}$	13.17	12.33	11.49	13.43	12.25	11.07	8.23	7.38	6.53

TABLE II. Experimental excitation energies E (in keV) and spins K of tentative γ vibrational bandheads in ^{162}Dy. The rotational constants A (in keV) are deduced from the first level spacing of the rotational band. The theoretical result $A_{\rm theo}$ (in keV) for the rotational constant is determined by a fit to the ground-state band ($K = 0$) and the γ band ($K = 2$). It is a prediction for the tentative $\gamma\gamma$ bands with $K = 4$ ($n_0 = n_2 = 0$ and $l_2 = K/2$) and with $K = 0$ ($n_0 = 0$, $n_2 = 1$, and $l_2 = K/2$).

	^{162}Dy			
E	0	888	1536	1400
K	0	2	4	0
A	13.45	12.34	9.87	8.87
$A_{\rm theo}$	13.45	12.34	11.23	11.23

These terms will affect wave functions at the considered order and spectra at the next higher order.

IV. COMPARISON BETWEEN THEORY AND DATA

Let us confront our predictions with data. The effective theory we derived allows us to describe small deviations in the moment of inertia of the β band and the γ band with $K = 2$ by a fit of R and S, respectively. The theory is thus sufficiently flexible to accommodate the small differences between the observed rotational constants for the ground-state band and the β and γ bands of a deformed nucleus. This overcomes a deficiency of the collective models (see, e.g., Refs. [38,40,42,43,46]). Table I in Ref. [44] shows that a_β is positive for most deformed nuclei. Once the low-energy constants C_0, R, and S (or $A_{\rm g.s.}$, a_β, and a_γ) are determined from the ground-state, the β, and the γ bands, the effective theory predicts that the difference between the rotational constants of multi-phonon vibrations and the ground-state band depends linearly on the number of excited phonons. There are only a few candidates for two-phonon excitations in deformed nuclei (see Refs. [53,54] for a summary of the status of the field in the early 1990s). Due to experimental advances, there is now robust evidence for two-phonon γ vibrational excitations in ^{168}Er [55–57], ^{166}Er [58,59], and ^{232}Th [60,61]. For earlier theoretical discussions on multi-phonon states in ^{168}Er, we refer the reader to Refs. [38,49,62–64]. A microscopic computation of the moments of inertia of γ vibrational bands in erbium isotopes can be found in Ref. [19].

Table I summarizes our results for ^{168,166}Er and ^{232}Th, respectively. The table shows the excitation energy E of the bandhead, its spin K, and the rotational constant A. The latter was determined by computing the first level spacing of the respective rotational bands according to the formula $A[I(I+1) - K^2]$. For each nucleus, the theoretical rotational constants $A_{\rm theo}$ are determined by adjusting the low-energy constants $A_{\rm g.s.}$ and a_γ of Eq. (34) to the rotational constants of the ground-state band and the γ band. This yields $a_\gamma = 0.84$ keV, $a_\gamma = 1.18$ keV, and $a_\gamma = 0.85$ keV for ^{168}Er, ^{166}Er, and ^{232}Th, respectively. These corrections are much smaller (i.e., by about a factor ξ/Ω) than the rotational constants $A_{\rm g.s.} = 13.17$ keV, $A_{\rm g.s.} = 13.43$ keV, and $A_{\rm theo} = 8.23$ keV of the respective ground-state bands. For $K = 4$, $A_{\rm theo}$ is a prediction. These predictions are in good quantitative agreement with data for ^{168}Er and in semiquantitative agreement with the data for ^{166}Er and ^{232}Th. More precisely, for ^{168}Er, the difference between data and theory is about 10% of a_γ and thus consistent with neglected higher-order corrections [which are of $O(\xi/\Omega)$]. For ^{166}Er, the difference between data and theory is about 43% of a_γ. This difference is probably at the limit of what one expects from estimates within the effective theory. For ^{232}Th, the difference between data and theory is about 87% of a_γ and clearly larger than expected. Here, the effective theory only describes correctly the trend that the rotational constants decrease with increasing spin K of the bandhead.

Let us also discuss the nucleus ^{162}Dy and follow Aprahamian *et al.* [2]. This nucleus exhibits a γ vibrational band at an excitation energy of $E = 888$ keV and two candidate $\gamma\gamma$ phonons at energies $E = 1536$ keV (with $K = 4$) and $E = 1400$ keV (with $K = 0$). The candidate $\gamma\gamma$ vibrations have quantum numbers $(n_0 = 0 = n_2, l_2 = K/2 = 2)$ and $(n_0 = 0, n_2 = 1, \quad l_2 = K/2 = 0)$, respectively. As expected for harmonic vibrations, these two levels are approximately degenerate and reasonably close to twice the excitation of the γ bandhead. However, only 12.5% of the $E2$ transition strength from the $K = 4$ bandhead can be associated with a double-phonon vibration. The candidate $\gamma\gamma$ vibration with $K = 0$ exhibits only about 15% of two-phonon strength. Let us nevertheless apply our effective theory to this case and compare experiment and theory for the rotational constants of the tentative $\gamma\gamma$ bands. Table II shows that the effective theory correctly predicts the trend of decreasing rotational constants, while the difference between theory and data is larger than expected on theoretical grounds. As discussed in Ref. [2], the evidence for $\gamma\gamma$ phonons in ^{162}Dy is not strong, and it is thus not too surprising that the effective theory does not quantitatively apply to this case.

Note that—at the considered order in the effective theory—the variation in the rotational constants is not affected by the omission of next-to-next-to-leading order corrections in the potential of the vibrational degrees of freedom (φ_0, φ_2). Those corrections introduce anharmonicities in the vibrational spectrum (i.e., the energies of the bandheads), but they do not influence the moments of inertia. Note also that the effective theory—at the here considered order—yields the rotational bands of the rigid rotor (which are proportional to $I(I+1) - K^2$). At the next higher order, i.e., at order (ξ^3/Ω^2), corrections proportional to $[I(I+1) - K^2]^2$ enter [25].

V. ODD-MASS NUCLEI AT NEXT-TO-LEADING ORDER

Odd-mass nuclei have half-integer spins in their ground states. We want to describe these nuclei in terms of vibrations and rotations alone. The elimination of the odd nucleon as an active degree of freedom leads to an important change in the symmetry properties of the Lagrangian for the rotations and vibrations. Due to the finite ground-state spin, the Lagrangians of odd-mass nuclei are not invariant under time reversal, and terms that are odd under time reversal need to be included in the description. In Ref. [25], the effective theory for the Nambu-Goldstone modes of odd-mass nuclei was considered at leading order. Here, we go one step further and include the vibrational degrees of freedom and consider the effective theory for deformed odd-mass nuclei at next-to-leading order.

Let us start with the vibrational degrees of freedom. The time-odd and rotationally invariant terms $\phi_0 D_t \phi_0$ and $\phi_2 D_t \phi_{-2}$ and its complex conjugate enter as additional building blocks of the Lagrangian. Instead of decomposing ϕ_2 in polar coordinates as in even-even nuclei, we here decompose it in Cartesian coordinates (mostly for its simplicity in gauge transformation, which we will see later):

$$\phi_2 = x + iy. \tag{35}$$

Hence,

$$\begin{aligned} \phi_2 D_t \phi_{-2} &= x\dot{x} + y\dot{y} - i(x\dot{y} - y\dot{x}) + 2iE_z(x^2 + y^2), \\ \phi_0 D_t \phi_0 &= \phi_0 \dot{\phi}_0 = \tfrac{1}{2}\partial_t\left(\phi_0^2\right). \end{aligned} \tag{36}$$

The power counting equation (7) yields the scaling

$$\phi_2 D_t \phi_{-2} \sim \phi_{-2} D_t \phi_2 \sim \phi_0 D_t \phi_0 \sim O(1). \tag{37}$$

All leading-order terms of the Lagrangian of even-even nuclei [Eq. (9)] also enter for odd-mass nuclei. The leading-order Lagrangian for odd-mass nuclei thus becomes

$$\begin{aligned} L_{\mathrm{LO}}^{(\mathrm{odd})} &= (D_t\phi_2)(D_t\phi_{-2}) + \frac{1}{2}\dot{\varphi}_0^2 + \frac{A}{2}\partial_t(\phi_0^2) \\ &\quad + \frac{\tilde{A}}{2}(\phi_2 D_t\phi_{-2} + \phi_{-2} D_t\phi_2) \\ &\quad + \frac{iB}{2}(\phi_2 D_t\phi_{-2} - \phi_{-2} D_t\phi_2). \end{aligned} \tag{38}$$

Here the parameters B, $\widetilde{A}$, and A scale as

$$B \sim \widetilde{A} \sim A \sim \Omega. \tag{39}$$

Note that $\phi_2 D_t \phi_{-2}$ and $\phi_{-2} D_t \phi_2$ are complex conjugate to each other, so they appear as linear combinations to yield real values. The terms proportional to A and $\tilde{A}$ are total time derivatives and can thus be dropped from the Lagrangian. However, it is instructive to keep them for a moment, and we will soon eliminate them by a gauge transformation. We employ Eq. (36) and find in leading order

$$\begin{aligned} L_{\mathrm{LO}}^{(\mathrm{odd})} &= \dot{x}^2 + \dot{y}^2 + \frac{1}{2}\dot{\varphi}_0^2 + B(x\dot{y} - y\dot{x}) \\ &\quad + \frac{A}{2}\partial_t\left(\phi_0^2\right) + \frac{\tilde{A}}{2}\partial_t(x^2 + y^2). \end{aligned} \tag{40}$$

Clearly, the nontrivial part of the Lagrangian describes a particle in three dimensions in a constant magnetic field with strength proportional to B. A Legendre transformation yields the Hamiltonian

$$\begin{aligned} H_{\mathrm{LO}}^{(\mathrm{odd})} &= \tfrac{1}{2}(p_0 - A\phi_0)^2 + \tfrac{1}{4}(p_x - \tilde{A}x + By)^2 \\ &\quad + \tfrac{1}{4}(p_y - \tilde{A}y - Bx)^2. \end{aligned} \tag{41}$$

Let us employ a gauge transformation with the phase function

$$\lambda(x, y, \phi_0) = \frac{\tilde{A}}{2}(x^2 + y^2) + \frac{A}{2}\phi_0^2 \tag{42}$$

and gradient

$$\vec{\nabla}\lambda = (\tilde{A}x, \tilde{A}y, A\phi_0) \tag{43}$$

to gauge away the trivial terms proportional to A and $\tilde{A}$. This yields

$$H_{\mathrm{LO}}^{(\mathrm{odd})} = \tfrac{1}{2}p_0^2 + \tfrac{1}{4}(p_x + By)^2 + \tfrac{1}{4}(p_y - Bx)^2. \tag{44}$$

At leading order, we thus have free motion in the direction of φ_0 and quantized Landau levels in the xy plane.

At next-to-leading order, the Lagrangian is

$$\begin{aligned} L_{\mathrm{NLO}}^{(\mathrm{odd})} &= L_{\mathrm{LO}}^{(\mathrm{odd})} + \frac{C_0}{2}E_+E_- + qE_z \\ &= \frac{1}{2}\dot{\varphi_0}^2 + \dot{x}^2 + \dot{y}^2 + B(x\dot{y} - y\dot{x}) \\ &\quad + \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) \\ &\quad - [q - 4(x\dot{y} - y\dot{x})]\dot{\alpha}\cos\beta. \end{aligned} \tag{45}$$

Here, we have dropped the irrelevant terms proportional to A and $\tilde{A}$ in $L_{\mathrm{LO}}^{\mathrm{odd}}$. We identify again the Lagrangian of a particle on the sphere and note that the term $qE_z = -q\dot{\alpha}\cos\beta$ is technically a Wess-Zumino term. Under rotations, this term remains invariant up to a total derivative, and the parameter q is related to the ground-state spin [25]. The coupling between rotations and vibrations in the Lagrangian (45) stems from the covariant derivative that appears in the leading-order Lagrangian (40), and higher-order terms have been neglected.

Let us discuss the coupling of the nuclear spin to the vibrations and rotations which is due to the time-odd terms in the Lagrangian. The coupling of the ground-state spin to the Euler angles can be viewed as a particle on the sphere coupled to a magnetic monopole with charge $2q$ [65]. Technically, the vibrations couple to the ground-state spin via an effective magnetic field B that is generated by the ground-state spin. Note that our approach takes the spin of the ground state as a static quantity and not as a degree of freedom. This is an approximation that we expect to be valid only for sizeable spins and low energies. At higher energies, or for small ground-state spins, the spin is a dynamical quantity and only the total spin, i.e., the sum of ground-state spin and the spin I associated with the Euler angles, is conserved. Our approach excludes terms such as the "Coriolis coupling" [12] from the Lagrangian, and it is well known that this coupling has an important, i.e., leading-order, contribution for ground states (or bandheads) with spin 1/2 [5].

At this point, we add a leading-order harmonic potential

$$V_{\mathrm{LO}} = \frac{\omega_0^2}{2}\varphi_0^2 \tag{46}$$

in the φ_0 vibrational degree of freedom (and the magnetic field B is the leading-order contribution to the ϕ_2 degrees of freedom), and we perform the Legendre transformation to obtain the Hamiltonian. One finds

$$H_{\rm NLO}^{\rm (odd)} = \frac{1}{2C_0}\left[p_\beta^2 + \frac{1}{\sin^2\beta}[p_\alpha + (q-2l_2)\cos\beta]^2\right] + \frac{1}{4}\left(p_x^2+p_y^2\right) + \frac{B^2}{4}(x^2+y^2) - \frac{B}{2}l_2 + \frac{1}{2}p_0^2 + \frac{\omega_0^2}{2}\varphi_0^2. \quad (47)$$

Note that $l_2 = (xp_y - yp_x)$ is an angular momentum. In the φ_0 degree of freedom we have a harmonic oscillation. Upon quantization, one finds the usual levels of the one-dimensional harmonic oscillator. The $\phi_2 = x + iy$ degrees of freedom correspond to a charged particle moving in a plane perpendicular to a strong magnetic field. This yields Landau levels upon quantization. On top of each of these "vibrational" states, one finds a rotational band due to the Euler angles.

The spectrum of the Hamiltonian for odd-mass nuclei at next-to-leading order thus is

$$E_{\rm NLO}^{\rm (odd)} = \omega_0\left(n_0+\frac{1}{2}\right) + \frac{|B|}{2}(2n_2+|l_2|+1) - \frac{B}{2}l_2 + \frac{1}{2C_0}[I(I+1)-(q-2l_2)^2]. \quad (48)$$

The quantum numbers are $n_0 = 0, 1, 2, \ldots$ for the harmonic oscillation of φ_0, $n_2 = 0, 1, 2, \ldots$, $l_2 = 0, \pm1, \pm2, \ldots$, from the Landau levels, and $I = |q-2l_2|, |q-2l_2|+1, |q-2l_2|+2, \ldots$ for the rotational bands. The eigenfunctions are essentially as in Eq. (18) for the even-even nuclei, but with modification of the indices of the Wigner D function (and again by rewriting $\phi_2 = x + iy = \varphi_2 e^{i\gamma}$).

Thus, the spectrum exhibits a large level density close to the ground state, in qualitative agreement with experimental observations for odd-mass nuclei. The large degeneracy of the lowest Landau level is split by the l_2-dependent shift of the bandhead. Next-to-leading order corrections to the vibrational potential (that we neglected for convenience) would further modify this picture. Note that q must be a positive or negative half integer, and the ground state with spin $|q-2l_2|$ is obtained for the value of l_2 that minimizes $|q-2l_2|$ for fixed q. For negative values of q (and positive values of B), this is achieved for $l_2 = 0$ in the lowest Landau level, and the spin of the ground state is $|q|$. For positive values of q (again under the assumption of positive B), the ground state has spin $1/2$, and l_2 is such that $|q-2l_2| = 1/2$. We repeat that the effective theory derived in this section is not valid for bandheads with spin $1/2$ because the assumption of a static spin is only warranted for sizable spins.

Thus, the effective theory for odd nuclei is quite similar to the effective theory for even-even nuclei. Both theories predict a number of low-lying bandheads that are collective vibrations. The comparison with experimental spectra shows that considerable anharmonicities are required in practice; i.e., next-to-leading order corrections to the vibrational Lagrangian must be significant. Within the effective theory, the higher level density in odd deformed nuclei arises due to magnetic effects and Landau-level physics.

It would of course be interesting to consider the spin as a dynamic degree of freedom and to drive the effective theory for odd-mass nuclei also to next-to-next-to-leading order. However, many more time-odd terms contribute, and many new parameters will appear, and this makes the description of spectra less challenging. Instead, it might be more interesting to couple electromagnetic fields to the effective theory and confront low-order results with the considerable amount of available data.

Note finally that the assumption of a static ground-state spin is probably not valid for odd-odd nuclei due to the weak coupling between the odd proton and neutron. Thus, one cannot simply let q assume integer values and apply the theory derived in this section to odd-odd nuclei.

VI. CONCLUSION

In summary, we computed higher-order corrections in the effective theory for deformed nuclei, and we focused particularly on the kinetic terms that couple rotations and vibrations. In even-even nuclei, the next-to-next-to-leading order corrections yield small corrections to the moments of inertia that are linear in the number of excited phonons. When applied to ^{166,168}Er, the effective theory largely explains the observed variations of the rotational constants of the two-phonon γ vibrations. In ^{232}Th, the theory explains the trend that rotational constants decrease with increasing spin of the bandhead. For odd nuclei, the effective theory at next-to-leading order includes time-odd terms in the Lagrangian. This approach introduces effective magnetic fields into the Hamiltonian and qualitatively explains observed features such as the high level densities.

ACKNOWLEDGMENTS

The authors thank M. Caprio, W. Nazarewicz, and N. Pietralla for discussions. This work has been supported by the US Department of Energy under Grants No. DE-FG02-96ER40963 (University of Tennessee) and No. DE-AC05-00OR22725 with UT-Battelle, LLC (Oak Ridge National Laboratory).

[1] W. F. Davidson *et al.*, J. Phys. G **7**, 455 (1981).
[2] A. Aprahamian *et al.*, Nucl. Phys. A **764**, 42 (2006).
[3] A. Bohr, Dan. Mat. Fys. Medd. **26**(14) (1952).
[4] A. Bohr and B. R. Mottelson, Dan. Mat. Fys. Medd. **27**(16) (1953).
[5] A. Bohr and B. R. Mottelson, *Nuclear Structure*, *Volume II: Nuclear Deformation* (Benjamin, Reading, MA, 1975).
[6] A. Faessler, W. Greiner, and R. K. Sheline, Nucl. Phys. **70**, 33 (1965).

[7] J. M. Eisenberg and W. Greiner, *Nuclear Models: Collective and Single-Particle Phenomena* (North-Holland, London, 1970).
[8] G. Gneuss and W. Greiner, Nucl. Phys. A **171**, 449 (1971).
[9] P. Hess, M. Seiwert, J. Maruhn, and W. Greiner, Z. Phys. A **296**, 147 (1980).
[10] A. Arima and F. Iachello, Phys. Rev. Lett. **35**, 1069 (1975).
[11] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, UK, 1987).
[12] A. K. Kerman, Dan. Mat. Fys. Medd. **30**(15) (1956).
[13] F. Iachello and O. Scholten, Phys. Rev. Lett. **43**, 679 (1979).
[14] S. Åberg, H. Flocard, and W. Nazarewicz, Annu. Rev. Nucl. Part. Sci. **40**, 439 (1990).
[15] S. Frauendorf, Rev. Mod. Phys. **73**, 463 (2001).
[16] C. Vargas, J. G. Hirsch, T. Beuschel, and J. P. Draayer, Phys. Rev. C **61**, 031301 (2000).
[17] E. C. Halbert and W. Nazarewicz, Phys. Rev. C **48**, R2158 (1993).
[18] G. Dracoulis, F. Kondev, and P. Walker, Phys. Lett. B **419**, 7 (1998).
[19] J. A. Sheikh, G. H. Bhat, Y. Sun, G. B. Vakil, and R. Palit, Phys. Rev. C **77**, 034313 (2008).
[20] Z. Zhang, X. Wu, Y. Lei, and J. Zeng, Nucl. Phys. A **816**, 19 (2009).
[21] X. Wu, Z. H. Zhang, J. Y. Zeng, and Y. A. Lei, Phys. Rev. C **83**, 034323 (2011).
[22] D. J. Rowe, Nucl. Phys. A **735**, 372 (2004).
[23] D. J. Rowe, T. A. Welsh, and M. A. Caprio, Phys. Rev. C **79**, 054304 (2009).
[24] K. Matsuyanagi, M. Matsuo, T. Nakatsukasa, N. Hinohara, and K. Sato, J. Phys. G **37**, 064018 (2010).
[25] T. Papenbrock, Nucl. Phys. A **852**, 36 (2011).
[26] S. Weinberg, Phys. Lett. B **251**, 288 (1990).
[27] S. Weinberg, Nucl. Phys. B **363**, 3 (1991).
[28] U. van Kolck, Phys. Rev. C **49**, 2932 (1994).
[29] P. F. Bedaque and U. van Kolck, Annu. Rev. Nucl. Part. Sci. **52**, 339 (2002).
[30] E. Epelbaum, Prog. Part. Nucl. Phys. **57**, 654 (2006).
[31] R. Machleidt and D. Entem, Phys. Rep. **503**, 1 (2011).
[32] C. Bertulani, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A **712**, 37 (2002).
[33] R. Higa, H.-W. Hammer, and U. van Kolck, Nucl. Phys. A **809**, 171 (2008).
[34] M. Marini, F. Pistolesi, and G. Strinati, Eur. Phys. J. B **1**, 151 (1998).
[35] T. Papenbrock and G. F. Bertsch, Phys. Rev. C **59**, 2052 (1999).
[36] H.-W. Hammer and R. Furnstahl, Nucl. Phys. A **678**, 277 (2000).
[37] R. Furnstahl, H.-W. Hammer, and S. Puglia, Ann. Phys. (NY) **322**, 2703 (2007).
[38] D. D. Warner, R. F. Casten, and W. F. Davidson, Phys. Rev. C **24**, 1713 (1981).
[39] E. Grosse *et al*., Phys. Scr. **24**, 337 (1981).
[40] S. Kuyucak and I. Morrison, Phys. Rev. C **38**, 2482 (1988).
[41] M. A. Caprio, Phys. Rev. C **72**, 054323 (2005).
[42] M. Caprio, Phys. Lett. B **672**, 396 (2009).
[43] M. A. Caprio, Phys. Rev. C **83**, 064309 (2011).
[44] K. Heyde and J. L. Wood, Rev. Mod. Phys. **83**, 1467 (2011).
[45] F. Iachello, Phys. Rev. Lett. **87**, 052502 (2001).
[46] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **87**, 052503 (2001).
[47] N. Pietralla and O. M. Gorbachenko, Phys. Rev. C **70**, 011304 (2004).
[48] S. Weinberg, *The Quantum Theory of Fields*, Vol. II (Cambridge University Press, Cambridge, UK, 1996).
[49] A. Bohr and B. R. Mottelson, Phys. Scr. **25**, 28 (1982).
[50] R. Fukuda, Phys. Rev. Lett. **61**, 1549 (1988).
[51] T. Inagaki and R. Fukuda, Phys. Rev. B **46**, 10931 (1992).
[52] R. Fukuda, T. Kotani, Y. Suzuki, and S. Yokojima, Prog. Theor. Phys. **92**, 833 (1994).
[53] P. Sood, D. Headly, and R. Sheline, At. Data Nucl. Data Tables **47**, 89 (1991).
[54] P. Sood, D. Headly, and R. Sheline, At. Data Nucl. Data Tables **51**, 273 (1992).
[55] H. G. Börner, J. Jolie, S. J. Robinson, B. Krusche, R. Piepenbring, R. F. Casten, A. Aprahamian, and J. P. Draayer, Phys. Rev. Lett. **66**, 691 (1991).
[56] M. Oshima, T. Morikawa, Y. Hatsukawa, S. Ichikawa, N. Shinohara, M. Matsuo, H. Kusakari, N. Kobayashi, M. Sugawara, and T. Inamura, Phys. Rev. C **52**, 3492 (1995).
[57] T. Härtlein, M. Heinebrodt, D. Schwalm, and C. Fahlander, Eur. Phys. J. A **2**, 253 (1998).
[58] C. Fahlander, A. Axelsson, M. Heinebrodt, T. Härtlein, and D. Schwalm, Phys. Lett. B **388**, 475 (1996).
[59] P. E. Garrett, M. Kadi, M. Li, C. A. McGrath, V. Sorokin, M. Yeh, and S. W. Yates, Phys. Rev. Lett. **78**, 4545 (1997).
[60] W. Korten, T. Härtlein, J. Gerl, D. Habs, and D. Schwalm, Phys. Lett. B **317**, 19 (1993).
[61] A. Martin, P. E. Garrett, M. Kadi, N. Warr, M. T. McEllistrem, and S. W. Yates, Phys. Rev. C **62**, 067302 (2000).
[62] M. Matsuo, Prog. Theor. Phys. **72**, 666 (1984).
[63] M. Matsuo and K. Matsuyanagi, Prog. Theor. Phys. **74**, 1227 (1985).
[64] R. Piepenbring and M. Jammari, Nucl. Phys. A **481**, 81 (1988).
[65] T. T. Wu and C. N. Yang, Nucl. Phys. B **107**, 365 (1976).